KAN SUBDIVISION AND PRODUCTS OF SIMPLICIAL SETS

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Abstract. The canonical map from the Kan subdivision of a product of finite simplicial sets to the product of the Kan subdivisions is a simple map, in the sense that its geometric realization has contractible point inverses.

1. Introduction

Kan's normal subdivision [5, §7] is a functor Sd from simplicial sets to simplicial sets. It agrees with barycentric subdivision when applied to (ordered) simplicial complexes. The maps induced by applying Kan subdivision to the projections $X \times Y \to X$ and $X \times Y \to Y$ combine to a canonical map $\kappa\colon \operatorname{Sd}(X \times Y) \to \operatorname{Sd} X \times \operatorname{Sd} Y$.

Definition 1.1. A map $f\colon A \to B$ of finite simplicial sets is said to be *simple* [9, 2.1.1] if its geometric realization $|f|\colon |A| \to |B|$ has contractible point inverses, meaning that the preimage $|f|^{-1}(b)$ is contractible for each point $b \in |B|$. We write $A \xrightarrow{\simeq_s} B$ to denote a simple map.

Theorem 1.2. *Let X and Y be finite simplicial sets. The canonical map*

$$\kappa\colon \operatorname{Sd}(X \times Y) \xrightarrow{\simeq_s} \operatorname{Sd} X \times \operatorname{Sd} Y\,,$$

from the Kan subdivision of the product $X \times Y$ to the product of the Kan subdivisions, is a simple map.

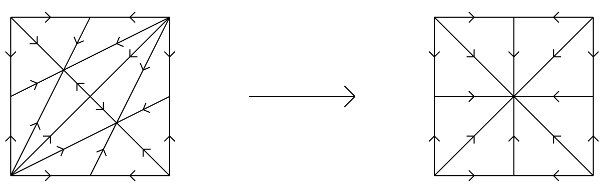
The theorem follows from the special cases when $X = \Delta[m]$ is a standard simplex, for some $m \geq 0$, by an induction on the dimension and number of top-dimensional cells in the CW complex $|X|$. A second induction, over the cells of $|Y|$, allows us to specialize further to the cases when $Y = \Delta[n]$, for some $n \geq 0$. Hence our real task is to prove the following result.

Proposition 1.3. *The canonical map*

$$\kappa\colon \operatorname{Sd}(\Delta[m] \times \Delta[n]) \xrightarrow{\simeq_s} \operatorname{Sd}\Delta[m] \times \operatorname{Sd}\Delta[n]$$

is simple, for each $m \geq 0$ and $n \geq 0$.

Example 1.4. For $m = n = 1$, we have the following picture.



The map κ is the identity on the boundary, and takes the three interior vertices on the left hand side to the single interior vertex on the right hand side. The rhombus on the left is collapsed to the diagonal on the right. The point inverses of $|\kappa|$ are

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points or closed intervals. Hence $|\kappa|$ is not a homeomorphism, but all preimages of points are contractible. Note that $|\kappa|$ does not admit a continuous section.

Remark 1.5. Simple maps have a direction: given a simple map $f\colon A \to B$ there might be no simple map $g\colon B \to A$. Simple-homotopy theory [2] concerns the equivalence relation on finite simplicial sets (or simplicial complexes) generated by simple maps. Piecewise-linear topology [6] concerns the properties of simplicial complexes that are invariant under linear subdivisions. It is often convenient and sufficient to only consider the linear subdivisions that arise by iterated barycentric subdivision [7, §3.5, §3.6]. Our theorem gives a strong version of the statement that the simple-homotopy type of a product is a well-defined piecewise-linear notion, also with respect to this more restricted notion of subdivision.

The weaker statement that the map κ in the theorem is a simple-homotopy equivalence is easily proved: there is a natural *last vertex map* $d_X\colon \operatorname{Sd} X \to X$, which is simple for all finite X [9, 2.2.17], and $(d_X \times d_Y) \circ \kappa = d_{X\times Y}$. However, this argument does not suffice to prove that κ is a simple map.

In Section 2 we present an application of our main result. In Section 3 we show how to deduce Theorem 1.2 from Proposition 1.3, and outline the proof of the latter result. The details of that proof occupy Sections 4, 5 and 6.

2. An application to the improvement functor

A simplicial set X is said to be *non-singular* [9, 1.2.2] if for each non-degenerate n-simplex x in X the representing map $\bar{x}\colon \Delta[n] \to X$ is a cofibration, or equivalently, if each non-degenerate n-simplex has $n+1$ distinct vertices, for each $n \geq 0$. This implies that the geometric realization $|X|$ has a preferred piecewise-linear structure. Let $\mathcal{C}$ be the category of finite simplicial sets and simplicial maps, let $\mathcal{D}$ be its full subcategory of finite non-singular simplicial sets, and let $s\mathcal{C}$ and $s\mathcal{D}$ denote the respective subcategories of simple maps.

Our main result has an application to the multiplicative properties of the *improvement functor* $I\colon \mathcal{C} \to \mathcal{D}$ constructed in [9, 2.5.2]. This functor associates to each finite simplicial set X a finite non-singular simplicial set $I(X)$, together with a natural simple map $s_X\colon I(X) \to X$. Hence the restricted functor $sI\colon s\mathcal{C} \to s\mathcal{D}$ induces a homotopy equivalence of classifying spaces, and the localized functor $I[s^{-1}]\colon \mathcal{C}[s^{-1}] \to \mathcal{D}[s^{-1}]$ (inverting the respective collections of simple maps [4, 1.1]) is an equivalence of categories. This shows that the simple-homotopy theory of finite simplicial sets is equivalent to the simple-homotopy theory of finite non-singular simplicial sets. For a useful relative version of this statement, see [9, 1.2.5]. The application we have in mind concerns the compatibility of this equivalence with the categorical products.

Proposition 2.1. *Let X and Y be finite simplicial sets, and let I denote the improvement functor from* [9, §2.5]*. The canonical map*

$$I(X \times Y) \xrightarrow{\simeq_s} I(X) \times I(Y)\,,$$

induced by the projections $X \times Y \to X$ and $X \times Y \to Y$, is simple.

Before showing how to deduce this from our main theorem, we recall some notation and terminology. Each simplex x of a simplicial set X is a degeneration of a unique non-degenerate simplex, which we denote $x^\#$. If x is non-degenerate then $x = x^\#$. Let $X^\#$ be the set of non-degenerate simplices in X, partially ordered by letting $x \leq y$ if x is a face of y. We call the nerve $B(X) = N(X^\#)$ of this partially ordered set the *Barratt nerve* of X, due to its early appearance in [1]. A simplicial map $f\colon X \to Y$ induces an order-preserving function $f^\#\colon X^\# \to Y^\#$, taking x to $f(x)^\#$, and a functorial map of nerves $B(f) = N(f^\#)\colon B(X) \to B(Y)$. The Kan subdivision $\mathrm{Sd}(X)$ is defined as the left Kan extension of $[n] \mapsto B(\Delta^n)$ along the Yoneda embedding of Δ into simplicial sets [9, 2.2.7], so there is a natural map $b_X \colon \mathrm{Sd}(X) \to B(X)$, which is an isomorphism for non-singular X [9, 2.2.11]. The improvement functor I is defined as the composite

$$I(X) = B(\mathrm{Sd}(X)) .$$

(The use of the opposite subdivision in [9] plays no explicit role in our arguments, and will be suppressed.)

Remark 2.2. The improvement functor I appears implicitly in the work of Thomason [8], as the composite $N \circ c\,\mathrm{Sd}^2$ of the nerve functor N (from small categories to simplicial sets) and the categorified double subdivision $c\,\mathrm{Sd}^2$ (from simplicial sets to categories). The latter functor is the left adjoint in Thomason's Quillen equivalence between these two (closed) model categories. By [8, Lemma 5.6] the category $c\,\mathrm{Sd}^2(X)$ is a partially ordered set, for any simplicial set X, and this makes it easy to identify it with the partially ordered set $\mathrm{Sd}(X)^\#$ of non-degenerate simplices in $\mathrm{Sd}(X)$.

Proof of Proposition 2.1. There are canonical maps

$$\mathrm{Sd}(\mathrm{Sd}(X \times Y)) \to \mathrm{Sd}(\mathrm{Sd}(X) \times \mathrm{Sd}(Y)) \to \mathrm{Sd}(\mathrm{Sd}(X)) \times \mathrm{Sd}(\mathrm{Sd}(Y)) .$$

The left hand map, $\mathrm{Sd}(\kappa)$, is simple by our Theorem 1.2, combined with the fact that Kan subdivision preserves simple maps [9, 2.3.3]. The right hand map is simple by the same theorem applied for the finite simplicial sets $\mathrm{Sd}(X)$ and $\mathrm{Sd}(Y)$. Hence the composite map is also simple [9, 2.1.3(a)].

The natural map $b_{\mathrm{Sd}(X)} \colon \mathrm{Sd}(\mathrm{Sd}(X)) \to B(\mathrm{Sd}(X))$ is simple by [9, 2.5.5, 2.5.8]. Hence the vertical maps are simple in the commutative square

$$\begin{tikzcd}
\mathrm{Sd}(\mathrm{Sd}(X \times Y)) \arrow[r, "\simeq_s"] \arrow[d, "\simeq_s"'] & \mathrm{Sd}(\mathrm{Sd}(X)) \times \mathrm{Sd}(\mathrm{Sd}(Y)) \arrow[d, "\simeq_s"] \\
B(\mathrm{Sd}(X \times Y)) \arrow[r] & B(\mathrm{Sd}(X)) \times B(\mathrm{Sd}(Y)) .
\end{tikzcd}$$

We have just argued that the upper horizontal map is simple, and this implies that the lower horizontal map is simple, by the right cancellation property of simple maps [9, 2.1.3(b)]. □

3. Outline of argument

Proof of Theorem 1.2 assuming Proposition 1.3. To reduce Theorem 1.2 to the case $X = \Delta[m]$, we write $X = \Delta[m] \cup_{\partial\Delta[m]} X'$ and think of $\kappa \colon \mathrm{Sd}(X \times Y) \to \mathrm{Sd}\,X \times \mathrm{Sd}\,Y$ as the vertical map of horizontal pushouts in the diagram

$$\begin{tikzcd}
\mathrm{Sd}(\Delta[m] \times Y) \arrow[d, "\kappa"'] & \mathrm{Sd}(\partial\Delta[m] \times Y) \arrow[l, tail] \arrow[r] \arrow[d, "\kappa"'] & \mathrm{Sd}(X' \times Y) \arrow[d, "\kappa"] \\
\mathrm{Sd}\,\Delta[m] \times \mathrm{Sd}\,Y & \mathrm{Sd}\,\partial\Delta[m] \times \mathrm{Sd}\,Y \arrow[l, tail] \arrow[r] & \mathrm{Sd}\,X' \times \mathrm{Sd}\,Y .
\end{tikzcd}$$

This uses the fact that Sd preserves colimits and cofibrations [9, 2.2.9]. By induction we may assume that κ is simple for $\partial\Delta[m]$ and X'. The case of general X then follows from the case $X = \Delta[m]$, by the gluing lemma for simple maps [9, 2.1.3(d)].

