RING COMPLETION OF RIG CATEGORIES

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Abstract. We offer a solution to the long-standing problem of group completing within the context of rig categories (also known as bimonoidal categories). Given a rig category $\mathcal{R}$ we construct a natural additive group completion $\bar{\mathcal{R}}$ that retains the multiplicative structure, hence has become a ring category. If we start with a commutative rig category $\mathcal{R}$ (also known as a symmetric bimonoidal category) the additive group completion $\bar{\mathcal{R}}$ will be a commutative ring category. In an accompanying paper [5] we show how to use this construction to prove the conjecture from [3] that the algebraic K-theory of the connective topological K-theory ring spectrum ku is equivalent to the algebraic K-theory of the rig category $\mathcal{V}$ of complex vector spaces.

1. Introduction and main result

Multiplicative structure in algebraic K-theory is a delicate matter. In 1980 Thomason [17] demonstrated that, after additive group completion, the most obvious approaches to multiplicative pairings cease to make sense. For instance, let us write $(-\mathcal{M})\mathcal{M}$ for the Grayson–Quillen [8] model for the algebraic K-theory of a symmetric monoidal category $\mathcal{M}$, written additively. An object in $(-\mathcal{M})\mathcal{M}$ is a pair (a, b) of objects of $\mathcal{M}$, thought of as representing the difference "$a - b$". The naïve guess for how to multiply elements is then dictated by the rule that $(a - b)(c - d) = (ac + bd) - (ad + bc)$. This, however, does not lead to a decent multiplicative structure: the resulting product is in most situations not functorial.

Several ways around this problem have been developed, but they all involve first passing to spectra or infinite loop spaces by one of the equivalent group completion machines, for instance the functor Spt from symmetric monoidal categories to spectra defined in [18, Appendix]. The original problem has remained unanswered: can one additively group complete and simultaneously keep the multiplicative structure, within the context of symmetric monoidal categories?

We answer this question affirmatively. Our motivation came from an outline of proof in [3] of the conjecture that 2-vector bundles give rise to a geometric cohomology theory of the same sort as elliptic cohomology, or more precisely, to the algebraic K-theory of connective topological K-theory, which by work of Ausoni and the fourth author [1], [2] is a spectrum of telescopic complexity 2. The solution of the ring completion problem given here enters as a step in our proof in [5] of that conjecture. For this application the alternatives provided in spectra were insufficient.

Before stating our main result, let us fix some terminology.

Definition 1.1. Let $|\mathcal{C}|$ denote the classifying space of a small category $\mathcal{C}$, that is, the geometric realization of its nerve $N\mathcal{C}$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ will be called an *unstable equivalence* if it induces a homotopy equivalence of classifying spaces $|F|\colon |\mathcal{C}| \to |\mathcal{D}|$, and will usually be denoted $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$. A lax symmetric monoidal functor $F\colon \mathcal{M} \to \mathcal{N}$ of symmetric monoidal categories, with or without zeros, is a *stable equivalence* if it induces a stable equivalence of spectra $\operatorname{Spt} F\colon \operatorname{Spt}\mathcal{M} \to \operatorname{Spt}\mathcal{N}$.

We note that the functor Spt takes values in connective spectra, and any lax symmetric monoidal functor whose underlying functor is an unstable equivalence is a stable equivalence.

Unstable equivalences are often called homotopy equivalences, or weak equivalences. We use "unstable" to emphasize the contrast with stable equivalences. These definitions readily extend to simplicial categories and functors between them.

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By a rig (resp. commutative rig) we mean a ring (resp. commutative ring) in the algebraic sense, except that negative elements are not assumed to exist. By a *rig category* (resp. *commutative rig category*), also known as a bimonoidal category (resp. symmetric bimonoidal category), we mean a category $\mathcal{R}$ with two binary operations $\oplus$ and $\otimes$, satisfying the axioms of a rig (resp. commutative rig) up to coherent natural isomorphisms. By a *bipermutative category* (resp. a *strictly bimonoidal category*) we mean a commutative rig category (resp. a rig category) where as many of the coherence isomorphisms as one can reasonably demand are identities. See Definitions 2.1 and 2.4 below for precise lists of axioms.

A morphism of simplicial rig categories is an unstable equivalence (resp. stable equivalence) if the underlying map of simplicial symmetric monoidal categories is.

By a *(simplicial) ring category* we mean a (simplicial) rig category $\bar{\mathcal{R}}$ such that $\pi_0|\bar{\mathcal{R}}|$ is a ring in the usual sense, with additive inverses.

Theorem 1.2. *Let $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, \otimes, 1_{\mathcal{R}})$ be a small simplicial rig category. There are simplicial rig categories $Z\mathcal{R}$ and $\bar{\mathcal{R}}$, natural in $\mathcal{R}$, and natural morphisms*

$$\mathcal{R} \xleftarrow{\sim} Z\mathcal{R} \longrightarrow \bar{\mathcal{R}}$$

of simplicial rig categories, such that

1. *$\bar{\mathcal{R}}$ is a simplicial ring category,*
2. *$\mathcal{R} \xleftarrow{\sim} Z\mathcal{R}$ is an unstable equivalence, and*
3. *$Z\mathcal{R} \longrightarrow \bar{\mathcal{R}}$ is a stable equivalence.*
4. *If furthermore (a) $\mathcal{R}$ is a groupoid, and (b) for every object X in $\mathcal{R}$ the translation functor $X \oplus (-)$ is faithful, then there is a natural chain of unstable equivalences of $Z\mathcal{R}$-modules connecting $\bar{\mathcal{R}}$ to the Grayson–Quillen model $(-\mathcal{R})\mathcal{R}$ for the additive group completion of $\mathcal{R}$.*

Addendum 1.3. *Let $\mathcal{R}$ be a small simplicial commutative rig category. There are natural morphisms*

$$\mathcal{R} \xleftarrow{\sim} Z\mathcal{R} \longrightarrow \bar{\mathcal{R}}$$

of simplicial commutative rig categories, such that all four statements of the theorem above hold.

In particular, the induced maps $\operatorname{Spt}\mathcal{R} \leftarrow \operatorname{Spt} Z\mathcal{R} \rightarrow \operatorname{Spt}\bar{\mathcal{R}}$ are stable equivalences of ring spectra, but the point is that $\bar{\mathcal{R}}$ is ring complete, before passing to spectra. Here are some examples of rig categories that can be ring completed by this method.

- If R is a rig, then the discrete category $\mathcal{R}$ with the elements of R as objects, and only identity morphisms, is a small rig category. When R is commutative, so is $\mathcal{R}$. The spectrum $\operatorname{Spt}\mathcal{R}$ is the Eilenberg–Mac Lane spectrum of the algebraic ring completion of R.
- There is a small commutative rig category $\mathcal{E}$ of finite sets, with objects the finite sets $\mathbf{n} = \{1, \dots, n\}$ for $n \geqslant 0$. In particular $\mathbf{0}$ is the empty set. There are no other morphisms in $\mathcal{E}$ than the automorphisms, and the automorphism group of $\mathbf{n}$ is the symmetric group Σ_n. Disjoint union and cartesian product of sets induce the operations $\oplus$ and $\otimes$, and $\operatorname{Spt}\mathcal{E}$ is equivalent to the sphere spectrum.
- For each commutative ring A there is a small commutative rig category $\mathcal{F}(A)$ of finitely generated free A-modules. The objects of $\mathcal{F}(A)$ are the free A-modules $A^n = \bigoplus_{i=1}^n A$ for $n \geqslant 0$. There are no other morphisms in $\mathcal{F}(A)$ than the automorphisms, and the automorphism group of A^n is the general linear group $GL_n(A)$. Direct sum and tensor product of A-modules induce the operations $\oplus$ and $\otimes$, and $\operatorname{Spt}\mathcal{F}(A)$ is the (free) algebraic K-theory spectrum of the ring A.
- Let $\mathcal{V}$ be the topological commutative rig category of complex (Hermitian) vector spaces. It has one object $\mathbb{C}^n$ for each $n \geqslant 0$, with automorphism space equal to the unitary group $U(n)$. There are no other morphisms. The spectrum $\operatorname{Spt}\mathcal{V}$ is a model for the connective topological K-theory spectrum ku. The case relevant to [3] and [5] is the 2-category of 2-vector spaces of Kapranov and Voevodsky [9], viewed as finitely generated free $\mathcal{V}$-modules. We can functorially convert $\mathcal{V}$ to a simplicial commutative rig category by replacing each morphism space with its singular simplicial set.

1.1. **Outline of proof.** The problem should be approached with some trepidation, since the reasons for the failure of the obvious attempts at a solution to this long-standing problem in algebraic K-theory are fairly well hidden. The standard approaches to additive group completion yield models that are symmetric monoidal categories with respect to an additive structure, but which have no meaningful multiplicative structure [17]. The failure comes about essentially because commutativity for addition only holds up to isomorphism. We therefore need to make a model that provides enough room to circumvent this difficulty.

Our solution comes in the form of a graded construction, $G\mathcal{R}$, related to iterations of the Grayson–Quillen model. It is a J-shaped diagram of symmetric monoidal categories, where the indexing category $J = I \int \mathcal{Q}$ is a certain permutative category over the category I of finite sets $\mathbf{n} = \{1, \dots, n\}$ and injective functions. Its definition can be motivated in a few steps. First, we use Thomason's homotopy colimit [18] of the diagram

$$0 \longleftarrow \mathcal{R} \xrightarrow{\Delta} \mathcal{R} \times \mathcal{R}$$

in symmetric monoidal categories as a model for the additive group completion of $\mathcal{R}$. An object (a, b) in the right hand category $\mathcal{R} \times \mathcal{R}$ represents the difference $a - b$, while an object a in the middle category $\mathcal{R}$ represents the relation $a - a = 0$, since a maps to (a, a) on the right hand side, and to zero in the left hand category.

Group completion is a homotopy idempotent process, meaning that we may repeat it any positive number of times and always obtain unstably equivalent results. For each $n \geqslant 0$ we realize the n-fold iterated group completion of $\mathcal{R}$ as the homotopy colimit of a $\mathcal{Q}\mathbf{n}$-shaped diagram in symmetric monoidal categories, where $\mathcal{Q}\mathbf{1}$ is the three-object category indexing the diagram displayed above, and in general $\mathcal{Q}\mathbf{n}$ is isomorphic to the product of n copies of $\mathcal{Q}\mathbf{1}$. One distinguished entry in the $\mathcal{Q}\mathbf{n}$-shaped diagram is the product of 2^n copies of $\mathcal{R}$. Its objects are given by 2^n objects of $\mathcal{R}$, which we regard as being located at the corners of an n-dimensional cube. These represent an alternating sum in $\bar{\mathcal{R}}$ of terms in $\mathcal{R}$, with signs determined by the position in the n-cube. The other entries in the $\mathcal{Q}\mathbf{n}$-shaped diagram are diagonally embedded subcubes of the n-cube, or the zero category, and encode cancellation laws in the group completion.

As regards the multiplicative structure, there is a natural pairing from the n-fold and the m-fold group completion to the $(n + m)$-fold group completion, with all possible $\otimes$-products of the entries in the two original cubes being spread out over the bigger cube. For instance, the product of the two 1-cubes (a, b) and (c, d) is a 2-cube $\left(\begin{smallmatrix} ac & ad \\ bc & bd \end{smallmatrix}\right)$, where for brevity we write ac for $a \otimes c$, and so on. Rather than trying to turn any single n-fold group completion into a ring category, we instead pass to the homotopy colimit over of all of them. To allow the homotopy colimit to retain the multiplicative structure, we proceed as in [6] and index the iterated group completions by the permutative category I, instead of the (non-symmetric) monoidal category of finite sets and inclusions that indexes sequential colimits. For each morphism $\mathbf{m} \to \mathbf{n}$ in I there is a preferred functor from $\mathcal{Q}\mathbf{m}$-shaped to $\mathcal{Q}\mathbf{n}$-shaped diagrams, involving extension by zero. For instance, the unique morphism $\mathbf{0} \to \mathbf{1}$ takes a in $\mathcal{R}$ (for $m = 0$) to $(a, 0)$ in $\mathcal{R} \times \mathcal{R}$ in the display above (for $n = 1$). See section 3 for further examples and pictures in low dimensions.

The resulting homotopy colimit, modulo a technical point about zero objects, gives the desired ring category $\bar{\mathcal{R}}$. As described, this is the homotopy colimit of an I-shaped diagram, whose entry at $\mathbf{n}$ is the homotopy colimit of a $\mathcal{Q}\mathbf{n}$-shaped diagram, for each $n \geqslant 0$. Such a double homotopy colimit can be condensed into a single homotopy colimit over a larger category, namely the Grothendieck construction $J = I \int \mathcal{Q}$. In the end we therefore prefer to present the ring category $\bar{\mathcal{R}}$ as the one-step homotopy colimit of a J-shaped diagram $G\mathcal{R}$. The graded multiplication

$$G\mathcal{R}(x) \times G\mathcal{R}(y) \to G\mathcal{R}(x + y)$$

for x, y in J is defined as above, by multiplying two cubes together to get a bigger cube, and makes $G\mathcal{R}$ a *J-graded rig category*. The difficulty one usually encounters does not appear, essentially because we have spread the product terms out over the vertices of the cubes, and not attempted to add together the "positive" and "negative" entries in some order or another.

From a homotopy theoretic point of view, the crucial information lies in the fact that for each $n \geqslant 0$, the homotopy colimit of the spectra associated to the $\mathcal{Q}\mathbf{n}$-shaped part of the $G\mathcal{R}$-diagram is stably equivalent to the spectrum associated with $\mathcal{R}$. For instance, the homotopy colimit of the diagram

$$* = \operatorname{Spt} 0 \longleftarrow \operatorname{Spt} \mathcal{R} \xrightarrow{\Delta} \operatorname{Spt}(\mathcal{R} \times \mathcal{R})$$

(for $n = 1$) is the "mapping cone of the diagonal", hence is again a model for the spectrum associated with $\mathcal{R}$. From a categorical point of view, the possibility of interchanging the factors in $\mathcal{R} \times \mathcal{R}$ tells us that the passage to spectra is unnecessary, since this flip induces the desired "negative path components", without having to stabilize.

We use Thomason's homotopy colimit in symmetric monoidal categories to transform the J-graded rig category $G\mathcal{R}$ into the rig category $\bar{\mathcal{R}}$, see Proposition 3.2 and Lemma 5.2. The technical point alluded to above is that zero objects are troublesome (few symmetric monoidal categories are "well pointed"), and must be handled with care. This gives rise to the intermediate simplicial rig category $Z\mathcal{R}$ that appears in Theorem 1.2.

1.2. **Plan.** The structure of the paper is as follows. After replacing the starting commutative rig (resp. rig) category $\mathcal{R}$ by an equivalent bipermutative (resp. strictly bimonoidal) category, we discuss graded versions of bipermutative and strictly bimonoidal categories and their morphisms in section 2. In section 3 we introduce the construction $G\mathcal{R}$ mentioned above, and show that it is a J-graded bipermutative (resp. strictly bimonoidal) category.

Thomason's homotopy colimit of symmetric monoidal categories is defined in a non-unital (or zeroless) setting. We extend this to the unital setting by constructing a derived version of it in section 4, and in section 5 we show that the homotopy colimit of a graded bipermutative (resp. graded strictly bimonoidal) category is almost a bipermutative (resp. strictly bimonoidal) category—it only lacks a zero. Section 6 describes how the results obtained so far combine to lead to an additive group completion within the framework of (symmetric) bimonoidal categories. This ring completion construction is given in Theorem 6.5.

Most of this paper appeared earlier as part of a preprint [4] with the title "Two-vector bundles define a form of elliptic cohomology". Some readers thought that title was hiding the result on rig categories explained in the current paper. We therefore now offer the ring completion result separately, and ask those readers interested in our main application to also turn to [5]. One should note that there was a mathematical error in the earlier preprint: the map T in the purported proof of Lemma 3.7(2) is not well defined, and so the version of the iterated Grayson–Quillen model used there might not have the right homotopy type.

A piece of notation: if $\mathcal{C}$ is any small category, then the expression $X \in \mathcal{C}$ is short for "X is an object of $\mathcal{C}$" and likewise for morphisms and diagrams.

We want to thank the referee for doing a thorough job.

2. Graded bipermutative categories

2.1. **Permutative categories.** A monoidal category (resp. symmetric monoidal category) is a category $\mathcal{M}$ with a binary operation $\oplus$ satisfying the axioms of a monoid (resp. commutative monoid), *i.e.*, a group (resp. abelian group) without negatives, up to coherent natural isomorphisms. A permutative category is a symmetric monoidal category where the associativity and the left and right unitality isomorphisms (but usually not the commutativity isomorphism), are identities. For the explicit definition of a permutative category see for instance [7, 3.1] or [12, §4]; compare also [11, XI.1]. Since our permutative categories are typically going to be the underlying additive symmetric monoidal categories of categories with some further multiplicative structure, we call the neutral element "zero", or simply 0.

We consider two kinds of functors between permutative categories $(\mathcal{M}, \oplus, 0_\mathcal{M}, \tau_\mathcal{M})$ and $(\mathcal{N}, \oplus, 0_\mathcal{N}, \tau_\mathcal{N})$, namely lax and strict symmetric monoidal functors. A *lax symmetric monoidal functor* is a functor F in the sense of [11, XI.2], *i.e.*, there are morphisms

$$f(a,b)\colon F(a) \oplus F(b) \to F(a \oplus b)$$

for all objects $a, b \in \mathcal{M}$, which are natural in a and b, there is a morphism

$$n\colon 0_\mathcal{N} \to F(0_\mathcal{M}),$$

and these structure maps fulfill the coherence conditions that are spelled out in [11, XI.2]; in particular

$$\begin{tikzcd} F(a) \oplus F(b) \arrow[r, "f(a,b)"] \arrow[d, "\tau_\mathcal{N}(F(a),F(b))"'] & F(a \oplus b) \arrow[d, "F(\tau_\mathcal{M}(a,b))"] \\ F(b) \oplus F(a) \arrow[r, "f(b,a)"] & F(b \oplus a) \end{tikzcd}$$

commutes for all $a, b \in \mathcal{M}$. Let Perm be the category of small permutative categories and lax symmetric monoidal functors.

