Chapter One

The stable parametrized *h*-cobordism theorem

1.1. The manifold part

We write DIFF for the category of C^∞ smooth manifolds, PL for the category of piecewise-linear manifolds, and TOP for the category of topological manifolds. We generically write CAT for any one of these geometric categories. Let $I = [0,1]$ and J be two fixed closed intervals in $\mathbb{R}$. We will form collars using I and stabilize manifolds and polyhedra using J.

In this section, as well as in Chapter 4, we let $\Delta^q = \{(t_0, \dots, t_q) \mid \sum_{i=0}^q t_i = 1, t_i \geq 0\}$ be the standard affine q-simplex.

By a **CAT bundle** $\pi \colon E \to \Delta^q$ we mean a CAT locally trivial family, i.e., a map such that there exists an open cover $\{U_\alpha\}$ of Δ^q and a CAT isomorphism over U_α (= a local trivialization) from $\pi^{-1}(U_\alpha) \to U_\alpha$ to a product bundle, for each α. For π to be a CAT bundle **relative to** a given product subbundle, we also ask that each local trivialization restricts to the identity on the product subbundle. We can always shrink the open cover to a cover by compact subsets $\{K_\alpha\}$, whose interiors still cover Δ^q, and this allows us to only work with compact polyhedra in the PL case.

Definition 1.1.1. (a) Let M be a compact CAT manifold, with empty or nonempty boundary. We define the **CAT *h*-cobordism space** $H(M) = H^{CAT}(M)$ of M as a simplicial set. Its 0-simplices are the compact CAT manifolds W that are h-cobordisms on M, i.e., the boundary

$$\partial W = M \cup N$$

is a union of two codimension zero submanifolds along their common boundary $\partial M = \partial N$, and the inclusions

$$M \subset W \supset N$$

are homotopy equivalences. For each $q \geq 0$, a q-simplex of $H(M)$ is a CAT bundle $\pi \colon E \to \Delta^q$ relative to the trivial subbundle $pr \colon M \times \Delta^q \to \Delta^q$, such that each fiber $W_p = \pi^{-1}(p)$ is a CAT h-cobordism on $M \cong M \times p$, for $p \in \Delta^q$.

(b) We also define a **collared** CAT h-cobordism space $H(M)^c = H^{CAT}(M)^c$, whose 0-simplices are h-cobordisms W on M equipped with a choice of collar, i.e., a CAT embedding

$$c \colon M \times I \to W$$

that identifies $M \times 0$ with M in the standard way, and takes $M \times [0,1)$ to an open neighborhood of M in W. A q-simplex of $H(M)^c$ is a CAT bundle

$\pi\colon E \to \Delta^q$ relative to an embedded subbundle $pr\colon M \times I \times \Delta^q \to \Delta^q$, such that each fiber is a collared CAT h-cobordism on M. The map $H(M)^c \to H(M)$ that forgets the choice of collar is a weak homotopy equivalence, because spaces of collars are contractible.

Remark 1.1.2. To ensure that these collections of simplices are really sets, we might assume that each bundle $E \to \Delta^q$ is embedded in $\mathbb{R}^\infty \times \Delta^q \to \Delta^q$. The simplicial operator associated to $\alpha\colon \Delta^p \to \Delta^q$ takes $E \to \Delta^q$ to the image of the pullback $\alpha^*(E) \subset \Delta^p \times_{\Delta^q} (\mathbb{R}^\infty \times \Delta^q)$ under the canonical identification $\Delta^p \times_{\Delta^q} (\mathbb{R}^\infty \times \Delta^q) \cong \mathbb{R}^\infty \times \Delta^p$. See [HTW90, 2.1] for a more detailed solution. To smooth any corners that arise, we interpret DIFF manifolds as coming equipped with a smooth normal field, as in [Wa82, §6]. The emphasis in this book will be on the PL case.

To see that the space of CAT collars on M in W is contractible, we note that [Ar70, Thm. 2] proves that any two TOP collars are ambient isotopic (relative to the boundary), and the argument generalizes word-for-word to show that any two parametrized families of collars (over the same base) are connected by a family of ambient isotopies, which proves the claim for TOP. In the PL category, the same proof works, once PL isotopies are chosen to replace the TOP isotopies F_s and G_s given on page 124 of [Ar70]. The proof in the DIFF case is different, using the convexity of the space of inward pointing normal fields.

Definition 1.1.3. (a) The **stabilization map**

$$\sigma\colon H(M) \to H(M \times J)$$

takes an h-cobordism W on M to the h-cobordism $W \times J$ on $M \times J$. It is well-defined, because $M \times J \subset W \times J$ and $(N \times J) \cup (W \times \partial J) \subset W \times J$ are homotopy equivalences. The **stable h-cobordism space** of M is the colimit

$$\mathcal{H}^{CAT}(M) = \operatorname*{colim}_k H^{CAT}(M \times J^k)$$

over $k \geq 0$, formed with respect to the stabilization maps. Each stabilization map is a cofibration of simplicial sets, so the colimit has the same homotopy type as the corresponding homotopy colimit, or mapping telescope.

(b) In the collared case, the stabilization map $\sigma\colon H(M)^c \to H(M \times J)^c$ takes a collared h-cobordism (W, c) on M to the h-cobordism $W \times J$ on $M \times J$ with collar

$$M \times I \times J \xrightarrow{c \times id} W \times J\,.$$

Each codimension zero CAT embedding $M \to M'$ induces a map $H(M)^c \to H(M')^c$ that takes (W, c) to the h-cobordism

$$W' = M' \times I \cup_{M \times I} W\,,$$

with the obvious collar $c'\colon M' \times I \to W'$. This makes $H(M)^c$ and $\mathcal{H}^{CAT}(M)^c = \operatorname{colim}_k H(M \times J^k)^c$ covariant **functors** in M, for codimension zero embeddings of CAT manifolds. The forgetful map $\mathcal{H}^{CAT}(M)^c \to \mathcal{H}^{CAT}(M)$ is also a weak homotopy equivalence.

We must work with the collared h-cobordism space when functoriality is required, but will often (for simplicity) just refer to the plain h-cobordism space. To extend the functoriality from codimension zero embeddings to general continuous maps $M \to M'$ of topological spaces, one can proceed as in [Ha78, Prop. 1.3] or [Wa82, p. 152], to which we refer for details.

Remark 1.1.4. For a cobordism to become an h-cobordism after suitable stabilization, weaker homotopical hypotheses suffice. For example, let $X \subset V$ be a codimension zero inclusion and homotopy equivalence of compact CAT manifolds. Let $c_0 \colon \partial X \times I \to X$ be an interior collar on the boundary of X, let $M_0 = c_0(\partial X \times 1)$ and $W_0 = c_0(\partial X \times I) \cup (V \setminus X)$. Then W_0 is a cobordism from M_0 to $N_0 = \partial V$, and the inclusion $M_0 \subset W_0$ is a homology equivalence by excision, but W_0 is in general not an h-cobordism on M_0. However, if we stabilize the inclusion $X \subset V$ three times, and perform the corresponding constructions, then the resulting cobordism is an h-cobordism.

In more detail, we have a codimension zero inclusion and homotopy equivalence $X \times J^3 \subset V \times J^3$. Choosing an interior collar $c \colon \partial(X \times J^3) \times I \to X \times J^3$ on the boundary of $X \times J^3$, we let $M = c(\partial(X \times J^3) \times 1)$, $N = \partial(V \times J^3)$ and

$$W = c(\partial(X \times J^3) \times I) \cup (V \times J^3 \setminus X \times J^3) .$$

Then W is a cobordism from M to N. The three inclusions $M \subset X \times J^3$, $N \subset V \times J^3$ and $W \subset V \times J^3$ are all π_1-isomorphisms (because any null-homotopy in $V \times J^3$ of a loop in N can be deformed away from the interior of V times some interior point of J^3, and then into N, and similarly in the two other cases). Since $X \times J^3 \subset V \times J^3$ is a homotopy equivalence, it follows that both $M \subset W$ and $N \subset W$ are π_1-isomorphisms. By excision, it follows that $M \subset W$ is a homology equivalence, now with arbitrary local coefficients. By the universal coefficient theorem, and Lefschetz duality for the compact manifold W, it follows that $N \subset W$ is a homology equivalence, again with arbitrary local coefficients. Hence both $M \subset W$ and $N \subset W$ are homotopy equivalences, and W is an h-cobordism on M.

In the following definitions, we specify one model $s\widetilde{\mathcal{E}}^h_\bullet(M)$ for the stable PL h-cobordism space $\mathcal{H}^{PL}(M)$, based on a category of compact polyhedra and simple maps. In the next two sections we will re-express this polyhedral model: first in terms of a category of finite simplicial sets and simple maps, and then in terms of the algebraic K-theory of spaces.