The reduction to the case $Y = \Delta[n]$ is similar. Write $Y = \Delta[n] \cup_{\partial\Delta[n]} Y'$ and think of $\kappa\colon \operatorname{Sd}(\Delta[m] \times Y) \to \operatorname{Sd}\Delta[m] \times \operatorname{Sd} Y$ as the map of pushouts in the diagram

$$\begin{tikzcd}
\operatorname{Sd}(\Delta[m] \times \Delta[n]) \arrow[d, "\kappa"'] & \operatorname{Sd}(\Delta[m] \times \partial\Delta[n]) \arrow[l, tail] \arrow[r] \arrow[d, "\kappa"'] & \operatorname{Sd}(\Delta[m] \times Y') \arrow[d, "\kappa"] \\
\operatorname{Sd}\Delta[m] \times \operatorname{Sd}\Delta[n] & \operatorname{Sd}\Delta[m] \times \operatorname{Sd}\partial\Delta[n] \arrow[l, tail] \arrow[r] & \operatorname{Sd}\Delta[m] \times \operatorname{Sd} Y' .
\end{tikzcd}$$

By induction we may assume that κ is simple for $\partial\Delta[n]$ and Y', so the main theorem follows from the case $Y = \Delta[n]$, i.e., from Proposition 1.3. □

Outline of proof of Proposition 1.3. To show that the canonical map

$$\kappa\colon \operatorname{Sd}(\Delta[m] \times \Delta[n]) \longrightarrow \operatorname{Sd}\Delta[m] \times \operatorname{Sd}\Delta[n]$$

is simple, it suffices to show that it is simple over the interior of each non-degenerate r-simplex $(z, w) = (z_0 \le \dots \le z_r, w_0 \le \dots \le w_r)$ of the target, for $r \ge 0$, cf. Notation 4.2. By Lemmas 4.3, 4.7 and 4.11, κ agrees with the nerve of the order-preserving function $\pi\colon P(\phi_r, \dots, \phi_1) \to [r]$ over that interior. Here

$$F_r \xrightarrow{\phi_r} \dots \xrightarrow{\phi_1} F_0$$

is a diagram of partially ordered sets and order-preserving functions, specified in Notations 4.6 and 4.8, and $P(\phi_r, \dots, \phi_1) = F_r \sqcup_{\phi_r} \dots \sqcup_{\phi_1} F_0$ is as in Definition 4.10.

Let

$$NF_r \xrightarrow{f_r} \dots \xrightarrow{f_1} NF_0$$

be the induced diagram of nerves. According to Definition 5.4, the nerve of π equals the reduced coordinate projection $\pi\colon M(f_r, \dots, f_1) \to \Delta[r]$, where $M(f_r, \dots, f_1)$ is the r-fold iterated *reduced mapping cylinder*. As we explain in Remark 5.11, there is a commutative diagram

$$\begin{tikzcd}
T(f_r, \dots, f_1) \arrow[r, "\mathrm{red}"] \arrow[d] & M(f_r, \dots, f_1) \arrow[d, "\pi"] \\
T^r \arrow[r, "\mathrm{red}"] & \Delta[r] ,
\end{tikzcd}$$

where $T(f_r, \dots, f_1)$ is the r-fold iterated ordinary mapping cylinder, cf. Definition 5.6. The reduction map $\mathrm{red}\colon T(f_r, \dots, f_1) \to M(f_r, \dots, f_1)$ is simple by Lemma 5.18, and the map $\mathrm{red}\colon T^r \to \Delta[r]$ is simple by Lemma 5.10.

By the composition and right cancellation properties of simple maps [9, 2.1.3], the reduced coordinate projection π will be simple if the ordinary cylinder projection $T(f_r, \dots, f_1) \to T^r$ is simple. Each point inverse of the geometric realization $|T(f_r, \dots, f_1)| \to |T^r|$ is homeomorphic to one of the spaces $|NF_i|$, for $0 \le i \le r$, so it suffices to prove that each of the partially ordered sets F_i has contractible classifying space. Using Notation 6.1, each F_i is of the form $P^{z',w'}$ for a suitable i-simplex (z', w') of $\operatorname{Sd}\Delta[m] \times \operatorname{Sd}\Delta[n]$. Hence the required contractibility is a consequence of our final technical result, Proposition 6.7. □

4. Partially ordered sets of paths

As a first step towards the proof of Proposition 1.3, we unravel the definition of the canonical map κ in that special case, and identify the part of κ that sits over a non-degenerate r-simplex in its target.

The m-simplex $\Delta[m] = N([m])$ is the nerve of the totally ordered set $[m] = \{0 < 1 < \dots < m\}$. Its k-simplices are the order-preserving functions $[k] \to [m]$, which we will refer to as *operators*, following [3, §4.1]. Injective operators are called *face operators*, and surjective operators are called *degeneracy operators*. The non-degenerate simplices of $\Delta[m]$ are the face operators $\mu\colon [k] \to [m]$, which correspond to the non-empty subsets $\operatorname{im}(\mu)$ of $[m]$. As in Section 2, we partially order the set $\Delta[m]^\#$ of non-degenerate simplices by setting $\mu \le \zeta$ if μ is a face of ζ, or equivalently, if the image of μ is a subset of the image of ζ. The Kan subdivision $\operatorname{Sd}\Delta[m] = N(\Delta[m]^\#)$ is the nerve of that partially ordered set. Hence an r-simplex of $\operatorname{Sd}\Delta[m]$ is a chain $\zeta_0 \le \zeta_1 \le \cdots \le \zeta_r$ of face operators $\zeta_i\colon [k_i] \to [m]$ for $0 \le i \le r$.

The product $\Delta[m] \times \Delta[n]$ can be regarded as the nerve of $[m] \times [n]$ with the product partial ordering. Its k-simplices are the order-preserving functions $[k] \to [m] \times [n]$, and its non-degenerate k-simplices are the injective order-preserving functions

$$\gamma\colon [k] \to [m] \times [n]\,,$$

which correspond to the non-empty totally ordered subsets $\operatorname{im}(\gamma)$ of $[m] \times [n]$. The set $C = (\Delta[m] \times \Delta[n])^\#$ of non-degenerate simplices is partially ordered by setting $\beta \le \gamma$ if β is a face of γ, or equivalently, if the image of β is a subset of the image of γ. The product $\Delta[m] \times \Delta[n]$ is a non-singular simplicial set, so its Kan subdivision $\operatorname{Sd}(\Delta[m] \times \Delta[n]) = N((\Delta[m] \times \Delta[n])^\#)$ is the nerve of the partially ordered set of its non-degenerate simplices. An r-simplex is a chain $\gamma_0 \le \gamma_1 \le \cdots \le \gamma_r$ of injective order-preserving functions $\gamma_i\colon [k_i] \to [m] \times [n]$ for $0 \le i \le r$.

The first projection $\mathrm{pr}_1\colon \Delta[m] \times \Delta[n] \to \Delta[m]$ induces the order-preserving function $\mathrm{pr}_1^\#$ that takes each injective order-preserving function $\gamma\colon [k] \to [m] \times [n]$ to the non-degenerate part $(\mathrm{pr}_1 \circ \gamma)^\#\colon [m_1] \to [m]$ of the composite $\mathrm{pr}_1 \circ \gamma$. This is the non-degenerate simplex with image $\operatorname{im}(\mathrm{pr}_1^\#(\gamma)) = \operatorname{im}(\mathrm{pr}_1 \circ \gamma)$. Similarly the second projection $\mathrm{pr}_2\colon \Delta[m] \times \Delta[n] \to \Delta[n]$ induces an order-preserving function $\mathrm{pr}_2^\#$ that takes γ to the non-degenerate simplex $[n_1] \to [n]$ with image $\operatorname{im}(\mathrm{pr}_2 \circ \gamma)$. We can display these order-preserving functions in the following diagram, where a feathered arrow denotes an injection and a two-headed arrow denotes a surjection.

$$\begin{tikzcd}
{[m_1]} \arrow[d, tail, "\mathrm{pr}_1^\#(\gamma)"'] & {[k]} \arrow[l, two heads] \arrow[r, two heads] \arrow[d, tail, "\gamma"'] & {[n_1]} \arrow[d, tail, "\mathrm{pr}_2^\#(\gamma)"] \\
{[m]} & {[m] \times [n]} \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] & {[n]}
\end{tikzcd}$$

The functorial map $\operatorname{Sd}(\mathrm{pr}_1)\colon \operatorname{Sd}(\Delta[m] \times \Delta[n]) \to \operatorname{Sd}\Delta[m]$ equals the map of nerves $N(\mathrm{pr}_1^\#)$ induced by $\mathrm{pr}_1^\#$, and similarly for the second projection. Hence the canonical map $\kappa = (\operatorname{Sd}(\mathrm{pr}_1), \operatorname{Sd}(\mathrm{pr}_2))$ equals the map $(N(\mathrm{pr}_1^\#), N(\mathrm{pr}_2^\#))$, which in turn can be identified with the map of nerves induced by

$$(\mathrm{pr}_1^\#, \mathrm{pr}_2^\#)\colon (\Delta[m] \times \Delta[n])^\# \to \Delta[m]^\# \times \Delta[n]^\#\,,$$

since the diagram

$$\begin{tikzcd}
N((\Delta[m] \times \Delta[n])^\#) \arrow[d, "N(\mathrm{pr}_1^\#, \mathrm{pr}_2^\#)"'] \arrow[dr, "\kappa"] & \\
N(\Delta[m]^\# \times \Delta[n]^\#) \arrow[r, "\cong"'] & N(\Delta[m]^\#) \times N(\Delta[n]^\#)
\end{tikzcd}$$

commutes.

Definition 4.1. Let $\mu \in \Delta[m]^\#$ and $\nu \in \Delta[n]^\#$ be non-degenerate simplices. A non-degenerate simplex $\gamma \in (\Delta[m] \times \Delta[n])^\#$ maps under $(\mathrm{pr}_1^\#, \mathrm{pr}_2^\#)$ to (μ, ν) if and only if $\operatorname{im}(\mathrm{pr}_1 \circ \gamma) = \operatorname{im}(\mu)$ and $\operatorname{im}(\mathrm{pr}_2 \circ \gamma) = \operatorname{im}(\nu)$. In this case we say that γ is a *(μ, ν)-path*. We write $P^{\mu,\nu}$ for the set of (μ, ν)-paths, partially ordered as a subset of $C = (\Delta[m] \times \Delta[n])^\#$.

This terminology (being a (μ,ν)-path) will be important throughout this paper. Note that being a (μ,ν)-path is a stronger condition than just being a path with image in $\mathrm{im}(\mu)\times\mathrm{im}(\nu)$.

