Topological Hochschild homology of topological modular forms

Bob Bruner[1] John Rognes[2]

[1]Department of Mathematics
Wayne State University

[2]Department of Mathematics
University of Oslo

Nordic Topology Meeting, NTNU, November 2008

Outline

Outline

1. Hyperelliptic Cohomology
2. $K(n)$-Local $THH(tmf)$
3. Homology of $THH(tmf)$
4. Homotopy of $THH(tmf)$

Chromatic Red-Shift

Algebraic K-theory often increases chromatic complexity by one.

- Algebraic K-theory of a finite field is a form of integral cohomology.
- Algebraic K-theory of the integers is a form of topological K-theory.
- Algebraic K-theory of topological K-theory is a form of elliptic cohomology.

We study algebraic K-theory of elliptic cohomology, $K(tmf)$, expecting to find a form of a v_3-periodic cohomology theory, tentatively called hyperelliptic cohomology.

Periodic Families

With increasing chromatic complexity, more of the stable homotopy groups of spheres is detected.

- Rational cohomology detects the 0-stem $\pi_0(S)$.
- Topological K-theory detects the image-of-J summand in $\pi_*(S)$. This includes all classes in dimensions $* \le 5$.
- Elliptic cohomology detects the v_2-periodic families in $\pi_*(S)$. For $p = 2$, this includes all classes in dimensions $* \le 30$.

With $K(tmf)$ we may hope to show that $\eta\theta_4$ in the 31-stem, or certain classes in the 39- to 41-stems, are part of v_3-periodic families. No such periodic family is presently known for $p = 2$.

Trace Invariants of Algebraic K-Theory

- We study the algebraic K-theory of an S-algebra B by the Bökstedt–Hsiang–Madsen trace maps

$$tr\colon K(B) \xrightarrow{trc} TC(B;p) \longrightarrow THH(B)\,.$$

- The right hand map factors through the S^1-homotopy fixed points

$$THH(B)^{hS^1} = F(S^\infty_+, THH(B))^{S^1}$$

and the approximate S^1-homotopy fixed points

$$THH(B)^{aS^1} = F(S^3_+, THH(B))^{S^1}\,.$$

σ-operator

- The cyclic structure on $THH(B)$ gives a circle action

$$S^1_+ \wedge THH(B) \to THH(B)\,.$$

- The σ-operator

$$\sigma \colon H_* THH(B) \to H_{*+1} THH(B)$$

is induced by circle action and the fundamental class in $H_1(S^1_+)$.

Summary of Results

Let $p = 2$ and $B = tmf$, the topological modular forms spectrum. We can:

- Compute the Morava $K(n)$-localizations $L_{K(n)} THH(tmf)$ for $0 \leq n \leq 2$.
- Describe $H_* THH(tmf)$ as an A_*-comodule algebra.
- Give a (quite complete) calculation of $\pi_* THH(tmf)$.

To pass from homological to homotopical calculations, we use the Adams spectral sequence.

Plans for Further Work

Jointly with Sverre Lunøe-Nielsen we plan to:

- Determine $H_* THH(tmf)^{aS^1}$ as an A_*-comodule algebra
- Compute $\pi_* THH(tmf)^{aS^1}$ (in a range)
- Use this to detect potential v_3-periodic classes in $\pi_*(S)$

Similar results for $THH(tmf)^{hS^1}$ or the S^1-Tate construction $THH(tmf)^{tS^1}$ would establish v_3-periodicity.

Outline

Rational $THH(tmf)$

In rational (= $K(0)$-local) homotopy

$$\pi_*(tmf) \otimes \mathbb{Q} = \mathbb{Q}[c_4, c_6]$$

equals elliptic modular forms, with $|c_i| = 2i$.

Theorem

$\pi_* THH(tmf) \otimes \mathbb{Q}$ is an exterior algebra over $\pi_*(tmf) \otimes \mathbb{Q}$ on two algebra generators σc_4 and σc_6 in dimensions 9 and 13.

$K(1)$-Local $THH(tmf)$

By Hopkins and Laures, the KO_*-algebra unit map for *tmf* factors

$$KO_* \longrightarrow KO_*[x] \xrightarrow{f} KO_* tmf$$

where f is étale.

Theorem

$\pi_* L_{K(1)} THH(tmf)$ is an exterior algebra over $\pi_* L_{K(1)} tmf = KO_*[j^{-1}]$ on one generator σf in dimension 1.