Definition 1.1.5. A PL map $f \colon K \to L$ of compact polyhedra will be called a **simple map** if it has contractible point inverses, i.e., if $f^{-1}(p)$ is contractible for each point $p \in L$. (A space is contractible if it is homotopy equivalent to a one-point space. It is, in particular, then non-empty.)

In this context, M. Cohen [Co67, Thm. 11.1] has proved that simple maps (which he called contractible mappings) are simple homotopy equivalences.

Two compact polyhedra are thus of the same simple homotopy type if and only if they can be linked by a finite chain of simple maps. The composite of two simple maps is always a simple map. This follows from Proposition 2.1.3 in Chapter 2, in view of the possibility of triangulating polyhedra and PL maps. Thus we can interpret the simple homotopy types of compact polyhedra as the path components of (the nerve of) a category of polyhedra and simple maps.

Definition 1.1.6. Let K be a compact polyhedron. We define a simplicial category $s\widetilde{\mathcal{E}}^h_\bullet(K)$ of compact polyhedra containing K as a deformation retract, and simple PL maps between these. In simplicial degree 0, the objects are compact polyhedra L equipped with a PL embedding and homotopy equivalence $K \to L$. The morphisms $f\colon L \to L'$ are the simple PL maps that restrict to the identity on K, via the given embeddings. A deformation retraction $L \to K$ exists for each object, but a choice of such a map is not part of the structure.

In simplicial degree q, the objects of $s\widetilde{\mathcal{E}}^h_q(K)$ are **PL Serre fibrations** (= PL maps whose underlying continuous map of topological spaces is a Serre fibration) of compact polyhedra $\pi\colon E \to \Delta^q$, with a PL embedding and homotopy equivalence $K \times \Delta^q \to E$ over Δ^q from the product fibration $pr\colon K \times \Delta^q \to \Delta^q$. The morphisms $f\colon E \to E'$ of $s\widetilde{\mathcal{E}}^h_q(K)$ are the simple PL fiber maps over Δ^q that restrict to the identity on $K \times \Delta^q$, via the given embeddings.

Each PL embedding $K \to K'$ of compact polyhedra induces a (forward) functor $s\widetilde{\mathcal{E}}^h_\bullet(K) \to s\widetilde{\mathcal{E}}^h_\bullet(K')$ that takes $K \to L$ to $K' \to K' \cup_K L$, and similarly in parametrized families. The pushout $K' \cup_K L$ exists as a polyhedron, because both $K \to K'$ and $K \to L$ are PL embeddings. This makes $s\widetilde{\mathcal{E}}^h_\bullet(K)$ a covariant functor in K, for PL embeddings. There is a natural **stabilization map**

$$\sigma\colon s\widetilde{\mathcal{E}}^h_\bullet(K) \to s\widetilde{\mathcal{E}}^h_\bullet(K \times J)$$

that takes $K \to L$ to $K \times J \to L \times J$, and similarly in parametrized families. It is a homotopy equivalence by Lemma 4.1.12 in Chapter 4.

As in the following definition, we often regard a simplicial set as a simplicial category with only identity morphisms, a simplicial category as the bisimplicial set given by its degreewise nerve (Definition 2.2.1), and a bisimplicial set as the simplicial set given by its diagonal. A map of categories, i.e., a functor, is a homotopy equivalence if the induced map of nerves is a weak homotopy equivalence. See [Se68, §2], [Qu73, §1] or [Wa78a, §5] for more on these conventions.

Definition 1.1.7. Let M be a compact PL manifold. There is a natural map of simplicial categories

$$u\colon H^{PL}(M)^c \to s\widetilde{\mathcal{E}}^h_\bullet(M \times I)$$

that takes (W, c) to the underlying compact polyhedron of the h-cobordism W, with the PL embedding and homotopy equivalence provided by the collar $c\colon M \times I \to W$, and views PL bundles over Δ^q as being particular cases of

PL Serre fibrations over Δ^q. It commutes with the stabilization maps, and therefore induces a natural map

$$u\colon \mathcal{H}^{PL}(M)^c \to \operatorname*{colim}_k s\widetilde{\mathcal{E}}^h_\bullet(M \times I \times J^k)\,.$$

Here is the PL manifold part of the stable parametrized h-cobordism theorem.

Theorem 1.1.8. *Let M be a compact PL manifold. There is a natural homotopy equivalence*

$$\mathcal{H}^{PL}(M) \simeq s\widetilde{\mathcal{E}}^h_\bullet(M)\,.$$

More precisely, there is a natural chain of homotopy equivalences

$$\mathcal{H}^{PL}(M)^c = \operatorname*{colim}_k H^{PL}(M \times J^k)^c \xrightarrow[\simeq]{u} \operatorname*{colim}_k s\widetilde{\mathcal{E}}^h_\bullet(M \times I \times J^k) \xleftarrow[\simeq]{\sigma} s\widetilde{\mathcal{E}}^h_\bullet(M)\,,$$

and $\mathcal{H}^{PL}(M)^c \simeq \mathcal{H}^{PL}(M)$.

By the argument of [Wa82, p. 175], which we explain below, it suffices to prove Theorem 1.1.8 when M is a codimension zero submanifold of Euclidean space, or a little more generally, when M is stably framed (see Definition 4.1.2). The proof of the stably framed case will be given in Chapter 4, and is outlined in Section 4.1. Cf. diagram (4.1.13).

Remark 1.1.9 (Reduction of Theorem 1.1.8 to the stably framed case). Here we use a second homotopy equivalent model $H(M)^r$ for the h-cobordism space of M, where each h-cobordism W comes equipped with a choice of a CAT retraction $r\colon W \to M$, and similarly in parametrized families. The forgetful map $H(M)^r \to H(M)$ is a weak homotopy equivalence, because each inclusion $M \subset W$ is a cofibration and a homotopy equivalence. For each CAT disc bundle $\nu\colon N \to M$ there is a **pullback map** $\nu^!\colon H(M)^r \to H(N)^r$, which takes an h-cobordism W on M with retraction $r\colon W \to M$ to the pulled-back h-cobordism $N \times_M W$ on N, with the pulled-back retraction.

$$\begin{array}{ccccc}
M \times J^k & \xrightarrow{\simeq} & W \times J^k & \xrightarrow{r \times id} & M \times J^k \\
\downarrow{\scriptstyle \tau} & & \downarrow & & \downarrow{\scriptstyle \tau} \\
N & \xrightarrow{\simeq} & N \times_M W & \longrightarrow & N \\
\downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle r^*\nu} & & \downarrow{\scriptstyle \nu} \\
M & \xrightarrow{\simeq} & W & \xrightarrow{r} & M
\end{array}$$

If $\tau\colon M \times J^k \to N$ is a second CAT disc bundle, so that the composite $\nu\tau$ equals the projection $pr\colon M \times J^k \to M$, then $(\nu\tau)^!$ equals the k-fold stabilization map $\tau^!\nu^! = \sigma^k$. Hence there is a commutative diagram

$$\begin{array}{ccccc}
H(M)^r & \xrightarrow{\nu^!} & H(N)^r & \xrightarrow{\tau^!} & H(M \times J^k)^r \\
\downarrow{\scriptstyle \simeq} & & & & \downarrow{\scriptstyle \simeq} \\
H(M) & & \xrightarrow{\sigma^k} & & H(M \times J^k)\,.
\end{array}$$

According to Haefliger–Wall [HW65, Cor. 4.2], each compact PL manifold M admits a stable normal disc bundle $\nu \colon N \to M$, with N embedded with codimension zero in some Euclidean n-space. Furthermore, PL disc bundles admit stable inverses. Let $\tau \colon M \times J^k \to N$ be the disc bundle in such a stable inverse to ν, such that $\nu\tau$ is isomorphic to the product k-disc bundle over M, and $\tau(\nu \times id)$ is isomorphic to the product k-disc bundle over N. By the diagram above, pullback along ν and τ define homotopy inverse maps

$$\mathcal{H}^{PL}(M) \xrightarrow{\nu^!} \mathcal{H}^{PL}(N) \xrightarrow{\tau^!} \mathcal{H}^{PL}(M)$$

after stabilization.