Notation 4.2. Let $(z,w)=(z_0\le\cdots\le z_r, w_0\le\cdots\le w_r)$ be a non-degenerate r-simplex in $\mathrm{Sd}\,\Delta[m]\times\mathrm{Sd}\,\Delta[n]$, with characteristic map $\overline{(z,w)}\colon\Delta[r]\to\mathrm{Sd}\,\Delta[m]\times\mathrm{Sd}\,\Delta[n]$, so that $z_i\in\Delta[m]^\#$ and $w_i\in\Delta[n]^\#$ are non-degenerate simplices for $0\le i\le r$. Let $P_i=P^{z_i,w_i}$ denote the partially ordered set of (z_i,w_i)-paths.

The (z_i,w_i) are pairwise distinct, so the P_i are pairwise disjoint as sets, for $0\le i\le r$. Let $P_0\cup\cdots\cup P_r$ denote the union of these sets, partially ordered as a subset of $C=(\Delta[m]\times\Delta[n])^\#$, and let $\lambda\colon P_0\cup\cdots\cup P_r\to[r]$ be the order-preserving function taking the elements of P_i to i, for each $0\le i\le r$.

By the restriction of a map $f\colon A\to B$ over a simplicial subset $Z\subseteq B$, we mean the associated map $f^{-1}(Z)\to Z$, given by restricting f to $f^{-1}(Z)\subseteq A$, and corestricting the target to Z.

Lemma 4.3. *The restriction of κ over the simplicial subset of $\mathrm{Sd}\,\Delta[m]\times\mathrm{Sd}\,\Delta[n]$ generated by (z,w) is isomorphic to the map $N\lambda\colon N(P_0\cup\cdots\cup P_r)\to\Delta[r]$.*

Proof. Nerves commute with pullbacks, which means we have a pullback square

$$\begin{tikzcd} N(P_0\cup\cdots\cup P_r) \arrow[r, tail] \arrow[d, "N\lambda"'] & \mathrm{Sd}(\Delta[m]\times\Delta[n]) \arrow[d, "\kappa"] \\ \Delta[r] \arrow[r, tail, "\overline{(z,w)}"] & \mathrm{Sd}\,\Delta[m]\times\mathrm{Sd}\,\Delta[n]\,. \end{tikzcd}$$

□

Corollary 4.4. *Let (μ,ν) be a 0-simplex in $\mathrm{Sd}\,\Delta[m]\times\mathrm{Sd}\,\Delta[n]$. The preimage under κ of the simplicial subset generated by (μ,ν) equals the nerve of the partially ordered set $P^{\mu,\nu}$ of (μ,ν)-paths. Hence the preimage under $|\kappa|$ of the corresponding 0-cell in $|\mathrm{Sd}\,\Delta[m]\times\mathrm{Sd}\,\Delta[n]|$ equals the classifying space $|NP^{\mu,\nu}|$ of that partially ordered set.*

We now explain how only a part of the union $P_0\cup\cdots\cup P_r$ plays a role over the interior of the r-cell $|\Delta[r]|$.

Definition 4.5. Let (μ,ν), $(\zeta,\eta)\in\Delta[m]^\#\times\Delta[n]^\#$ be such that $\mu\le\zeta$ and $\nu\le\eta$, and let $L=\mathrm{im}(\mu)\times\mathrm{im}(\nu)\subseteq[m]\times[n]$. We say that a (ζ,η)-path $\gamma\colon[k]\to[m]\times[n]$ is (μ,ν)-*full* if some face β of γ is a (μ,ν)-path. In this case there is a greatest such face β, with image $\mathrm{im}(\beta)=\mathrm{im}(\gamma)\cap L$. We write $\beta=\gamma\cap L$ for this greatest (μ,ν)-path.

This terminology (being (μ,ν)-full) will also be important throughout this paper. In general, not every (ζ,η)-path will be (μ,ν)-full.

Notation 4.6. For each $0\le i\le r$, let F_i denote the set of (z_i,w_i)-paths that are (z_j,w_j)-full for every $0\le j<i$. Partially order F_i as a subset of P_i, and partially order $F_0\cup\cdots\cup F_r$ as a subset of $P_0\cup\cdots\cup P_r$. The F_i are pairwise disjoint as sets, since this holds for the P_i. Let $\xi\colon F_0\cup\cdots\cup F_r\to[r]$ be the order-preserving function that takes each element of F_i to i.

The notations P_i and F_i are chosen for their brevity. They always depend on the implicit choice of a non-degenerate r-simplex (z,w), as above.

Lemma 4.7. *The maps $|N\xi|$ and $|N\lambda|$ agree over the interior of $|\Delta[r]|$.*

Proof. By the definition of ξ and λ, the triangle

$$\begin{tikzcd} N(F_0 \cup \dots \cup F_r) \arrow[r, tail] \arrow[dr, "N\xi"'] & N(P_0 \cup \dots \cup P_r) \arrow[d, "N\lambda"] \\ & \Delta[r] \end{tikzcd}$$

commutes. Let $x \in N(P_0 \cup \dots \cup P_r)$ be a p-simplex that does not lie over the boundary $\partial\Delta[r]$ of $\Delta[r]$. In other words, x is an order-preserving function such that the composite

$$[p] \xrightarrow{x} P_0 \cup \dots \cup P_r \xrightarrow{\lambda} [r]$$

is surjective. We claim that x factors through $F_0 \cup \dots \cup F_r$. To prove this, let $u \in [p]$ be arbitrary, fix i so that $x(u) \in P_i$, and consider any $0 \le j < i$. The surjectivity of $\lambda \circ x$ tells us that there is a $v \in [p]$ with $x(v) \in P_j$. We cannot have $v \ge u$, since $\lambda \circ x$ is order-preserving, so $v < u$ and the (z_j, w_j)-path $x(v)$ must be a face of $x(u)$. This shows that $x(u)$ is (z_j, w_j)-full, which implies that $x(u) \in F_i$.

It follows that $|N(F_0 \cup \dots \cup F_r)| \to |N(P_0 \cup \dots \cup P_r)|$ restricts to the identity over the interior of $|\Delta[r]|$. □

Notation 4.8. Let $L_i = \mathrm{im}(z_i) \times \mathrm{im}(w_i)$, for each $0 \le i \le r$, and let $\phi_i \colon F_i \to F_{i-1}$ be the order-preserving function $\gamma \mapsto \gamma \cap L_{i-1}$, for each $0 < i \le r$.

As before, these notations depend on the implicit choice of a non-degenerate r-simplex (z, w). To conclude this section, we clarify how the partially ordered set $F_0 \cup \dots \cup F_r$ is determined by the diagram

$$F_r \xrightarrow{\phi_r} \dots \xrightarrow{\phi_1} F_0$$

of partially ordered sets and order-preserving functions.

Definition 4.9 ([9, 2.4.3])**.** Given an order-preserving function $\varphi \colon V \to W$ of partially ordered sets, let $P(\varphi) = V \sqcup_\varphi W$ be the disjoint union of the sets V and W, with the partial ordering generated by the relations in V, the relations in W, and the relation $\varphi(v) < v$ for all $v \in V$.

In other words, for $v \in V$ and $w \in W$ we have $w \le v$ in $P(\varphi)$ if and only if $w \le \varphi(v)$ in W. We write $\varphi \vee 1 \colon P(\varphi) \to W$ for the order-preserving function that restricts to φ on V and to the identity on W, and $\pi \colon P(\varphi) \to [1] = \{0 < 1\}$ for the order-preserving function that maps V to 1 and W to 0.

Definition 4.10 ([9, 2.4.13])**.** For a sequence of r order-preserving functions

$$V_r \xrightarrow{\varphi_r} \dots \xrightarrow{\varphi_1} V_0$$

let $P(\varphi_r, \dots, \varphi_1) = V_r \sqcup_{\varphi_r} \dots \sqcup_{\varphi_1} V_0$ be the disjoint union of the sets $V_0, \dots, V_r$, with the partial ordering generated by the relations in V_i for $0 \le i \le r$, and the relations $\varphi_i(v) < v$ for all $v \in V_i$ and $0 < i \le r$. We write $\pi \colon P(\varphi_r, \dots, \varphi_1) \to [r]$ for the order-preserving function that takes V_i to i, for each $0 \le i \le r$.

Alternatively, for $r \ge 2$ we can construct $P(\varphi_r, \dots, \varphi_1)$ as an instance $P(\psi_1)$ of the previous definition, by an induction on the length of the sequence of functions. To start the induction, let $\psi_r = \varphi_r$. Thereafter, for $1 \le j < r$, let ψ_j be the composite

$$P(\varphi_r, \dots, \varphi_{j+1}) \xrightarrow{\psi_{j+1} \vee 1} V_j \xrightarrow{\varphi_j} V_{j-1} \,.$$

Then $P(\varphi_r, \dots, \varphi_j) = P(\psi_j)$ as partially ordered sets.

Lemma 4.11. *Let $F_r \xrightarrow{\phi_r} \dots \xrightarrow{\phi_1} F_0$ be as in Notation 4.8. Then $F_0 \cup \dots \cup F_r = P(\phi_r, \dots, \phi_1)$ as partially ordered sets, and the functions $\xi \colon F_0 \cup \dots \cup F_r \to [r]$ and $\pi \colon P(\phi_r, \dots, \phi_1) \to [r]$ coincide.*

Proof. The disjoint unions $F_0 \cup \dots \cup F_r$ and $F_r \sqcup_{\phi_r} \dots \sqcup_{\phi_1} F_0$ agree as subsets of $C = (\Delta[m] \times \Delta[n])^{\#}$. The lemma asserts that the subset partial ordering on $F_0 \cup \dots \cup F_r$ inherited from C equals the partial ordering generated by the subset partial orderings on the individual subsets F_i for $0 \le i \le r$, together with the relations $\phi_i(\gamma) < \gamma$ for $\gamma \in F_i$ and $0 < i \le r$.

We prove that $F_{j-1} \cup \dots \cup F_r$ and $P(\phi_r, \dots, \phi_j) = F_r \sqcup_{\phi_r} \dots \sqcup_{\phi_j} F_{j-1}$ agree as partially ordered subsets of C, by a descending induction on j. This is clear for $j = r+1$, when we interpret both expressions as F_r. Thereafter, for $1 \le j \le r$, we can inductively assume that $F_j \cup \dots \cup F_r = P(\phi_r, \dots, \phi_{j+1})$ as partially ordered subsets of C.

The order-preserving function $\psi_j \colon P(\phi_r, \dots, \phi_{j+1}) \to F_{j-1}$ is then equal to the function $F_j \cup \dots \cup F_r \to F_{j-1}$ given by $\gamma \mapsto \gamma \cap L_{j-1}$. The partial ordering on $P(\phi_r, \dots, \phi_j) = P(\psi_j)$ agrees with the subset partial ordering when restricted to $F_j \cup \dots \cup F_r = P(\phi_r, \dots, \phi_{j+1})$, and when restricted to F_{j-1}. It remains to check that, for $\gamma \in F_j \cup \dots \cup F_r$ and $\beta \in F_{j-1}$, we have $\beta \le \gamma$ in the subset partial ordering on $F_{j-1} \cup (F_j \cup \dots \cup F_r)$ if and only if $\beta \le \gamma$ in $P(\psi_j)$. But this is clear, since $\beta \le \gamma$ in $P(\psi_j)$ if and only if $\beta \le \psi_j(\gamma)$ in F_{j-1}, and $\psi_j(\gamma) = \gamma \cap L_{j-1}$ is the greatest face of γ that lies in F_{j-1}. □

Definition 4.12 ([9, 2.1.9])**.** A map $f \colon A \to B$ of finite simplicial sets is *simple over* U, for a given subset $U \subseteq |B|$, if for each $b \in U$ the preimage $|f|^{-1}(b)$ is contractible.