We might say that f is a binatural transformation, *i.e.*, a natural transformation of functors $\mathcal{M} \times \mathcal{M} \to \mathcal{N}$. Here "bi-" refers to the two variables, and should not be confused with the "bi-" in bipermutative, which refers to the two operations $\oplus$ and $\otimes$.

A *strict symmetric monoidal functor* has furthermore to satisfy that the morphisms $f(a,b)$ and n are identities, so that

$$F(a \oplus b) = F(a) \oplus F(b) \quad \text{and} \quad F(0_\mathcal{M}) = 0_\mathcal{N}$$

[11, XI.2]. We denote the category of small permutative categories and strict symmetric monoidal functors by Strict.

A natural transformation $\nu \colon F \Rightarrow G$ of lax symmetric monoidal functors, with components $\nu_a \colon F(a) \to G(a)$, is required to be compatible with the structure morphisms, so that $\nu_{a\oplus b} \circ f(a,b) = g(a,b) \circ (\nu_a \oplus \nu_b)$ and $\nu_{0_{\mathcal{M}}} \circ n = n$. Similarly for natural transformations of strict symmetric monoidal functors.

Since any symmetric monoidal category is naturally equivalent to a permutative category, we lose no generality by only considering permutative categories. We mostly consider the unital situation, except for the places in sections 4 and 5 where we explicitly state that we are in the zeroless situation.

2.2. **Bipermutative categories.** Roughly speaking, a rig category $\mathcal{R}$ consists of a symmetric monoidal category $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, \tau_{\mathcal{R}})$ together with a functor $\mathcal{R} \times \mathcal{R} \to \mathcal{R}$ called "multiplication" and denoted by $\otimes$ or juxtaposition. Note that the multiplication is not a map of monoidal categories. The multiplication has a unit $1_{\mathcal{R}} \in \mathcal{R}$, multiplying by $0_{\mathcal{R}}$ is the zero map, multiplying by $1_{\mathcal{R}}$ is the identity map, and the multiplication is (left and right) distributive over $\oplus$ up to appropriately coherent natural isomorphisms. If we pose the additional requirement that our rig categories are commutative, then this coincides with what is often called a symmetric bimonoidal category. Laplaza spelled out the coherence conditions in [10, pp. 31–35].

According to [13, VI, Proposition 3.5] any commutative rig category is naturally equivalent in the appropriate sense to a bipermutative category, and a similar rigidification result holds for rig categories. Our main theorem (resp. its addendum) is therefore equivalent to the corresponding statement where we assume that $\mathcal{R}$ is a strictly bimonoidal category (resp. a bipermutative category). We will focus on the bipermutative case in the course of this paper, and indicate what has to be adjusted in the strictly bimonoidal case.

The reader can recover the axioms for a bipermutative category from Definition 2.1 below as the special case of a "0-graded bipermutative category", where 0 is the one-morphism category. Otherwise one may for instance consult [7, 3.6]. One word of warning: Elmendorf and Mandell's left distributivity law is precisely what we (and [13, VI, Definition 3.3]) call the right distributivity law. Note that we demand strict right distributivity, and that this implies both cases of condition 3.3(b) in [7], in view of condition 3.3(c).

If $\mathcal{R}$ is a small rig category such that $\pi_0|\mathcal{R}|$ is a ring (has additive inverses), then we call $\mathcal{R}$ a *ring category*. Elmendorf and Mandell's ring categories are not ring categories in our sense, but non-commutative rig categories. In the course of this paper we have to resolve rig categories simplicially. If $\mathcal{R}$ is a small simplicial rig category such that $\pi_0|\mathcal{R}|$ is a ring, then we call $\mathcal{R}$ a *simplicial ring category* (even though it is usually not a simplicial object in the category of ring categories).

If $\mathcal{R}$ is a strictly bimonoidal category, a *left $\mathcal{R}$-module* is a permutative category $\mathcal{M}$ together with a multiplication $\mathcal{R} \times \mathcal{M} \to \mathcal{M}$ that is strictly associative and coherently distributive, as spelled out in [7, 9.1.1].

2.3. **J-graded bipermutative categories and strictly bimonoidal categories.** The following definition of a J-graded bipermutative category is designed to axiomatize the key properties of the functor $G\mathcal{R}$ described in section 3, and simultaneously to generalize the definition of a bipermutative category (as the case $J = 0$). More generally, we could have introduced J-graded rig categories (resp. J-graded commutative rig categories), generalizing rig categories (resp. commutative rig categories), but this would have led to an even more cumbersome definition. We will therefore always assume that the input $\mathcal{R}$ to our machinery has been transformed to an equivalent bipermutative or strictly bimonoidal category before we start.

Definition 2.1. Let $(J, +, 0, \chi)$ be a small permutative category. A *J-graded bipermutative category* is a functor

$$\mathcal{C} \colon J \longrightarrow \mathrm{Strict}$$

from J to the category Strict of small permutative categories and strict symmetric monoidal functors, together with data $(\otimes, 1, \gamma_\otimes)$ as specified below, and subject to the following conditions. The permutative structure of $\mathcal{C}(x)$ will be denoted $(\mathcal{C}(x), \oplus, 0_x, \gamma_\oplus)$.

(1) There are composition functors

$$\otimes \colon \mathcal{C}(x) \times \mathcal{C}(y) \to \mathcal{C}(x+y)$$

for all $x, y \in J$, that are natural in x and y. More explicitly, for each pair of objects $a \in \mathcal{C}(x)$, $b \in \mathcal{C}(y)$ there is an object $a \otimes b$ in $\mathcal{C}(x+y)$, and for each pair of morphisms $f \colon a \to a'$, $g \colon b \to b'$

there is a morphism $f \otimes g\colon a \otimes b \to a' \otimes b'$, satisfying the usual associativity and unitality requirements. For each pair of morphisms $k\colon x \to z$, $\ell\colon y \to w$ in J the diagram

$$\begin{tikzcd} \mathcal{C}(x) \times \mathcal{C}(y) \arrow[r, "\otimes"] \arrow[d, "\mathcal{C}(k)\times\mathcal{C}(\ell)"'] & \mathcal{C}(x+y) \arrow[d, "\mathcal{C}(k+\ell)"] \\ \mathcal{C}(z) \times \mathcal{C}(w) \arrow[r, "\otimes"] & \mathcal{C}(z+w) \end{tikzcd}$$

commutes.

(2) There is a unit object $1 \in \mathcal{C}(0)$, such that $1 \otimes (-)\colon \mathcal{C}(y) \to \mathcal{C}(y)$ and $(-) \otimes 1\colon \mathcal{C}(x) \to \mathcal{C}(x)$ are the identity functors for all $x, y \in J$. More precisely, the inclusion $\{1\} \times \mathcal{C}(y) \to \mathcal{C}(0) \times \mathcal{C}(y)$ composed with $\otimes\colon \mathcal{C}(0) \times \mathcal{C}(y) \to \mathcal{C}(0+y) = \mathcal{C}(y)$ equals the projection isomorphism $\{1\} \times \mathcal{C}(y) \cong \mathcal{C}(y)$, and likewise for the functor from $\mathcal{C}(x) \times \{1\}$.

(3) For each pair of objects $a \in \mathcal{C}(x)$, $b \in \mathcal{C}(y)$ there is a twist isomorphism

$$\gamma_\otimes = \gamma_\otimes^{a,b}\colon a \otimes b \longrightarrow \mathcal{C}(\chi^{y,x})(b \otimes a)$$

in $\mathcal{C}(x+y)$, where $\chi^{y,x}\colon y + x \to x + y$ is the commutativity isomorphism in J, such that

$$\begin{tikzcd} a \otimes b \arrow[r, "\gamma_\otimes^{a,b}"] \arrow[d, "f\otimes g"'] & \mathcal{C}(\chi^{y,x})(b \otimes a) \arrow[d, "\mathcal{C}(\chi^{y,x})(g\otimes f)"] \\ a' \otimes b' \arrow[r, "\gamma_\otimes^{a',b'}"] & \mathcal{C}(\chi^{y,x})(b' \otimes a') \end{tikzcd}$$

commutes for any f, g as above, and

$$\mathcal{C}(k+\ell)(\gamma_\otimes^{a,b}) = \gamma_\otimes^{\mathcal{C}(k)(a),\mathcal{C}(\ell)(b)},$$

for any k, ℓ as above. We require that $\mathcal{C}(\chi^{y,x})(\gamma_\otimes^{b,a}) \circ \gamma_\otimes^{a,b}$ is equal to the identity on $a \otimes b$ for all objects a and b:

$$\begin{tikzcd} a \otimes b \arrow[rr, "\mathrm{id}_{a\otimes b}"] \arrow[dr, "\gamma_\otimes^{a,b}"'] & & \mathcal{C}(\chi^{y,x})\mathcal{C}(\chi^{x,y})(a \otimes b) \\ & \mathcal{C}(\chi^{y,x})(b \otimes a) \arrow[ur, "\mathcal{C}(\chi^{y,x})(\gamma_\otimes^{b,a})"'] & \end{tikzcd}$$

and that $\gamma_\otimes^{a,1}$ and $\gamma_\otimes^{1,a}$ are equal to the identity on a for all objects a.

(4) The composition $\otimes$ is strictly associative, and the diagram

$$\begin{tikzcd} a \otimes b \otimes c \arrow[r, "\gamma_\otimes^{a\otimes b,c}"] \arrow[d, "\mathrm{id}\otimes\gamma^{b,c}"'] & \mathcal{C}(\chi^{z,x+y})(c \otimes a \otimes b) \arrow[d, "\mathcal{C}(\chi^{z,x+y})(\gamma^{c,a}\otimes\mathrm{id})"] \\ \mathcal{C}(\mathrm{id}+\chi^{z,y})(a \otimes c \otimes b) \arrow[r, equal] & \mathcal{C}(\chi^{z,x+y})\mathcal{C}(\chi^{x,z}+\mathrm{id})(a \otimes c \otimes b) \end{tikzcd}$$

commutes for all objects a, b and c (compare [11, p. 254, (7a)]).

(5) Multiplication with the zero object 0_x annihilates everything, for each $x \in J$. More precisely, the inclusion $\{0_x\} \times \mathcal{C}(y) \to \mathcal{C}(x) \times \mathcal{C}(y)$ composed with $\otimes\colon \mathcal{C}(x) \times \mathcal{C}(y) \to \mathcal{C}(x+y)$ is the constant functor to 0_{x+y}, and likewise for the composite functor from $\mathcal{C}(x) \times \{0_y\}$.

(6) Right distributivity holds strictly, *i.e.*,

$$\begin{tikzcd} (\mathcal{C}(x) \times \mathcal{C}(x)) \times \mathcal{C}(y) \arrow[r, "\oplus\times\mathrm{id}"] \arrow[d, "\Delta"'] & \mathcal{C}(x) \times \mathcal{C}(y) \arrow[dd, "\otimes"] \\ (\mathcal{C}(x) \times \mathcal{C}(y)) \times (\mathcal{C}(x) \times \mathcal{C}(y)) \arrow[d, "\otimes\times\otimes"'] & \\ \mathcal{C}(x+y) \times \mathcal{C}(x+y) \arrow[r, "\oplus"] & \mathcal{C}(x+y) \end{tikzcd}$$

commutes, where $\oplus$ is the monoidal structure and Δ is the diagonal on $\mathcal{C}(y)$ combined with the identity on $\mathcal{C}(x) \times \mathcal{C}(x)$, followed by a twist. We denote these instances of identities by d_r, so $d_r = \mathrm{id}\colon \oplus \circ (\otimes \times \otimes) \circ \Delta \to \otimes \circ (\oplus \times \mathrm{id})$.

(7) The left distributivity transformation, d_ℓ, is given in terms of d_r and $\gamma_\otimes$ as

$$d_\ell = \gamma_\otimes \circ d_r \circ (\gamma_\otimes \oplus \gamma_\otimes)\,.$$

(Here we suppress the twist $\mathcal{C}(\chi)$ from the notation.) More explicitly, for all $x, y \in J$ and $a \in \mathcal{C}(x)$, $b, b' \in \mathcal{C}(y)$ the following diagram defines d_ℓ:

$$\begin{tikzcd}
a \otimes b \oplus a \otimes b' \arrow[r, "\gamma_\otimes^{a,b} \oplus \gamma_\otimes^{a,b'}"] \arrow[d, "d_\ell"'] & \mathcal{C}(\chi^{y,x})(b \otimes a) \oplus \mathcal{C}(\chi^{y,x})(b' \otimes a) \arrow[d, equal] \\
a \otimes (b \oplus b') \arrow[d, equal] & \mathcal{C}(\chi^{y,x})(b \otimes a \oplus b' \otimes a) \arrow[d, "\mathcal{C}(\chi^{y,x})(d_r) = \mathrm{id}"] \\
\mathcal{C}(\chi^{y,x})\mathcal{C}(\chi^{x,y})(a \otimes (b \oplus b')) & \mathcal{C}(\chi^{y,x})((b \oplus b') \otimes a) \arrow[l, "\mathcal{C}(\chi^{y,x})(\gamma_\otimes^{b \oplus b', a})"']
\end{tikzcd}$$

(8) The diagram

$$\begin{tikzcd}
(a \otimes b) \oplus (a \otimes b') \arrow[r, "d_\ell"] \arrow[d, "\gamma_\oplus"'] & a \otimes (b \oplus b') \arrow[d, "\mathrm{id} \otimes \gamma_\oplus"] \\
(a \otimes b') \oplus (a \otimes b) \arrow[r, "d_\ell"] & a \otimes (b' \oplus b)
\end{tikzcd}$$

commutes for all objects. The analogous diagram for d_r also commutes. Due to the definition of d_ℓ in terms of $\gamma_\otimes$ and the identity d_r, it suffices to demand that $\gamma_\oplus \circ (\gamma_\otimes \oplus \gamma_\otimes) = (\gamma_\otimes \oplus \gamma_\otimes) \circ \gamma_\oplus$ and $(\gamma_\oplus \otimes \mathrm{id}) \circ \gamma_\otimes = \gamma_\otimes \circ (\mathrm{id} \otimes \gamma_\oplus)$.

(9) The distributivity transformations are associative, *i.e.*, the diagram

$$\begin{tikzcd}
(a \otimes b \otimes c) \oplus (a \otimes b \otimes c') \arrow[d, "d_\ell"'] \arrow[dr, "d_\ell"] & \\
a \otimes ((b \otimes c) \oplus (b \otimes c')) \arrow[r, "\mathrm{id} \otimes d_\ell"'] & a \otimes b \otimes (c \oplus c')
\end{tikzcd}$$

commutes for all objects.

(10) The following pentagon diagram commutes

$$\begin{tikzcd}
& (a \otimes (b \oplus b')) \oplus (a' \otimes (b \oplus b')) \arrow[ddr, "d_r"] & \\
(a \otimes b) \oplus (a \otimes b') \oplus (a' \otimes b) \oplus (a' \otimes b') \arrow[ur, "d_\ell \oplus d_\ell"] \arrow[dd, "\mathrm{id} \oplus \gamma_\oplus \oplus \mathrm{id}"'] & & \\
& & (a \oplus a') \otimes (b \oplus b') \\
(a \otimes b) \oplus (a' \otimes b) \oplus (a \otimes b') \oplus (a' \otimes b') \arrow[dr, "d_r \oplus d_r"'] & & \\
& ((a \oplus a') \otimes b) \oplus ((a \oplus a') \otimes b') \arrow[uur, "d_\ell"'] &
\end{tikzcd}$$

for all objects $a, a' \in \mathcal{C}(x)$ and $b, b' \in \mathcal{C}(y)$.

Remark 2.2. In Definition 2.1, condition (1) says that we have a binatural transformation

$$\otimes\colon \mathcal{C} \times \mathcal{C} \Rightarrow \mathcal{C} \circ (+)$$

of bifunctors $J \times J \to \mathrm{Cat}$, where Cat denotes the category of small categories. Condition (3) demands that there be a modification [11, p. 278]

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, Rightarrow, "c_{\mathrm{Cat}}"] \arrow[d, Rightarrow, "\otimes"] \arrow[dr, phantom, "\overset{\gamma_\otimes}{\Rrightarrow}"] & (\mathcal{C} \times \mathcal{C}) \circ tw_J \arrow[d, Rightarrow, "\otimes"] \\ \mathcal{C} \circ (+) & \mathcal{C} \circ (+) \circ tw_J \arrow[l, Rightarrow, "\mathcal{C}(\chi)"] \end{tikzcd}$$

where c_{Cat} is the symmetric structure on Cat (with respect to product) and tw_J is the interchange of factors on $J \times J$.

In the following we will denote a J-graded bipermutative category $\mathcal{C}\colon J \to \mathrm{Strict}$ by $\mathcal{C}^\bullet$ if the category J is clear from the context. For the one-morphism category $J = 0$, a J-graded bipermutative category is the same as a bipermutative category. Thus every J-graded bipermutative category $\mathcal{C}^\bullet$ comes with a bipermutative category $\mathcal{C}(0)$, and $\mathcal{C}^\bullet$ can be viewed as a functor $J \to \mathcal{C}(0)$-modules.

Example 2.3. We consider the small bipermutative category of finite sets, whose objects are the finite sets of the form $\mathbf{n} = \{1, \ldots, n\}$ for $n \geqslant 0$, and whose morphisms $\mathbf{m} \to \mathbf{n}$ are all functions $\{1, \ldots, m\} \to \{1, \ldots, n\}$.