Likewise, there is a homotopy equivalent variant $s\widetilde{\mathcal{E}}^h_\bullet(M)^r$ of $s\widetilde{\mathcal{E}}^h_\bullet(M)$, with a (contractible) choice of PL retraction $r \colon L \to M$ for each polyhedron L containing M, and similarly in parametrized families. There is a simplicial functor $\nu^! \colon s\widetilde{\mathcal{E}}^h_\bullet(M)^r \to s\widetilde{\mathcal{E}}^h_\bullet(N)^r$, by the pullback property of simple maps (see Proposition 2.1.3). It is a homotopy equivalence, because each stabilization map σ is a homotopy equivalence by Lemma 4.1.12. Thus it suffices to prove Theorem 1.1.8 for N, which is stably framed, in place of M.

Remark 1.1.10. A similar argument lets us reduce the stable parametrized TOP h-cobordism theorem to the PL case. By [Mi64] and [Ki64] each compact TOP manifold M admits a normal disc bundle $\nu \colon N \to M$ in some Euclidean space, and ν admits a stable inverse. As a codimension zero submanifold of Euclidean space, N can be given a PL structure. By the argument above, $\nu^! \colon \mathcal{H}^{TOP}(M) \to \mathcal{H}^{TOP}(N)$ is a homotopy equivalence. Furthermore, $H^{PL}(N) \to H^{TOP}(N)$ is a homotopy equivalence for $n = \dim(N) \geq 5$, by triangulation theory [BL74, Thm. 6.2] and [KS77, V.5.5]. Thus $\mathcal{H}^{PL}(N) \simeq \mathcal{H}^{TOP}(N)$, and the TOP case of Theorem 0.1 follows from the PL case.

Remark 1.1.11. There are further possible variations in the definition of the h-cobordism space $H(M)$. For a fixed h-cobordism W on M, the path component of $H(M)$ containing W is a classifying space for CAT bundles with fiber W, relative to the product bundle with fiber M. A homotopy equivalent model for this classifying space is the bar construction $BCAT(W \operatorname{rel} M)$ of the simplicial group of CAT automorphisms of W relative to M. Hence there is a homotopy equivalence

$$H(M) \simeq \coprod_{[W]} BCAT(W \operatorname{rel} M)\,,$$

where $[W]$ ranges over the set of isomorphism classes of CAT h-cobordisms on M.

In particular, when $W = M \times I$ is the product h-cobordism on $M \cong M \times 0$, we are led to the simplicial group

$$C(M) = CAT(M \times I, M \times 1) \tag{1.1.12}$$

of **CAT concordances** (= pseudo-isotopies) on M. By definition, these are the CAT automorphisms of $M \times I$ that pointwise fix the complement of $M \times 1$

in $\partial(M \times I)$. More generally, we follow the convention of [WW01, 1.1.2] and write $CAT(W, N)$ for the simplicial group of CAT automorphisms of W that agree with the identity on the complement of N in ∂W. Here N is assumed to be a codimension zero CAT submanifold of the boundary ∂W. When N is empty we may omit it from the notation, so that $CAT(W) = CAT(W \operatorname{rel} \partial W)$.

The concordances that commute with the projection to $I = [0, 1]$ are the same as the isotopies of $M \operatorname{rel} \partial M$ that start from the identity, but concordances are not required to commute with this projection, hence the name pseudo-isotopy. The inclusion $C(M) \to CAT(M \times I \operatorname{rel} M \times 0)$ is a homotopy equivalence, so the path component of $H(M)$ that contains the trivial h-cobordisms is homotopy equivalent to the bar construction $BC(M)$. In general, $H(M)$ is a non-connective delooping of the CAT concordance space $C(M)$.

By the s-cobordism theorem, the set of path components of $H(M)$ is in bijection with the Whitehead group $\mathrm{Wh}_1(\pi) = K_1(\mathbb{Z}[\pi])/(\pm\pi)$, when $d = \dim(M) \geq 5$ and M is connected with fundamental group π. For disconnected M, the Whitehead group should be interpreted as the sum of the Whitehead groups associated to its individual path components. For each element $\tau \in \mathrm{Wh}_1(\pi)$, we write $H(M)_\tau$ for the path component of $H(M)$ that consists of the h-cobordisms with Whitehead torsion τ. For example, $H(M)_0 \simeq BC(M)$ is the s-cobordism space.

Still assuming $d \geq 5$, we can find an h-cobordism W_1 from M to M_τ, with prescribed Whitehead torsion τ relative to M, and a second h-cobordism W_2 from M_τ to M, with Whitehead torsion $-\tau$ relative to M_τ. Then $W_1 \cup_{M_\tau} W_2 \cong M \times I$ and $W_2 \cup_M W_1 \cong M_\tau \times I$, by the sum formula for Whitehead torsion and the s-cobordism theorem. Gluing with W_2 at M, and with W_1 at M_τ, define homotopy inverse maps

$$H(M)_\tau \to H(M_\tau)_0 \to H(M)_\tau \,.$$

Hence

$$H(M) = \coprod_\tau H(M)_\tau \simeq \coprod_\tau H(M_\tau)_0 \simeq \coprod_\tau BC(M_\tau)\,, \tag{1.1.13}$$

where $\tau \in \mathrm{Wh}_1(\pi)$.

1.2. The non-manifold part

In this section, as well as in Chapters 2 and 3, we let Δ^q be the simplicial q-simplex, the simplicial set with geometric realization $|\Delta^q|$ the standard affine q-simplex.

Definition 1.2.1. A simplicial set X is **finite** if it is generated by finitely many simplices, or equivalently, if its geometric realization $|X|$ is compact. A map $f\colon X \to Y$ of finite simplicial sets will be called a **simple map** if its geometric realization $|f|\colon |X| \to |Y|$ has contractible point inverses, i.e., if for each $p \in |Y|$ the preimage $|f|^{-1}(p)$ is contractible.

A map $f\colon X \to Y$ of simplicial sets is a **weak homotopy equivalence** if its geometric realization $|f|$ is a homotopy equivalence. A map $f\colon X \to Y$ of simplicial sets is a **cofibration** if it is injective in each degree, or equivalently, if its geometric realization $|f|$ is an embedding. We say that f is a **finite cofibration** if, furthermore, Y is generated by the image of X and finitely many other simplices.

We shall see in Section 2.1 that simple maps are weak homotopy equivalences, and that the composite of two simple maps is a simple map. In particular, the simple maps of finite simplicial sets form a category.

Definition 1.2.2. By the Yoneda lemma, there is a one-to-one correspondence between the n-simplices x of a simplicial set X and the simplicial maps $\bar{x}\colon \Delta^n \to X$. We call $\bar{x}$ the **representing map** of x. A simplicial set X will be called **non-singular** if for each non-degenerate simplex $x \in X$ the representing map $\bar{x}\colon \Delta^n \to X$ is a cofibration.

In any simplicial set X, the geometric realization $|\bar{x}|\colon |\Delta^n| \to |X|$ of the representing map of a non-degenerate simplex x restricts to an embedding of the interior of $|\Delta^n|$. The additional condition imposed for non-singular simplicial sets is that this map is required to be an embedding of the whole of $|\Delta^n|$. It amounts to the same to ask that the images of the $(n+1)$ vertices of $|\Delta^n|$ in $|X|$ are all distinct.

When viewed as simplicial sets, ordered simplicial complexes provide examples of non-singular simplicial sets, but not all non-singular simplicial sets arise this way. For example, the union $\Delta^1 \cup_{\partial\Delta^1} \Delta^1$ of two 1-simplices along their boundary is a non-singular simplicial set, but not an ordered simplicial complex.

Definition 1.2.3. For any simplicial set X, let $\mathcal{C}(X)$ be the category of finite cofibrations $y\colon X \to Y$. The morphisms from y to $y'\colon X \to Y'$ are the simplicial maps $f\colon Y \to Y'$ under X, i.e., those satisfying $fy = y'$.

For finite X, let $s\mathcal{C}^h(X) \subset \mathcal{C}(X)$ be the subcategory with objects such that $y\colon X \to Y$ is a weak homotopy equivalence, and morphisms such that $f\colon Y \to Y'$ is a simple map. Let $\mathcal{D}(X) \subset \mathcal{C}(X)$ and $s\mathcal{D}^h(X) \subset s\mathcal{C}^h(X)$ be the full subcategories generated by the objects $y\colon X \to Y$ for which Y is non-singular. Let $i\colon s\mathcal{D}^h(X) \to s\mathcal{C}^h(X)$ be the inclusion functor.

The definition of $s\mathcal{C}^h(X)$ only makes sense, as stated, for finite X, because we have not defined what it means for $f\colon Y \to Y'$ to be a simple map when Y or Y' are not finite. We will extend the definition of $s\mathcal{C}^h(X)$ to general simplicial sets X in Definition 3.1.12, as the colimit of the categories $s\mathcal{C}^h(X_\alpha)$ where X_α ranges over the finite simplicial subsets of X. The categories $\mathcal{D}(X)$ and $s\mathcal{D}^h(X)$ are only non-empty when X itself is non-singular, because there can only be a cofibration $y\colon X \to Y$ to a non-singular simplicial set Y when X is also non-singular.