To summarize what we have learned so far, let κ be as in Proposition 1.3, let (z, w) be as in Notation 4.2, and let $\pi \colon P(\phi_r, \dots, \phi_1) \to [r]$ be as in Notation 4.8 and Definition 4.10.

Lemma 4.13. *The map κ is simple over the open r-cell corresponding to (z, w) in the CW structure of $|\operatorname{Sd}\Delta[m] \times \operatorname{Sd}\Delta[n]|$, if and only if $N\pi$ is simple over the interior of $|\Delta[r]|$.*

5. Mapping cylinders

The (backward) reduced mapping cylinder, as introduced in [9, §2.4], is a functor that to each simplicial map $f \colon X \to Y$ associates a simplicial set $M(f)$, factoring f as an inclusion $\mathrm{in} \colon X \to M(f)$ followed by a simple projection map $\mathrm{pr} \colon M(f) \to Y$. There is a natural reduction map $\mathrm{red} \colon T(f) \to M(f)$ from the ordinary mapping cylinder $T(f) = X \times \Delta[1] \cup_X Y$ to the reduced mapping cylinder, and the ordinary coordinate projection $T(f) \to \Delta[1]$ factors naturally through a reduced coordinate projection $\pi \colon M(f) \to \Delta[1]$:

$$\begin{tikzcd}
X \arrow[d, tail] \arrow[dr, tail, "\mathrm{in}"] \arrow[drr, bend left, "f"] & & \\
T(f) \arrow[r, "\mathrm{red}"] \arrow[dr] & M(f) \arrow[r, "\mathrm{pr}", "\simeq_s"'] \arrow[d, "\pi"] & Y \\
& \Delta[1] \, . &
\end{tikzcd}$$

More generally, given a sequence $X_r \xrightarrow{f_r} \dots \xrightarrow{f_1} X_0$ of maps of simplicial sets, there is an r-fold iterated reduced mapping cylinder $M(f_r, \dots, f_1)$, a reduction map $\mathrm{red} \colon T(f_r, \dots, f_1) \to M(f_r, \dots, f_1)$ from the r-fold iterated ordinary mapping cylinder, and compatible coordinate projection maps:

$$\begin{tikzcd}
T(f_r, \dots, f_1) \arrow[r, "\mathrm{red}"] \arrow[dr] & M(f_r, \dots, f_1) \arrow[d, "\pi"] \\
& \Delta[r] \,.
\end{tikzcd}$$

In this section, we shall recognize the map $N\pi\colon N(P(\phi_r, \dots, \phi_1)) \to \Delta[r]$ as the reduced coordinate projection map $\pi\colon M(f_r, \dots, f_1) \to \Delta[r]$ for the sequence of simplicial sets $X_i = NF_i$ and maps $f_i = N\phi_i$. Thereafter, we use the second author's criterion [9, 2.4.16] to show that the iterated reduction map $T(f_r, \dots, f_1) \to M(f_r, \dots, f_1)$ is simple in this case. This reduces the problem of showing that π is simple to the problem of showing that the classifying spaces $|NF_i|$ are contractible, which we will address in the following, final, section.

Definition 5.1 ([9, 2.4.5])**.** Given an order-preserving function $\varphi\colon V \to W$ of partially ordered sets, let $\mathrm{in}\colon V \to P(\varphi) = V \sqcup_\varphi W$ and $\mathrm{in}'\colon W \to P(\varphi)$ be the front and back inclusions, respectively, and let $\mathrm{pr} = \varphi \vee 1\colon P(\varphi) \to W$ and $\pi\colon P(\varphi) \to [1]$ be as before. Consider the commutative diagram

$$\begin{tikzcd}
& V \arrow[dl, "i_1"'] \arrow[dr, "\mathrm{in}"] \arrow[drr, bend left, "\varphi"] & & \\
V \times [1] \arrow[rr, "\rho"] & & P(\varphi) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi"] & W \\
V \arrow[u, "i_0"] \arrow[r, "\varphi"] & W \arrow[ur, "\mathrm{in}'"] & {[1]} &
\end{tikzcd}$$

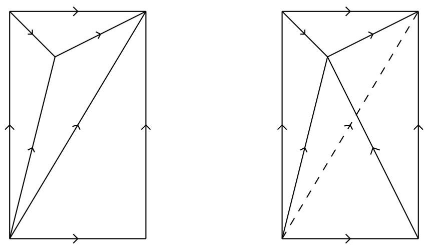
of order-preserving functions, where $i_t(v) = (v, t)$, $\rho(v, 1) = \mathrm{in}(v)$ and $\rho(v, 0) = \mathrm{in}'(\varphi(v))$. Applying nerves, we get the commutative diagram

$$\begin{tikzcd}
& NV \arrow[dl, "i_1"'] \arrow[d] \arrow[dr, "\mathrm{in}"] \arrow[drr, bend left, "N\varphi"] & & \\
NV \times \Delta[1] \arrow[r] & T(N\varphi) \arrow[r, "\mathrm{red}"] & N(P(\varphi)) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi"] & NW \\
NV \arrow[u, "i_0"] \arrow[r, "N\varphi"] & NW \arrow[u] \arrow[ur, "\mathrm{in}'"] & \Delta[1] \,, &
\end{tikzcd}$$

where the lower left hand square is the pushout defining $T(N\varphi) = NV \times \Delta[1] \cup_{NV} NW$, and the reduction map red is induced by $N\rho$ and in'. We define the *reduced mapping cylinder* of the simplicial map $f = N\varphi\colon NV \to NW$ to be the nerve $M(f) = N(P(\varphi))$ of the partially ordered set $P(\varphi) = V \sqcup_\phi W$, with inclusion, projection, reduction and coordinate projection maps as in the diagram above.

Example 5.2. Let $[2] = \{0 < 1 < 2\}$ and $[1]' = \{0' < 1'\}$. The reduced mapping cylinders of the nerves of the elementary degeneracy operators $\sigma_0\colon [2] \to [1]'; 0, 1 \mapsto$

$0'$ (left) and $\sigma_1\colon [2] \to [1]$; $1, 2 \mapsto 1'$ (right) are illustrated in the pictures below.



The extra relation $1' < 1$ in $P(\sigma_1) = [2] \sqcup_{\sigma_1} [1]'$, which does not hold in $P(\sigma_0) = [2] \sqcup_{\sigma_0} [1]'$, leads to the additional 3-simplex $0' < 1' < 1 < 2$ on the right hand side.

The reduced mapping cylinder $M(f)$ for a general simplicial map $f\colon X \to Y$ can be defined as a left Kan extension from these special cases, essentially as in [9, 2.4.4], but the definition just given suffices for our purposes. See [9, 2.4.12] for the compatibility of the two definitions.

We have already alluded to the following result.

Lemma 5.3. *Let $f\colon X \to Y$ be any map of finite simplicial sets. The projection* $\mathrm{pr}\colon M(f) \to Y$ *is a simple map.*

Proof. See [9, 2.4.8]. □

The process of making reduced mapping cylinders can be iterated.

Definition 5.4. Suppose given a sequence $V_r \xrightarrow{\varphi_r} \dots \xrightarrow{\varphi_1} V_0$ of partially ordered sets and order-preserving functions. Define the *iterated reduced mapping cylinder* of the sequence

$$X_r \xrightarrow{f_r} \dots \xrightarrow{f_1} X_0$$

of simplicial sets and maps, with $X_i = NV_i$ and $f_i = N\varphi_i$, to be the nerve

$$M(f_r, \dots, f_1) = N(P(\varphi_r, \dots, \varphi_1))$$

of the partially ordered set $P(\varphi_r, \dots, \varphi_1) = V_r \sqcup_{\varphi_r} \cdots \sqcup_{\varphi_1} V_0$. Let the *reduced coordinate projection* $\pi\colon M(f_r, \dots, f_1) \to \Delta[r]$ be the nerve of the order-preserving function $\pi\colon P(\varphi_r, \dots, \varphi_1) \to [r]$ that takes V_i to i, for each $0 \le i \le r$.

The iterated reduced mapping cylinder $M(f_r, \dots, f_1)$ of a general sequence $X_r \to \cdots \to X_0$ of simplicial sets and maps can, again, be defined as a left Kan extension from these special cases, but the definition given will suffice for this paper.

Lemma 5.5. *The iterated reduced mapping cylinder $M(f_r, \dots, f_1)$ is naturally isomorphic to the reduced mapping cylinder of the composite map*

$$M(f_r, \dots, f_2) \xrightarrow{\mathrm{pr}} X_1 \xrightarrow{f_1} X_0\,,$$

where the iterated projection $\mathrm{pr}\colon M(f_r, \dots, f_1) \to X_0$ *corresponds to the projection* $\mathrm{pr}\colon M(f_1 \circ \mathrm{pr}) \to X_0$.

Proof. Recall the notation from Definition 4.10. The composite of the two order-preserving functions

$$P(\varphi_r, \dots, \varphi_2) \xrightarrow{\psi_2 \vee 1} V_1 \xrightarrow{\varphi_1} V_0\,,$$

equals ψ_1. Hence the composite map $f_1 \circ \mathrm{pr}$ equals $N\psi_1$, and $M(f_r, \dots, f_1) = N(P(\varphi_r, \dots, \varphi_1)) = N(P(\psi_1))$ equals $M(N\psi_1) = M(f_1 \circ \mathrm{pr})$, as asserted. □

Definition 5.6. Let $X_r \xrightarrow{f_r} \dots \xrightarrow{f_1} X_0$ be any sequence of simplicial sets and maps. Define the *iterated ordinary mapping cylinder* $T(f_r, \dots, f_1) = T(f_1 \circ \mathrm{pr})$ to be the ordinary mapping cylinder of the composite map

$$T(f_r, \dots, f_2) \xrightarrow{\mathrm{pr}} X_1 \xrightarrow{f_1} X_0 \,,$$

and let the iterated projection $\mathrm{pr}\colon T(f_r, \dots, f_1) \to X_0$ be the ordinary projection $\mathrm{pr}\colon T(f_1 \circ \mathrm{pr}) \to X_0$.