$K(2)$-Local $THH(tmf)$

By the Morava change-of-rings theorem the Hopkins–Miller spectrum $L_{K(2)} tmf = EO_2$ is a pro-étale extension of $L_{K(2)} S$.

Theorem

$\pi_* L_{K(2)} THH(tmf)$ is isomorphic to $\pi_* L_{K(2)} tmf = \pi_* EO_2$.

Chromatic Assembly Problem

$THH(tmf)$ is

- $K(0)$-locally like four $= 2^2$ copies of tmf,
- $K(1)$-locally like two $= 2^1$ copies of tmf, and
- $K(2)$-locally like one $= 2^0$ copy of tmf.

What is the global picture?

Outline

The Steenrod Algebra

- Let $A = \langle Sq^i \mid i \geq 1 \rangle$ be the mod 2 Steenrod algebra and let

$$A_* = P(\bar{\xi}_k \mid k \geq 1)$$

be the dual Steenrod algebra, with $|\bar{\xi}_k| = 2^k - 1$.

- The coproduct on A_* is given by

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{2^i}\,.$$

Homology of *tmf*

- In cohomology

$$H^*(tmf) = A \otimes_{A(2)} \mathbb{F}_2$$

  where $A(2) = \langle Sq^1, Sq^2, Sq^4 \rangle$.

- In homology

$$H_*(tmf) = P(\bar{\xi}_1^8, \bar{\xi}_2^4, \bar{\xi}_3^2, \bar{\xi}_k \mid k \geq 4)$$

  is an A_*-comodule subalgebra of A_*.

The Bökstedt Spectral Sequence

The Bökstedt spectral sequence

$$E^2_{**} = HH_*(H_*(tmf)) \Longrightarrow H_*(THH(tmf))$$

collapses at

$$E^2_{**} = H_*(tmf) \otimes E(\sigma\bar{\xi}_1^8, \sigma\bar{\xi}_2^4, \sigma\bar{\xi}_3^2, \sigma\bar{\xi}_k \mid k \geq 4)$$

since the algebra generators are in filtration ≤ 1.

The Homology of $THH(tmf)$

The multiplicative extensions are determined by the Dyer–Lashof operations.

Theorem

$$H_* THH(tmf) = H_*(tmf) \otimes E^3 P_*$$

as an A_*-comodule algebra, where

$$E^3 P_* = E(\sigma\bar{\xi}_1^8, \sigma\bar{\xi}_2^4, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4)\,.$$

The Adams Spectral Sequence for $\pi_* THH(tmf)$, I

- The E_2-term of the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^* THH(tmf), \mathbb{F}_2) \Longrightarrow \pi_{t-s} THH(tmf)_2^\wedge$$

can, by change-of-rings, be rewritten as

$$E_2^{**} = \mathrm{Ext}_{A(2)}^{**}(E^3 P^*, \mathbb{F}_2) = \mathrm{Ext}_{A(2)_*}^{**}(\mathbb{F}_2, E^3 P_*) \,.$$

- We must understand $E^3 P_*$ as an $A(2)_*$-comodule.

A_*-Coaction

The A_*-coaction is generated by

$$\begin{aligned}
\sigma\bar{\xi}_1^8 &\mapsto 1 \otimes \sigma\bar{\xi}_1^8 \\
\sigma\bar{\xi}_2^4 &\mapsto 1 \otimes \sigma\bar{\xi}_2^4 + \bar{\xi}_1^4 \otimes \sigma\bar{\xi}_1^8 \\
\sigma\bar{\xi}_3^2 &\mapsto 1 \otimes \sigma\bar{\xi}_3^2 + \bar{\xi}_1^2 \otimes \sigma\bar{\xi}_2^4 + \bar{\xi}_2^2 \otimes \sigma\bar{\xi}_1^8 \\
\sigma\bar{\xi}_4 &\mapsto 1 \otimes \sigma\bar{\xi}_4 + \bar{\xi}_1 \otimes \sigma\bar{\xi}_3^2 + \bar{\xi}_2 \otimes \sigma\bar{\xi}_2^4 + \bar{\xi}_3 \otimes \sigma\bar{\xi}_1^8
\end{aligned}$$

so the square $(\sigma\bar{\xi}_4)^2$ in dimension 32 is A_*-comodule primitive.