Definition 1.2.4. The geometric realization $|X|$ of a finite non-singular simplicial set X is canonically a compact polyhedron, which we call the **polyhedral**

realization of X. Its polyhedral structure is characterized by the condition that $|\bar{x}|\colon |\Delta^n| \to |X|$ is a PL map for each (non-degenerate) simplex x of X. The geometric realization $|f|\colon |X| \to |Y|$ of a simplicial map of finite non-singular simplicial sets is then a PL map.

For any compact polyhedron K, let $s\mathcal{E}^h(K)$ be the category of PL embeddings $\ell\colon K \to L$ of compact polyhedra, and simple PL maps $f\colon L \to L'$ under K. For any finite non-singular simplicial set X let $r\colon s\mathcal{D}^h(X) \to s\mathcal{E}^h(|X|)$ be the polyhedral realization functor that takes $y\colon X \to Y$ to $|y|\colon |X| \to |Y|$, and similarly for morphisms. Let $\tilde{n}\colon s\mathcal{E}^h(K) \to s\widetilde{\mathcal{E}}^h_\bullet(K)$ be the simplicial functor that includes $s\mathcal{E}^h(K)$ as the 0-simplices in $s\widetilde{\mathcal{E}}^h_\bullet(K)$, as introduced in Definition 1.1.6.

See Definition 3.4.1 for more on compact polyhedra, PL maps and the polyhedral realization functor. The non-manifold parts of the stable parametrized h-cobordism theorem follow.

Theorem 1.2.5. *Let X be a finite non-singular simplicial set. The full inclusion functor*

$$i\colon s\mathcal{D}^h(X) \to s\mathcal{C}^h(X)$$

is a homotopy equivalence.

Theorem 1.2.5 will be proved as part of Proposition 3.1.14. Cf. diagram (3.1.15).

Theorem 1.2.6. *Let X be a finite non-singular simplicial set. The composite*

$$\tilde{n} \circ r\colon s\mathcal{D}^h(X) \to s\widetilde{\mathcal{E}}^h_\bullet(|X|)$$

of the polyhedral realization functor r and the 0-simplex inclusion $\tilde{n}$, is a homotopy equivalence.

Theorem 1.2.6 is proved at the end of Section 3.5. Cf. diagram (3.5.4). We do not claim that the individual functors $r\colon s\mathcal{D}^h(X) \to s\mathcal{E}^h(|X|)$ and $\tilde{n}\colon s\mathcal{E}^h(|X|) \to s\widetilde{\mathcal{E}}^h_\bullet(|X|)$ are homotopy equivalences, only their composite. The proof involves factoring the composite in a different way, through a simplicial category $s\widetilde{\mathcal{D}}^h_\bullet(X)$, to be introduced in Definition 3.1.7(d).

The construction $X \mapsto s\mathcal{C}^h(X)$ is covariantly functorial in the simplicial set X. It is homotopy invariant in the sense that any weak homotopy equivalence $x\colon X \to X'$ induces a homotopy equivalence $x_*\colon s\mathcal{C}^h(X) \to s\mathcal{C}^h(X')$. Union along X defines a sum operation on $s\mathcal{C}^h(X)$ that makes it a grouplike monoid, with $\pi_0 s\mathcal{C}^h(X)$ isomorphic to the Whitehead group of $\pi_1(X)$. See Definition 3.1.11, Corollary 3.2.4 and Proposition 3.2.5 for precise statements and proofs.

1.3. Algebraic K-theory of spaces

For any simplicial set X, let $\mathcal{R}_f(X)$ be the category of finite retractive spaces over X, with objects (Y, r, y) where $y\colon X \to Y$ is a finite cofibration of simplicial sets and $r\colon Y \to X$ is a retraction, so that $ry = id_X$. A morphism from (Y, r, y) to (Y', r', y') is a simplicial map $f\colon Y \to Y'$ over and under X, so that $r = r'f$ and $fy = y'$. There is a functor $\mathcal{R}_f(X) \to \mathcal{C}(X)$ that forgets the structural retractions. (The category $\mathcal{C}(X)$ was denoted $\mathcal{C}_f(X)$ in [Wa78b] and [Wa85], but in this book we omit the subscript to make room for a simplicial direction.)

The two subcategories $co\mathcal{R}_f(X)$ and $h\mathcal{R}_f(X)$ of $\mathcal{R}_f(X)$, of maps $f\colon Y \to Y'$ that are cofibrations and weak homotopy equivalences, respectively, make $\mathcal{R}_f(X)$ a category with cofibrations and weak equivalences in the sense of [Wa85, §1.1 and §1.2]. The $S_\bullet$-construction $S_\bullet\mathcal{R}_f(X)$ is then defined as a simplicial category (with cofibrations and weak equivalences), see [Wa85, §1.3], and the **algebraic K-theory** of the space X is defined to be the loop space

$$A(X) = \Omega|hS_\bullet\mathcal{R}_f(X)|\,.$$

Any weak homotopy equivalence $X \to X'$ induces a homotopy equivalence $A(X) \to A(X')$, and we can write $A(M)$ for $A(X)$ when $M = |X|$.

The $S_\bullet$-construction can be iterated, and the sequence of spaces

$$\{\ n \mapsto |h\underbrace{S_\bullet \cdots S_\bullet}_{n}\mathcal{R}_f(X)|\ \}$$

(with appropriate structure maps) defines a spectrum $\mathbf{A}(X)$, which has $A(X)$ as its underlying infinite loop space. Let $\mathbf{S} = \{n \mapsto S^n\}$ be the **sphere spectrum**. In the special case $X = *$ there is a unit map

$$\eta\colon \mathbf{S} \to \mathbf{A}(*)\,,$$

adjoint to the based map $S^0 \to |h\mathcal{R}_f(*)|$ that takes the non-base point to the 0-simplex corresponding to the object (Y, r, y) with $Y = S^0$.

These spectra can be given more structure. By [GH99, Prop. 6.1.1] each $\mathbf{A}(X)$ is naturally a symmetric spectrum [HSS00], with the symmetric group Σ_n acting on the n-th space by permuting the $S_\bullet$-constructions. Furthermore, the smash product of finite based simplicial sets induces a multiplication $\mu\colon \mathbf{A}(*) \wedge \mathbf{A}(*) \to \mathbf{A}(*)$ that, together with the unit map η, makes $\mathbf{A}(*)$ a commutative symmetric ring spectrum. Each spectrum $\mathbf{A}(X)$ is naturally an $\mathbf{A}(*)$-module spectrum.

For based and connected X, there is a homotopy equivalent definition of $A(X)$ as the algebraic K-theory $K(\mathbf{S}[\Omega X])$ of the spherical group ring $\mathbf{S}[\Omega X]$. Here ΩX can be interpreted as the Kan loop group of X, see [Wa96], and $\mathbf{S}[\Omega X]$ is its unreduced suspension spectrum $\Sigma^\infty(\Omega X)_+$, viewed as a symmetric ring spectrum, or any other equivalent notion.

Remark 1.3.1. The CAT Whitehead spaces can be defined in several, mostly equivalent, ways. In early papers on the subject [Wa78b, pp. 46–47], [Wa82, p. 144], [WW88, pp. 575–576], $\mathrm{Wh}^{CAT}(M)$ is defined for compact CAT manifolds M as a delooping of the stable h-cobordism space $\mathcal{H}^{CAT}(M)$, making

Theorem 0.1 a definition rather than a theorem. With that definition in mind, the reader might justifiably wonder what this book is all about.

On the other hand, in [Ha75, p. 102] and [Ha78, p. 15] the PL Whitehead space $\mathrm{Wh}^{PL}(K)$ is defined for polyhedra K as a delooping of the classifying space of the category of simple maps that we denote by $s\mathcal{E}^h(K)$. (In Hatcher's first cited paper, there is no delooping.) In [Wa85, Prop. 3.1.1] the PL Whitehead space $\mathrm{Wh}^{PL}(X)$ is defined for simplicial sets X as the delooping $|s\mathcal{N}_\bullet\mathcal{C}^h(X)|$ of the classifying space of the category $s\mathcal{C}^h(X)$. We do not know that Hatcher and Waldhausen's definitions are equivalent for $K = |X|$, but they do become equivalent if $s\mathcal{E}^h(K)$ is expanded to the simplicial category $s\widetilde{\mathcal{E}}^h_\bullet(K)$, see diagram (3.1.8) and Remark 3.1.10.