Definition 5.7. Suppose given any sequence $V_r \xrightarrow{\varphi_r} \dots \xrightarrow{\varphi_1} V_0$ of partially ordered sets and order-preserving functions. Let the *r-fold iterated reduction map*

$$\mathrm{red}\colon T(f_r, \dots, f_1) \to M(f_r, \dots, f_1)$$

be defined as the composite map

$$T(T(f_r, \dots, f_2) \to X_0) \to T(M(f_r, \dots, f_2) \to X_0) \xrightarrow{\mathrm{red}} M(M(f_r, \dots, f_2) \to X_0)\,.$$

Here the left hand map is induced by the $(r-1)$-fold iterated reduction map $\mathrm{red}\colon T(f_r, \dots, f_2) \to M(f_r, \dots, f_2)$ and the identity map of X_0, while the right hand map is the reduction map for $f_1 \circ \mathrm{pr}\colon M(f_r, \dots, f_2) \to X_0$.

Notation 5.8. Consider the terminal sequence $[0] \to \dots \to [0]$ of partially ordered sets and order-preserving functions. The order-preserving function $\pi\colon [0] \sqcup_{\varphi_r} \dots \sqcup_{\varphi_1} [0] \to [r]$ is a bijection, so the r-fold iterated reduced mapping cylinder

$$M^r = M(\Delta[0] \to \dots \to \Delta[0]) = N([0] \sqcup_{\varphi_r} \dots \sqcup_{\varphi_1} [0])$$

maps isomorphically to $\Delta[r]$ under the reduced coordinate projection. Let $T^r = T(\Delta[0] \to \dots \to \Delta[0])$ be the r-fold iterated ordinary mapping cylinder. Then $T^0 \cong \Delta[0]$, $T^1 \cong \Delta[1]$, and $T^r \cong T^{r-1} \times \Delta[1] \cup_{T^{r-1}} \Delta[0]$ for $r \geq 2$. The iterated reduction map $\mathrm{red}\colon T^r \to M^r$ agrees, up to the isomorphism π, with the ordinary cylinder projection $T^r \to \Delta[r]$.

The map $T^r \to M^r$ is an isomorphism for $r = 0$ and 1. To address the case $r \geq 2$, the following terminology will be convenient.

Definition 5.9 ([9, 2.4.7])**.** A map $f\colon A \to B$ of finite simplicial sets is a *homotopy equivalence over the target* if it has a section $s\colon B \to A$ such that $|sf|$ is homotopic to the identity map on $|A|$, by a homotopy over $|B|$. The homotopy provides a contraction of each point inverse of $|f|$, so such a map f is simple.

Lemma 5.10. *The iterated reduction map* $\mathrm{red}\colon T^r \to M^r$ *is simple.*

Proof. By induction, we may assume that $r \geq 2$ and that $T^{r-1} \to M^{r-1}$ is simple. We must prove that the composite

$$T(T^{r-1} \to \Delta[0]) \to T(M^{r-1} \to \Delta[0]) \xrightarrow{\mathrm{red}} M(M^{r-1} \to \Delta[0])$$

is simple. The left hand map is simple by the inductive hypothesis and the gluing lemma for simple maps [9, 2.1.3(d)]. The right hand map is isomorphic to the reduction map for the unique map $N\epsilon\colon \Delta[r-1] \to \Delta[0]$, which is the canonical map from the pushout to the lower right hand entry in the commutative square

$$\begin{tikzcd}
\Delta[r-1] \arrow[r, "N\epsilon"] \arrow[d, "i_0"'] & \Delta[0] \arrow[d, "\mathrm{in}'"] \\
\Delta[r-1] \times \Delta[1] \arrow[r, "N\rho"] & N([r-1] \sqcup_\epsilon [0])\,.
\end{tikzcd}$$

By the gluing lemma, and the fact that $N\epsilon$ is simple, it suffices to prove that $N\rho$ is simple. There is an order-preserving bijection $[r-1] \sqcup_\epsilon [0] \cong [r]$, taking $0 \in [0]$

to $0 \in [r]$ and $i \in [r-1]$ to $i+1 \in [r]$. Under this identification, $N\rho$ corresponds to $N\theta$, where $\theta\colon [r-1] \times [1] \to [r]$ is given by $\theta(i,0) = 0$ and $\theta(i,1) = i+1$.

To prove that $N\theta\colon \Delta[r-1] \times \Delta[1] \to \Delta[r]$ is simple, it suffices to show that it is a homotopy equivalence over the target. The order-preserving function $\sigma\colon [r] \to [r-1] \times [1]$, given by $\sigma(0) = (0,0)$ and $\sigma(j) = (j-1,1)$ for $1 \le j \le r$, is a section of θ, and $(\sigma \circ \theta)(i,t) \le (i,t)$ for all $(i,t) \in [r-1] \times [1]$. Hence $N\sigma$ is a section of $N\theta$, and there is a simplicial homotopy from $N\sigma \circ N\theta$ to the identity, covering the identity of $\Delta[r]$. □

Remark 5.11. Naturality of the iterated reduction map, with respect to the terminal morphism

$$\begin{tikzcd} X_r \arrow[r, "f_r"] \arrow[d] & \dots \arrow[r, "f_1"] & X_0 \arrow[d] \\ \Delta[0] \arrow[r] & \dots \arrow[r] & \Delta[0] \end{tikzcd}$$

in the category of r-tuples of composable maps, leads to the commutative diagram

$$\begin{tikzcd} T(f_r, \dots, f_1) \arrow[r, "\mathrm{red}"] \arrow[d] & M(f_r, \dots, f_1) \arrow[d] \arrow[dr, "\pi"] & \\ T^r \arrow[r, "\mathrm{red}", "\simeq_s"'] & M^r \arrow[r, "\cong"'] & \Delta[r]\,. \end{tikzcd}$$

The lower reduction map $T^r \to M^r$ is simple, by the previous lemma. Each point inverse of the geometric realization $|T(f_r, \dots, f_1)| \to |T^r|$ of the left hand vertical map is homeomorphic to one of the spaces $|X_0|, \dots, |X_r|$. This is obvious for $r = 0$ and 1, and follows by an easy induction for $r \ge 2$. Hence the map $T(f_r, \dots, f_1) \to T^r$ is simple if (and only if) these spaces are contractible.

The left hand picture in Example 5.2 illustrates that the point inverses of the map $|\pi|\colon |M(f_r, \dots, f_1)| \to |\Delta[r]|$ need not be homeomorphic to one of the spaces $|X_0|, \dots, |X_r|$. This is why we introduce ordinary mapping cylinders and reduction maps, as technical tools.

In the remainder of this section we shall prove that the upper reduction map $T(f_r, \dots, f_1) \to M(f_r, \dots, f_1)$ is simple in the case when $X_i = NF_i$ and $f_i = N\phi_i$ are as in Notation 4.8. In the following section we shall prove that the classifying spaces $|NF_i| = |X_i|$ are contractible in that case. By the diagram above, and the right cancellation property for simple maps, it will then follow that $\pi\colon M(f_r, \dots, f_1) \to \Delta[r]$ is simple. In view of Lemma 4.13, this will complete the proof of our main theorem.

The reduction map $\mathrm{red}\colon T(f) \to M(f)$ is not always simple, as the restriction $f\colon \partial\Delta[2] \to \Delta[1]$ of $N\sigma_0\colon \Delta[2] \to \Delta[1]$ illustrates; see the left hand figure in Example 5.2. When $f = N\varphi\colon NV \to NW$ is the nerve of an order-preserving function of partially ordered sets, we have a useful criterion explained in Proposition 5.14 below.

Definition 5.12. Let V be a partially ordered set. A subset $I \subseteq V$ is called a *left ideal* if $v \in I$ and $u \le v$ in V implies $u \in I$. It is called a *right ideal* if $v \in I$ and $v \le w$ in V implies $w \in I$. We always give I the subset partial ordering.

Definition 5.13. Suppose given a partially ordered set V and an element $v \in V$. Let V/v denote the left ideal in V consisting of all elements $u \in V$ with $u \le v$. Any order-preserving function $\varphi\colon V \to W$ restricts to an order-preserving function $\varphi/v\colon V/v \to W/\varphi(v)$.

Proposition 5.14. *Let $\varphi\colon V \to W$ be an order-preserving function of finite partially ordered sets. Then the following conditions are equivalent:*

(1) *The nerve of $\varphi/v\colon V/v \to W/\varphi(v)$ is simple, for each element $v \in V$.*
(2) *The reduction map $\mathrm{red}\colon T(N\varphi) \to M(N\varphi)$ is simple.*

Proof. See [9, 2.4.16]. □

Lemma 5.15. *Let $\varphi\colon V \to W$ and $\mathrm{pr} = \varphi \vee 1\colon P(\varphi) = V \sqcup_\varphi W \to W$ be as above. The nerve of the order-preserving function $\mathrm{pr}/v\colon P(\varphi)/v \to W/\mathrm{pr}(v)$ is simple, for each $v \in P(\varphi)$.*

Proof. If $v \in W \subseteq P(\varphi)$ then $P(\varphi)/v = W/v$, $\mathrm{pr}(v) = v$ and $\mathrm{pr}/v\colon W/v \to W/v$ is the identity, whose nerve is obviously simple.

Otherwise, if $v \in V \subseteq P(\varphi)$ then $P(\varphi)/v = (V \sqcup_\varphi W)/v$ is equal to $P(\varphi/v) = V/v \sqcup_{\varphi/v} W/\varphi(v)$, and $\mathrm{pr}(v) = \varphi(v)$, so we can identify pr/v with the order-preserving function $\varphi/v \vee 1\colon P(\varphi/v) \to W/\varphi(v)$. Its nerve is the projection map from the reduced mapping cylinder of $N(\varphi/v)$ to its target, which is simple by [9, 2.4.8], recalled above as Lemma 5.3. □

We now return to the context of the previous section, where $(z_0, w_0) < \cdots < (z_r, w_r)$ is a non-degenerate r-simplex in $\mathrm{Sd}\,\Delta[m] \times \mathrm{Sd}\,\Delta[n]$, $L_i = \mathrm{im}(z_i) \times \mathrm{im}(w_i) \subset [m] \times [n]$, F_i is the set of (z_i, w_i)-paths that are (z_j, w_j)-full for every $0 \le j < i$, and $\phi_i\colon F_i \to F_{i-1}$ maps γ to $\phi_i(\gamma) = \gamma \cap L_{i-1}$.

Definition 5.16. The bijective correspondence between the elements $\gamma\colon [k] \to [m] \times [n]$ of $C = (\Delta[m] \times \Delta[n])^\#$ and the non-empty, totally ordered subsets $\mathrm{im}(\gamma) \subseteq [m] \times [n]$ lets us define two further operations:

First, there is an order-preserving function $C/\gamma \times C/\gamma \to C/\gamma$ that takes two paths $\alpha, \beta \in C/\gamma$ to the path $\alpha \cup \beta \in C/\gamma$ with $\mathrm{im}(\alpha \cup \beta) = \mathrm{im}(\alpha) \cup \mathrm{im}(\beta)$.

Second, there is an order-preserving function $F_i \to C$ taking $\beta \in F_i$ to the path $\beta \setminus L_{i-1}$ with image $\mathrm{im}(\beta) \setminus L_{i-1}$.