A_*-Comodule Decomposition of E^3P_*

Definition

$$L[1]_* = \mathbb{F}_2\{\sigma\bar{\xi}_1^8, \sigma\bar{\xi}_2^4, \sigma\bar{\xi}_3^2, \sigma\bar{\xi}_4\}$$

with exterior powers the layer comodules

$$L[j]_* = \Lambda^j L[1]_* \qquad \text{for } 0 \leq j \leq 4.$$

Lemma

$$E^3P_* = (L[0]_* \oplus \cdots \oplus L[4]_*) \otimes P((\sigma\bar{\xi}_4)^2)$$

is the direct sum of the terms $\Sigma^{32i}L[j]_*$ *for* $i \geq 0$, $0 \leq j \leq 4$.

The A_*-Comodules $L[j]_*$, I

The bottom and top exterior powers

$$L[0]_* = \mathbb{F}_2\{1\} \qquad 0$$

$$L[4]_* = \mathbb{F}_2\{\sigma\bar{\xi}_1^8\ \sigma\bar{\xi}_2^4\ \sigma\bar{\xi}_3^2\ \sigma\bar{\xi}_4\} \qquad 53$$

are concentrated in dimensions 0 and 53.

The A_*-Comodules $L[j]_*$, II

The generating comodule

$$\begin{tikzcd}
L[1]_* & 9 \arrow[r, bend left, "Sq^4"] & 13 \arrow[r, bend left, "Sq^2"] & 15 \arrow[r, "Sq^1"] & 16
\end{tikzcd}$$

is dual to the third exterior power

$$\begin{tikzcd}
L[3]_* & 37 \arrow[r, "Sq^1"] & 38 \arrow[r, bend left, "Sq^2"] & 40 \arrow[r, bend left, "Sq^4"] & 44
\end{tikzcd}$$

The A_*-Comodules $L[j]_*$, III

The middle exterior power $L[2]_*$

$$\begin{tikzcd}
22 \arrow[r, bend right, "Sq^2"'] & 24 \arrow[r, "Sq^1"'] \arrow[rr, bend left, "Sq^4"] & 25 \arrow[rr, bend right, "Sq^4"'] & 28 \arrow[r, "Sq^1"] & 29 \arrow[r, bend left, "Sq^2"] & 31
\end{tikzcd}$$

is self-dual.

A Realization Lemma

Lemma

For each $0 \leq j \leq 4$ there exists a finite CW spectrum $L[j]$ with

$$H_* L[j] = L[j]_*$$

as A_*-comodules. This determines $L[j]$ uniquely up to 2-adic equivalence.

A Linear Ordering

Each A_*-comodule $\Sigma^{32i} L[j]_*$ in the sum decomposition of

$$E^3 P_* = E(\sigma\bar{\xi}_1^8, \sigma\bar{\xi}_2^4, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4)$$

has a unique A_*-comodule primitive.
We linearly order the summands according to the dimension of this primitive:

$$L[0]_* \,,\; L[1]_* \,,\; L[2]_* \,,\; \Sigma^{32} L[0]_* \,,\; L[3]_* \,,\; \Sigma^{32} L[1]_* \,,\; L[4]_* \,,\; \Sigma^{32} L[2]_* \,,\; \Sigma^{64} L[0]_* \,,\; \dots$$

A *tmf*-Module Filtration

Lemma

There is a filtration of tmf-module spectra

$$tmf = T^0 \to \cdots \to T^{k-1} \to T^k \to \cdots \to THH(tmf)$$

with homotopy cofiber sequences

$$T^{k-1} \to T^k \to tmf \wedge \Sigma^{32i} L[j]$$

such that

$$H_* T^k = \bigoplus H_*(tmf) \otimes \Sigma^{32i} L[j]_*$$

is the sum of terms 0 *through* k *in the linear ordering.*

Comment on Proof

This is approximately the *tmf*-module filtration generated by a skeleton filtration.

When the E^3P_*-summands overlap, as for $L[3]_*$ and $\Sigma^{32}L[1]_*$, the proof is incomplete, due to a possible attachment of a cell of the "lower" piece to a cell of the "higher" piece by η^2. This first plays a role in dimension 44, and can probably be resolved by the $K(1)$-local calculation.