With Waldhausen's cited definition, the PL case of Theorem 0.1 becomes the main result established in this book, asserting that there is a natural equivalence $\mathcal{H}^{PL}(|X|) \simeq s\mathcal{C}^h(X)$ for finite combinatorial manifolds X, by the proof outlined in diagram (0.4). This definition has the advantage that it provides notation for stating Theorem 0.1 in the PL case, but it has the disadvantage that it does not also cover the DIFF case.

To obtain the given statement of Theorem 0.1, and to directly connect the main result about h-cobordism spaces to algebraic K-theory, we therefore choose to redefine the CAT Whitehead spaces $\mathrm{Wh}^{CAT}(X)$ directly in terms of the functor $A(X)$, by analogy with the definition of the Whitehead group $\mathrm{Wh}_1(\pi)$ as a quotient of the algebraic K-group $K_1(\mathbb{Z}[\pi])$. The role of the geometric category CAT is not apparent in the resulting definition of $\mathrm{Wh}^{CAT}(X)$, so the superscript in the notation is only justified once Theorem 0.1 has been proved.

That the K-theoretic definition in the PL case agrees with Waldhausen's cited definition is the content of [Wa85, Thm. 3.1.7] and [Wa85, Thm. 3.3.1]. The correctness of the redefinition in the DIFF case (which is the real content of the DIFF case of Theorem 0.1) is a consequence of smoothing theory and a vanishing theorem, and is explained at the end of this section.

By [Wa85, Thm. 3.2.1] and a part of [Wa85, Thm. 3.3.1] (recalled in diagram (1.4.7) below), there is a natural map

$$\alpha\colon h(X; A(*)) \to A(X)$$

of homotopy functors in X, where

$$h(X; A(*)) = \Omega^\infty(\mathbf{A}(*) \wedge X_+)$$

is the unreduced homological functor associated to the spectrum $\mathbf{A}(*)$. The natural map α is a homotopy equivalence for $X = *$, which characterizes it up to homotopy equivalence as the **assembly map** associated to the homotopy functor $A(X)$, see [WW95, §1]. The assembly map extends to a map

$$\alpha\colon \mathbf{A}(*) \wedge X_+ \to \mathbf{A}(X)$$

of (symmetric) spectra, as is seen from [Wa85, Thm. 3.3.1] by iterating the $S_\bullet$-construction.

Definition 1.3.2. For each simplicial set X, let the **PL Whitehead spectrum** $\mathbf{Wh}^{PL}(X)$ be defined as the homotopy cofiber of the spectrum level assembly map, so that there is a natural cofiber sequence of spectra

$$\mathbf{A}(*) \wedge X_+ \xrightarrow{\alpha} \mathbf{A}(X) \to \mathbf{Wh}^{PL}(X) .$$

Let the **PL Whitehead space** $\mathrm{Wh}^{PL}(X)$ be defined as the underlying infinite loop space $\mathrm{Wh}^{PL}(X) = \Omega^\infty \mathbf{Wh}^{PL}(X)$.

Let the **TOP Whitehead spectrum** and **TOP Whitehead space** be defined in the same way, as $\mathbf{Wh}^{TOP}(X) = \mathbf{Wh}^{PL}(X)$ and $\mathrm{Wh}^{TOP}(X) = \mathrm{Wh}^{PL}(X)$, respectively.

With this (revised) definition, there is obviously a natural homotopy fiber sequence

$$h(X; A(*)) \xrightarrow{\alpha} A(X) \to \mathrm{Wh}^{PL}(X) \tag{1.3.3}$$

of homotopy functors in X. Continuing the homotopy fiber sequence one step to the left, we get an identification of the looped PL Whitehead space $\Omega\,\mathrm{Wh}^{PL}(X)$ with the homotopy fiber of the space level assembly map $\alpha\colon h(X; A(*)) \to A(X)$, without needing to refer to the previously mentioned spectrum level constructions.

Summary of proof of the PL case of Theorem 0.1, and Theorem 0.2. By [Wa85, Thm. 3.1.7] and [Wa85, 3.3.1], the revised definition of $\mathrm{Wh}^{PL}(X)$ agrees up to natural homotopy equivalence with the one given in [Wa85, Prop. 3.1.1]. In particular, there is a natural chain of homotopy equivalences

$$s\mathcal{C}^h(X) \simeq \Omega\,\mathrm{Wh}^{PL}(X) ,$$

also with the revised definition. The proof of [Wa85, Thm. 3.1.7] contains some forward references to results proved in the present book, which we have summarized in Remark 1.4.5.

By our Theorems 1.1.8, 1.2.5 and 1.2.6, proved in Sections 4.1–4.3, 3.1 and 3.5, respectively, there is a natural chain of homotopy equivalences

$$\mathcal{H}^{PL}(M) \xrightarrow{\simeq} s\widetilde{\mathcal{E}}^h_\bullet(M) \xleftarrow{\simeq} s\mathcal{D}^h(X) \xrightarrow{\simeq} s\mathcal{C}^h(X)$$

for each compact PL manifold M, triangulated as $|X|$. This establishes the homotopy equivalence of Theorem 0.1 in the PL case. The homotopy fiber sequence of Theorem 0.2 is the Puppe sequence obtained by continuing (1.3.3) one step to the left. □

The unit map $\eta\colon \mathbf{S} \to \mathbf{A}(*)$ induces a natural map of unreduced homological functors

$$Q(X_+) = \Omega^\infty(\mathbf{S} \wedge X_+) \xrightarrow{\eta} \Omega^\infty(\mathbf{A}(*) \wedge X_+) = h(X; A(*)) .$$

We define the spectrum map $\iota\colon \Sigma^\infty X_+ \to \mathbf{A}(X)$ as the composite

$$\Sigma^\infty X_+ = \mathbf{S} \wedge X_+ \xrightarrow{\eta \wedge id} \mathbf{A}(*) \wedge X_+ \xrightarrow{\alpha} \mathbf{A}(X)$$

and let $\iota = \alpha \circ \eta\colon Q(X_+) \to A(X)$ be the underlying map of infinite loop spaces.

Definition 1.3.4. For each simplicial set X let the **DIFF Whitehead spectrum** $\mathbf{Wh}^{DIFF}(X)$ be defined as the homotopy cofiber of the spectrum map ι, so that there is a natural cofiber sequence of spectra

$$\Sigma^\infty X_+ \xrightarrow{\iota} \mathbf{A}(X) \to \mathbf{Wh}^{DIFF}(X)\,.$$

Let the **DIFF Whitehead space** $\mathrm{Wh}^{DIFF}(X)$ be defined as the underlying infinite loop space $\mathrm{Wh}^{DIFF}(X) = \Omega^\infty \mathbf{Wh}^{DIFF}(X)$.

There is obviously a natural homotopy fiber sequence

$$Q(X_+) \xrightarrow{\iota} A(X) \to \mathrm{Wh}^{DIFF}(X) \tag{1.3.5}$$

of homotopy functors in X. Continuing the homotopy fiber sequence one step to the left, we get an identification of the looped DIFF Whitehead space $\Omega\,\mathrm{Wh}^{DIFF}(X)$ with the homotopy fiber of the space level map $\iota\colon Q(X_+) \to A(X)$. However, in this case the splitting of ι leads to the attractive formula $A(X) \simeq Q(X_+) \times \mathrm{Wh}^{DIFF}(X)$, which is one reason to focus on the unlooped Whitehead space.

Proof of the DIFF case of Theorem 0.1, and Theorem 0.3. We can deduce Theorem 0.3 and the DIFF case of Theorem 0.1 from Theorem 0.2. The argument was explained in [Wa78b, §3] and [Wa82, §2], but we review and comment on it here for the reader's convenience.

We consider homotopy functors F from spaces to based spaces, such that there is a natural map $F(M) \to \mathrm{hofib}(F(M_+) \to F(*))$. The **stabilization** F^S of F (not related to the other kind of stabilization that we use) is an unreduced homological functor, with

$$F^S(M) \simeq \operatorname*{colim}_n \Omega^n\, \mathrm{hofib}(F(\Sigma^n(M_+)) \to F(*))\,.$$

In the notation of [Go90b], $F^S(M) = D_*F(M_+)$, where D_*F is the differential of F at $*$. There is a natural map $F(M) \to F^S(M)$, which is a homotopy equivalence whenever F itself is a homological functor. This form of stabilization preserves natural homotopy fiber sequences.