Lemma 5.17. *The nerve of the order-preserving function*

$$\phi_i/\gamma\colon F_i/\gamma \to F_{i-1}/\phi_i(\gamma)$$

is simple, for each $\gamma \in F_i$. Therefore the reduction map $T(N\phi_i) \to M(N\phi_i)$ is simple, for each $1 \le i \le r$.

Proof. We prove that $N(\phi_i/\gamma)$ is a homotopy equivalence over the target, in the sense of Definition 5.9. Using the operations mentioned above we can define an order-preserving function

$$\sigma\colon F_{i-1}/\phi_i(\gamma) \to F_i/\gamma$$

by $\alpha \mapsto \alpha \cup (\gamma \setminus L_{i-1})$. It is straightforward to check that σ is a section of ϕ_i/γ. If $\beta \in F_i/\gamma$, we see that

$$\beta = \phi_i(\beta) \cup (\beta \setminus L_{i-1}) \le \phi_i(\beta) \cup (\gamma \setminus L_{i-1}) = (\sigma \circ \phi_i/\gamma)(\beta)\,.$$

Hence there is a simplicial homotopy from the identity of $N(F_i/\gamma)$ to $N\sigma \circ N(\phi_i/\gamma)$, covering the identity of $N(F_{i-1}/\phi_i(\gamma))$.

The second assertion now follows from Proposition 5.14. □

Lemma 5.18. *The r-fold iterated reduction map*

$$\mathrm{red}\colon T(N\phi_r, \dots, N\phi_1) \to M(N\phi_r, \dots, N\phi_1)$$

is simple, for each $r \ge 0$.

Proof. This is trivial for $r = 0$, and was proved in Lemma 5.17 for $r = 1$. We may therefore suppose $r \geq 2$, and assume, for some $1 \leq i \leq r$, that the $(r-i)$-fold iterated reduction map $\mathrm{red}\colon T(N\phi_r, \dots, N\phi_{i+1}) \to M(N\phi_r, \dots, N\phi_{i+1})$ is simple. We must prove that the $(r-i+1)$-fold iterated reduction map $\mathrm{red}\colon T(N\phi_r, \dots, N\phi_i) \to M(N\phi_r, \dots, N\phi_i)$ is simple. By definition, this is the composite of two further maps, each of which will be shown to be simple.

The first of the two is the map of ordinary mapping cylinders

$$T(T(N\phi_r, \dots, N\phi_{i+1}) \to NF_{i-1}) \to T(M(N\phi_r, \dots, N\phi_{i+1}) \to NF_{i-1})$$

induced by the $(r-i)$-fold iterated reduction map and the identity on NF_{i-1}. It is simple by the inductive hypothesis and the gluing lemma for simple maps.

The second of the two is the reduction map

$$T(M(N\phi_r, \dots, N\phi_{i+1}) \to NF_{i-1}) \to M(M(N\phi_r, \dots, N\phi_{i+1}) \to NF_{i-1})$$

associated to $N\phi_i \circ \mathrm{pr}\colon M(N\phi_r, \dots, N\phi_{i+1}) \to NF_{i-1}$. The latter map equals the nerve of the composite order-preserving function

$$P(\phi_r, \dots, \phi_{i+1}) \xrightarrow{\psi_{i+1} \vee 1} F_i \xrightarrow{\phi_i} F_{i-1} ,$$

which we denote as $\psi_i = \phi_i \circ (\psi_{i+1} \vee 1)$. Recall from the proof of Lemma 5.5 that $P(\phi_r, \dots, \phi_{i+1}) = P(\psi_{i+1})$, where $\psi_{i+1}\colon F_r \sqcup_{\phi_r} \dots \sqcup_{\phi_{i+2}} F_{i+1} \to F_i$ is given by the composite $\phi_{i+1} \circ \dots \circ \phi_j$ on F_j, so that $\psi_{i+1} \vee 1 = \mathrm{pr}\colon P(\psi_{i+1}) \to F_i$ and $\psi_i = \phi_i \circ \mathrm{pr}$. We will use the criterion in Proposition 5.14 to show that $\mathrm{red}\colon T(N\psi_i) \to M(N\psi_i)$ is simple.

Consider any $\gamma \in P(\phi_r, \dots, \phi_{i+1}) = P(\psi_{i+1})$. We must show that the nerve of ψ_i/γ is simple. This order-preserving function is the composite of pr/γ and $\phi_i/\mathrm{pr}(\gamma)$. The nerve of $\mathrm{pr}/\gamma\colon P(\psi_{i+1})/\gamma \to F_i/\mathrm{pr}(\gamma)$ is simple by Lemma 5.15, for the order-preserving function ψ_{i+1} and the element $\gamma \in P(\psi_{i+1})$. The nerve of $\phi_i/\mathrm{pr}(\gamma)$ is simple by Lemma 5.17, for the element $\mathrm{pr}(\gamma) \in F_i$. Hence the composite $N(\phi_i/\mathrm{pr}(\gamma)) \circ N(\mathrm{pr}/\gamma)$ is also simple, as we needed to prove. □

6. Contracting sets of paths

As in the previous sections, we consider the map of nerves $\kappa\colon \mathrm{Sd}(\Delta[m] \times \Delta[n]) \to \mathrm{Sd}\,\Delta[m] \times \mathrm{Sd}\,\Delta[n]$ induced by the order-preserving function $(\mathrm{pr}_1^\#, \mathrm{pr}_2^\#)\colon (\Delta[m] \times \Delta[n])^\# \to \Delta[m]^\# \times \Delta[n]^\#$. We consider an r-simplex (z, w) in the target of κ, represented by a chain $(z_0 \leq \dots \leq z_r, w_0 \leq \dots \leq w_r)$, where the z_i and w_i are faces of $\Delta[m]$ and $\Delta[n]$, respectively. However, unlike in the previous sections, we no longer assume that (z, w) is non-degenerate. This added generality will be convenient for our inductive proofs.

Notation 6.1. For any r-simplex $(z, w) = (z_0 \leq \dots \leq z_r, w_0 \leq \dots \leq w_r)$ in the nerve of the partially ordered set $\Delta[m]^\# \times \Delta[n]^\#$, let $P^{z,w}$ be the partially ordered set of (z_r, w_r)-paths $\gamma\colon [k] \to [m] \times [n]$ that are (z_j, w_j)-full for each $0 \leq j < r$, partially ordered as a subset of $(\Delta[m] \times \Delta[n])^\#$.

When (z, w) is non-degenerate, $P^{z,w}$ is equal to the partially ordered set F_r from Notation 4.6. More generally, for each $0 \leq i \leq r$ the partially ordered set F_i is equal to $P^{z',w'}$, where $(z', w') = (z_0 \leq \dots \leq z_i, w_0 \leq \dots \leq w_i)$ is the front i-face of (z, w). As explained in Remark 5.11, the task that remains is to prove that each classifying space $|NF_i|$ is contractible. Hence it will suffice to prove that $|NP^{z,w}|$ is contractible, for each simplex (z, w). We shall prove this by induction on the dimension $r \geq 0$ of that simplex.

We begin with the case $r = 0$, when $(z, w) = (z_0, w_0) = (\mu, \nu)$ for some faces $\mu \in \Delta[m]^\#$ and $\nu \in \Delta[n]^\#$. The proof we present for the following proposition introduces some ideas that will reappear in the cases $r \geq 1$.

Proposition 6.2. *The classifying space $|NP^{\mu,\nu}|$ is contractible, for each pair of faces $\mu \in \Delta[m]^\#$ and $\nu \in \Delta[n]^\#$.*

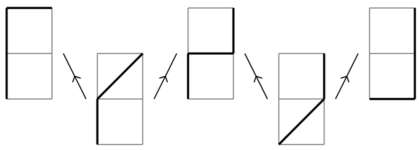
Proof. Let $\iota_m \colon [m] \to [m]$ and $\iota_n \colon [n] \to [n]$ be the identity morphisms. In this proof it will be convenient to refer to (ι_m, ι_n)-paths as (m,n)*-paths*, and to write $P^{m,n} = P^{\iota_m,\iota_n}$ for the partially ordered set formed by these paths. It consists of the injective order-preserving functions $\gamma \colon [k] \to [m] \times [n]$ such that $\mathrm{pr}_1 \circ\gamma \colon [k] \to [m]$ and $\mathrm{pr}_2 \circ\gamma \colon [k] \to [n]$ are surjective, and is partially ordered so that $\beta \le \gamma$ if and only if $\mathrm{im}(\beta) \subseteq \mathrm{im}(\gamma)$.

Represent the faces μ and ν as injective order-preserving functions $\mu \colon [m_1] \to [m]$ and $\nu \colon [n_1] \to [n]$, respectively. Then there is an order-preserving bijection $P^{m_1,n_1} \cong P^{\mu,\nu}$, taking $\gamma \colon [k] \to [m_1] \times [n_1]$ to $(\mu \times \nu)\gamma \colon [k] \to [m] \times [n]$. It will therefore suffice to prove that the classifying space of P^{m_1,n_1} is contractible, for each $m_1, n_1 \ge 0$. Changing the notation, we will prove that each partially ordered set $P^{m,n}$ of (m,n)-paths has contractible classifying space. This will be done by induction on $m \ge 0$.

Note first that any (m,n)-path $\gamma \colon [k] \to [m] \times [n]$ must pass through $\gamma(0) = (0,0)$ and $\gamma(k) = (m,n)$, due to the assumption that $\mathrm{pr}_1 \circ\gamma$ and $\mathrm{pr}_2 \circ\gamma$ are surjective.

For $m = 0$, $P^{0,n}$ consists of a single path $[n] \to [0] \times [n]$, taking i to $(0,i)$, so $|NP^{0,n}|$ is a single point.

For $m = 1$, $P^{1,n}$ consists of n "short" paths $[n] \to [1] \times [n]$ (for each $0 \le j < n$ there is one such path mapping j and $j+1$ to $(0,j)$ and $(1,j+1)$, respectively), and $n+1$ "long" paths $[n+1] \to [1] \times [n]$ (for each $0 \le j \le n$ there is one such path mapping j and $j+1$ to $(0,j)$ and $(1,j)$, respectively). These are partially ordered as a zig-zag of length $2n$, as illustrated below for $n = 2$.



Hence the nerve $NP^{1,n}$ is a union of $2n$ edges, with alternating orientations, and the classifying space $|NP^{1,n}|$ is homeomorphic to an interval.

To handle the general case, fix some $m \ge 1$ and $n \ge 0$, and suppose inductively that $|NP^{m_1,n_1}|$ is contractible whenever $0 \le m_1 \le m$ and $n_1 \ge 0$. We shall prove that the classifying space of $P^{m+1,n}$ is contractible.