Outline

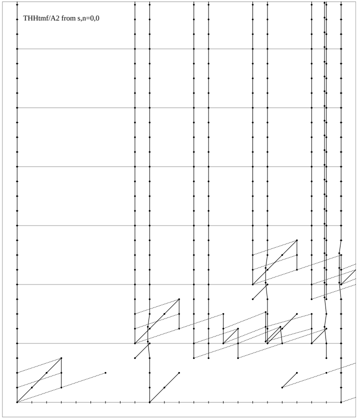
The Adams Spectral Sequence for $\pi_* THH(tmf)$, II

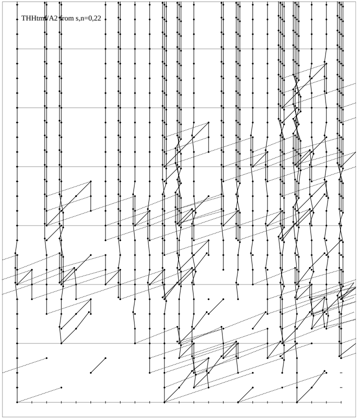
- The Adams spectral sequence E_2-term

$$\begin{aligned} E_2^{**} &= \mathrm{Ext}_A^{**}(H^* THH(tmf), \mathbb{F}_2) \\ &= \mathrm{Ext}_{A(2)}^{**}(E^3 P^*, \mathbb{F}_2) \Longrightarrow \pi_* THH(tmf)_2^\wedge \end{aligned}$$

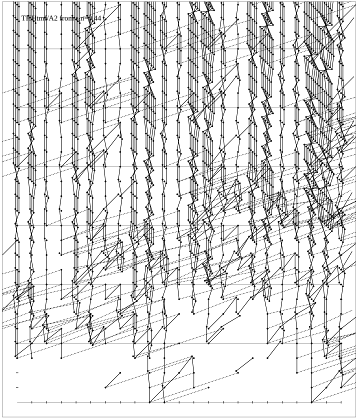
  is machine computable using Bruner's `ext`-program.
- It gets crowded after the 30-stem.

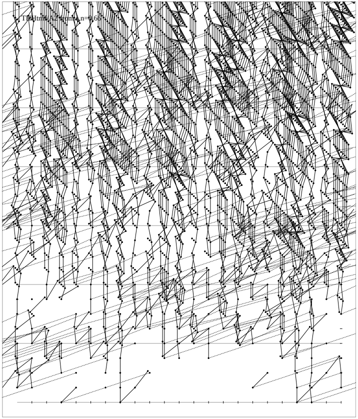
Adams Chart for $\pi_* THH(tmf)$, $0 \leq * \leq 44$





Adams Chart for $\pi_* THH(tmf)$, $44 \leq * \leq 88$





Plan for the Calculation of $\pi_* THH(tmf)$

To clarify we use the *tmf*-module filtration:

$$\begin{tikzcd} tmf \arrow[r] & \cdots \arrow[r] & T^{k-1} \arrow[r] & T^k \arrow[r] \arrow[d] & \cdots \arrow[r] & THH(tmf) \\ & & & tmf \wedge \Sigma^{32i} L[j] & & \end{tikzcd}$$

- First calculate homotopy $tmf_*(\Sigma^{32} L[j])$ of the filtration quotients, for $0 \leq j \leq 4$.
- Then assemble homotopy $\pi_*(T^k)$ of filtration stages, for $k \geq 0$.

Adams Spectral Sequence for Filtration Layers

- The Adams spectral sequence for the (i, j)-th layer

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_A^{s,t}(H^*(tmf \wedge \Sigma^{32i} L[j]), \mathbb{F}_2) \\ &= \mathrm{Ext}_{A(2)}^{s,t}(\Sigma^{32i} L[j]^*, \mathbb{F}_2) \Longrightarrow (tmf)_{t-s}(\Sigma^{32i} L[j]) \end{aligned}$$

  is practically independent of i.
- Reduces to the five cases $0 \le j \le 4$.

Adams E_2-Term for $\pi_* tmf$

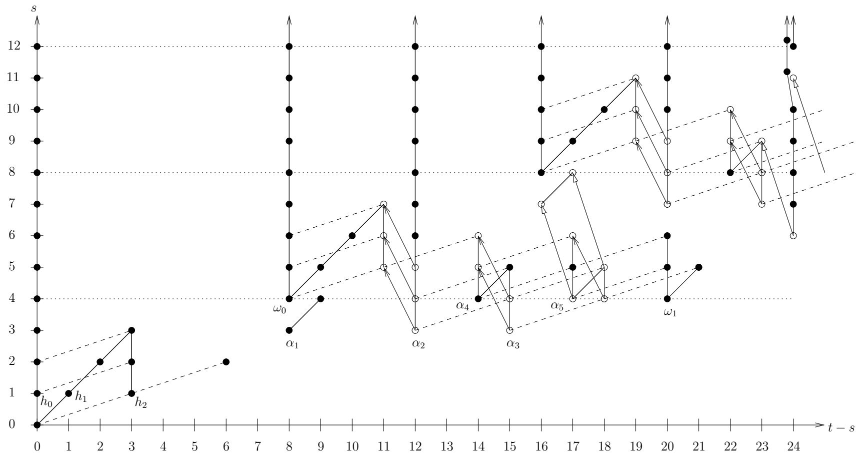
For $j = 0$, $L[0] = S$ and we are computing $\pi_* tmf$.
The Adams E_2-term