Each term in the homotopy fiber sequence of Theorem 0.2 is such a homotopy functor. Hence there is a natural homotopy equivalence

$$\Omega\,\mathrm{hofib}(A(M) \to A^S(M)) \xrightarrow{\simeq} \mathrm{hofib}(\mathcal{H}^{PL}(M) \to \mathcal{H}^{PL,S}(M))\,.$$

The stable h-cobordism space $\mathcal{H}^{DIFF}(M)$ can also be extended to such a homotopy functor. By Morlet's disjunction lemma [BLR75, §1], cf. [Ha78, Lem. 5.4], the stabilized functor $\mathcal{H}^{DIFF,S}(M)$ is contractible. By smoothing theory, also known as Morlet's comparison theorem, the homotopy fiber of the natural map $\mathcal{H}^{DIFF}(M) \to \mathcal{H}^{PL}(M)$ is a homological functor [BL77, §4]. Hence there is a natural chain of homotopy equivalences

$$\begin{aligned}\mathcal{H}^{DIFF}(M) &\xleftarrow{\simeq} \mathrm{hofib}(\mathcal{H}^{DIFF}(M) \to \mathcal{H}^{DIFF,S}(M)) \\ &\xrightarrow{\simeq} \mathrm{hofib}(\mathcal{H}^{PL}(M) \to \mathcal{H}^{PL,S}(M))\,.\end{aligned}$$

The composite map $Q(M_+) \xrightarrow{\iota} A(M) \to A^S(M)$ is a homotopy equivalence, by the "vanishing of the mystery homology theory" [Wa87a, Thm.]. Alternatively, this can be deduced from B. I. Dundas' theorem on relative K-theory [Du97, p. 224], which implies that the cyclotomic trace map induces a profinite homotopy equivalence $A^S(M) \simeq TC^S(M)$, together with the calculation $TC^S(M) \simeq Q(M_+)$ of [He94]. The rational result was obtained in [Wa78b, Prop. 2.9] from work by A. Borel [Bo74], F. T. Farrell and W.-C. Hsiang [FH78]. Either way, it follows that the composite natural map

$$\mathrm{hofib}(A(M) \to A^S(M)) \to A(M) \to \mathrm{Wh}^{DIFF}(M)$$

is a homotopy equivalence. In combination, we obtain a natural chain of homotopy equivalences that induces the homotopy equivalence

$$\mathcal{H}^{DIFF}(M) \simeq \Omega\,\mathrm{Wh}^{DIFF}(M)$$

claimed in Theorem 0.1. The homotopy fiber sequence of Theorem 0.3 is the Puppe sequence obtained by continuing (1.3.5) one step to the left. The stabilization map $A(M) \to A^S(M)$ provides a natural splitting of $\iota\colon Q(M_+) \to A(M)$, up to homotopy, and together with the map $A(M) \to \mathrm{Wh}^{DIFF}(M)$ it defines the natural homotopy factorization of the theorem. □

1.4. Relation to other literature

The main assertion in Hatcher's paper [Ha75] is his Theorem 9.1, saying that there is a k-connected map from the PL h-cobordism space $H^{PL}(M)$ to a classifying space $\mathcal{S}(M)$ for "PL Serre fibrations with homotopy fiber M and a fiber homotopy trivialization," provided that $n = \dim(M) \ge 3k+5$. The model for $\mathcal{S}(M)$ chosen by Hatcher equals the simplicial set of objects in our simplicial category $s\widetilde{\mathcal{E}}^h_\bullet(M)$. In Hatcher's Proposition 3.1, this space is asserted to be homotopy equivalent to the nerve of $s\mathcal{E}^h(M)$. That particular claim appears to be difficult to prove in the polyhedral context, because the proposed argument for his Proposition 2.5 makes significant use of chosen triangulations. However, it follows from [St86, Thm. 1] and our Theorem 1.2.6 that $\mathcal{S}(M)$ is homotopy equivalent to the nerve of the simplicial category $s\widetilde{\mathcal{E}}^h_\bullet(M)$, so in essence, Hatcher's Theorem 9.1 claims that the map $H^{PL}(M) \to s\widetilde{\mathcal{E}}^h_\bullet(M)$ is about $(n/3)$-connected, for $n = \dim(M)$. Stabilizing with respect to the dimension, this amounts to the manifold part Theorem 1.1.8 of our stable parametrized h-cobordism theorem. Thus the stable form of Hatcher's main assertion is correct.

The relevance of simple maps to the study of PL homeomorphisms of manifolds may be motivated by the following theorem of M. Cohen [Co70, Thm. 1]: For closed PL n-manifolds M and N with $n \ge 5$ each simple PL map $M \to N$ can be uniformly approximated by a PL homeomorphism $M \cong N$. A similar result in the TOP category was proved by L. Siebenmann [Si72].

The first author's paper [Wa78b] (from the 1976 Stanford conference) contains in its Section 5 the assertion that Hatcher's polyhedral model $s\widetilde{\mathcal{E}}^h_\bullet(M)$ for $\mathcal{H}^{PL}(M)$ is homotopy equivalent to the model $s\mathcal{C}^h(X)$ that is defined in terms of simplicial sets, where $M = |X|$ as usual. This translation is the content of our non-manifold Theorems 1.2.5 and 1.2.6. Furthermore, Section 5 of that paper contains the homotopy fiber sequences of Theorems 0.2 and 0.3. Modulo some forward references to the present work, their proofs appeared in [Wa85], except for the result that $A^S(M) \simeq Q(M_+)$, which appeared in [Wa87a]. For more on these forward references, see Remark 1.4.5.

Hatcher's paper [Ha78] in the same proceedings surveys, among other things, how concordance spaces (with their canonical involution) measure the difference between the "honest" automorphism groups of manifolds and the block automorphism groups of manifolds, which are determined by surgery theory [Wa70, §17.A]. The spectral sequence of [Ha78, Prop. 2.1] makes this precise in the concordance stable range. In [WW88, Thm. A], M. Weiss and B. Williams express this spectral sequence as coming from the $\mathbb{Z}/2$-homotopy orbit spectral sequence of an involution on the stable h-cobordism space, with its infinite loop space structure. Their later survey [WW01] explains, among many other things, how this contribution from concordance and h-cobordism spaces also measures the difference between the "honest" moduli space parametrizing bundles of compact manifolds and the block moduli space given by the surgery classification of manifolds.

In the meantime, M. Steinberger's paper [St86] appeared, whose Theorem 1 proves that (the nerve of) $s\mathcal{D}^h(X)$ is a classifying space for "PL Serre fibrations with homotopy fiber $|X|$ and a fiber homotopy trivialization." Thus $s\mathcal{D}^h(X) \simeq \mathcal{S}(M)$, which is close to our Theorem 1.2.6. His main tool for proving this is a special category of finite convex cell complexes in Euclidean space, and certain piecewise linear maps between these.

Steinberger's Theorem 2 is the same as our Theorem 1.2.5, but his proof leaves a significant part to be discovered by the reader. His argument [St86, p. 19] starts out just as our first (non-functorial) proof of Proposition 3.1.14, and relies on a result similar to our Proposition 2.5.1. At that point, he appeals to an analog $C(h)$ of Cohen's PL mapping cylinder, but defined for general maps h of simplicial sets. However, he does not establish the existence of this construction, nor its relevant properties. Presumably the intended $C(h)$ is our backward reduced mapping cylinder $M(Sd(h))$ of the normal subdivision of h, and the required properties are those established in our Sections 2.1 through 2.4.

The following year, T. A. Chapman's paper [Ch87] appeared. His Theorem 3 proves the stable form of Hatcher's main claim, that a version of $\mathcal{H}^{PL}(M)$ is homotopy equivalent to the classifying space $\mathcal{S}(M)$. Modulo the identification of $\mathcal{S}(M)$ with $s\widetilde{\mathcal{E}}^h_\bullet(M)$, this is equivalent to our Theorem 1.1.8. Combining Chapman's Theorem 3 with Steinberger's Theorems 1 and 2 one obtains a homotopy equivalence $\mathcal{H}^{PL}(M) \simeq s\mathcal{C}^h(X)$, for $M = |X|$. When combined with the homotopy equivalence $s\mathcal{C}^h(X) \simeq \Omega \operatorname{Wh}^{PL}(X)$ from [Wa85, §3], bringing algebraic K-theory into the picture, one recovers the PL case of our Theorem 0.1. In a similar way, Chapman's Theorem 2 is analogous to our main geometric Theorem 4.1.14, except that Chapman works with manifolds embedded with codimension zero in some Euclidean space, whereas we have chosen to work with stably framed manifolds. His main tool is a stable fibered controlled h-cobordism theorem.