Disjoint union of sets gives rise to a permutative structure

$$\mathbf{n} \oplus \mathbf{m} := \mathbf{n} \sqcup \mathbf{m}$$

and we identify $\mathbf{n} \sqcup \mathbf{m}$ with $\mathbf{n} + \mathbf{m}$. For functions $f\colon \mathbf{n} \to \mathbf{n}'$ and $g\colon \mathbf{m} \to \mathbf{m}'$ we define $f \oplus g$ as the map on the disjoint union $f \sqcup g$ which we will denote by $f + g$. The additive twist $c_\oplus$ is given by the shuffle maps

$$\chi(n,m)\colon \mathbf{n} + \mathbf{m} \longrightarrow \mathbf{m} + \mathbf{n}$$

with

$$\chi(n,m)(i) = \begin{cases} m + i & \text{for } i \leqslant n \\ i - n & \text{for } i > n. \end{cases}$$

Multiplication of sets is defined via

$$\mathbf{n} \otimes \mathbf{m} := \mathbf{nm}\,.$$

If we identify the element $(i-1) \cdot m + j$ in $\mathbf{nm}$ with the pair (i,j) with $i \in \mathbf{n}$ and $j \in \mathbf{m}$, then the function $f \otimes g$ is given by

$$(i,j) \mapsto (f(i), g(j)),$$

and the multiplicative twist

$$c_\otimes\colon \mathbf{n} \otimes \mathbf{m} \longrightarrow \mathbf{m} \otimes \mathbf{n}$$

sends (i,j) to (j,i). The empty set $\mathbf{0}$ is a strict zero for the addition and the singleton set $\mathbf{1}$ is a strict unit for the multiplication. Right distributivity is the identity and the left distributivity law is given by the resulting permutation

$$d_\ell = c_\otimes \circ d_r \circ (c_\otimes \oplus c_\otimes)\,.$$

For later reference we denote this instance of d_ℓ by ξ.

Considering only the subcategory of bijections, instead of arbitrary functions, results in the bipermutative category of finite sets $\mathcal{E}$ that we referred to in the introduction. Later, we will make use of the zeroless bipermutative category of finite nonempty sets and surjective functions.

Definition 2.4. A *J-graded strictly bimonoidal category* is a functor $\mathcal{C}\colon J \to \mathrm{Strict}$ to the category of permutative categories and strict symmetric monoidal functors, satisfying the conditions of Definition 2.1, except that we do not require the existence of the natural isomorphism $\gamma_\otimes$, and the left distributivity isomorphism d_ℓ is not given in terms of d_r. Axiom (7) of Definition 2.1 has to be replaced by the following condition:

(7') The diagram

$$\begin{tikzcd} a \otimes b \otimes c \oplus a \otimes b' \otimes c \arrow[r, "d_r"] \arrow[d, "d_\ell"] & (a \otimes b \oplus a \otimes b') \otimes c \arrow[d, "d_\ell \otimes \mathrm{id}"] \\ a \otimes (b \otimes c \oplus b' \otimes c) \arrow[r, "\mathrm{id} \otimes d_r"] & a \otimes (b \oplus b') \otimes c \end{tikzcd}$$

commutes for all objects.

In the J-graded bipermutative case condition (7') follows from the other axioms.

Definition 2.5. A *lax morphism of bipermutative categories*, $F\colon \mathcal{C} \to \mathcal{D}$, is a pair of lax symmetric monoidal functors $(\mathcal{C}, \oplus, 0_{\mathcal{C}}, c_\oplus) \to (\mathcal{D}, \oplus, 0_{\mathcal{D}}, c_\oplus)$ and $(\mathcal{C}, \otimes, 1_{\mathcal{C}}, c_\otimes) \to (\mathcal{D}, \otimes, 1_{\mathcal{D}}, c_\otimes)$, with the same underlying functor $F\colon \mathcal{C} \to \mathcal{D}$, that respect the left and right distributivity laws.

In other words, we have a binatural transformation from $\oplus \circ (F \times F)$ to $F \circ \oplus$:

$$\eta_\oplus = \eta_\oplus(a,b)\colon F(a) \oplus F(b) \to F(a \oplus b)$$

for $a, b \in \mathcal{C}$, as well as a binatural transformation from $\otimes \circ (F \times F)$ to $F \circ \otimes$:

$$\eta_\otimes = \eta_\otimes(a,b)\colon F(a) \otimes F(b) \to F(a \otimes b)$$

for $a, b \in \mathcal{C}$, plus morphisms $0_{\mathcal{D}} \to F(0_{\mathcal{C}})$ and $1_{\mathcal{D}} \to F(1_{\mathcal{C}})$. We require that these commute with $c_\oplus$ and $c_\otimes$, respectively, and that the following diagram (and the analogous one for d_ℓ) commutes

$$\begin{tikzcd}
F(a) \otimes F(b) \oplus F(a') \otimes F(b) \arrow[r, equal, "d_r = \mathrm{id}"] \arrow[d, "\eta_\otimes \oplus \eta_\otimes"'] & (F(a) \oplus F(a')) \otimes F(b) \arrow[r, "\eta_\oplus \otimes \mathrm{id}"] & F(a \oplus a') \otimes F(b) \arrow[d, "\eta_\otimes"] \\
F(a \otimes b) \oplus F(a' \otimes b) \arrow[r, "\eta_\oplus"'] & F(a \otimes b \oplus a' \otimes b) \arrow[r, equal, "F(d_r) = \mathrm{id}"'] & F((a \oplus a') \otimes b)
\end{tikzcd}$$

for all objects $a, a', b \in \mathcal{C}$, *i.e.*, we have

$$\eta_\oplus \circ (\eta_\otimes \oplus \eta_\otimes) = \eta_\otimes \circ (\eta_\oplus \otimes \mathrm{id})$$

and

$$F(\gamma_\otimes \circ (\gamma_\otimes \oplus \gamma_\otimes)) \circ \eta_\oplus \circ (\eta_\otimes \oplus \eta_\otimes) = \eta_\otimes \circ (\mathrm{id} \otimes \eta_\oplus) \circ \gamma_\otimes \circ (\gamma_\otimes \oplus \gamma_\otimes)\,.$$

For a *lax morphism of strictly bimonoidal categories* we demand that F be lax monoidal with respect to $\otimes$, lax symmetric monoidal with respect to $\oplus$, and that

$$F(d_\ell) \circ \eta_\oplus \circ (\eta_\otimes \oplus \eta_\otimes) = \eta_\otimes \circ (\mathrm{id} \otimes \eta_\oplus) \circ d_\ell \quad \text{and} \quad F(d_r) \circ \eta_\oplus \circ (\eta_\otimes \oplus \eta_\otimes) = \eta_\otimes \circ (\eta_\oplus \otimes \mathrm{id}) \circ d_r.$$

Definition 2.6. A *lax morphism of J-graded bipermutative categories*, $F\colon \mathcal{C}^\bullet \to \mathcal{D}^\bullet$, consists of a natural transformation F from $\mathcal{C}^\bullet$ to $\mathcal{D}^\bullet$ that is compatible with the bifunctors $\oplus$, $\otimes$ and the units. Additively, we require a transformation $\eta_\oplus$ from $\oplus \circ (F \times F)$ to $F \circ \oplus$:

$$\begin{tikzcd}
\mathcal{C}(x) \times \mathcal{C}(x) \arrow[r, "\oplus"] \arrow[d, "F_x \times F_x"'] & \mathcal{C}(x) \arrow[d, "F_x"] \\
\mathcal{D}(x) \times \mathcal{D}(x) \arrow[r, "\oplus"'] \arrow[ur, phantom, "\Nearrow \eta_\oplus^x"] & \mathcal{D}(x)
\end{tikzcd}$$

that commutes with $\gamma_\oplus$, is binatural with respect to morphisms in $\mathcal{C}(x) \times \mathcal{C}(x)$, and is natural with respect to x. Multiplicatively, we require a transformation $\eta_\otimes$ from $\otimes \circ (F \times F)$ to $F \circ \otimes$:

$$\begin{tikzcd}
\mathcal{C}(x) \times \mathcal{C}(y) \arrow[r, "\otimes"] \arrow[d, "F_x \times F_y"'] & \mathcal{C}(x+y) \arrow[d, "F_{x+y}"] \\
\mathcal{D}(x) \times \mathcal{D}(y) \arrow[r, "\otimes"'] \arrow[ur, phantom, "\Nearrow \eta_\otimes^{x,y}"] & \mathcal{D}(x+y)
\end{tikzcd}$$

that commutes with $\gamma_\otimes$, is binatural with respect to morphisms in $\mathcal{C}(x) \times \mathcal{C}(y)$, and is natural with respect to x and y. The functor F must respect the distributivity constraints in that it fulfills

$$\eta_\oplus \circ (\eta_\otimes \oplus \eta_\otimes) = \eta_\otimes \circ (\eta_\oplus \otimes \mathrm{id})$$

and

$$F(d_\ell) \circ \eta_\oplus \circ (\eta_\otimes \oplus \eta_\otimes) = \eta_\otimes \circ (\mathrm{id} \otimes \eta_\oplus) \circ d_\ell\,.$$

For a *lax morphism of J-graded strictly bimonoidal categories* there is no requirement on (F and) $\eta_\otimes$ concerning the multiplicative twist $\gamma_\otimes$.

3. A cubical construction on (bi-)permutative categories

We remodel the Grayson–Quillen construction [8] of the group completion of a permutative category to suit our multiplicative needs. The naïve product $(ac \oplus bd, ad \oplus bc)$ of two pairs (a, b) and (c, d) in their model will be replaced by the quadruple $\left(\begin{smallmatrix} ac & ad \\ bc & bd \end{smallmatrix}\right)$, where no order of adding terms is chosen. This avoids the "phoniness" of the multiplication [17], but requires that we keep track of n-cubical diagrams of objects, of varying dimensions $n \geqslant 0$. We start by introducing the indexing category $I \int \mathcal{Q}$ for all of these diagrams, and then describe the $I \int \mathcal{Q}$-shaped diagram $G\mathcal{M}$ associated to a permutative category $\mathcal{M}$. If we start with a bipermutative category $\mathcal{R}$, the result will be an $I \int \mathcal{Q}$-graded bipermutative category $G\mathcal{R}$.

3.1. **An indexing category.** Let I be the usual skeleton of the category of finite sets and injective functions, *i.e.*, its objects are the finite sets $\mathbf{n} = \{1, \dots, n\}$ for $n \geqslant 0$, and its morphisms are the injective functions $\varphi \colon \mathbf{m} \to \mathbf{n}$. We define the sum of two objects $\mathbf{n}$ and $\mathbf{m}$ to be $\mathbf{n} + \mathbf{m}$ and use the twist maps $\chi(n, m)$ defined in Example 2.3. Then $(I, +, \mathbf{0}, \chi)$ is a permutative category.

For each $n \geqslant 0$, let $\mathcal{Q}\mathbf{n}$ be the category whose objects are subsets T of

$$\{\pm 1, \dots, \pm n\} = \{-n, \dots, -1, 1, \dots, n\}$$

such that the absolute value function $T \to \mathbb{Z}$ is injective. In other words, we may have $i \in T$ or $-i \in T$, but not both, for each $1 \leqslant i \leqslant n$. Morphisms in $\mathcal{Q}\mathbf{n}$ are inclusions $S \subseteq T$ of subsets. (The objects could equally well be described as pairs (T, w) where $T \subseteq \mathbf{n}$ and w is a function $T \to \{\pm 1\}$, and similarly for the morphisms.) Let $\mathcal{P}\mathbf{n} \subseteq \mathcal{Q}\mathbf{n}$ be the full subcategory generated by the subsets of $\mathbf{n} = \{1, \dots, n\}$, *i.e.*, the T with only positive elements.

For example, the category $\mathcal{Q}\mathbf{2}$ can be depicted as:

$$\begin{tikzcd}
\{-1, 2\} & \{2\} \arrow[l, hook'] \arrow[r, hook] & \{1, 2\} \\
\{-1\} \arrow[u, hook] \arrow[d, hook] & \varnothing \arrow[l, hook'] \arrow[r, hook] \arrow[u, hook] \arrow[d, hook] & \{1\} \arrow[u, hook] \arrow[d, hook] \\
\{-1, -2\} & \{-2\} \arrow[l, hook'] \arrow[r, hook] & \{1, -2\}
\end{tikzcd}$$

and $\mathcal{P}\mathbf{2}$ is given by the upper right hand square. We shall use $\mathcal{P}\mathbf{n}$ and $\mathcal{Q}\mathbf{n}$ to index n-dimensional cubical diagrams with 2^n and 3^n vertices, respectively.

For each morphism $\varphi \colon \mathbf{m} \to \mathbf{n}$ in I we define a functor $\mathcal{Q}\varphi \colon \mathcal{Q}\mathbf{m} \to \mathcal{Q}\mathbf{n}$ as follows. First, let $C\varphi = \mathbf{n} \setminus \varphi(\mathbf{m})$ be the complement of the image of the injective function φ. Then extend φ to an odd function $\{\pm 1, \dots, \pm m\} \to \{\pm 1, \dots, \pm n\}$, which we also call φ, and let

$$(\mathcal{Q}\varphi)(S) = \varphi(S) \sqcup C\varphi$$

for each object $S \in \mathcal{Q}\mathbf{m}$. For example, if $\varphi \colon \mathbf{1} \to \mathbf{2}$ is given by $\varphi(1) = 2$, then $C\varphi = \{1\}$ and $\mathcal{Q}\varphi$ is the functor

$$\begin{tikzcd}
\{-1\} \arrow[d, mapsto] & \varnothing \arrow[l, hook'] \arrow[r, hook] \arrow[d, mapsto] & \{1\} \arrow[d, mapsto] \\
\{1, -2\} & \{1\} \arrow[l, hook'] \arrow[r, hook] & \{1, 2\}
\end{tikzcd}$$

embedding $\mathcal{Q}\mathbf{1}$ into the right hand column of $\mathcal{Q}\mathbf{2}$. Similarly, the function $\varphi \colon \mathbf{1} \to \mathbf{2}$ with $\varphi(1) = 1$ embeds $\mathcal{Q}\mathbf{1}$ into the upper row of $\mathcal{Q}\mathbf{2}$.

If $S \subseteq T$ then clearly $(\mathcal{Q}\varphi)(S) \subseteq (\mathcal{Q}\varphi)(T)$. If $\psi \colon \mathbf{k} \to \mathbf{m}$ is a second morphism in I, we see that $\mathcal{Q}\varphi \circ \mathcal{Q}\psi = \mathcal{Q}(\varphi\psi)$, and so $\mathbf{n} \mapsto \mathcal{Q}\mathbf{n}$ defines a functor $\mathcal{Q} \colon I \to \mathrm{Cat}$. Restricting to sets with only positive entries, we get a subfunctor $\mathcal{P} \subseteq \mathcal{Q}$ that may be easier to grasp: if $\varphi \colon \mathbf{m} \to \mathbf{n} \in I$, then $\mathcal{P}\varphi \colon \mathcal{P}\mathbf{m} \to \mathcal{P}\mathbf{n}$ is the functor sending $S \subseteq \mathbf{m}$ to $\varphi(S) \sqcup C\varphi$, where $C\varphi = \mathbf{n} \setminus \varphi(\mathbf{m})$ is the complement of the image of φ.

Our main indexing category will be the Grothendieck construction $J = I \int \mathcal{Q}$. This is the category with objects pairs $x = (\mathbf{m}, S)$ with $\mathbf{m} \in I$ and $S \in \mathcal{Q}\mathbf{m}$, and with morphisms $x = (\mathbf{m}, S) \to (\mathbf{n}, T) = y$ consisting of pairs (φ, ι) with $\varphi \colon \mathbf{m} \to \mathbf{n}$ a morphism in I and $\iota \colon (\mathcal{Q}\varphi)(S) \subseteq T$ an inclusion. To give a functor $\mathcal{C}$ from $I \int \mathcal{Q}$ to any category is equivalent to giving a functor $\mathcal{C}_{\mathbf{n}}$ from $\mathcal{Q}\mathbf{n}$ for each $n \geqslant 0$, together with natural transformations $\mathcal{C}_\varphi \colon \mathcal{C}_{\mathbf{m}} \Rightarrow \mathcal{C}_{\mathbf{n}} \circ \mathcal{Q}\varphi$ for all $\varphi \colon \mathbf{m} \to \mathbf{n}$ in I, which must be compatible with identities and composition in I.

Consider the functor $+\colon \mathcal{Q}\mathbf{n}\times\mathcal{Q}\mathbf{m}\to\mathcal{Q}(\mathbf{n}+\mathbf{m})$ defined as follows. The injective functions $in_1\colon \mathbf{n}\to\mathbf{n}+\mathbf{m}$ and $in_2\colon \mathbf{m}\to\mathbf{n}+\mathbf{m}$ are given by $in_1(i)=i$ and $in_2(j)=n+j$. Extending to odd functions we define $T+S$ to be the disjoint union of images

$$in_1(T)\sqcup in_2(S)\subseteq\{\pm1,\dots,\pm(n+m)\}\,.$$

For example, if $T=\{-1,2\}\subseteq\{\pm1,\pm2,\pm3\}$ and $S=\{1,-2\}\subseteq\{\pm1,\pm2\}$, then $T+S=\{-1,2,4,-5\}\subseteq\{\pm1,\dots,\pm5\}$.

These functors, for varying $n,m\geqslant 0$, combine to an *addition functor* on $I\int\mathcal{Q}$. For each pair of objects $(\mathbf{n},T),(\mathbf{m},S)\in I\int\mathcal{Q}$ we define $(\mathbf{n},T)+(\mathbf{m},S)=(\mathbf{n}+\mathbf{m},T+S)$, and likewise on morphisms.