$$E_2^{s,t} = \mathrm{Ext}_{A(2)}^{**}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(tmf)_2^\wedge$$

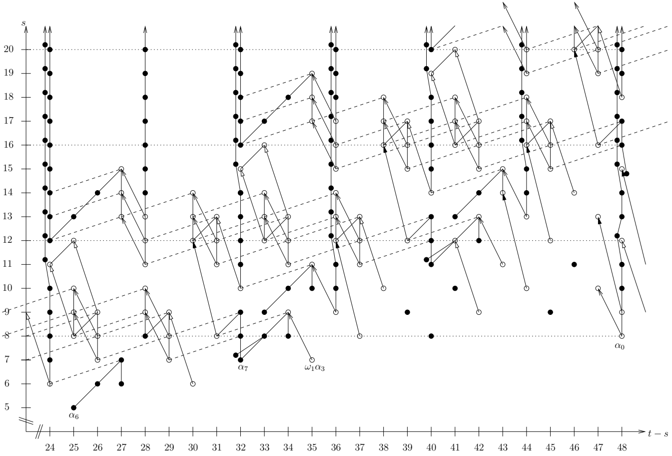
was computed by Iwai–Shimada. It has algebra generators:

- h_0, h_1, h_2
- $\alpha_0 = v_2^8, \alpha_1, \alpha_2, \ldots, \alpha_6, \alpha_7$
- $\omega_0 = v_1^4, \omega_1$

Adams Chart for $\pi_* tmf$, $0 \leq * \leq 24$



Adams Chart for $\pi_* tmf$, $24 \leq * \leq 48$



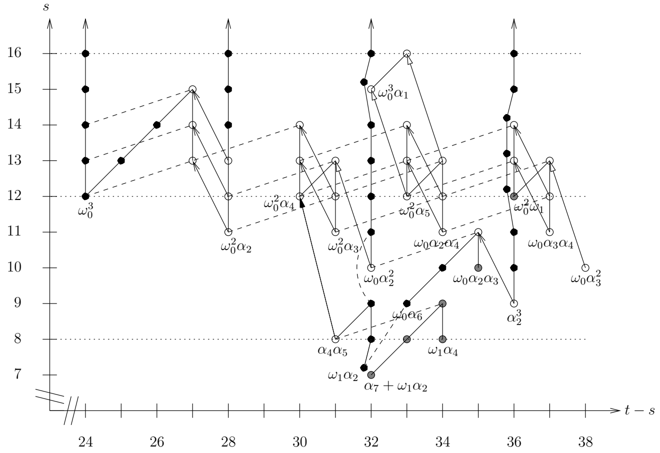
Adams Differentials for $\pi_* tmf$

Hopkins–Mahowald computed these Adams differentials. Permanent cycles are black. Dead classes are white.
To describe the differentials, write the E_2-term as the sum of two pieces:

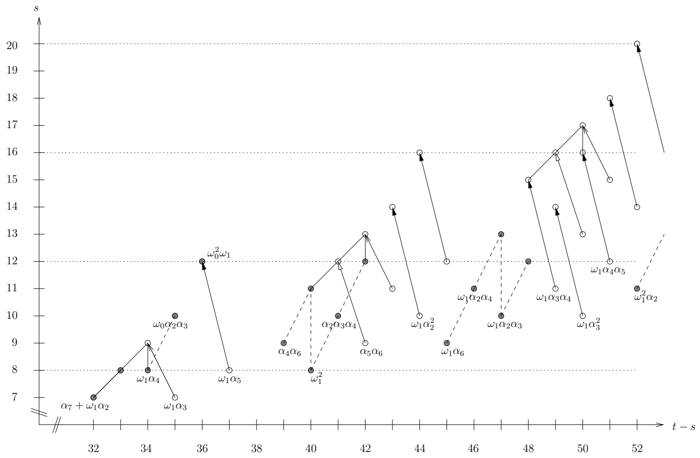
- The Bott periodic part: free over $P(\omega_0, \alpha_0) = P(v_1^4, v_2^8)$.
- The Mahowald–Tangora wedge: free of rank one over $P(v_1, w, \alpha_0)$ on $\omega_1\alpha_3$ in dimension 35.