Chapman's paper omits proofs of several results, because of their similarity with other results in the literature (his Propositions 2.2 and 2.3), and only discusses the absolute case of some inductive proofs that rely on a relative statement for their inductive hypotheses (his Theorems 3.2 and 5.2). Furthermore, some arguments involving careful control estimates are only explained over the 0- and 1-skeleta of a parameter domain, and it is left to the reader to extend these over all higher skeleta.

Since Theorem 1.1.8, 1.2.5 and 1.2.6 are fundamental results for the relation between the stable h-cobordism spaces and the Whitehead spaces, we prefer to provide proofs that do not leave too many constructions, generalizations or relativizations to be discovered or filled in by the reader. The tools used in our presentation are close to those of [Wa85], which provides the connection onwards from the Whitehead spaces to the algebraic K-theory of spaces. Taken together, these two works complete the bridge connecting geometric topology to algebraic K-theory.

The present book is also needed to justify the forward references from [Wa85], including Theorem 2.3.2 and its consequence Proposition 2.3.3, which were used in [Wa85, §3.1] on the way to Theorem 0.2. Hence these results from our Chapter 2 are also required for Theorem 0.3 and the DIFF case of Theorem 0.1, neither of which are covered by Steinberger and Chapman's papers.

Returning to Hatcher's original paper, the unstable form of the main assertion would imply not only the stable conclusion, but also a PL concordance stability result [Ha78, Cor. 9.2], to the effect that a suspension map $\sigma\colon C^{PL}(M) \to C^{PL}(M \times J)$ is about $(n/3)$-connected, for $n = \dim(M)$. Delooping once, this would imply that the stabilization map $\sigma\colon H^{PL}(M) \to H^{PL}(M \times J)$ is also about $(n/3)$-connected. As we discuss in Remark 4.2.3, our methods are essentially stable. In particular, we do not attempt to prove these PL concordance stability results. However, working in the DIFF category, K. Igusa proved the following concordance stability result in [Ig88], using Hatcher's PL argument as an outline for the proof.

Theorem 1.4.1 (Igusa). *The suspension map*

$$\sigma\colon C^{DIFF}(M) \to C^{DIFF}(M \times J)$$

is k-connected, for all compact smooth n-manifolds M with $n \geq \max\{2k+7, 3k+4\}$.

Delooping once, and iterating, it follows that the infinite stabilization map $H^{DIFF}(M) \to \mathcal{H}^{DIFF}(M)$ is $(k+1)$-connected, for M, n and k as in the theorem. When combined with Theorem 0.3 and calculations of the algebraic K-theory of spaces $A(M)$, this leads to concrete results on the homotopy groups $\pi_i C^{DIFF}(M)$ and $\pi_i H^{DIFF}(M)$, for i up to about $n/3$.

For example, in the case $M = D^n \simeq *$ there is a rational homotopy equivalence $A(M) \simeq A(*) \to K(\mathbb{Z})$, and the striking consequences for $\pi_i DIFF(D^n) \otimes \mathbb{Q}$ of Borel's calculation [Bo74] of $K_i(\mathbb{Z}) \otimes \mathbb{Q}$ were explained in [Wa78b, Thm. 3.2] and [Ig88, p. 7]. Recall from Remark 1.1.11 the convention that $DIFF(D^n) = DIFF(D^n \operatorname{rel} S^{n-1})$. Analogous rational results for Euclidean and spherical space forms were obtained in [FH78], [HJ82] and [HJ83]. Calculations of the p-torsion in $\pi_i A(*)$ were made in [Ro02] for $p = 2$ and [Ro03] for odd regular primes, and some consequences concerning the p-torsion in $\pi_i DIFF(D^n)$ were drawn in Section 6 of the latter paper.

D. Burghelea and R. Lashof [BL77, Thm. C] used smoothing theory and Morlet's disjunction lemma to show that the PL concordance stability theorem stated by Hatcher would imply a DIFF concordance stability theorem, in about half the PL concordance stable range. T. Goodwillie has improved on this argument, using his multiple disjunction lemma from [Go90a], to establish a DIFF concordance stable range only three less than such an assumed PL concordance stable range.

However, no proof of a concordance stability theorem for general PL manifolds seems to be known. In the absence of a PL proof, it was observed by Burghelea and by Goodwillie that for smoothable manifolds M one can deduce a PL concordance stability theorem from Igusa's DIFF concordance stability theorem, with the same concordance stable range. The following argument was explained to us by Goodwillie. It implies that the optimal DIFF concordance stable range and the optimal PL concordance stable range for smoothable manifolds are practically the same.

Corollary 1.4.2 (Burghelea, Goodwillie). *The suspension map*

$$\sigma \colon C^{PL}(M) \to C^{PL}(M \times J)$$

is k-connected, for compact smoothable n-manifolds M with $n \geq \max\{2k+7, 3k+4\}$.

Proof. Let M be a compact DIFF n-manifold and let $P \to M$ be its frame bundle, i.e., the principal O_n-bundle associated to the tangent bundle of M. By smoothing theory [BL74, Thm. 4.2] there is a homotopy fiber sequence

$$DIFF(M) \to PL(M) \to \Gamma(M; PL_n/O_n)\,.$$

Here $\Gamma(M; PL_n/O_n)$ denotes the space of sections s in the fiber bundle associated to $P \to M$ with fiber PL_n/O_n, such that $s|\partial M$ maps to the base point in each fiber. (The precise statement requires a detour via spaces of piecewise differentiable maps, which we suppress.) For concordance spaces [BL77, (2.4)] there is a similar homotopy fiber sequence

$$C^{DIFF}(M) \to C^{PL}(M) \to \Gamma(M; C_n)\,, \tag{1.4.3}$$

where $\Gamma(M; C_n)$ is the space of sections in a bundle over M with fiber

$$C_n = \operatorname{hofib}(PL_n/O_n \to PL_{n+1}/O_{n+1}),$$

with prescribed behavior on ∂M. (Burghelea–Lashof use the notation F_n for the $TOP/DIFF$ analog of this homotopy fiber.)

Let

$$F^{CAT}(M) = \operatorname{hofib}(C^{CAT}(M) \xrightarrow{\sigma} C^{CAT}(M \times J))$$

be the homotopy fiber of the suspension map for CAT concordances. By [BL77, Thm. A] the concordance suspension maps are compatible with a suspension map $\varphi\colon C_n \to \Omega C_{n+1}$, so there is a homotopy fiber sequence

(1.4.4) $$F^{DIFF}(M) \to F^{PL}(M) \to \Gamma(M; F_n),$$

where $\Gamma(M; F_n)$ is the space of sections in a bundle with fiber

$$F_n = \operatorname{hofib}(C_n \xrightarrow{\varphi} \Omega C_{n+1}),$$

still with prescribed behavior on ∂M.

The columns in the following diagram are homotopy fiber sequences:

$$\begin{array}{ccc} C_n & \xrightarrow{\varphi} & \Omega C_{n+1} \\ \downarrow & & \downarrow \\ O_{n+1}/O_n & \xrightarrow{\varphi^{DIFF}} & \Omega(O_{n+2}/O_{n+1}) \\ \downarrow & & \downarrow \\ PL_{n+1}/PL_n & \xrightarrow{\varphi^{PL}} & \Omega(PL_{n+2}/PL_{n+1}). \end{array}$$

The lower vertical arrows are $(n+2)$-connected for $n \geq 5$, by the $PL/DIFF$ stability theorem [KS77, V.5.2]. Hence C_n, ΩC_{n+1} and the upper horizontal map φ are all $(n+1)$-connected, and the homotopy fiber F_n is at least n-connected.

Igusa's theorem implies that $F^{DIFF}(M)$ is $(k-1)$-connected, for M, n and k as in the statement of the corollary. In addition, $\sigma\colon C^{DIFF}(M) \to C^{DIFF}(M \times J)$ is 0-connected (for $n \geq 7$). Since $\Gamma(M; \Omega C_{n+1})$ is 1-connected, it follows from (1.4.3) that $\sigma\colon C^{PL}(M) \to C^{PL}(M \times J)$ is at least 0-connected.

Now consider the special case $M = D^n$. The spaces $PL(D^n)$, $C^{PL}(D^n)$ and $F^{PL}(D^n)$ are all contractible, by the Alexander trick. The tangent bundle of D^n is trivial, so $\Gamma(D^n; F_n) = \Omega^n F_n$. Igusa's theorem and (1.4.4) then imply that $\Omega^n F_n$ is k-connected. It follows that F_n is $(n+k)$-connected, because we saw from the $PL/DIFF$ stability theorem that F_n is at least n-connected.