Lemma 3.1. *Addition makes $I\int\mathcal{Q}$ and $I\int\mathcal{P}$ into permutative categories.*

Proof. The zero object is $(\mathbf{0},\mathbf{0})$, and the isomorphism $(\chi(n,m),\mathrm{id})\colon(\mathbf{n}+\mathbf{m},T+S)\to(\mathbf{m}+\mathbf{n},S+T)$ provides the symmetric structure. ☐

3.2. **The cube construction.** Let $\mathcal{M}$ be a permutative category (with zero). Define a functor

$$\mathcal{M}_{\mathbf{n}}\colon\mathcal{P}\mathbf{n}\to\mathrm{Strict}$$

for each $n\geqslant 0$, by sending a subset $T\subseteq\mathbf{n}$ to $\mathcal{M}_{\mathbf{n}}(T)=\mathcal{M}^{\mathcal{P}T}$, the permutative category of functions from the **set** $\mathcal{P}T$ of subsets of T to $\mathcal{M}$, *i.e.*, the product of one copy of $\mathcal{M}$ for each subset of T. If $\iota\colon S\subseteq T$ we get a strict symmetric monoidal functor $\mathcal{M}_{\mathbf{n}}(\iota)\colon\mathcal{M}^{\mathcal{P}S}\to\mathcal{M}^{\mathcal{P}T}$ by sending the object $a=(a_U\mid U\subseteq S)\in\mathcal{M}^{\mathcal{P}S}$ to $(a_{V\cap S}\mid V\subseteq T)$, and likewise with morphisms. These are diagonal functors, since each a_U gets repeated once for each V with $V\cap S=U$.

For $n=0,1,2$ the diagrams $\mathcal{M}_{\mathbf{n}}$ have the following shapes:

$$\mathcal{M}\ ,\qquad \mathcal{M}\to\mathcal{M}\times\mathcal{M}\qquad\text{and}\qquad \begin{tikzcd} \mathcal{M}\times\mathcal{M} \arrow[r] & \mathcal{M}^{\times 4} \\ \mathcal{M} \arrow[u] \arrow[r] & \mathcal{M}\times\mathcal{M} \arrow[u] \end{tikzcd}$$

where the morphisms are the appropriate diagonals. Note that the two diagonal $\mathcal{M}\times\mathcal{M}\to\mathcal{M}^{\times4}$ in $\mathcal{M}_{\mathbf{2}}$ differ by a twist map. One of the diagonals sends the pair $(a_\emptyset,a_{\{1\}})$ to $\left(\begin{smallmatrix} a_\emptyset & a_{\{1\}}\\ a_\emptyset & a_{\{1\}}\end{smallmatrix}\right)$, and the other sends $(a_\emptyset,a_{\{2\}})$ to $\left(\begin{smallmatrix} a_\emptyset & a_\emptyset\\ a_{\{2\}} & a_{\{2\}}\end{smallmatrix}\right)$.

In general, $\mathcal{M}_{\mathbf{n}}(\mathbf{n})$ is the product of 2^n copies of $\mathcal{M}$, viewed as spread out over the corners of an n-dimensional cube.

For $\varphi\colon\mathbf{m}\to\mathbf{n}$ we define a natural transformation $\mathcal{M}_\varphi\colon\mathcal{M}_{\mathbf{m}}\Rightarrow\mathcal{M}_{\mathbf{n}}\circ\mathcal{P}\varphi$: for $S\in\mathcal{P}\mathbf{m}$ we let $\mathcal{M}_\varphi(S)$ be the composite

$$\mathcal{M}_{\mathbf{m}}(S)=\mathcal{M}^{\mathcal{P}S}\cong\mathcal{M}^{\mathcal{P}(\varphi(S))}\to\mathcal{M}^{\mathcal{P}(\varphi(S)\sqcup C\varphi)}=\mathcal{M}_{\mathbf{n}}((\mathcal{P}\varphi)(S))$$

where the isomorphism is just the reindexation induced by φ, and the functor $\mathcal{M}^{\mathcal{P}(\varphi(S))}\to\mathcal{M}^{\mathcal{P}(\varphi(S)\sqcup C\varphi)}$ is the identity on factors indexed by subsets of $\varphi(S)$ and zero on the factors that are not hit by φ. Explicitly,

$$\mathcal{M}_\varphi(S)(f)_V=\begin{cases} f_{\varphi^{-1}(V)} & \text{if } V\subseteq\varphi(S)\\ 0 & \text{otherwise}\end{cases}$$

for any morphism $f\colon a\to b\in\mathcal{M}^{\mathcal{P}S}$ and $V\subseteq\varphi(S)\sqcup C\varphi$. These are extension by zero functors, not diagonals. Each f_U gets repeated exactly once, as $\mathcal{M}_\varphi(S)(f)_V$ for $V=\varphi(U)$.

For instance, if $\varphi\colon\mathbf{1}\to\mathbf{2}$ is given by $\varphi(1)=2$, then $\mathcal{M}_{\mathbf{1}}(\varnothing)=\mathcal{M}\to\mathcal{M}\times\mathcal{M}=\mathcal{M}_{\mathbf{2}}(\{1\})$ and $\mathcal{M}_{\mathbf{1}}(\mathbf{1})=\mathcal{M}\times\mathcal{M}\to\mathcal{M}^{\times4}\cong\mathcal{M}_{\mathbf{2}}(\mathbf{2})$ are given by appropriate inclusions onto factors in products. For both morphisms $\varphi\colon\mathbf{1}\to\mathbf{2}$ the associated functors $\mathcal{M}\to\mathcal{M}\times\mathcal{M}$ are the inclusion onto the $\varnothing$-factor, whereas the two functors $\mathcal{M}\times\mathcal{M}\to\mathcal{M}^{\times4}$ include onto either the $\varnothing$ and $\{1\}$ factors, or the $\varnothing$ and $\{2\}$ factors, depending on φ.

We see that for all $S\subseteq T\subseteq\mathbf{m}$ and $\varphi\colon\mathbf{m}\to\mathbf{n}$, the diagram

$$\begin{tikzcd} \mathcal{M}^{\mathcal{P}S} \arrow[r] \arrow[d, "\mathcal{M}_\varphi(S)"'] & \mathcal{M}^{\mathcal{P}T} \arrow[d, "\mathcal{M}_\varphi(T)"] \\ \mathcal{M}^{\mathcal{P}(\varphi(S)\sqcup C\varphi)} \arrow[r] & \mathcal{M}^{\mathcal{P}(\varphi(T)\sqcup C\varphi)} \end{tikzcd}$$

commutes, sending $a\in\mathcal{M}^{\mathcal{P}S}$ both ways to $W\mapsto a_{\varphi^{-1}(W)\cap S}$ if $W\subseteq\varphi(T)$ and 0 otherwise.

If $\psi\colon \mathbf{k} \to \mathbf{m} \in I$, then we have an equality $\mathcal{M}_\varphi \mathcal{M}_\psi = \mathcal{M}_{\varphi\psi}$ of natural transformations $\mathcal{M}_{\mathbf{k}} \Rightarrow \mathcal{M}_{\mathbf{n}} \circ \mathcal{P}(\varphi\psi)$:

$$
\begin{tikzcd}
\mathcal{P}\mathbf{k} \arrow[d, "\mathcal{P}\psi"'] \arrow[dr, bend left, "\mathcal{M}_{\mathbf{k}}"] \arrow[dd, bend right=60, "P(\varphi\psi)"'] \arrow[d, phantom, shift left=6ex, "\Downarrow \mathcal{M}_\psi"] & \\
\mathcal{P}\mathbf{m} \arrow[r, "\mathcal{M}_{\mathbf{m}}"] \arrow[d, "\mathcal{P}\varphi"'] \arrow[d, phantom, shift left=6ex, "\Downarrow \mathcal{M}_\varphi"] & \mathrm{Strict} \\
\mathcal{P}\mathbf{n} \arrow[ur, bend right, "\mathcal{M}_{\mathbf{n}}"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{P}\mathbf{k} \arrow[dr, bend left, "\mathcal{M}_{\mathbf{k}}"] \arrow[dd, bend right=60, "P(\varphi\psi)"'] & \\
\Downarrow \mathcal{M}_{\varphi\psi} & \mathrm{Strict} \\
\mathcal{P}\mathbf{n} \arrow[ur, bend right, "\mathcal{M}_{\mathbf{n}}"'] &
\end{tikzcd}
$$

Both natural transformations are represented by the functors $\mathcal{M}^{\mathcal{P}S} \to \mathcal{M}^{\mathcal{P}(\varphi\psi(S) \sqcup C(\varphi\psi))}$ sending a to $V \mapsto a_{(\varphi\psi)^{-1}V}$ if $V \subseteq \varphi\psi(S)$ and 0 otherwise. Thus $\mathcal{M}$ can be viewed as a left lax transformation from the functor $\mathcal{P}\colon I \to \mathrm{Cat}$ to the constant functor at Strict. (We recall the definition of a left lax transformation in subsection 4.1 below.)

This left lax transformation $\mathcal{M}\colon \mathcal{P} \Rightarrow \mathrm{Strict}$ extends to a left lax transformation $\mathcal{M}\colon \mathcal{Q} \Rightarrow \mathrm{Strict}$ by declaring that $\mathcal{M}_{\mathbf{n}}(T) = 0$ if T contains negative elements.

The first three diagrams now look like:

$$
\mathcal{M}\ , \qquad 0 \leftarrow \mathcal{M} \to \mathcal{M} \times \mathcal{M} \qquad \text{and} \qquad
\begin{tikzcd}
0 & \mathcal{M} \times \mathcal{M} \arrow[l] \arrow[r] & \mathcal{M}^{\times 4} \\
0 \arrow[u] \arrow[d] & \mathcal{M} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathcal{M} \times \mathcal{M} \arrow[u] \arrow[d] \\
0 & 0 \arrow[l] \arrow[r] & 0
\end{tikzcd}
$$

Another way of saying that we have a left lax transformation $\mathcal{Q} \Rightarrow \mathrm{Strict}$ is to say that we have a functor $I \textstyle\int \mathcal{M}\colon I \textstyle\int \mathcal{Q} \to I \textstyle\int \mathrm{Strict} \cong I \times \mathrm{Strict}$ (over I). Projecting to the second factor, $I \textstyle\int \mathcal{M}$ gives rise to a functor

$$G\mathcal{M}\colon I \textstyle\int \mathcal{Q} \to \mathrm{Strict}\,.$$

Explicitly, $G\mathcal{M}(\mathbf{n}, T) = \mathcal{M}_{\mathbf{n}}(T)$, which is $\mathcal{M}^{\mathcal{P}T}$ if T contains no negatives and 0 otherwise. If $\varphi\colon \mathbf{m} \to \mathbf{n} \in I$ and $\iota\colon (\mathcal{Q}\varphi)(S) \subseteq T \in \mathcal{Q}\mathbf{n}$, then $G\mathcal{M}(\varphi, \iota)\colon \mathcal{M}_{\mathbf{m}}(S) \to \mathcal{M}_{\mathbf{n}}(T)$ is the composite of $G\mathcal{M}(\varphi, \mathrm{id}) = \mathcal{M}_\varphi S\colon \mathcal{M}_{\mathbf{m}}(S) \to \mathcal{M}_{\mathbf{n}}((\mathcal{Q}\varphi)(S))$ and $G\mathcal{M}(\mathrm{id}, \iota) = \mathcal{M}_{\mathbf{n}}(\iota)\colon \mathcal{M}_{\mathbf{n}}((\mathcal{Q}\varphi)(S)) \to \mathcal{M}_{\mathbf{n}}(T)$.

3.3. **Multiplicative structure.** Since the diagram $G\mathcal{M}\colon I \textstyle\int \mathcal{Q} \to \mathrm{Strict}$ is so simple, only consisting of diagonals and inclusions onto factors in products, algebraic structure on $\mathcal{M}$ is easily transferred to $G\mathcal{M}$.

Proposition 3.2. *If $\mathcal{R}$ is a strictly bimonoidal category, then $G\mathcal{R}$ is an $I \textstyle\int \mathcal{Q}$-graded strictly bimonoidal category. If $\mathcal{R}$ is a bipermutative category, then $G\mathcal{R}$ is an $I \textstyle\int \mathcal{Q}$-graded bipermutative category.*

Proof. We must specify composition functors

$$\otimes\colon G\mathcal{R}(\mathbf{n}, T) \times G\mathcal{R}(\mathbf{m}, S) \to G\mathcal{R}(\mathbf{n} + \mathbf{m}, T + S)$$

for all $(\mathbf{n}, T), (\mathbf{m}, S) \in I \textstyle\int \mathcal{Q}$. Let $a \in G\mathcal{R}(\mathbf{n}, T)$ and $b \in G\mathcal{R}(\mathbf{m}, S)$. If S and T only contain positive elements, then $a \otimes b \in G\mathcal{R}(\mathbf{n} + \mathbf{m}, T + S)$ is defined by

$$(a \otimes b)_{V+U} = a_V \otimes b_U\ ,$$

where the $\otimes$-product on the right is formed in $\mathcal{R}$. As V and U range over all the subsets of T and S, respectively, $V + U$ ranges over all the subsets of $T + S$. If T or S contain negative elements, we set $a \otimes b = 0$. Likewise for morphisms in $G\mathcal{R}(\mathbf{n}, T)$ and $G\mathcal{R}(\mathbf{m}, S)$. These composition functors are clearly natural in $(\mathbf{n}, T)$ and $(\mathbf{m}, S)$.

The unit object 1 of $G\mathcal{R}(\mathbf{0}, \mathbf{0}) \cong \mathcal{R}$ corresponds to the unit object $1_{\mathcal{R}}$ of $\mathcal{R}$. In the bipermutative case, the twist isomorphism

$$\gamma_\otimes\colon a \otimes b \longrightarrow G\mathcal{R}(\chi(m, n), \mathrm{id})(b \otimes a)$$

has components

$$(a \otimes b)_{V+U} = a_V \otimes b_U \xrightarrow{\gamma_\otimes^{\mathcal{R}}} b_U \otimes a_V = (b \otimes a)_{U+V}$$

for all $V \subseteq T$ and $U \subseteq S$, where $\gamma_\otimes^{\mathcal{R}}$ is the twist isomorphism in $\mathcal{R}$.

Since everything is defined pointwise, the multiplicative structure on $\mathcal{R}$ forces all the axioms of an $I \textstyle\int \mathcal{Q}$-graded strictly bimonoidal category (or $I \textstyle\int \mathcal{Q}$-graded bipermutative category) to hold for $G\mathcal{R}$. $\square$

4. Hocolim-lemmata

We recall Thomason's homotopy colimit construction in the case of a J-shaped diagram of zeroless permutative categories, and then construct a derived version of this construction for permutative categories with zero.

4.1. **The case without zeros.** Let $\mathrm{Perm}_{\mathrm{nz}}$ be the category of permutative categories without zero objects, and lax symmetric monoidal functors. Let $\mathrm{Strict}_{\mathrm{nz}}$ be the subcategory with the same objects, but with strict symmetric monoidal functors as morphisms. There are forgetful functors $V\colon \mathrm{Strict}_{\mathrm{nz}} \to \mathrm{Perm}_{\mathrm{nz}}$ and $U\colon \mathrm{Perm}_{\mathrm{nz}} \to \mathrm{Cat}$, with composite $W = UV\colon \mathrm{Strict}_{\mathrm{nz}} \to \mathrm{Cat}$.

For any small category J, let Cat^J be the category of functors $J \to \mathrm{Cat}$ and *left lax transformations*. Recall that for functors $\mathcal{C}, \mathcal{D}\colon J \to \mathrm{Cat}$, a left lax transformation $F\colon \mathcal{C} \to \mathcal{D}$ assigns to each object $x \in J$ a functor $F_x\colon \mathcal{C}(x) \to \mathcal{D}(x)$, and to each morphism $k\colon x \to y$ in J a natural transformation $\nu^k\colon \mathcal{D}(k) \circ F_x \Rightarrow F_y \circ \mathcal{C}(k)$ of functors $\mathcal{C}(x) \to \mathcal{D}(y)$:

$$\begin{tikzcd} \mathcal{C}(x) \arrow[r, "\mathcal{C}(k)"] \arrow[d, "F_x"'] & \mathcal{C}(y) \arrow[d, "F_y"] \\ \mathcal{D}(x) \arrow[r, "\mathcal{D}(k)"'] \arrow[ur, Rightarrow, "\nu^k"] & \mathcal{D}(y) \end{tikzcd}$$

These must be compatible with composition in J, so that $\nu^{\mathrm{id}} = \mathrm{id}$ and $\nu^{\ell k} = \nu^\ell \mathcal{C}(k) \circ \mathcal{D}(\ell)\nu^k$ for $\ell\colon y \to z$ in J. If each $\nu^k = \mathrm{id}$, we have a natural transformation in the usual sense.

Similarly, let $\mathrm{Perm}_{\mathrm{nz}}{}^J$ be the category of functors $J \to \mathrm{Perm}_{\mathrm{nz}}$ and left lax transformations. In this case, the categories $\mathcal{C}(x), \mathcal{C}(y), \mathcal{D}(x), \mathcal{D}(y)$ etc. are symmetric monoidal without zero, the functors $\mathcal{C}(k), \mathcal{D}(k), F_x, F_y$ etc. are lax symmetric monoidal, and ν^k is a natural transformation of lax symmetric monoidal functors. Finally, let $\mathrm{Strict}_{\mathrm{nz}}{}^J$ be the category of functors $J \to \mathrm{Strict}_{\mathrm{nz}}$ and left lax transformations. In this case, all of the symmetric monoidal functors are strict.