The first piece comes in several stages: Infantile, Puerile, Juvenile, Virile, Senile.

Adams Chart for *tmf* — The Bott Periodic Part



Adams Chart for *tmf* — The Wedge Part



Adams Spectral Sequence for *tmf* — Summary

- The Adams E_2-term for *tmf* is completely known, including cup and Massey products, by machine computation.
- The Adams differentials are completely known, using E_∞ structure and/or the Adams–Novikov spectral sequence.
- The additive extensions of $\pi_*(tmf)$ are completely known, using Massey products and Moss' theorem.

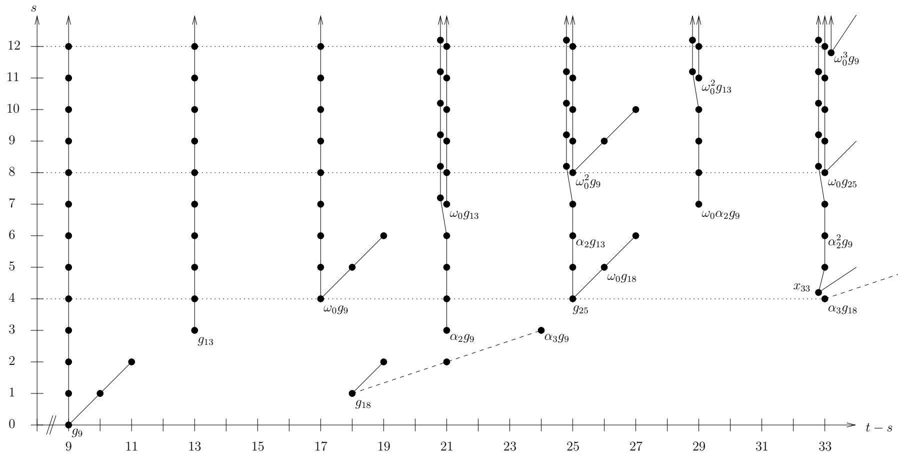
Adams E_2-Term for $tmf_*(L[1])$

- For $j = 1$, $L[1] = S^9 \cup_\nu e^{13} \cup_\eta e^{15} \cup_2 e^{16}$.
- The Adams E_2-term

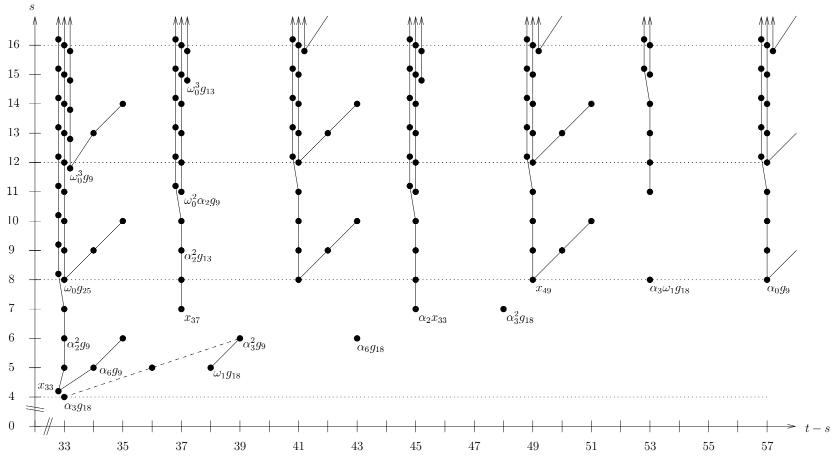
$$E_2^{s,t} = \mathrm{Ext}^{**}_{A(2)}(L[1]^*, \mathbb{F}_2) \Longrightarrow tmf_*(L[1])^\wedge_2$$

  was computed by Davis–Mahowald.
- This spectral sequence is a module over the *tmf* spectral sequence.
- We write g_n (or x_n) for a module generator in dimension $n = t - s$.