Returning to the case of a general smoothable n-manifold M, the section space $\Gamma(M; F_n)$ is k-connected by obstruction theory. Hence $F^{PL}(M)$ is $(k-1)$-connected by Igusa's theorem and (1.4.4). It follows that the PL concordance stabilization map $\sigma\colon C^{PL}(M) \to C^{PL}(M \times J)$ is k-connected, because we saw from Igusa's theorem and (1.4.3) that it is at least 0-connected. □

In a relative way, Igusa's theorem improves on the cited $PL/DIFF$ stability theorem, by showing that the PL suspension map

$$PL_{n+1}/PL_n \xrightarrow{\varphi^{PL}} \Omega(PL_{n+2}/PL_{n+1})$$

is at least $(n+k+2)$-connected, when $n \geq \max\{2k+7, 3k+4\}$. For comparison, the DIFF suspension map

$$S^n \cong O_{n+1}/O_n \xrightarrow{\varphi^{DIFF}} \Omega(O_{n+2}/O_{n+1}) \cong \Omega S^{n+1}$$

is precisely $(2n-1)$-connected, by Freudenthal's theorem.

Remark 1.4.5. There are some forward references in [Wa85, §3.1] concerning simple maps to (an earlier version of) the present work. For the reader's convenience, we make these explicit here. The claim that simple maps form a category, and satisfy a gluing lemma [Wa85, p. 401] is contained in our Proposition 2.1.3. The claim that the H-space $s\mathcal{C}^h(X)$ is grouplike [Wa85, p. 402] is our Corollary 3.2.4.

The proof of [Wa85, Lem. 3.1.4] contains three forward references. The "well known argument" was implicit in [GZ67], and is made explicit in our Lemma 3.2.14. The result that the last vertex map is simple is our Proposition 2.2.18. The fact that subdivision preserves simple maps is our Proposition 2.3.3. In our proof, the full strength of our Theorem 2.3.2 is used. Thus that result, on the quasi-naturality of the Fritsch–Puppe homeomorphism, is presently required for the identification $s\mathcal{C}^h(X) \simeq \Omega \operatorname{Wh}^{PL}(X)$, and thus for Theorems 0.1, 0.2 and 0.3.

On top of page 405 of [Wa85], use is made of a simplicial deformation retraction of $[n] \mapsto X^{\Delta^n \times \Delta^n}$ onto $[n] \mapsto X^{\Delta^n}$, where X is a simplicial set. The relevant inclusion is induced by the projection $pr_1\colon \Delta^n \times \Delta^n \to \Delta^n$, and we take the retraction to be induced by the diagonal map $diag\colon \Delta^n \to \Delta^n \times \Delta^n$. Then $diag \circ pr_1$ is the nerve of the composite functor (= order-preserving function) $f\colon [n] \times [n] \to [n] \times [n]$ that takes (i,j) to (i,i), for $i,j \in [n]$. There is a chain of natural transformations

$$(i,i) \leq (i, \max(i,j)) \geq (i,j)$$

relating f to the identity on $[n] \times [n]$, which is natural in $[n]$. Taking nerves, we get a chain of simplicial homotopies relating $diag \circ pr_1$ to the identity on $\Delta^n \times \Delta^n$, which is still natural in $[n]$. Forming mapping spaces into X, we obtain the required chain of simplicial homotopies.

There are two references on page 406 of [Wa85] to our Proposition 3.2.5, i.e., the fact that the functor $X \mapsto s\mathcal{C}^h(X)$ respects weak homotopy equivalences.

The concluding reformulation [Wa85, Prop. 3.3.2] of [Wa85, Thm. 3.3.1] is not correct as stated. In the definition of the simplicial category $\mathcal{R}_f(X)_\bullet$, the condition on the objects in simplicial degree q, that the composite map

$$Y \xrightarrow{r} X \times \Delta^q \xrightarrow{pr} \Delta^q$$

is locally fiber homotopy trivial, should be replaced with the stronger condition that the map is a Serre fibration. This leads to the following definition and corrected proposition.

Definition 1.4.6. To each simplicial set X we associate a simplicial category $\widetilde{\mathcal{R}}_\bullet(X)$. In simplicial degree q, it is the full subcategory of $\mathcal{R}_f(X \times \Delta^q)$ generated by the objects (Y, r, y) for which the composite map

$$Y \xrightarrow{r} X \times \Delta^q \xrightarrow{pr} \Delta^q$$

is a Serre fibration. Let $\widetilde{\mathcal{R}}^h_\bullet(X)$ be the full simplicial subcategory with objects such that $y\colon X \times \Delta^q \to Y$ is also a weak homotopy equivalence, and let s- and h-prefixes indicate the subcategories of simple maps and weak homotopy equivalences, respectively.

By [Wa85, Thm. 3.3.1], there is a homotopy cartesian square

$$\begin{array}{ccc} sS_\bullet\mathcal{R}^h_f(X^{\Delta^\bullet}) & \longrightarrow & sS_\bullet\mathcal{R}_f(X^{\Delta^\bullet}) \\ \downarrow & & \downarrow \\ hS_\bullet\mathcal{R}^h_f(X^{\Delta^\bullet}) & \longrightarrow & hS_\bullet\mathcal{R}_f(X^{\Delta^\bullet}) \end{array} \tag{1.4.7}$$

where the entries have the following meaning. Let X^K denote the mapping space $\mathrm{Map}(K, X)$, with p-simplices the maps $\Delta^p \times K \to X$. For each q, let X^{Δ^q} denote this mapping space (that is, take $K = \Delta^q$). Then the entries in the diagram need to be taken in the following slightly tricky sense: for each fixed q evaluate the functor in question on X^{Δ^q}, and then take the simplicial object that results by varying q.

The upper left hand term is one model for $\mathrm{Wh}^{PL}(X)$, the lower left hand term is contractible, and the loop spaces of the right hand terms are homotopy equivalent to $h(X; A(*))$ and $A(X)$, respectively. The homotopy fiber sequence (1.3.3) is part of the Puppe fiber sequence derived from this homotopy cartesian square.

Proposition 1.4.8. *There is a homotopy cartesian square*

$$\begin{array}{ccc} sS_\bullet\widetilde{\mathcal{R}}^h_\bullet(X) & \longrightarrow & sS_\bullet\widetilde{\mathcal{R}}_\bullet(X) \\ \downarrow & & \downarrow \\ hS_\bullet\widetilde{\mathcal{R}}^h_\bullet(X) & \longrightarrow & hS_\bullet\widetilde{\mathcal{R}}_\bullet(X) \end{array}$$

and it is homotopy equivalent to the square (1.4.7) by a natural map.

Proof. The natural map of homotopy cartesian squares is induced by the functor $\mathcal{R}_f(X^{\Delta^\bullet}) \to \widetilde{\mathcal{R}}_\bullet(X)$ given in simplicial degree q by the composite

$$\mathcal{R}_f(X^{\Delta^q}) \to \mathcal{R}_f(X^{\Delta^q} \times \Delta^q) \to \mathcal{R}_f(X \times \Delta^q)\,,$$

where the first map is given by product with Δ^q, and the second map is functorially induced by

$$(ev, pr)\colon X^{\Delta^q} \times \Delta^q \to X \times \Delta^q$$

where ev is the evaluation map. When applied to a retractive space Y over X^{Δ^q} the result is a Serre fibration over $X \times \Delta^q$, by the fiber gluing lemma for Serre fibrations, Lemma 2.7.10. The proof then proceeds as in [Wa85, p. 418], up to the claim that $s\mathcal{C}^h(X) \to s\widetilde{\mathcal{C}}^h_\bullet(X)$ is a homotopy equivalence. This will be proved as Corollary 3.5.2.

Actually, the fiber gluing lemma for Serre fibrations was also used in the verification that the simplicial category $\widetilde{\mathcal{R}}_\bullet(X)$ in Definition 1.4.6 is a simplicial category with cofibrations; in particular, that for every q the category $\widetilde{\mathcal{R}}_q(X)$ is a category with cofibrations, as is required for the use of the $S_\bullet$-construction. □

In the "manifold approach" paper [Wa82, Prop. 5.1], a similar (approximately) homotopy cartesian square is constructed, where the entries are simplicial (sets or) categories of CAT manifolds. As discussed in [Wa82, pp. 178–180], there is a chain of homotopy equivalences relating the manifold square to the square of Proposition 1.4.8. This is how one can deduce [Wa82, Prop. 5.5], for CAT = PL, asserting that the manifold functor that corresponds to

$$sS_\bullet\mathcal{R}_f(X^{\Delta^\bullet}) \simeq sS_\bullet\widetilde{\mathcal{R}}_\bullet(X)$$

is indeed a homological functor in X.