Spaces of Manifolds

John Rognes

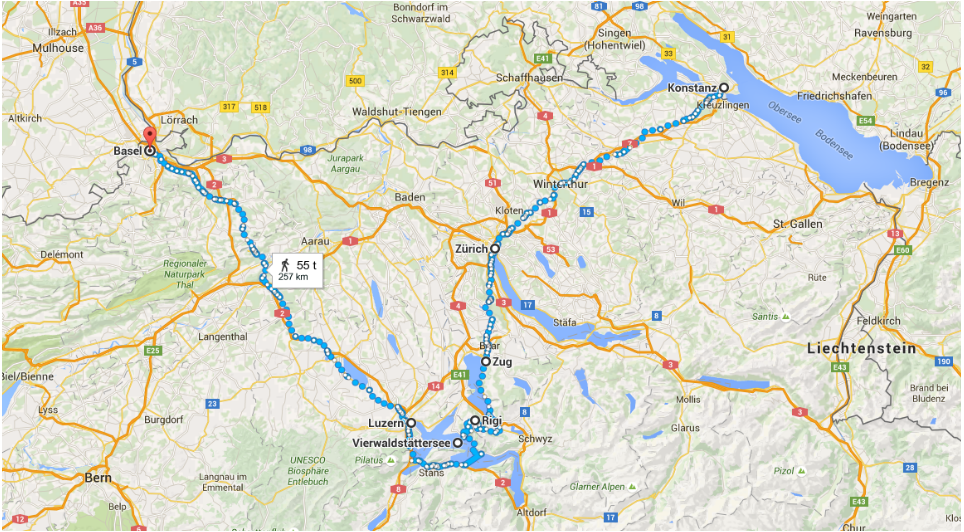
University of Oslo, Norway

Abel in Zürich, 2016

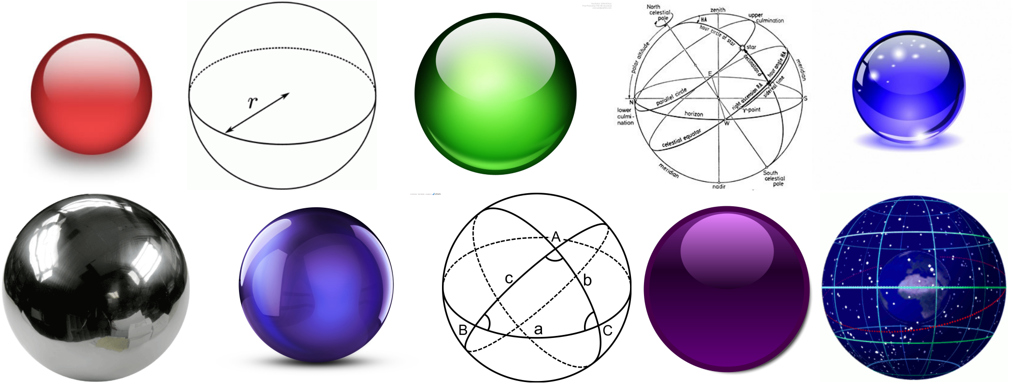
Outline

1. Flavors of Manifolds
2. Automorphism Groups
3. L- and K-Theory
4. Analysis and Algebra

I truthfully do not reget the little detour (Abel, July 1826)