For each $0 \le j \le n$, let $Q_j \subset P^{m+1,n}$ be the partially ordered subset consisting of the paths $\gamma \colon [k] \to [m+1] \times [n]$ that pass through (m,j), i.e., so that $(m,j) \in \mathrm{im}(\gamma)$. Each $(m+1,n)$-path γ has to pass through some (m,j), since $\mathrm{pr}_2 \circ\gamma \colon [k] \to [m+1]$ is surjective, so the Q_j cover $P^{m+1,n}$. Furthermore, these are right ideals in $P^{m+1,n}$, so their nerves NQ_j also cover $NP^{m+1,n}$, and

$$|NP^{m+1,n}| = |NQ_0| \cup \cdots \cup |NQ_n|$$

is a finite union of CW complexes. By a Mayer–Vietoris type argument, it will therefore suffice to prove that each ℓ-fold intersection

$$|NQ_{j_1}| \cap \cdots \cap |NQ_{j_\ell}| = |N(Q_{j_1} \cap \cdots \cap Q_{j_\ell})|$$

is contractible, for $0 \le j_1 < \cdots < j_\ell \le n$ and $\ell \ge 1$. In fact, we only need to consider single and double intersections, since $Q_{j_1} \cap \cdots \cap Q_{j_\ell} = Q_{j_1} \cap Q_{j_\ell}$ in our context.

In the first case, there is an order-preserving bijection

$$Q_j \cong P^{m,j} \times P^{1,n-j}$$

for each $0 \le j \le n$, breaking an $(m+1,n)$-path γ passing through (m,j) into two pieces: γ^1 in $[m] \times [j]$ and γ^2 in $\{m < m+1\} \times \{j < \cdots < n\} \cong [1] \times [n-j]$.

Hence $|NQ_j| \cong |NP^{m,j}| \times |NP^{1,n-j}|$ is a product of two contractible spaces, by our inductive hypothesis.

In the second case, there is an order-preserving bijection

$$Q_i \cap Q_j \cong P^{m,i} \times P^{0,j-i} \times P^{1,n-j}$$

for each $0 \le i < j \le n$, breaking a path γ passing through (m, i) and (m, j) into three pieces: γ^1 in $[m] \times [i]$, γ^2 in $\{m\} \times \{i < \cdots < j\} \cong [0] \times [j-i]$ and γ^3 in $\{m < m+1\} \times \{j < \cdots < n\} \cong [1] \times [n-j]$. Hence $|N(Q_i \cap Q_j)| \cong |NP^{m,i}| \times |NP^{0,j-i}| \times |NP^{1,n-j}|$ is a product of three contractible spaces, by the inductive hypothesis. □

To handle the cases $r \ge 1$, we will use the following notation.

Notation 6.3. For $m, n \ge 0$, let $C^{m,n} = (\Delta[m] \times \Delta[n])^\#$ be the partially ordered set of paths γ, previously denoted C, where $\gamma\colon [k] \to [m] \times [n]$ is order-preserving and injective. Let $D^{m,n} = \Delta[m]^\# \times \Delta[n]^\#$ be the partially ordered set of pairs of faces (μ, ν), where $\mu\colon [m_1] \to [m]$ and $\nu\colon [n_1] \to [n]$ are order-preserving and injective. The order-preserving function $(\mathrm{pr}_1^\#, \mathrm{pr}_2^\#)\colon C^{m,n} \to D^{m,n}$ induces the canonical map κ upon passage to nerves.

For any $(p, q) \in [m] \times [n]$, let $C^{m,n}_{(p,q)}$ be the subset of paths $\gamma \in C^{m,n}$ that pass through (p, q), so that $\gamma(j) = (p, q)$ for some $j \in [k]$. Let $D^{m,n}_{(p,q)}$ be the subset of pairs $(\mu, \nu) \in D^{m,n}$ such that $(p, q) \in \mathrm{im}(\mu) \times \mathrm{im}(\nu)$. When $(p, q) \in \mathrm{im}(z_0) \times \mathrm{im}(w_0)$, let $P^{z,w}_{(p,q)}$ be the subset of (z, w)-paths $\gamma \in P^{z,w}$ that pass through (p, q), in the same sense as above. Each of these subsets is given the subset partial ordering.

Lemma 6.4. *(a) There is an order-preserving bijection*

$$C^{m,n}_{(p,q)} \cong C^{p,q}_{(p,q)} \times C^{m-p,n-q}_{(0,0)}$$

taking $\gamma\colon [k] \to [m] \times [n]$ with $\gamma(j) = (p, q)$ to (γ^1, γ^2), where $\gamma^1\colon [j] \to [p] \times [q]$ and $\gamma^2\colon [k-j] \to [m-p] \times [n-q]$ are given by $\gamma^1(i) = \gamma(i)$ for $i \in [j]$ and $\gamma^2(i) + (p, q) = \gamma(i + j)$ for $i \in [k-j]$.

(b) There is an order-preserving bijection

$$D^{m,n}_{(p,q)} \cong D^{p,q}_{(p,q)} \times D^{m-p,n-q}_{(0,0)}$$

taking (μ, ν) to $((\mu^1, \nu^1), (\mu^2, \nu^2))$, where (if $\mu\colon [m_1] \to [m]$ and $\mu(j) = p$) $\mu^1\colon [j] \to [p]$ and $\mu^2\colon [m_1 - j] \to [m-p]$ are given by $\mu^1(i) = \mu(i)$ and $\mu^2(i) + p = \mu(i + j)$, and similarly for ν, ν^1 and ν^2.

(c) Suppose that $(p, q) \in \mathrm{im}(z_0) \times \mathrm{im}(w_0)$. The bijection in (a) restricts to an order-preserving bijection

$$P^{z,w}_{(p,q)} \cong P^{z^1,w^1} \times P^{z^2,w^2}$$

taking a (z, w)-path γ passing through (p, q) to (γ^1, γ^2), where γ^1 is a (z^1, w^1)-path and γ^2 is a (z^2, w^2)-path. Here the bijection in (b) takes z_i to (z^1_i, z^2_i) and w_i to (w^1_i, w^2_i) for each i, and $(z^1, w^1) = (z^1_0 \le \cdots \le z^1_r, w^1_0 \le \cdots \le w^1_r)$ and $(z^2, w^2) = (z^2_0 \le \cdots \le z^2_r, w^2_0 \le \cdots \le w^2_r)$.

Proof. Part (a) is clear, since any path $\gamma\colon [k] \to [m] \times [n]$ with $\gamma(j) = (p, q)$ must map $[j]$ into $[p] \times [q]$ and $\{j < \cdots < k\} \cong [k-j]$ into $\{p < \cdots < m\} \times \{q < \cdots < n\} \cong [m-p] \times [n-q]$.

Part (b) is also clear, since any face $\mu\colon [m_1] \to [m]$ with $\mu(j) = p$ must map $[j]$ into $[p]$ and $\{j < \cdots < k\} \cong [k-j]$ into $\{p < \cdots < m\} \cong [m-p]$, and likewise for ν.

To prove (c), note that a path γ passing through (p, q) is a (z_r, w_r)-path if and only if γ^1 (passing through (p, q)) is a (z^1_r, w^1_r)-path and γ^2 (passing through $(0, 0)$)

is a (z^2_r, w^2_r)-path. The condition that γ^1 passes through (p, q) is automatic, since $(p, q) \in \mathrm{im}(z^1_0) \times \mathrm{im}(w^1_0) \subseteq \mathrm{im}(z^1_r) \times \mathrm{im}(w^1_r)$, so a (z^1_r, w^1_r)-path in $[p] \times [q]$ must end at (p, q), and similarly for the condition that γ^2 passes through $(0, 0)$.

Finally, for each $0 \le j < r$, the (z_r, w_r)-path γ passing through (p, q) is (z_j, w_j)-full if and only if γ^1 is (z^1_j, w^1_j)-full and γ^2 is (z^2_j, w^2_j)-full. This follows from the result in the previous paragraph, together with the observation that the path $\gamma \cap \mathrm{im}(z_j) \times \mathrm{im}(w_j)$, with image $\mathrm{im}(\gamma) \cap \mathrm{im}(z_j) \times \mathrm{im}(w_j)$, passes through (p, q), and corresponds under the bijection in (a) with the pair of paths $\gamma^1 \cap \mathrm{im}(z^1_j) \times \mathrm{im}(w^1_j)$ and $\gamma^2 \cap \mathrm{im}(z^2_j) \times \mathrm{im}(w^2_j)$. □

Notation 6.5. For $(p, q) \le (p', q')$ in $[m] \times [n]$, let $C^{m,n}_{(p,q),(p',q')} = C^{m,n}_{(p,q)} \cap C^{m,n}_{(p',q')}$ be the set of paths that pass through both (p, q) and (p', q'). Let $D^{m,n}_{(p,q),(p',q')} = D^{m,n}_{(p,q)} \cap D^{m,n}_{(p',q')}$ be the set of pairs (μ, ν) such that (p, q) and (p', q') lie in $\mathrm{im}(\mu) \times \mathrm{im}(\nu)$. When $(p, q), (p', q') \in \mathrm{im}(z_0) \times \mathrm{im}(w_0)$, let $P^{z,w}_{(p,q),(p',q')} = P^{z,w}_{(p,q)} \cap P^{z,w}_{(p',q')}$ be the set of (z, w)-paths that pass through both (p, q) and (p', q').

Lemma 6.6. *Suppose that $(p, q) \le (p', q')$ in $\mathrm{im}(z_0) \times \mathrm{im}(w_0)$.*

(a) There is an order-preserving bijection

$$D^{m,n}_{(p,q),(p',q')} \cong D^{p,q}_{(p,q)} \times D^{p'-p,q'-q}_{(0,0),(p'-p,q'-q)} \times D^{m-p',n-q'}_{(0,0)}$$

taking (μ, ν) to $((\mu^1, \nu^1), (\mu^2, \nu^2), (\mu^3, \nu^3))$.

(b) There is an order-preserving bijection

$$P^{z,w}_{(p,q),(p',q')} \cong P^{z^1,w^1} \times P^{z^2,w^2} \times P^{z^3,w^3}$$

taking the (z, w)-paths γ that pass through both (p, q) and (p', q'), to the triples $(\gamma^1, \gamma^2, \gamma^3)$, where γ^ℓ is a (z^ℓ, w^ℓ)-path for each $1 \le \ell \le 3$.

Proof. This follows by two applications of the previous lemma, or by a direct proof. In (b), $(z^\ell, w^\ell) = (z^\ell_0 \le \cdots \le z^\ell_r, w^\ell_0 \le \cdots \le w^\ell_r)$ for each $1 \le \ell \le 3$, where z_i corresponds to (z^1_i, z^2_i, z^3_i) as in (a), and likewise for w_i. □

We can now prove our main technical result.

Proposition 6.7. *The classifying space $|NP^{z,w}|$ is contractible, for each simplex (z, w) in $\mathrm{Sd}\,\Delta[m] \times \mathrm{Sd}\,\Delta[n]$.*

Proof. The proof involves a double induction. We begin with an *outer induction* on the dimension $r \ge 0$ of the simplex (z, w). The initial case $r = 0$ was handled in Proposition 6.2. For the outer inductive step, we fix an $r \ge 0$ and assume that $|NP^{z,w}|$ is contractible for each r-simplex (z, w). We shall prove that $|NP^{x,y}|$ is contractible for each $(r + 1)$-simplex

$$(x, y) = (x_0 \le \cdots \le x_{r+1}, y_0 \le \cdots \le y_{r+1})$$

in the nerve of $\Delta[m]^\# \times \Delta[n]^\#$.