Let $\Delta\colon \mathrm{Cat} \to \mathrm{Cat}^J$ be the constant J-shaped diagram functor. Given a functor $\mathcal{C}\colon J \to \mathrm{Cat}$, the Grothendieck construction $J \wr \mathcal{C}$ is a model for the homotopy colimit in Cat [16]. We recall that an object in $J \wr \mathcal{C}$ is a pair (x, X) where $x \in J$ and $X \in \mathcal{C}(x)$ are objects, while a morphism $(x, X) \to (y, Y)$ is a pair (k, f) where $k\colon x \to y \in J$ and $f\colon \mathcal{C}(k)(X) \to Y \in \mathcal{C}(y)$ are morphisms. This construction defines a functor $J \wr (-)\colon \mathrm{Cat}^J \to \mathrm{Cat}$, which is left adjoint to $\Delta\colon \mathrm{Cat} \to \mathrm{Cat}^J$. Here it is, of course, important that we are allowing left lax transformations as morphisms in Cat^J, since otherwise the left adjoint would be the categorical colimit.

Thomason's homotopy colimit of permutative categories [18] is constructed to have a similar universal property with respect to the composite $\Delta V\colon \mathrm{Strict}_{\mathrm{nz}} \to \mathrm{Perm}_{\mathrm{nz}}{}^J$, where V is as above and $\Delta\colon \mathrm{Perm}_{\mathrm{nz}} \to \mathrm{Perm}_{\mathrm{nz}}{}^J$ is the constant J-shaped diagram functor. We briefly recall the explicit description.

Definition 4.1. Let $\mathcal{C}\colon J \to \mathrm{Perm}_{\mathrm{nz}}$ be a functor. An object in $\mathrm{hocolim}_J \mathcal{C}$ is an expression

$$n[(x_1, X_1), \dots, (x_n, X_n)]$$

where $n \geqslant 1$ is a natural number, each x_i is an object of J, and each X_i is an object of $\mathcal{C}(x_i)$. A morphism from $n[(x_1, X_1), \dots, (x_n, X_n)]$ to $m[(y_1, Y_1), \dots, (y_m, Y_m)]$ consists of three parts: a surjective function $\psi\colon \mathbf{n} \to \mathbf{m}$, morphisms $\ell_i\colon x_i \to y_{\psi(i)}$ in J for each $1 \leqslant i \leqslant n$, and morphisms $\varrho_j\colon \bigoplus_{i \in \psi^{-1}(j)} \mathcal{C}(\ell_i)(X_i) \to Y_j$ in $\mathcal{C}(y_j)$ for each $1 \leqslant j \leqslant m$. We will write $(\psi, \ell_i, \varrho_j)$ to signify this morphism.

See [18, 3.22] for the definition of composition in the category $\mathrm{hocolim}_J \mathcal{C}$. This category is permutative, without a zero, if one defines addition to be given by concatenation [18, p. 1632]. Each left lax transformation $F\colon \mathcal{C} \to \mathcal{D}$ induces a strict symmetric monoidal functor $\mathrm{hocolim}_J F\colon \mathrm{hocolim}_J \mathcal{C} \to \mathrm{hocolim}_J \mathcal{D}$, so this construction defines a functor $\mathrm{hocolim}_J\colon \mathrm{Perm}_{\mathrm{nz}}{}^J \to \mathrm{Strict}_{\mathrm{nz}}$.

The universal property in [18, 3.21] says that $\mathrm{hocolim}_J\colon \mathrm{Perm}_{\mathrm{nz}}{}^J \to \mathrm{Strict}_{\mathrm{nz}}$ is left adjoint to $\Delta V\colon \mathrm{Strict}_{\mathrm{nz}} \to \mathrm{Perm}_{\mathrm{nz}}{}^J$. Again, it is critical that we are allowing left lax transformations as morphisms in $\mathrm{Perm}_{\mathrm{nz}}{}^J$.

Recall Definition 1.1 of unstable equivalences in Cat and stable equivalences in $\mathrm{Perm}_{\mathrm{nz}}$ and $\mathrm{Strict}_{\mathrm{nz}}$. We use the corresponding pointwise notions in diagram categories like Cat^J and $\mathrm{Perm}_{\mathrm{nz}}{}^J$, so a left lax transformation $F\colon \mathcal{C} \to \mathcal{D}$ between functors $\mathcal{C}, \mathcal{D}\colon J \to \mathrm{Perm}_{\mathrm{nz}}$ is a stable (resp. unstable) equivalence if every one of its components $F_x\colon \mathcal{C}(x) \to \mathcal{D}(x)$ is a stable (resp. unstable) equivalence, for $x \in J$.

Lemma 4.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a stable (resp. unstable) equivalence in* $\mathrm{Perm}_{\mathrm{nz}}{}^J$*. Then*

$$\mathrm{hocolim}_J F\colon \mathrm{hocolim}_J \mathcal{C} \to \mathrm{hocolim}_J \mathcal{D}$$

is a stable (resp. unstable) equivalence in $\mathrm{Strict}_{\mathrm{nz}}$.

If $\mathcal{C}\colon J \to \mathrm{Perm}_{\mathrm{nz}}$ *is a constant functor and* J *is contractible, then* $\mathcal{C}(x) \xrightarrow{\sim} \operatorname{hocolim}_J \mathcal{C}$ *is an unstable equivalence for each* $x \in J$.

Proof. The stable case follows from [18, 4.1], since homotopy colimits of spectra preserve stable equivalences. The unstable case follows by the same line of argument, see [18, p. 1635] for an overview.

First consider the strict case, when $F\colon \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ is a left lax transformation and unstable equivalence of functors $J \to \mathrm{Strict}_{\mathrm{nz}}$. The doubly forgetful functor $W\colon \mathrm{Strict}_{\mathrm{nz}} \to \mathrm{Cat}$ has a left adjoint, the free functor $P\colon \mathrm{Cat} \to \mathrm{Strict}_{\mathrm{nz}}$, with $P\mathcal{C} = \coprod_{n\geqslant 1} \widetilde{\Sigma_n} \times_{\Sigma_n} \mathcal{C}^{\times n}$, where $\widetilde{\Sigma_n}$ is the translation category of the symmetric group Σ_n.

The free–forgetful adjunction (P, W) generates a simplicial resolution $(PW)^{\bullet+1}\mathcal{C} = \{[q] \mapsto (PW)^{q+1}\mathcal{C}\}$ of $\mathcal{C}$ by free zeroless permutative categories. See [18, 1.2] for details. By Thomason's argument [18, p. 1641–1644], the augmentation $(PW)^{\bullet+1}\mathcal{C} \to \mathcal{C}$ induces an unstable equivalence

$$\operatorname{hocolim}_J V(PW)^{\bullet+1}\mathcal{C} \xrightarrow{\sim} \operatorname{hocolim}_J V\mathcal{C}\,,$$

and similarly for $\mathcal{D}$ and F. Hence it suffices to prove that $\operatorname{hocolim}_J V(PW)^{q+1}F$ is an unstable equivalence, for each $q \geqslant 0$.

Consider the functors $\mathcal{C}', \mathcal{D}'\colon J \to \mathrm{Cat}$ given by $\mathcal{C}' = W(PW)^q\mathcal{C}$ and $\mathcal{D}' = W(PW)^q\mathcal{D}$, and let $F' = W(PW)^qF$ be the resulting left lax transformation $\mathcal{C}' \to \mathcal{D}'$. Then F' is an unstable equivalence, by q applications of Lemma 4.3 below. We must prove that $VPF'\colon VP\mathcal{C}' \to VP\mathcal{D}'$ induces an unstable equivalence of homotopy colimits. This follows from Lemma 4.4 below, the fact that the Grothendieck construction $J \int F'\colon J \int \mathcal{C}' \to J \int \mathcal{D}'$ respects unstable equivalences, and one more application of Lemma 4.3.

Finally consider the lax case, when $F\colon \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ is a left lax transformation and unstable equivalence of functors $J \to \mathrm{Perm}_{\mathrm{nz}}$. For each $x \in J$ let $\hat{\mathcal{C}}(x) = \operatorname{hocolim}_0 \mathcal{C}(x)$ be the homotopy colimit of the functor $0 \to \mathrm{Perm}_{\mathrm{nz}}$ taking the unique object of 0 to $\mathcal{C}(x)$. By the universal property of $\operatorname{hocolim}_0$ this defines a functor $\hat{\mathcal{C}}\colon J \to \mathrm{Strict}_{\mathrm{nz}}$, and a natural transformation $\mathcal{C} \to V\hat{\mathcal{C}}$. It is an unstable equivalence by [14, 4.3]. Summation in the permutative categories $\mathcal{C}(x)$ induces a left lax natural transformation $V\hat{\mathcal{C}} \to \mathcal{C}$, such that the composite $\mathcal{C} \to V\hat{\mathcal{C}} \to \mathcal{C}$ equals the identity transformation.

By naturality of these constructions with respect to F, we see that $F\colon \mathcal{C} \to \mathcal{D}$ is a retract of $V\hat{F}\colon V\hat{\mathcal{C}} \to V\hat{\mathcal{D}}$ as a morphism in ${\mathrm{Perm}_{\mathrm{nz}}}^J$, where $\hat{F}\colon \hat{\mathcal{C}} \xrightarrow{\sim} \hat{\mathcal{D}}$ is a left lax transformation and unstable equivalence of functors $J \to \mathrm{Strict}_{\mathrm{nz}}$. By functoriality, $\operatorname{hocolim}_J F$ is a retract of $\operatorname{hocolim}_J V\hat{F}$, which is an unstable equivalence by the first case of the proof applied to $\hat{F}\colon \hat{\mathcal{C}} \to \hat{\mathcal{D}}$. It follows that $\operatorname{hocolim}_J F$ is also an unstable equivalence.

The claim in the case of a constant diagram follows by the same argument. □

Lemma 4.3. *The free functor* $P\colon \mathrm{Cat} \to \mathrm{Strict}_{\mathrm{nz}}$ *sends unstable equivalences to unstable equivalences.*

Proof. This follows from the natural homeomorphism of classifying spaces $|P\mathcal{C}| \cong \coprod_{n\geqslant 1} E\Sigma_n \times_{\Sigma_n} |\mathcal{C}|^{\times n}$ and the fact that $|\widetilde{\Sigma_n}| = E\Sigma_n$ is a free Σ_n-space. □

Lemma 4.4. *Let* $\mathcal{C}'\colon J \to \mathrm{Cat}$ *be any functor. There is a natural unstable equivalence*

$$P(J \textstyle\int \mathcal{C}') \xrightarrow{\sim} \operatorname{hocolim}_J VP\mathcal{C}'\,.$$

Proof. Thomason proved this in [18]. There the statement appears in the second paragraph on page 1639, in rather different–looking notation, and the proof starts with the last paragraph on page 1637. □

Lemma 4.5. *Let* I *be the category of finite sets and injective functions, and let* $\mathbf{m} \in I$. *If* $\mathcal{C}\colon I \to \mathrm{Perm}_{\mathrm{nz}}$ *is a functor that sends each* $\varphi\colon \mathbf{m} \to \mathbf{n} \in I$ *to a stable (resp. unstable) equivalence* $\mathcal{C}(\varphi)\colon \mathcal{C}(\mathbf{m}) \to \mathcal{C}(\mathbf{n})$, *then the canonical functor* $\mathcal{C}(\mathbf{m}) \to \operatorname{hocolim}_I \mathcal{C}$ *is a stable (resp. unstable) equivalence.*

Proof. This is a weak version of Bökstedt's lemma [6, 9.1], which holds for homotopy colimits in Cat since it holds for homotopy colimits in simplicial sets. By the argument above, using the resolution by free permutative categories, it also holds in $\mathrm{Perm}_{\mathrm{nz}}$. □

4.2. **The case with zero.** We shall need a version of the homotopy colimit for diagrams of permutative categories with zero. Thomason comments that such a homotopy colimit with zero is not a homotopy functor, unless the category is "well based". Hence we must derive our functor to get a homotopy invariant version. We do this by means of another simplicial resolution, this time generated by the free–forgetful adjunction between permutative categories with and without zeros.

The functor $R\colon \mathrm{Strict} \to \mathrm{Strict}_{\mathrm{nz}}$ that forgets the special role of the zero object has a left adjoint $L\colon \mathrm{Strict}_{\mathrm{nz}} \to \mathrm{Strict}$, given by adding a disjoint zero object: $L\mathcal{N} = \mathcal{N}_+$ for $\mathcal{N} \in \mathrm{Strict}_{\mathrm{nz}}$. Likewise, the forgetful functor $R'\colon \mathrm{Perm}^J \to {\mathrm{Perm}_{\mathrm{nz}}}^J$ has a left adjoint $L'\colon {\mathrm{Perm}_{\mathrm{nz}}}^J \to \mathrm{Perm}^J$, given by adding disjoint zeros pointwise: $L'\mathcal{C}\colon x \mapsto \mathcal{C}(x)_+$ for $\mathcal{C}\colon J \to \mathrm{Perm}_{\mathrm{nz}}$ and all $x \in J$.

Let $\mathrm{Strict}_{\mathrm{iz}} \subset \mathrm{Strict}$ be the full subcategory generated by objects of the form $L\mathcal{N} = \mathcal{N}_+$, *i.e.*, the permutative categories with an *isolated zero* object. Similarly, let ${\mathrm{Perm}_{\mathrm{iz}}}^J \subset \mathrm{Perm}^J$ be the full subcategory generated by objects of the form $L'\mathcal{C} = \mathcal{C}_+$.

In the statement and proof of following lemma we omit the forgetful functors R and R' from the notation, and write $\mathcal{N}_+$ and $\mathcal{C}_+$ for $L\mathcal{N}$ and $L'\mathcal{C}$, respectively. Note that $\Delta V(\mathcal{N}_+) = \Delta V(\mathcal{N})_+$, where $\Delta V\colon \mathrm{Strict}_{\mathrm{nz}} \to {\mathrm{Perm}_{\mathrm{nz}}}^J$ is as in subsection 4.1.

Lemma 4.6. *The functor $(\Delta V)_{\mathrm{iz}}\colon \mathrm{Strict}_{\mathrm{iz}} \to {\mathrm{Perm}_{\mathrm{iz}}}^J$, taking $\mathcal{N}_+$ to the constant diagram $x \mapsto \mathcal{N}_+$ for $x \in J$, has a left adjoint ${\mathrm{hocolim}^{\mathrm{iz}}}_J\colon {\mathrm{Perm}_{\mathrm{iz}}}^J \to \mathrm{Strict}_{\mathrm{iz}}$, satisfying*

$$ {\mathrm{hocolim}^{\mathrm{iz}}}_J(\mathcal{C}_+) = (\mathrm{hocolim}_J \mathcal{C})_+ $$

for all zeroless diagrams $\mathcal{C}\colon J \to \mathrm{Perm}_{\mathrm{nz}}$.

Proof. To define the functor ${\mathrm{hocolim}^{\mathrm{iz}}}_J$, we must specify a strict symmetric monoidal functor

$$ {\mathrm{hocolim}^{\mathrm{iz}}}_J F\colon (\mathrm{hocolim}_J \mathcal{C})_+ \longrightarrow (\mathrm{hocolim}_J \mathcal{D})_+ $$

for each left lax transformation $F\colon \mathcal{C}_+ \to \mathcal{D}_+$. The morphism

$$ \eta_+\colon \mathcal{D}_+ \to (\Delta V(\mathrm{hocolim}_J \mathcal{D}))_+ = \Delta V((\mathrm{hocolim}_J \mathcal{D})_+)\,, $$

where $\eta\colon \mathrm{id} \to \Delta V \circ \mathrm{hocolim}_J$ is the adjunction unit, induces a function

$$ \begin{aligned} \mathrm{Perm}^J(\mathcal{C}_+, \mathcal{D}_+) &\cong {\mathrm{Perm}_{\mathrm{nz}}}^J(\mathcal{C}, \mathcal{D}_+) \to {\mathrm{Perm}_{\mathrm{nz}}}^J(\mathcal{C}, \Delta V((\mathrm{hocolim}_J \mathcal{D})_+)) \\ &\cong \mathrm{Strict}_{\mathrm{nz}}(\mathrm{hocolim}_J \mathcal{C}, (\mathrm{hocolim}_J D)_+) \cong \mathrm{Strict}((\mathrm{hocolim}_J \mathcal{C})_+, (\mathrm{hocolim}_J D)_+)\,. \end{aligned} $$

We declare the image of $F\colon \mathcal{C}_+ \to \mathcal{D}_+$ to be ${\mathrm{hocolim}^{\mathrm{iz}}}_J F$. A diagram chase shows that ${\mathrm{hocolim}^{\mathrm{iz}}}_J(GF) = {\mathrm{hocolim}^{\mathrm{iz}}}_J G \circ {\mathrm{hocolim}^{\mathrm{iz}}}_J F$ for each $G\colon \mathcal{D}_+ \to \mathcal{E}_+$, so ${\mathrm{hocolim}^{\mathrm{iz}}}_J$ is a functor.

The adjunction property follows from the chain of natural bijections

$$ \begin{aligned} \mathrm{Strict}({\mathrm{hocolim}^{\mathrm{iz}}}_J(\mathcal{C}_+), \mathcal{N}_+) &= \mathrm{Strict}((\mathrm{hocolim}_J \mathcal{C})_+, \mathcal{N}_+) \cong \mathrm{Strict}_{\mathrm{nz}}(\mathrm{hocolim}_J \mathcal{C}, \mathcal{N}_+) \\ &\cong {\mathrm{Perm}_{\mathrm{nz}}}^J(\mathcal{C}, \Delta V(\mathcal{N}_+)) = {\mathrm{Perm}_{\mathrm{nz}}}^J(\mathcal{C}, (\Delta V)_{\mathrm{iz}}(\mathcal{N}_+)) \\ &\cong \mathrm{Perm}^J(\mathcal{C}_+, (\Delta V)_{\mathrm{iz}}(\mathcal{N}_+))\,. \end{aligned} $$

□

Definition 4.7. For each permutative category with zero $\mathcal{M} \in \mathrm{Perm}$ let $Z\mathcal{M} \in {\mathrm{Perm}_{\mathrm{iz}}}^{\Delta^{op}} \subset \mathrm{Perm}^{\Delta^{op}}$ be the simplicial object in permutative categories with isolated zeroes given by

$$ [q] \mapsto Z_q\mathcal{M} = (LR)^{q+1}(\mathcal{M})\,. $$

The face and degeneracy maps are induced by the adjunction counit $LR \to \mathrm{id}$ and unit $\mathrm{id} \to RL$, as usual. The counit also induces a natural augmentation map $\epsilon\colon Z\mathcal{M} \to \mathcal{M}$ of simplicial permutative categories with zero, where $\mathcal{M}$ is viewed as a constant simplicial object.