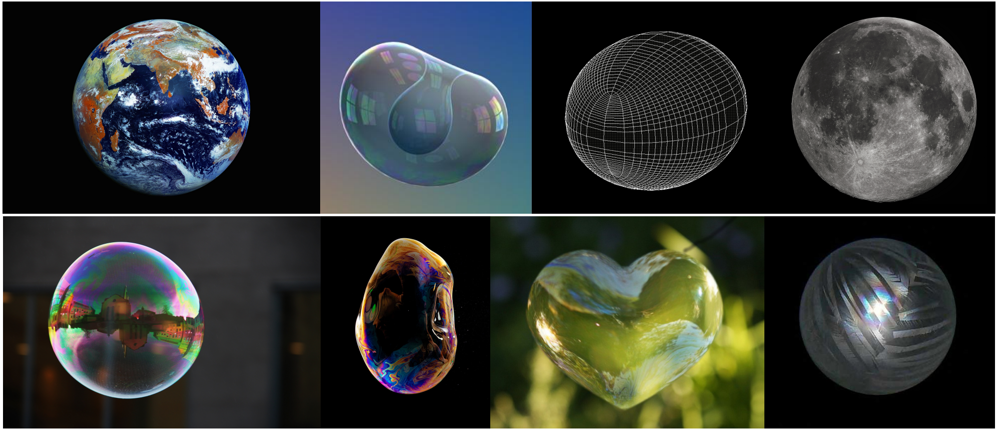


Geometric Spheres



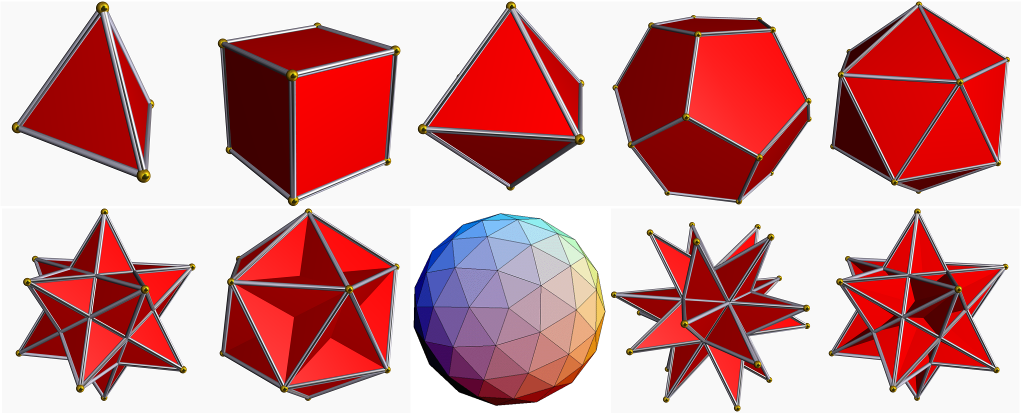
Determined by center and radius in a 3-space $\overset{\simeq}{\longleftrightarrow}$ a point in $BO(3)$.

Differentiable Spheres



Each differentiable sphere $\overset{\simeq}{\longleftrightarrow}$ a point in $BDIFF(S^2)$.

Piecewise-Linear Spheres



Each piecewise-linear sphere $\overset{\simeq}{\longleftrightarrow}$ a point in $BPL(S^2)$.

Topological Spheres

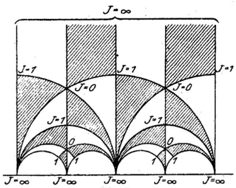
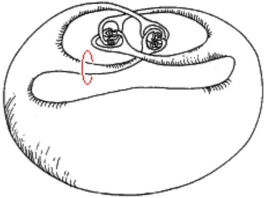
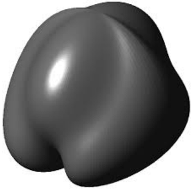
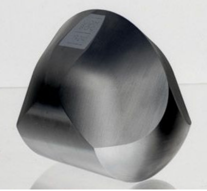


Fig. 7.



Each topological sphere $\overset{\simeq}{\longleftrightarrow}$ a point in $BTOP(S^2)$.

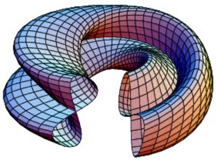
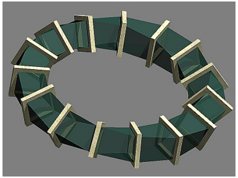
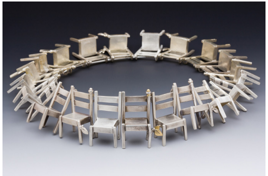
Relaxation

$$BO(3) \longrightarrow BDIFF(S^2) \longrightarrow BPD(S^2) \xleftarrow{\simeq} BPL(S^2) \longrightarrow BTOP(S^2)$$

Classifying Spaces

Each space $BCAT(M)$ of manifolds classifies bundles with such manifolds as fibers.

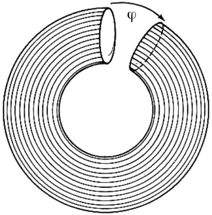

$$\begin{array}{ccc} E & \longrightarrow & ECAT(M) \times_{CAT(M)} M \\ \downarrow{\scriptstyle M} & & \downarrow \\ B & \xrightarrow{g} & BCAT(M) \end{array}$$

Automorphisms

Automorphism group of CAT symmetries:

$$CAT(M) := \{\varphi\colon M \xrightarrow{\cong} M\}$$

Mapping torus construction



induces homotopy equivalence:

$$CAT(M) \xrightarrow{\simeq} \Omega(BCAT(M)) := Map(S^1, BCAT(M))$$

Homotopy Groups

Main Goal: To understand the homotopy groups

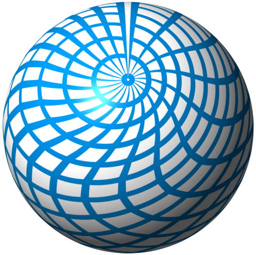
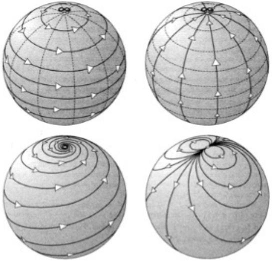
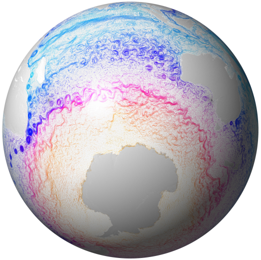
$$\pi_i CAT(M) \xrightarrow{\cong} \pi_{i+1} BCAT(M)$$

of these automorphism groups and classifying spaces.