Let the pair (ζ, η) be the 0-th vertex of (x, y), and let the r-simplex (z, w) be the 0-th face of (x, y). In other words, let $\zeta = x_0 \colon [s] \to [m]$ and $\eta = y_0 \colon [t] \to [n]$, for some $s, t \ge 0$, and let $z_i = x_{i+1}$ and $w_i = y_{i+1}$ for all $0 \le i \le r$. With this notation, an (x, y)-path is the same as a (z, w)-path that is (ζ, η)-full. We maintain this notational scheme throughout this proof.

We shall prove that the classifying space of $P^{x,y}$ is contractible, by an *inner induction* on the dimension $s \ge 0$ of the face ζ of $\Delta[m]$.

The inner induction begins with the case $s = 0$: When $\zeta \colon [0] \to [m]$ is 0-dimensional, an (x, y)-path is the same as a (z, w)-path that goes through both $(\zeta(0), \eta(0))$ and $(\zeta(0), \eta(t))$. Hence

$$P^{x,y} = P^{z,w}_{(\zeta(0),\eta(0)),(\zeta(0),\eta(t))} \cong P^{z^1,w^1} \times P^{z^2,w^2} \times P^{z^3,w^3}$$

by Lemma 6.6. Here $(z_i^1, w_i^1) \in D^{\zeta(0),\eta(0)}$, $(z_i^2, w_i^2) \in D^{0,\eta(t)-\eta(0)}$ and $(z_i^3, w_i^3) \in D^{m-\zeta(0),n-\eta(t)}$, for all $0 \le i \le r$. Each (z^ℓ, w^ℓ) is an r-simplex, for $1 \le \ell \le 3$, so $|NP^{x,y}|$ is a product of three contractible spaces by the outer inductive hypothesis, and is therefore contractible.

The inner induction continues with the case $s = 1$: When $\zeta\colon [1] \to [m]$ is 1-dimensional, any (ζ, η)-full path must go through $(\zeta(0), \eta(0))$ and $(\zeta(1), \eta(t))$. Hence

$$P^{x,y} \cong P^{x^1,y^1} \times P^{x^2,y^2} \times P^{x^3,y^3}$$

by Lemma 6.6. Here $(x_i^1, y_i^1) \in D^{\zeta(0),\eta(0)}$, $(x_i^2, y_i^2) \in D^{\zeta(1)-\zeta(0),\eta(t)-\eta(0)}$ and $(x_i^3, y_i^3) \in D^{m-\zeta(1),n-\eta(t)}$, for all $0 \le i \le r+1$.

The part $\zeta^1 = x_0^1$ of $\zeta\colon [1] \to [n]$ that lands in $[\zeta(0)]$ is 0-dimensional, and similarly for the part $\zeta^3 = x_0^3$. Hence P^{x^1,y^1} and P^{x^3,y^3} have contractible classifying spaces, by the already established case $s = 0$ of the inner induction. It therefore remains to prove that P^{x^2,y^2} has contractible classifying space. Replacing (x^2, y^2) with (x, y) in the notation, we may and will assume that $\zeta(0) = 0$, $\zeta(1) = m$, $\eta(0) = 0$ and $\eta(t) = n$, and seek to prove that $P^{x,y}$ has contractible classifying space.

By symmetry, the case $t = 0$ can be handled just like the case $s = 0$. We therefore assume $t \ge 1$. For each $0 \le j \le t-1$ let

$$Q_j = P^{x,y}_{(0,\eta(j)),(m,\eta(j+1))}$$

be the subset of $P^{x,y}$ of paths that go through both $(0, \eta(j))$ and $(m, \eta(j+1))$, where $\zeta(0) = 0$ and $\zeta(1) = m$. Each (ζ, η)-full path must go through these two points for some $0 \le j \le t-1$, so $P^{x,y} = Q_0 \cup \dots \cup Q_{t-1}$. Each Q_j is a right ideal, so $|NP^{x,y}| = |NQ_0| \cup \dots \cup |NQ_{t-1}|$.

We now argue that each $|NQ_j|$ is contractible. Since every path in Q_j goes through both $(0, \eta(j))$ and $(m, \eta(j+1))$, we have an order-preserving bijection

$$Q_j \cong P^{x^1,y^1} \times P^{x^2,y^2} \times P^{x^3,y^3}$$

with $(x_i^1, y_i^1) \in D^{0,\eta(j)}$, $(x_i^2, y_i^2) \in D^{m,\eta(j+1)-\eta(j)}$ and $(x_i^3, y_i^3) \in D^{0,n-\eta(j+1)}$. Once more, $\zeta^1 = x_0^1$ and $\zeta^3 = x_0^3$ are 0-dimensional, so P^{x^1,y^1} and P^{x^3,y^3} have contractible classifying spaces by the case $s = 0$. On the other hand, $\zeta^2 = x_0^2 = \zeta$ and $\eta^2 = y_0^2\colon [1] \to [\eta(j+1) - \eta(j)]$ are both 1-dimensional.

We claim that the classifying space of P^{x^2,y^2} is also contractible. Since both ζ^2 and η^2 are 1-dimensional, a (z^2, w^2)-path is (ζ^2, η^2)-full if and only if it goes through both $(\zeta^2(0), \eta^2(0)) = (0, 0)$ and $(\zeta^2(1), \eta^2(1)) = (m, \eta(j+1) - \eta(j))$. By assumption, $\zeta^2 = x_0^2 \le x_1^2 = z_0^2$ and $\eta^2 = y_0^2 \le y_1^2 = w_0^2$, so any (z_0^2, w_0^2)-full path in $[m] \times [\eta(j+1) - \eta(j)]$ will begin at $(0, 0)$ and end at $(m, \eta(j+1) - \eta(j))$. Hence

$$P^{x^2,y^2} = P^{z^2,w^2}_{(0,0),(m,\eta(j+1)-\eta(j))} = P^{z^2,w^2},$$

which has contractible classifying space by the outer inductive hypothesis, since (z^2, w^2) is an r-simplex. This completes the proof that each $|NQ_j|$ is contractible.

Next we consider the $Q_i \cap Q_j$ for $0 \le i < j \le t-1$. If $i+1 < j$ there are no paths that go through both $(\zeta(1), \eta(i+1))$ and $(\zeta(0), \eta(j))$, so in these cases $Q_i \cap Q_j = \emptyset$. This implies that all ℓ-fold intersections $Q_{j_1} \cap \dots \cap Q_{j_\ell} = \emptyset$ are empty, for $0 \le j_1 < \dots < j_\ell \le t-1$ and $\ell \ge 3$.

It remains to consider the double intersection $Q_{j-1} \cap Q_j$, for $1 \le j \le t-1$. It consists of the (x, y)-paths that go through the four points $(0, \eta(j-1))$, $(0, \eta(j))$, $(m, \eta(j))$ and $(m, \eta(j+1))$, where $\zeta(0) = 0$ and $\zeta(1) = m$. There is a unique such path, with image contained in the totally ordered subset

$$\{0\} \times [\eta(j)] \cup [m] \times \{\eta(j)\} \cup \{m\} \times \{\eta(j) < \dots < n\}$$

of $[m] \times [n]$. Hence $|NQ_{j-1}| \cap |NQ_j|$ is a single point.

It follows that the union $|NQ_0| \cup \dots \cup |NQ_{t-1}|$ is homotopy equivalent to the set of points $\{j \mid 0 \le j \le t-1\}$, connected by the intervals $[j-1, j]$ for $1 \le j \le t-1$. Their union is the interval $[0, t-1]$, which is contractible. This completes the proof for $s = 1$ and $t \ge 1$.

The inner induction ends with the inner inductive step, for $s \ge 2$: Assume inductively that $|NP^{x',y'}|$ is contractible for each $(r+1)$-simplex (x', y') such that the dimension of $\zeta' = x'_0$ is strictly less than s. We must prove that $P^{x,y}$ has contractible classifying space when the dimension of $\zeta = x_0$, as a face of $\Delta[m]$, equals s.

For each $0 \le j \le t$, where t is the dimension of $\eta = y_0$, let

$$Q_j = P^{x,y}_{(\zeta(s-1), \eta(j))}$$

be the partially ordered set of (x, y)-paths that go through $(\zeta(s-1), \eta(j))$. Each (x, y)-path is (ζ, η)-full, hence must pass through one of these points, so $P^{x,y} = Q_0 \cup \dots \cup Q_t$. Each Q_j is a right ideal, so $|NP^{x,y}| = |NQ_0| \cup \dots \cup |NQ_t|$ is a finite union of CW complexes.

By Lemma 6.4 there is an order-preserving bijection

$$Q_j \cong P^{x^1,y^1} \times P^{x^2,y^2}$$

where (x^1, y^1) and (x^2, y^2) are $(r+1)$-simplices, with $\zeta^1 = x^1_0$ of dimension $s-1$ and $\zeta^2 = x^2_0$ of dimension 1. Hence $|NP^{x^1,y^1}|$ and $|NP^{x^2,y^2}|$ are contractible by the inner inductive hypothesis, and this implies that $|NQ_j|$ is contractible.

For $0 \le i < j \le t$, the double intersection

$$Q_i \cap Q_j = P^{x,y}_{(\zeta(s-1), \eta(i)), (\zeta(s-1), \eta(j))}$$

consists of the (x, y)-paths that pass through $(\zeta(s-1), \eta(i))$ and $(\zeta(s-1), \eta(j))$. By Lemma 6.6 there is an isomorphism

$$Q_i \cap Q_j \cong P^{x^1,y^1} \times P^{x^2,y^2} \times P^{x^3,y^3}$$

of partially ordered sets, where $\zeta^1 = x^1_0$ is of dimension $s-1$, $\zeta^2 = x^2_0$ is of dimension 0, and $\zeta^3 = x^3_0$ is of dimension 1. Each of the three factors has contractible classifying space, by the inner inductive hypothesis, so $|NQ_i| \cap |NQ_j|$ is also contractible.

Finally, for any $0 \le j_1 < \dots < j_\ell \le t$ with $\ell \ge 3$, the ℓ-fold intersection

$$Q_{j_1} \cap \dots \cap Q_{j_\ell} = Q_{j_1} \cap Q_{j_\ell}$$

equals one of the double intersections considered above. This is because any (ζ, η)-full path that passes through $(\zeta(s-1), \eta(j_1))$ and $(\zeta(s-1), \eta(j_\ell))$ must pass through $(\zeta(s-1), \eta(i))$ for each $j_1 \le i \le j_\ell$.

Hence we have proved that each finite intersection $|NQ_{j_1}| \cap \dots \cap |NQ_{j_\ell}|$ is contractible, which readily implies that the union $|NP^{x,y}| = |NQ_0| \cup \dots \cup |NQ_t|$ is contractible. □

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