Lemma 4.8. *Let $\mathcal{M}$ be a permutative category. The augmentation map $\epsilon\colon Z\mathcal{M} \xrightarrow{\sim} \mathcal{M}$ is an unstable equivalence.*

Proof. The map $R\epsilon\colon RZ\mathcal{M} \to R\mathcal{M}$ of simplicial zeroless permutative categories admits a simplicial homotopy inverse, induced by the adjunction unit. Hence the map of classifying spaces $|\epsilon|\colon |Z\mathcal{M}| \to |\mathcal{M}|$ admits a homotopy inverse, since the classifying space only depends on the underlying category. □

We extend Z pointwise to define a simplicial resolution $\epsilon\colon Z\mathcal{C} \xrightarrow{\sim} \mathcal{C}$ for any $\mathcal{C}\colon J \to \mathrm{Perm}$, with $Z\mathcal{C}\colon x \mapsto Z\mathcal{C}(x)$ for all $x \in J$. This allows us to define a derived homotopy colimit for permutative categories with zero.

Definition 4.9. The *derived homotopy colimit*

$$ \mathrm{Dhocolim}_J\colon \mathrm{Perm}^J \to {\mathrm{Strict}_{\mathrm{iz}}}^{\Delta^{op}} \subset \mathrm{Strict}^{\Delta^{op}} $$

is defined by

$$ \mathrm{Dhocolim}_J \mathcal{C} = {\mathrm{hocolim}^{\mathrm{iz}}}_J(Z\mathcal{C}) = \{[q] \mapsto {\mathrm{hocolim}^{\mathrm{iz}}}_J(LR)^{q+1}\mathcal{C}\}\,. $$

The construction deserves its name:

Lemma 4.10. *Let $\mathcal{C} \to \mathcal{D}$ be a stable (resp. unstable) equivalence in Perm^J. Then $Z_q\mathcal{C} \to Z_q\mathcal{D}$ is a stable (resp. unstable) equivalence for each $q \geqslant 0$, so the induced functor*

$$\mathrm{Dhocolim}_J\,\mathcal{C} \longrightarrow \mathrm{Dhocolim}_J\,\mathcal{D}$$

is a stable (resp. unstable) equivalence, too.

Proof. The functor LR adds a disjoint base point to the classifying space, and the counit $LR \to \mathrm{id}$ induces a stable equivalence of spectra [18, 2.1]. Hence LR preserves both stable and unstable equivalences. Iterating $(q+1)$ times yields the assertion for Z_q. □

Lemma 4.11. *Let I be the category of finite sets and injective functions, and let $\mathbf{m} \in I$. If $\mathcal{C}\colon I \to \mathrm{Perm}$ is a functor that sends each $\varphi\colon \mathbf{m} \to \mathbf{n} \in I$ to a stable (resp. unstable) equivalence $\mathcal{C}(\varphi)\colon \mathcal{C}(\mathbf{m}) \to \mathcal{C}(\mathbf{n})$, then the canonical functors*

$$\mathcal{C}(\mathbf{m}) \xleftarrow{\sim} Z\mathcal{C}(\mathbf{m}) \longrightarrow \mathrm{Dhocolim}_I\,\mathcal{C}$$

are stable (resp. unstable) equivalences.

Proof. This follows from Lemmas 4.5 and 4.8. □

5. The homotopy colimit of a graded bipermutative category

We are now ready for a key proposition. To emphasize the graded nature of our constructions we again use the notation $\mathcal{C}^x$ for the value of $\mathcal{C}$ at x, where we have previously written $\mathcal{C}(x)$.

Proposition 5.1. *Let J be a permutative category, and let $\mathcal{C}^\bullet$ be a J-graded bipermutative category. Then $\mathrm{Dhocolim}_J\,\mathcal{C}^\bullet$ is a simplicial bipermutative category, and*

$$\mathcal{C}^0 \xleftarrow{\sim} Z\mathcal{C}^0 \longrightarrow \mathrm{Dhocolim}_J\,\mathcal{C}^\bullet$$

are lax morphisms of simplicial bipermutative categories. The same statements hold when replacing "bipermutative" by "strictly bimonoidal".

Furthermore, for each $x \in J$, the canonical functors

$$\mathcal{C}^x \xleftarrow{\sim} Z\mathcal{C}^x \longrightarrow \mathrm{Dhocolim}_J\,\mathcal{C}^\bullet$$

are maps of $Z\mathcal{C}^0$-modules.

Proof. Recall the adjoint pair (L', R') from subsection 4.2. If $\mathcal{C}^\bullet$ is a J-graded bipermutative category, then so is $L'R'\mathcal{C}^\bullet = \mathcal{C}^\bullet_+$, and $Z\mathcal{C}^\bullet$ becomes a simplicial J-graded bipermutative category. By Lemma 5.2, which we will prove below, we get that $\mathrm{hocolim}_J\,R'(L'R')^q\mathcal{C}^\bullet$ becomes a zeroless bipermutative category for each $q \geqslant 0$. Hence $\mathrm{hocolim}^{\mathrm{iz}}{}_J\,Z_q\mathcal{C}^\bullet = L\,\mathrm{hocolim}_J\,R'(L'R')^q\mathcal{C}^\bullet$ is a bipermutative category, and all the simplicial structure maps are lax morphisms of bipermutative categories. Therefore $\mathrm{Dhocolim}_J\,\mathcal{C}^\bullet$ becomes a simplicial bipermutative category.

Likewise, for each $q \geqslant 0$, Lemma 5.2 below guarantees that

$$Z_q\mathcal{C}^0 \to \mathrm{hocolim}^{\mathrm{iz}}{}_J\,Z_q\mathcal{C}^\bullet$$

is a lax morphism of bipermutative categories and that each

$$Z_q\mathcal{C}^x \to \mathrm{hocolim}^{\mathrm{iz}}{}_J\,Z_q\mathcal{C}^\bullet$$

is a map of $Z_q\mathcal{C}^0$-modules, so we are done by functoriality. □

We omit the forgetful functors R and R' in the statement and proof of the following lemma, which contains the most detailed diagram chasing required in this paper.

Lemma 5.2. *Let J be a permutative category. If $\mathcal{C}^\bullet$ is a J-graded bipermutative category, then Thomason's homotopy colimit of permutative categories $\mathrm{hocolim}_J\,\mathcal{C}^\bullet$ is a zeroless bipermutative category. The canonical functor $\mathcal{C}^0 \to \mathrm{hocolim}_J\,\mathcal{C}^\bullet$ is a lax morphism of zeroless bipermutative categories. Furthermore, for each $x \in J$, the canonical functor*

$$\mathcal{C}^x \longrightarrow \mathrm{hocolim}_J\,\mathcal{C}^\bullet$$

is a map of zeroless $\mathcal{C}^0$-modules.

If $\mathcal{C}^\bullet$ is a J-graded strictly bimonoidal category, then $\mathrm{hocolim}_J\,\mathcal{C}^\bullet$ is a zeroless strictly bimonoidal category with a lax morphism of zeroless strictly bimonoidal categories $\mathcal{C}^0 \to \mathrm{hocolim}_J\,\mathcal{C}^\bullet$, and zeroless $\mathcal{C}^0$-module maps $\mathcal{C}^x \to \mathrm{hocolim}_J\,\mathcal{C}^\bullet$.

Proof. Thomason showed that the homotopy colimit is a permutative category without zero. The additive twist isomorphism

$$\tau_\oplus\colon n[(x_1, X_1), \dots, (x_n, X_n)] \oplus m[(y_1, Y_1), \dots, (y_m, Y_m)] \xrightarrow{\cong} m[(y_1, Y_1), \dots, (y_m, Y_m)] \oplus n[(x_1, X_1), \dots, (x_n, X_n)]$$

is given by $(\chi(n, m), \mathrm{id}, \mathrm{id})$, where $\chi(n, m) \in \Sigma_{n+m}$ is as in Example 2.3. Let $[X]$ and $[Y]$ be shorthand notations for the objects $n[(x_1, X_1), \dots, (x_n, X_n)]$ and $m[(y_1, Y_1), \dots, (y_m, Y_m)]$, respectively. The twist isomorphism for $\oplus$ then appears as

$$\tau_\oplus\colon [X] \oplus [Y] \xrightarrow{\cong} [Y] \oplus [X] .$$

In order to distinguish the multiplicative structure of $\mathcal{C}^\bullet$ from the one on the homotopy colimit, we shall simply denote the composition functor $\otimes$ on $\mathcal{C}^\bullet$ by juxtaposition of objects, or by $\cdot$. The multiplicative bifunctor $\otimes$ on the homotopy colimit is then defined at the object level by

$$\begin{aligned} &n[(x_1, X_1), \dots, (x_n, X_n)] \otimes m[(y_1, Y_1), \dots, (y_m, Y_m)] \\ &\quad := nm[(x_1 + y_1, X_1Y_1), \dots, (x_1 + y_m, X_1Y_m), \dots, (x_n + y_1, X_nY_1), \dots, (x_n + y_m, X_nY_m)] , \end{aligned}$$

where the entries are ordered lexicographically. We will use the shorthand notation $[X] \otimes [Y]$ for this object.

The object $1 := 1[(0, 1)]$ is a unit for $\otimes$. With these definitions, as extended below to the morphism level, $(\mathrm{hocolim}_J\, \mathcal{C}^\bullet, \otimes, 1)$ is a strict monoidal category.

We will define the multiplicative twist map $\tau_\otimes\colon [X] \otimes [Y] \xrightarrow{\cong} [Y] \otimes [X]$ as a composite of two morphisms. First, we apply the twist map $\gamma_\otimes$ for the multiplication in $\mathcal{C}^\bullet$ to every entry of the form X_iY_j. The triple $(\mathrm{id}_{\mathbf{nm}}, \chi^{x_i,y_j}, \gamma_\otimes)$ defines a morphism

$$\begin{aligned} &nm[(x_1 + y_1, X_1Y_1), \dots, (x_1 + y_m, X_1Y_m), \dots, (x_n + y_1, X_nY_1), \dots, (x_n + y_m, X_nY_m)] \\ &\quad \to nm[(y_1 + x_1, Y_1X_1), \dots, (y_m + x_1, Y_mX_1), \dots, (y_1 + x_n, Y_1X_n), \dots, (y_m + x_n, Y_mX_n)] . \end{aligned}$$

(Here χ is the twist map in J. To be precise, the third coordinate of the morphism is really $\mathcal{C}(\chi^{x_i,y_j})(\gamma_\otimes)$, but we omit $\mathcal{C}(\chi^{x_i,y_j})$ from the notation.) Second, we use the permutation $\sigma_{n,m} \in \Sigma_{nm}$ that induces matrix transposition, that is, transposition in the lexicographic ordering of the entries. The triple $(\sigma_{n,m}, \mathrm{id}_{y_j+x_i}, \mathrm{id})$ defines a morphism

$$\begin{aligned} &nm[(y_1 + x_1, Y_1X_1), \dots, (y_m + x_1, Y_mX_1), \dots, (y_1 + x_n, Y_1X_n), \dots, (y_m + x_n, Y_mX_n)] \\ &\quad \to mn[(y_1 + x_1, Y_1X_1), \dots, (y_1 + x_n, Y_1X_n), \dots, (y_m + x_1, Y_mX_1), \dots, (y_m + x_n, Y_mX_n)] . \end{aligned}$$

Let the twist map for $\otimes$ be the composite morphism $\tau_\otimes = (\sigma_{n,m}, \mathrm{id}_{y_j+x_i}, \mathrm{id}) \circ (\mathrm{id}_{\mathbf{nm}}, \chi^{x_i,y_j}, \gamma_\otimes)$.

As matrix transposition squares to the identity, $\chi^{y_j,x_i} \circ \chi^{x_i,y_j} = \mathrm{id}$ and $\gamma_\otimes^2 = \mathrm{id}$, we obtain that $\tau_\otimes^2 = \mathrm{id}$. If $[X] = 1$ is the multiplicative unit, then we have that $\sigma_{1,m}$ is the identity in Σ_m and χ^{0,y_j} is the identity as well, so $\tau_\otimes\colon 1 \otimes [Y] \to [Y] \otimes 1$ is the identity. Similarly one shows that $\tau_\otimes$ gives the identity morphism if $[Y] = 1$ is the multiplicative unit.

We have now verified properties (1), (2) and (3) of Definition 2.1, at the level of objects. We leave to the reader to check property (4). Property (5) is disregarded in the zeroless situation.

Writing out $([X] \otimes [Y]) \oplus ([X'] \otimes [Y])$ and $([X] \oplus [X']) \otimes Y$ we get the same object, and we define the right distributivity d_r to be the identity morphism between these two expressions. The left distributivity d_ℓ involves a reordering of elements. It is a morphism

$$d_\ell\colon ([X] \otimes [Y]) \oplus ([X] \otimes [Y']) \longrightarrow [X] \otimes ([Y] \oplus [Y']) .$$

The source is

$$(nm + nm')[(x_1 + y_1, X_1Y_1), \dots, (x_n + y_m, X_nY_m), (x_1 + y'_1, X_1Y'_1), \dots, (x_n + y'_{m'}, X_nY'_{m'})] ,$$

while the target is

$$n(m + m')[(x_1 + y_1, X_1Y_1), \dots, (x_1 + y'_{m'}, X_1Y'_{m'}), \dots, (x_n + y_1, X_nY_1), \dots, (x_n + y'_{m'}, X_nY'_{m'})] .$$

The same terms $(x_i + y_j, X_iY_j)$ and $(x_i + y'_j, X_iY'_j)$ occur in both the source and target, but their ordering differs by a suitable permutation $\xi \in \Sigma_{nm+nm'}$. Thus we define the morphism d_ℓ by the triple $(\xi, \mathrm{id}, \mathrm{id})$. Note that ξ is the left distributivity isomorphism in the bipermutative category of finite sets and functions, as defined in Example 2.3.

We have to check that the so defined distributivity transformation d_ℓ coincides with $\tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes)$. The twist terms $\gamma_\otimes$ and χ occur twice in the composition, so they reduce to the identity. What is left is a permutation that is caused by $\tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes)$, and this is precisely ξ.

We have now verified properties (6) and (7) of Definition 2.1. Since the isomorphisms $\tau_\oplus$, d_r and d_ℓ are all of the form $(\sigma, \mathrm{id}, \mathrm{id})$ for suitable permutations σ, properties (8), (9) and (10) all follow from the corresponding ones in the bipermutative category of finite sets and functions.

This finishes the proof that the zeroless bipermutative category structure works fine on objects. It remains to establish that $\oplus$ and $\otimes$ are bifunctors on $\mathrm{hocolim}_J \mathcal{C}^\bullet$, that the various associativity and distributivity laws are natural, and that the additive and multiplicative twists are natural.

For $\oplus$ this is straightforward and can be found in [18]: suppose given two morphisms

$$(\psi, \ell_i, \varrho_j)\colon n[(x_1, X_1), \ldots, (x_n, X_n)] \to n'[(x'_1, X'_1), \ldots, (x'_{n'}, X'_{n'})]$$

and

$$(\varphi, k_i, \pi_j)\colon m[(y_1, Y_1), \ldots, (y_m, Y_m)] \to m'[(y'_1, Y'_1), \ldots, (y'_{m'}, Y'_{m'})]$$

in the homotopy colimit, with $\psi\colon \mathbf{n} \to \mathbf{n}'$, $\ell_i : x_i \to x'_{\psi(i)}$ and $\varrho_j : \bigoplus_{\psi(i)=j} \mathcal{C}(\ell_i)(X_i) \to X'_j$, and $\varphi\colon \mathbf{m} \to \mathbf{m}'$ with corresponding k_i and π_j. Then there is a surjection $\psi + \varphi$ from $\mathbf{n} + \mathbf{m}$ to $\mathbf{n}' + \mathbf{m}'$, and we can recycle the morphisms ℓ_i and k_i to give corresponding morphisms in J. In the third coordinate we can use the morphisms ϱ_j and π_j to get new ones, because the preimages of $\mathbf{n}'$ and $\mathbf{m}'$ under $\psi + \varphi$ are disjoint. Taken together, this results in a morphism from the sum $(n+m)[(x_1, X_1), \ldots, (y_m, Y_m)]$ to the sum $(n' + m')[(x'_1, X'_1), \ldots, (y'_{m'}, Y'_{m'})]$. It is straightforward to see that $\oplus$ defines a bifunctor, that the associativity law for $\oplus$ is natural, and that the additive twist $\tau_\oplus$ is natural.