Adams Chart for $tmf_*(L[1])$, $9 \leq * \leq 33$



Adams Chart for $tmf_*(L[1])$, $33 \leq * \leq 57$



Adams Differentials for $tmf_*(L[1]$

- This spectral sequence is quite sparse.
- The first nonzero differential is

$$d_3(\alpha_0^2 g_{18}) = \omega_1^4 \alpha_3 g_{18}$$

landing in dimension $t - s = 113$.
- This is well beyond the initial range of interest.

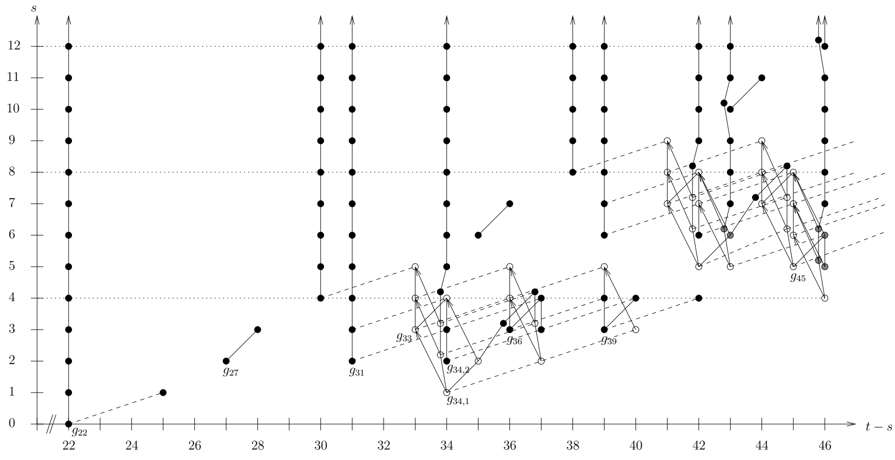
Adams E_2-Term for $tmf_*(L[2])$

- For $j = 2$, $L[2]$ is a self-dual 6-cell CW spectrum.
- The Adams E_2-term

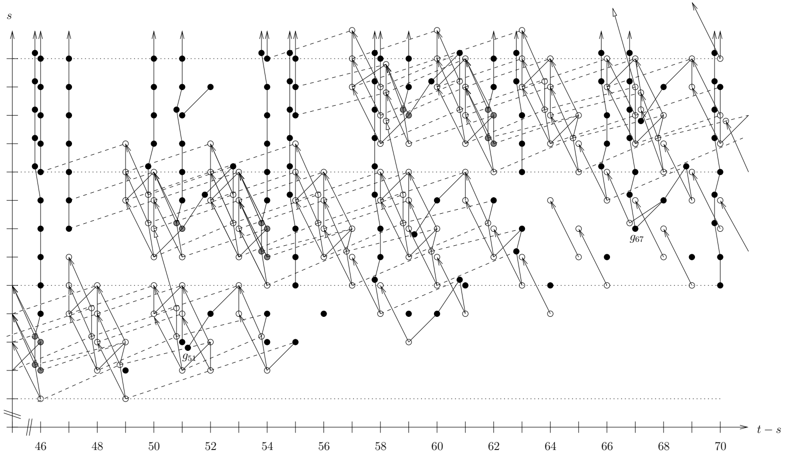
$$E_2^{s,t} = \mathrm{Ext}^{**}_{A(2)}(L[2]^*, \mathbb{F}_2) \Longrightarrow tmf_*(L[2])^\wedge_2$$

  is machine computable.
- This spectral sequence is a module over the *tmf* spectral sequence.
- We write g_n for a module generator in dimension $n = t - s$. The two generators in dimension 34 are called $g_{34,1}$ and $g_{34,2}$.

Adams Chart for $tmf_*(L[2])$, $22 \leq * \leq 46$



Adams Chart for $tmf_*(L[2])$, $46 \leq * \leq 70$

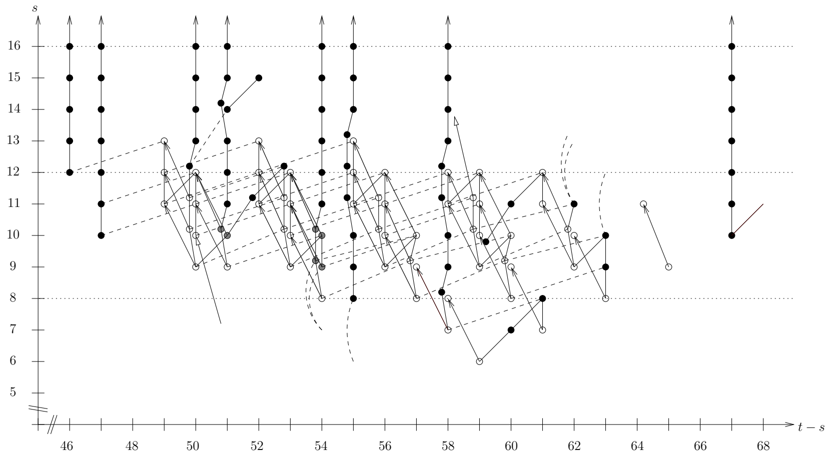


Adams Differentials for $tmf_*(L[2])$

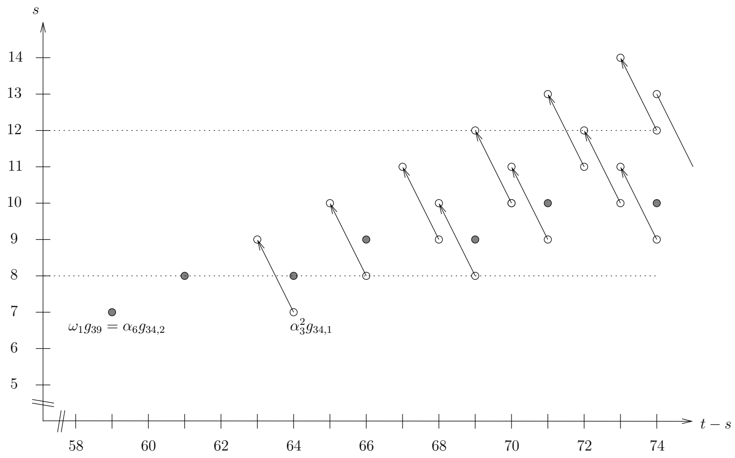
We have computed the Adams differentials for $tmf_*(L[2])$.
To describe the differentials, write the E_2-term as the sum of two pieces:

- A Bott periodic part, which is free over $P(\omega_0, \alpha_0)$.
- A double Mahowald–Tangora wedge, which is free of rank two over $P(v_1, w, \alpha_0)$ on $\omega_1 g_{39}$ and $\alpha_3^2 g_{34,1}$ in dimensions 59 and 64.

Adams Chart for $tmf_*(L[2])$ — The Bott Periodic Part



Adams Chart for $tmf_*(L[2])$ — The Wedge Part



Adams Spectral Sequence for $tmf_*(L[2])$ — Summary

- The Adams E_2-term for $tmf_*(L[2])$ is completely known, including cup and Massey products.
- The Adams differentials are completely known, using rational information and the tmf-module structure.
- The additive extensions of $tmf_*(L[2])$ are (almost) completely known.

Adams spectral sequences for $tmf_*(L[3])$, $tmf_*(L[4])$

- For $j = 3$, with
$$L[3] = S^{37} \cup_2 e^{38} \cup_\eta e^{40} \cup_\nu e^{44}$$
the Adams spectral sequence for $tmf_*(L[3])$ is sparse like the one for $tmf_*(L[1])$.
- For $j = 4$, with $L[4] = S^{53}$ the Adams spectral sequence for $tmf_*(L[4])$ is a shifted copy of the one for $\pi_*(tmf)$.

Assembling the Layers

- The zeroth layer $T^0 = tmf$ splits off from $THH(tmf)$.
- The second layer $tmf \wedge L[2]$ is nontrivially attached to the first layer $tmf \wedge L[1]$:

Theorem

There is a differential

$$d_2(g_{22}) = h_2 g_{18}$$

in the Adams spectral sequence for $\pi_* THH(tmf)$.

The String Orientation

The proof uses the string orientation

$$M\,\mathrm{String} = MO\langle 8\rangle \to tmf$$

and the induced map

$$tmf \wedge BBO\langle 8\rangle_+ = THH(MO\langle 8\rangle, tmf) \to THH(tmf)$$

to prove that $g_9^2 = g_{18}$ in $\pi_* THH(tmf)$.
From $h_2 g_9 = 0$ it follows that $h_2 g_{18}$ is a boundary.

The NeXT Step

The third layer $tmf \wedge S^{32}$ is nontrivially attached to the second layer:

Lemma

There is a nonzero differential

$$d_{k+2}(g_{32}) = h_0^k g_{31}$$

for some $k \geq 0$.

May need Pontryagin power operations in tmf-homology to determine k.

Christian Nassau's Big Ext Chart