For the remainder of this proof let us denote the elements in the set $\mathbf{nm} = \{1, \ldots, nm\}$ as pairs (i, j) with $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. The tensor product of the morphisms $(\psi, \ell_i, \varrho_j)$ and (φ, k_i, π_j) has three coordinates. On the first we take the product of the surjections, *i.e.*,

$$\mathbf{nm} \ni (i, j) \mapsto (\psi(i), \varphi(j)) \in \mathbf{n}'\mathbf{m}',$$

and on the second we take the sum $\ell_i + k_j\colon x_i + y_j \to x'_{\psi(i)} + y'_{\varphi(j)}$ of the morphisms $\ell_i, k_j \in J$. The third coordinate of the morphism $(\psi, \ell_i, \varrho_j) \otimes (\varphi, k_i, \pi_j)$ has to be a morphism

$$\bigoplus_{(\psi(i),\varphi(j))=(r,s)} \mathcal{C}(\ell_i + k_j)(X_i \cdot Y_j) = \bigoplus_{(\psi(i),\varphi(j))=(r,s)} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) \longrightarrow X'_r \cdot Y'_s$$

in $\mathcal{C}(x'_r + y'_s)$, for each $1 \leqslant r \leqslant n'$ and $1 \leqslant s \leqslant m'$. Here, the sum is taken with respect to the lexicographical ordering of the indices (i, j). Consider the following diagram:

$$\begin{tikzcd}
 & \bigoplus_{(\psi(i),\varphi(j))=(r,s)} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\sigma"] & \\
\bigoplus_{\psi(i)=r} \bigoplus_{\varphi(j)=s} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) \arrow[d, "\bigoplus_{\psi(i)=r} d_\ell"'] & & \bigoplus_{\varphi(j)=s} \bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) \arrow[d, "\bigoplus_{\varphi(j)=s} d_r"] \\
\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \cdot \Big(\bigoplus_{\varphi(j)=s} \mathcal{C}(k_j)(Y_j)\Big) \arrow[d, "\bigoplus_{\psi(i)=r} \mathrm{id} \cdot \pi_s"'] & & \bigoplus_{\varphi(j)=s} \Big(\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i)\Big) \cdot \mathcal{C}(k_j)(Y_j) \arrow[d, "\bigoplus_{\varphi(j)=s} \varrho_r \cdot \mathrm{id}"] \\
\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \cdot Y'_s \arrow[d, "d_r"'] & & \bigoplus_{\varphi(j)=s} X'_r \cdot \mathcal{C}(k_j)(Y_j) \arrow[d, "d_\ell"] \\
\Big(\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i)\Big) \cdot Y'_s \arrow[dr, "\varrho_r \cdot \mathrm{id}"'] & & X'_r \cdot \Big(\bigoplus_{\varphi(j)=s} \mathcal{C}(k_j)(Y_j)\Big) \arrow[dl, "\mathrm{id} \cdot \pi_s"] \\
 & X'_r \cdot Y'_s &
\end{tikzcd}$$

The isomorphism σ is an appropriate permutation of the summands. The distributivity laws in $\mathcal{C}^\bullet$ are natural with respect to morphisms in $\mathcal{C}^\bullet$, and therefore we have the identities:

$$d_r \circ \Big(\bigoplus_{\psi(i)=r} \mathrm{id}_{\mathcal{C}(\ell_i)(X_i)} \cdot \pi_s \Big) = \Big(\Big(\mathrm{id}_{\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i)} \Big) \cdot \pi_s \Big) \circ d_r$$
$$d_\ell \circ \Big(\bigoplus_{\varphi(j)=s} \varrho_r \cdot \mathrm{id}_{\mathcal{C}(k_j)(Y_j)} \Big) = \Big(\varrho_r \cdot \Big(\mathrm{id}_{\bigoplus_{\varphi(j)=s} \mathcal{C}(k_j)(Y_j)} \Big) \Big) \circ d_\ell$$

Combining these with the generalized pentagon equation

$$d_r \circ \bigoplus_{\psi(i)=r} d_\ell = d_\ell \circ \bigoplus_{\varphi(j)=s} d_r \circ \sigma$$

we see that the diagram commutes. We define the third coordinate in the tensor product morphism to be the composition given by either of the two branches.

Note that for $(\psi, \ell_i, \varrho_j) \otimes \mathrm{id}$ the definition reduces to $(\varrho_j \cdot \mathrm{id}) \circ d_r$, and similarly the third coordinate of $\mathrm{id} \otimes (\varphi, k_i, \pi_j)$ is $(\mathrm{id} \cdot \pi_j) \circ d_\ell$. In particular, the tensor product of identity morphisms is an identity morphism.

Compositions of morphisms in the homotopy colimit involve an additive twist [18, p. 1631]. For

$$(\psi', \ell'_i, \varrho'_j)\colon n'[(x'_1, X'_1), \ldots, (x'_{n'}, X'_{n'})] \to n''[(x''_1, X''_1), \ldots, (x''_{n''}, X''_{n''})]$$

the morphism $\bigoplus_{\psi'\psi(i)=r} \mathcal{C}(\ell'_{\psi(i)}\ell_i)(X_i) \to X''_r$ is given as a composition. First, one has to permute the summands

$$\sigma\colon \bigoplus_{\psi'\psi(i)=r} \mathcal{C}(\ell'_{\psi(i)}\ell_i)(X_i) \to \bigoplus_{\psi'(k)=r} \bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_k \ell_i)(X_i)\,.$$

Then, as we assumed that $\mathcal{C}$ is a functor to Strict, we know that

$$\bigoplus_{\psi'(k)=r} \bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_k\ell_i)(X_i) = \bigoplus_{\psi'(k)=r} \bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_k)\mathcal{C}(\ell_i)(X_i) = \bigoplus_{\psi'(k)=r} \mathcal{C}(\ell'_k)\Big(\bigoplus_{\psi(i)=k} \mathcal{C}(\ell_i)(X_i) \Big)\,.$$

Finally, we apply the morphism

$$\bigoplus_{\psi'(k)=r} \mathcal{C}(\ell'_k)(\varrho_k)\colon \bigoplus_{\psi'(k)=r} \mathcal{C}(\ell'_k)\Big(\bigoplus_{\psi(i)=k} \mathcal{C}(\ell_i)(X_i) \Big) \longrightarrow \bigoplus_{\psi'(k)=r} \mathcal{C}(\ell'_k)(X'_k)$$

and continue with ϱ'_r to end up in X''_r.

In order to prove that the tensor product actually defines a bifunctor, we will show that

$$(\psi, \ell_i, \varrho_j) \otimes (\varphi, k_i, \pi_j) = ((\psi, \ell_i, \varrho_j) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (\varphi, k_i, \pi_j)) = (\mathrm{id} \otimes (\varphi, k_i, \pi_j)) \circ ((\psi, \ell_i, \varrho_j) \otimes \mathrm{id})$$

and

$$((\psi', \ell'_i, \varrho'_j) \otimes \mathrm{id}) \circ ((\psi, \ell_i, \varrho_j) \otimes \mathrm{id}) = ((\psi', \ell'_i, \varrho'_j) \circ (\psi, \ell_i, \varrho_j)) \otimes \mathrm{id}\,,$$

and leave the check of the remaining identity to the reader.

The first equation is straightforward to see, because $((\psi, \ell_i, \varrho_j) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (\varphi, k_i, \pi_j))$ corresponds to the left branch of the diagram above and the other composition is given by the right branch.

For the second equation we have to check that $(((\varrho' \circ \varrho) \cdot \mathrm{id}) \circ d_r)_s = ((\varrho' \cdot \mathrm{id}) \circ d_r \circ (\varrho \cdot \mathrm{id}) \circ d_r)_s$. Both morphisms have source

$$\bigoplus_{\psi'\psi(i)=s} \mathcal{C}(\ell'_{\psi(i)}\ell_i + \mathrm{id})(X_i \cdot Y_j) = \bigoplus_{\psi'\psi(i)=s} \mathcal{C}(\ell'_{\psi(i)}\ell_i)(X_i) \cdot Y_j$$

and the left hand side corresponds to the left branch of the following diagram and the right hand side to the right branch.

$$\begin{tikzcd}
\bigoplus_{\psi'\psi(i)=s} \mathcal{C}(\ell'_{\psi(i)}\ell_i)(X_i)\cdot Y_j \arrow[rr, equals] \arrow[d, "d_r"'] & & \bigoplus_{\psi'\psi(i)=s} \mathcal{C}(\ell'_{\psi(i)}\ell_i+\mathrm{id})(X_i\cdot Y_j) \arrow[d, "\sigma"] \\
\left(\bigoplus_{\psi'\psi(i)=s} \mathcal{C}(\ell'_{\psi(i)}\ell_i)(X_i)\right)\cdot Y_j \arrow[d, "\sigma\cdot\mathrm{id}"'] & & \bigoplus_{\psi'(k)=s}\bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_{\psi(i)}+\mathrm{id})\mathcal{C}(\ell_i+\mathrm{id})(X_i\cdot Y_j) \arrow[d, equals] \\
\left(\bigoplus_{\psi'(k)=s}\bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_{\psi(i)})\mathcal{C}(\ell_i)(X_i)\right)\cdot Y_j \arrow[d, equals] & & \bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k+\mathrm{id})\left(\bigoplus_{\psi(i)=k} \mathcal{C}(\ell_i+\mathrm{id})(X_i\cdot Y_j)\right) \arrow[d, "\bigoplus_{\psi'(k)=s}\mathcal{C}(\ell'_k+\mathrm{id})((\varrho_k\cdot\mathrm{id})\circ d_r)"] \\
\left(\bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k)\left(\bigoplus_{\psi(i)=k} \mathcal{C}(\ell_i)(X_i)\right)\right)\cdot Y_j \arrow[d, "(\bigoplus_{\psi'(k)=s}\mathcal{C}(\ell'_k)(\varrho_k))\cdot\mathrm{id}"'] & & \bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k)(X'_k)\cdot Y_j \arrow[d, "d_r"] \\
\left(\bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k)(X'_k)\right)\cdot Y_j \arrow[rr, equals] \arrow[dr, "\varrho'_s\cdot\mathrm{id}"'] & & \left(\bigoplus_{\psi'(k)=s} \mathcal{C}(\ell'_k)(X'_k)\right)\cdot Y_j \arrow[dl, "\varrho'_s\cdot\mathrm{id}"] \\
& X''_s\cdot Y_j &
\end{tikzcd}$$

Naturality of d_r in $\mathcal{C}^\bullet$ ensures that d_r can change place with $\bigoplus_{\psi'(k)=s}\mathcal{C}(\ell'_k+\mathrm{id})(\varrho_k\cdot\mathrm{id})$ on the right branch. That $d_r\circ\sigma=(\sigma\cdot\mathrm{id})\circ d_r$ holds because $\mathcal{C}^\bullet$ satisfies property (8) from Definition 2.1, and hence the diagram commutes.

In order to show that the associativity identification is natural, we have to prove that

$$((\psi^1,\ell_i^1,\varrho_j^1)\otimes(\psi^2,\ell_i^2,\varrho_j^2))\otimes(\psi^3,\ell_i^3,\varrho_j^3)=(\psi^1,\ell_i^1,\varrho_j^1)\otimes((\psi^2,\ell_i^2,\varrho_j^2)\otimes(\psi^3,\ell_i^3,\varrho_j^3))$$

for morphisms in the homotopy colimit. The claim is obvious on the coordinates of the surjections and the morphisms in J.

For proving the identity in the third coordinate of morphisms, note that the naturality of $\otimes$ implies that we can write

$$\begin{aligned}
&((\psi^1,\ell_i^1,\varrho_j^1)\otimes(\psi^2,\ell_i^2,\varrho_j^2))\otimes(\psi^3,\ell_i^3,\varrho_j^3)\\
=&(((\psi^1,\ell_i^1,\varrho_j^1)\otimes\mathrm{id})\otimes\mathrm{id})\circ((\mathrm{id}\otimes(\psi^2,\ell_i^2,\varrho_j^2))\otimes\mathrm{id})\circ((\mathrm{id}\otimes\mathrm{id})\otimes(\psi^3,\ell_i^3,\varrho_j^3))\,.
\end{aligned}$$

Therefore, it suffices to prove the claim for each of the factors. We will show it for the middle one and leave the other ones to the curious reader. Recall that $\mathrm{id}\otimes(\psi^2,\ell_i^2,\varrho_j^2)$ has as third coordinate the composition $(\mathrm{id}\cdot\varrho_j^2)\circ d_\ell$ and therefore $(\mathrm{id}\otimes(\psi^2,\ell_i^2,\varrho_j^2))\otimes\mathrm{id}$ has third coordinate

$$(((\mathrm{id}\cdot\varrho_j^2)\circ d_\ell)\cdot\mathrm{id})\circ d_r=(\mathrm{id}\cdot\varrho_j^2\cdot\mathrm{id})\circ(\mathrm{id}\cdot d_\ell)\circ d_r\,.$$

But $(\mathrm{id}\cdot d_\ell)\circ d_r=(d_r\cdot\mathrm{id})\circ d_\ell$ (equation (7') of Definition 2.4) holds in $\mathcal{C}^\bullet$, and therefore the third coordinate equals

$$(\mathrm{id}\cdot\varrho_j^2\cdot\mathrm{id})\circ(\mathrm{id}\cdot d_\ell)\circ d_r=(\mathrm{id}\cdot\varrho_j^2\cdot\mathrm{id})\circ(d_r\cdot\mathrm{id})\circ d_\ell$$

which is the third coordinate of $\mathrm{id}\otimes((\psi^2,\ell_i^2,\varrho_j^2)\otimes\mathrm{id})$.

Naturality of the multiplicative twist map can be seen as follows. We have to show that

$$\tau_\otimes\circ((\psi,\ell_i,\varrho_j)\otimes(\varphi,k_i,\pi_j))=((\psi,\ell_i,\varrho_j)\otimes(\varphi,k_i,\pi_j))\circ\tau_\otimes\,.$$

On the first coordinate of the morphisms this reduces to the equality

$$\sigma_{n',m'}\circ(\psi,\varphi)(i,j)=(\varphi(j),\psi(i))=(\varphi,\psi)\circ\sigma_{n,m}(i,j),$$

and on the second coordinate we have the equation

$$\chi\circ(\ell_i+k_j)=(k_j+\ell_i)\circ\chi$$

because χ is natural. Thus, it remains to prove that the above equation holds in the third coordinate, which amounts to showing that the following diagram commutes.

$$\begin{tikzcd}
\bigoplus_{\psi(i)=r}\bigoplus_{\varphi(j)=s}\mathcal{C}(\ell_i)(X_i)\cdot\mathcal{C}(k_j)(Y_j) \arrow[r, "(\bigoplus\bigoplus\gamma_\otimes)\circ\sigma"] \arrow[dd, "\bigoplus_{\psi(i)=r} d_\ell"'] & \bigoplus_{\varphi(j)=s}\bigoplus_{\psi(i)=r}\mathcal{C}(k_j)(Y_j)\cdot\mathcal{C}(\ell_i)(X_i) \arrow[d, "\sigma^{-1}"] \\
 & \bigoplus_{\psi(i)=r}\bigoplus_{\varphi(j)=s}\mathcal{C}(k_j)(Y_j)\cdot\mathcal{C}(\ell_i)(X_i) \arrow[d, "\bigoplus_{\psi(i)=r} d_r"] \\
\bigoplus_{\psi(i)=r}\mathcal{C}(\ell_i)(X_i)\cdot\Big(\bigoplus_{\varphi(j)=s}\mathcal{C}(k_j)(Y_j)\Big) \arrow[r, dotted, "\bigoplus_{\psi(i)=r}\gamma_\otimes"] \arrow[d, "\bigoplus_{\psi(i)=r}\mathrm{id}\cdot\pi_s"'] & \bigoplus_{\psi(i)=r}\Big(\bigoplus_{\varphi(j)=s}\mathcal{C}(k_j)(Y_j)\Big)\cdot\mathcal{C}(\ell_i)(X_i) \arrow[d, "\bigoplus_{\psi(i)=r}\pi_s\cdot\mathrm{id}"] \\
\bigoplus_{\psi(i)=r}\mathcal{C}(\ell_i)(X_i)\cdot Y'_s \arrow[d, "d_r"'] & \bigoplus_{\psi(i)=r} Y'_s\cdot\mathcal{C}(\ell_i)(X_i) \arrow[d, "d_\ell"] \\
\Big(\bigoplus_{\psi(i)=r}\mathcal{C}(\ell_i)(X_i)\Big)\cdot Y'_s \arrow[d, "\varrho_r\cdot\mathrm{id}"'] & Y'_s\cdot\Big(\bigoplus_{\psi(i)=r}\mathcal{C}(\ell_i)(X_i)\Big) \arrow[d, "\mathrm{id}\cdot\varrho_r"] \\
X'_r\cdot Y'_s \arrow[r, "\gamma_\otimes"] & Y'_s\cdot X'_r
\end{tikzcd}$$

The top diagram commutes because d_ℓ is defined in terms of d_r and $\gamma_\otimes$. For the bottom diagram we apply the same argument together with the naturality of $\gamma_\otimes$.

We have to check that right distributivity is the identity on morphisms. Consider three morphisms as above. When we focus on the surjections $\psi^1\colon \mathbf{n}\to\mathbf{n}'$, $\psi^2\colon \mathbf{m}\to\mathbf{m}'$, and $\psi^3\colon \mathbf{l}\to\mathbf{l}'$, we see that a condition like $(\psi^1+\psi^2)\psi^3(i,j)=(r,s)$ only affects either the preimage of $\mathbf{n}'\mathbf{l}'$ or the preimage of $\mathbf{m}'\mathbf{l}'$ in $(\mathbf{n}+\mathbf{m})\mathbf{l}$, but never both. Therefore, the third coordinate of the morphism

$$((\psi^1,\ell_i^1,\varrho_j^1)\oplus(\psi^2,\ell_i^2,\varrho_j^2))\otimes(\psi^3,\ell_i^3,\varrho_j^3)$$

is either a third coordinate of $(\psi^1,\ell_i^1,\varrho_j^1)\otimes(\psi^3,\ell_i^3,\varrho_j^3)$ or of $(\psi^2,\ell_i^2,\varrho_j^2)\otimes(\psi^3,\ell_i^3,\varrho_j^3)$, and thus right distributivity is the identity on morphisms.

In the J-graded bipermutative case the naturality of the left distributivity isomorphism follows from the one of d_r and the multiplicative twist. In both the bipermutative and the strictly bimonoidal case left distributivity is given by $(\xi,\mathrm{id},\mathrm{id})$. Therefore naturality of d_ℓ in the bipermutative setting proves naturality in the strictly bimonoidal setting as well.

This finishes the proof that $\mathrm{hocolim}_J\,\mathcal{C}^\bullet$ is a bipermutative category without zero. We now prove the remaining statements of the lemma.

There is a natural functor $G\colon \mathcal{C}^0\to\mathrm{hocolim}_J\,\mathcal{C}^\bullet$ which sends $X\in\mathcal{C}^0$ to $G(X)=1[(0,X)]$. Note that the functor G is strict (symmetric) monoidal with respect to $\otimes$, because $G(1)=1[(0,1)]$ and

$$G(X)\otimes G(Y)=1[(0,X)]\otimes 1[(0,Y)]=1[(0+0,X\otimes Y)]=1[(0,X\otimes Y)]=G(X\otimes Y)\,.$$

However, G is only lax symmetric monoidal with respect to $\oplus$: there is a binatural transformation $\eta_\oplus=(\psi,\mathrm{id},\mathrm{id})$ from $G(X)\oplus G(X')=1[(0,X)]\oplus 1[(0,X')]=2[(0,X),(0,X')]$ to $G(X\oplus X')=1[(0,X\oplus X')]$, given by the canonical surjection $\psi\colon \mathbf{2}\to\mathbf{1}$ and identity morphisms in the other two components. This morphism is of course not an isomorphism.

We have to show that the functor G respects the distributivity constraints $d_r=\mathrm{id}$ and d_ℓ. In our situation we have that $\eta_\otimes=\mathrm{id}$, so we have to check that

$$\eta_\oplus=\eta_\oplus\otimes\mathrm{id}$$

and

$$G(\tau_\otimes\circ(\tau_\otimes\oplus\tau_\otimes))\circ\eta_\oplus=(\mathrm{id}\otimes\eta_\oplus)\circ\tau_\otimes\circ(\tau_\otimes\oplus\tau_\otimes)\,.$$

The first equation is just stating the fact that

$$\begin{tikzcd}
2[(0,X),(0,X')] \otimes 1[(0,Y)] \arrow[r, "\eta_\oplus \otimes \mathrm{id}"] \arrow[d, equal] & 1[(0, X \oplus X')] \otimes 1[(0,Y)] \arrow[d, equal] \\
2[(0, X \otimes Y), (0, X' \otimes Y)] \arrow[r, "\eta_\oplus"] & 1[(0, (X \oplus X') \otimes Y)]
\end{tikzcd}$$

commutes, in view of the identity $d_r \colon (X \otimes Y) \oplus (X' \otimes Y) = (X \oplus X') \otimes Y$.

For the left distributivity law we should observe that the multiplicative twist $\tau_\otimes$ on the homotopy colimit reduces to the morphism $(\mathrm{id}, \chi, \gamma_\otimes)$ in the case of elements of length 1 in the homotopy colimit, and that $\chi^{0,0} = \mathrm{id}$. Furthermore, $\mathrm{id} \otimes (\psi, \mathrm{id}, \mathrm{id}) = (\psi, \mathrm{id}, \mathrm{id})$ holds. Therefore

$$\begin{aligned}
(\mathrm{id} \otimes \eta_\oplus) \circ d_\ell &= (\mathrm{id} \otimes (\psi, \mathrm{id}, \mathrm{id})) \circ \tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes) \\
&= (\psi, \mathrm{id}, \mathrm{id}) \circ (\mathrm{id}, \mathrm{id}, \gamma_\otimes \circ (\gamma_\otimes \oplus \gamma_\otimes)) \\
&= (\mathrm{id}, \mathrm{id}, \gamma_\otimes \circ (\gamma_\otimes \oplus \gamma_\otimes)) \circ (\psi, \mathrm{id}, \mathrm{id}) = G(d_\ell) \circ \eta_\oplus \,.
\end{aligned}$$

The claim about the module structure is obvious.

As the left distributivity on the homotopy colimit is of the form $(\xi, \mathrm{id}, \mathrm{id})$, the above proof carries over to the strictly bimonoidal case. □

Lemma 5.3. *If $F \colon \mathcal{C}^\bullet \to \mathcal{D}^\bullet$ is a lax morphism of J-graded bipermutative categories (resp. J-graded strictly bimonoidal categories) then it induces a lax morphism $F_* \colon \mathrm{hocolim}_J\, \mathcal{C}^\bullet \to \mathrm{hocolim}_J\, \mathcal{D}^\bullet$ of zeroless bipermutative categories (resp. zeroless strictly bimonoidal categories).*

Proof. Of course, we define $F_* \colon \mathrm{hocolim}_J\, \mathcal{C}^\bullet \to \mathrm{hocolim}_J\, \mathcal{D}^\bullet$ on objects by

$$F_*(n[(x_1, X_1), \dots, (x_n, X_n)]) := n[(x_1, F(X_1)), \dots, (x_n, F(X_n))] \,.$$

Note that with this definition F_* is *strict* symmetric monoidal with respect to $\oplus$ even if F was only lax symmetric monoidal.

Given a morphism $(\psi, \ell_i, \varrho_j)$ from $n[(x_1, X_1), \dots, (x_n, X_n)]$ to $m[(y_1, Y_1), \dots, (y_m, Y_m)]$ we define the induced morphism

$$(\psi, \ell_i, \varrho_j^F) \colon F_*(n[(x_1, X_1), \dots, (x_n, X_n)]) \to F_*(m[(y_1, Y_1), \dots, (y_m, Y_m)])$$

as follows: we keep the surjection ψ and the morphisms ℓ_i, and for the third coordinate we take the composition

$$\varrho_j^F \colon \bigoplus_{\psi(i)=j} \mathcal{D}(\ell_i)(F(X_i)) = \bigoplus_{\psi(i)=j} F(\mathcal{C}(\ell_i)(X_i)) \xrightarrow{\eta_\oplus} F(\bigoplus_{\psi(i)=j} \mathcal{C}(\ell_i)(X_i)) \xrightarrow{F(\varrho_j)} F(Y_j) \,.$$

The naturality of $\eta_\oplus$ ensures that composition of morphisms is well-defined.

Let $n[(x_1, X_1), \dots, (x_n, X_n)]$ and $m[(y_1, Y_1), \dots, (y_m, Y_m)]$ be two objects in $\mathrm{hocolim}_J\, \mathcal{C}^\bullet$. Applying $\otimes \circ (F_*, F_*)$ yields

$$nm[(x_1 + y_1, F(X_1) \otimes F(Y_1)), \dots, (x_n + y_m, F(X_n) \otimes F(Y_m))]$$

whereas the composition $F_* \circ \otimes$ gives

$$nm[(x_1 + y_1, F(X_1 \otimes Y_1)), \dots, (x_n + y_m, F(X_n \otimes Y_m))] \,.$$

Thus, we can use $(\mathrm{id}, \mathrm{id}, \eta_\otimes)$ to obtain a natural transformation $\eta_\otimes^*$ from $\otimes \circ (F_*, F_*)$ to $F_* \circ \otimes$. This transformation inherits all properties from $\eta_\otimes$. In particular, $\eta_\otimes^*$ is lax symmetric monoidal if $\eta_\otimes$ was so.

It remains to check the properties concerning the distributivity laws. As d_r is the identity on the J-graded bipermutative category and on the homotopy colimit, and $\eta_\oplus$ is the identity on the homotopy colimit, the equalities reduce to

$$\eta_\otimes^* \oplus \eta_\otimes^* = \eta_\otimes^*$$

and

$$F_*(d_\ell) \circ (\eta_\otimes^* \oplus \eta_\otimes^*) = \eta_\otimes^* \circ d_\ell \,.$$

The first equation is straightforward to check.

The left distributivity law in the homotopy colimit is given by $d_\ell = (\xi, \mathrm{id}, \mathrm{id})$ and $\eta_\otimes^* \oplus \eta_\otimes^*$ is equal to

$$\eta_\otimes^* \oplus \eta_\otimes^* = (\mathrm{id}_{\mathbf{nm}}, \mathrm{id}, \eta_\otimes) \oplus (\mathrm{id}_{\mathbf{nm'}}, \mathrm{id}, \eta_\otimes) \,.$$

As addition in the homotopy colimit is given by concatenation, we can simplify the above expression to $(\mathrm{id}_{\mathbf{nm}+\mathbf{nm}'}, \mathrm{id}, \eta_\otimes)$. As d_ℓ differs from the identity only in the first coordinate, and $\eta_\otimes^* \oplus \eta_\otimes^*$ only in the third coordinate, these morphisms commute. □

6. A ring completion device

Recall from subsection 3.2 the construction $G\mathcal{M}\colon I \int \mathcal{Q} \to \mathrm{Strict}$.

Lemma 6.1. *Let $\mathcal{M}$ be a permutative category. Then*

(1) *the canonical functor $\mathcal{M} \to \mathrm{hocolim}_{I \int \mathcal{Q}} G\mathcal{M}$ is a stable equivalence,*
(2) $\mathrm{hocolim}_{I \int \mathcal{Q}} G\mathcal{M}$ *is group complete (*i.e., *its monoid of path components is a group), and*
(3) *the canonical functor* $\mathrm{hocolim}_{T \in \mathcal{Q}\mathbf{1}} G\mathcal{M}(\mathbf{1}, T) \xrightarrow{\sim} \mathrm{hocolim}_{I \int \mathcal{Q}} G\mathcal{M}$ *is an unstable equivalence.*

Proof. Recall that spectrification commutes with homotopy colimits [18, 4.1], *i.e.*, $\mathrm{hocolim}_J \mathrm{Spt}$ is equivalent to $\mathrm{Spt}\,\mathrm{hocolim}_J$. Given $\mathbf{n} \in I$, the homotopy colimit $\mathrm{hocolim}_{T \in \mathcal{Q}\mathbf{n}} \mathrm{Spt}\, G\mathcal{M}(\mathbf{n}, T)$ can be calculated by taking the homotopy colimit in each of the n directions of $\mathcal{Q}\mathbf{n}$ successively. Since all nontrivial maps involved are diagonal maps, we see that the homotopy colimit in the n-th direction can be identified with $\mathrm{hocolim}_{S \in \mathcal{Q}(\mathbf{n-1})} \mathrm{Spt}\, G\mathcal{M}(\mathbf{n-1}, S)$, through the inclusion $\mathbf{n-1} \to \mathbf{n}$ that skips n. By induction it follows that each morphism in the I-shaped diagram $\mathbf{n} \mapsto \mathrm{hocolim}_{T \in \mathcal{Q}\mathbf{n}} G\mathcal{M}(\mathbf{n}, T)$ is a stable equivalence. Lemma 4.5 then says that the functor $\mathcal{M} \to \mathrm{hocolim}_{\mathbf{n} \in I} \mathrm{hocolim}_{T \in \mathcal{Q}\mathbf{n}} G\mathcal{M}(\mathbf{n}, T)$ is a stable equivalence.

The claim that the functor $\mathcal{M} \to \mathrm{hocolim}_{I \int \mathcal{Q}} G\mathcal{M}$ is a stable equivalence follows, since by extending Thomason's proof [16] of $\mathrm{hocolim}_I |\mathcal{Q}| \simeq |I \int \mathcal{Q}| = \mathrm{hocolim}_{I \int \mathcal{Q}} *$ (for the trivial functor $*$) to allow for arbitrary functors from $I \int \mathcal{Q}$, we have an equivalence

$$\mathrm{hocolim}_{I \int \mathcal{Q}} \mathrm{Spt}\, G\mathcal{M} \simeq \mathrm{hocolim}_{\mathbf{n} \in I} \mathrm{hocolim}_{T \in \mathcal{Q}\mathbf{n}} \mathrm{Spt}\, G\mathcal{M}(\mathbf{n}, T)\,.$$

See also [15, 2.3] for a write-up in the dual situation.

That π_0 of $\mathrm{hocolim}_{I \int \mathcal{Q}} G\mathcal{M}$ is a group can be seen as follows. It is enough to show that elements of the form $1[((\mathbf{n}, S), a)]$ have negatives, for $n \geqslant 1$, since all other elements are sums of ones of this form. If $S \neq \mathbf{n}$ then there is an inclusion $S \subseteq T \in \mathcal{Q}\mathbf{n}$ with T containing a negative number, so there is a path

$$1[((\mathbf{n}, S), a)] \to 1[((\mathbf{n}, T), 0)] \leftarrow 1[((\mathbf{0}, \mathbf{0}), 0)]$$

in the homotopy colimit, and the element represents zero.

If $S = \mathbf{n}$, so that $a \in \mathcal{M}^{\mathcal{P}\mathbf{n}}$, let $b \in \mathcal{M}^{\mathcal{P}\mathbf{n}}$ be given by $b_U = a_V$, where $V = U \cup \{n\}$ if $n \notin U$ and $V = U \setminus \{n\}$ if $n \in U$, for all $U \subseteq \mathbf{n}$. Then $a \oplus b$ is isomorphic to $\mathcal{M}_{\mathbf{n}}(\iota)(c)$ for some $c \in \mathcal{M}^{\mathcal{P}S}$, where $S = \mathbf{n-1}$ and $\iota\colon S \subset \mathbf{n}$ is the inclusion. Hence there is a path

$$1[(\mathbf{n}, \mathbf{n}), a] \oplus 1[(\mathbf{n}, \mathbf{n}), b] \to 1[(\mathbf{n}, \mathbf{n}), a \oplus b] \leftrightarrow 1[(\mathbf{n}, \mathbf{n}), \mathcal{M}_{\mathbf{n}}(\iota)(c)] \leftarrow 1[(\mathbf{n}, S), c]$$

in the homotopy colimit, and, as we saw above, the right hand element represents zero.

Now, since stable equivalences between group complete symmetric monoidal categories are unstable equivalences, the third claim also follows. □

Lemma 6.2. *If $\mathcal{M}$ is a permutative category with zero, such that all morphisms are isomorphisms and each additive translation is faithful, then there is an unstable equivalence*

$$\mathrm{hocolim}_{S \in \mathcal{Q}\mathbf{1}} G\mathcal{M}(\mathbf{1}, S) \xrightarrow{\sim} (-\mathcal{M})\mathcal{M}\,.$$

Proof. This is entirely due to Thomason. Theorem 5.2 in [18] asserts that there is an unstable equivalence from $\mathrm{hocolim}_{S \in \mathcal{Q}\mathbf{1}} G\mathcal{M}(\mathbf{1}, S)$ to the "simplified double mapping cylinder", and his argument on pp. 1657–1658 exhibits an unstable equivalence from the simplified double mapping cylinder to $(-\mathcal{M})\mathcal{M}$. □

Remark 6.3. The unstable equivalence $\mathrm{hocolim}_{S \in \mathcal{Q}\mathbf{1}} G\mathcal{M}(\mathbf{1}, S) \xrightarrow{\sim} (-\mathcal{M})\mathcal{M}$ is the additive extension of the assignment that sends $1[\{-1\}, 0]$ and $1[\varnothing, a]$ to $(0, 0) \in (-\mathcal{M})\mathcal{M}$, and $1[\{1\}, (a, b)]$ to (a, b). The map on morphisms is straightforward, once one declares that the morphism $1[\varnothing, a] \to 1[\{1\}, (a, a)]$ is sent to $[\mathrm{id}_a, a]\colon (0, 0) \to (a, a) \in (-\mathcal{M})\mathcal{M}$.

Collecting Proposition 3.2, Lemma 5.2 and Lemma 6.1, we obtain our zeroless ring completion.

Corollary 6.4. *Let $\mathcal{R}$ be a bipermutative category (resp. a strictly bimonoidal category). The canonical lax morphism*

$$\mathcal{R} \longrightarrow \mathrm{hocolim}_{I \int \mathcal{Q}} G\mathcal{R}$$

is a stable equivalence of zeroless bipermutative categories (resp. zeroless strictly bimonoidal categories), and

$$\mathrm{hocolim}_{S \in \mathcal{Q}\mathbf{1}} G\mathcal{R}(\mathbf{1}, S) \xrightarrow{\sim} \mathrm{hocolim}_{I \int \mathcal{Q}} G\mathcal{R}$$

is an unstable equivalence of $\mathcal{R}$-modules.

Using Proposition 5.1 to add zeros, and tracing the action of $Z\mathcal{R}$, we have the main result:

Theorem 6.5. *If $\mathcal{R}$ is a commutative rig category (resp. a rig category), then*

$$\bar{\mathcal{R}} = \mathrm{Dhocolim}_{I \int \mathcal{Q}} G\mathcal{R}$$

is a simplicial commutative ring category (resp. a simplicial ring category). Here $G\mathcal{R}$ is the $I \int \mathcal{Q}$-graded bipermutative category (resp. $I \int \mathcal{Q}$-graded strictly bimonoidal category) of Proposition 3.2 applied to the bipermutative category (resp. strictly bimonoidal category) associated with $\mathcal{R}$.

The simplicial rig maps of Proposition 5.1

$$\mathcal{R} \xleftarrow{\sim} Z\mathcal{R} \longrightarrow \bar{\mathcal{R}}$$

are stable equivalences of $Z\mathcal{R}$-modules. Furthermore, if $\mathcal{R}$ is a groupoid with faithful additive translation, then the maps

$$(-\mathcal{R})\mathcal{R} \xleftarrow{\sim} Z(-\mathcal{R})\mathcal{R} \xleftarrow{\sim} Z\,\mathrm{hocolim}_{S \in \mathcal{Q}\mathbf{1}} G\mathcal{R}(\mathbf{1}, S) \xrightarrow{\sim} \bar{\mathcal{R}}$$

form a chain of unstable equivalences of $Z\mathcal{R}$-modules.

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