Spheres vs. Discs

Diffeomorphisms fixing one hemisphere, and the isometries, generate all diffeomorphisms of a sphere:

$$DIFF(D^n \operatorname{rel} \partial) \times O(n+1) \xrightarrow{\simeq} DIFF(S^n)$$

Bott Periodicity

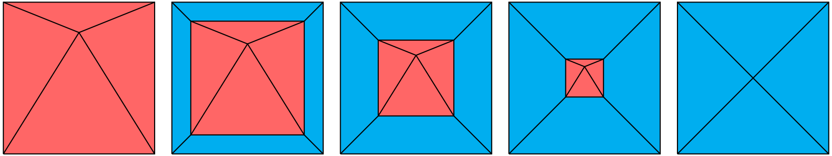
Bott (1957): For $i < n$ the homotopy groups

$$\pi_i O(n+1) \xrightarrow{\cong} \pi_i O \cong \pi_{i+1} BO$$

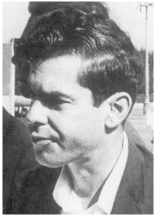
are well known.

The Alexander Trick

Alexander (1923): $TOP(D^n \operatorname{rel} \partial)$ and $PL(D^n \operatorname{rel} \partial)$ are contractible.



Surfaces



Smale (1958): $DIFF(D^2 \operatorname{rel} \partial)$ is contractible, so

$$O(3) \simeq DIFF(S^2) \simeq PL(S^2) \simeq TOP(S^2)$$

and

$$DIFF(M) \simeq PL(M) \simeq TOP(M)$$

for surfaces M.

Three-Manifolds

Hatcher (1981): $DIFF(D^3 \operatorname{rel} \partial)$ is contractible, so

$$O(4) \simeq DIFF(S^3) \simeq PL(S^3) \simeq TOP(S^3)$$

and

$$DIFF(M) \simeq PL(M) \simeq TOP(M)$$

for 3-manifolds M.

DIFF is a Little Different

Milnor (1956): Each exotic 7-sphere detects a nontrivial element in

$$\pi_0 DIFF(D^6 \operatorname{rel} \partial) \neq 0$$

(with Kervaire: This group is $\mathbb{Z}/28$).

Also in Higher Degrees

Novikov (1963): Exotic 9-sphere representing $\eta\epsilon$ in $cok(J)$ detects

$$\mathbb{Z}/2 \subset \pi_1 DIFF(D^7 \operatorname{rel} \partial)$$

via

$$\pi_i DIFF(D^n \operatorname{rel} \partial) \longrightarrow \pi_0 DIFF(D^{n+i} \operatorname{rel} \partial)$$

DIFF is Very Different

Farrell and Hsiang (1978): For $i \lesssim n/3$

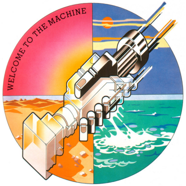
$$\pi_i DIFF(D^n \operatorname{rel} \partial) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{for } i \equiv 3 \bmod 4 \text{ and } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

so $\pi_i DIFF(D^n \operatorname{rel} \partial)$ is often not finite.
Question: How do they prove this?

Machines

$$\underbrace{CAT(M)}_{\text{CAT automorphisms}} \longrightarrow \underbrace{\widetilde{CAT}(M)}_{\text{block automorphisms}} \longrightarrow \underbrace{G(M)}_{\text{homotopy automorphisms}}$$

- Homotopy theory $\rightsquigarrow G(M)$;
- Surgery theory $\rightsquigarrow G(M)/\widetilde{CAT}(M) \rightsquigarrow \widetilde{CAT}(M)$;
- Pseudoisotopy theory $\rightsquigarrow \widetilde{CAT}(M)/CAT(M) \rightsquigarrow CAT(M)$;
- Algebraic K-theory $\rightsquigarrow$ stable pseudoisotopy theory.



Homotopy Automorphisms

The grouplike monoid

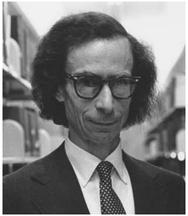
$$G(M) := \{\varphi\colon M \xrightarrow{\simeq} M\}$$

of homotopy automorphisms φ is a union of path components of $Map(M, M)$.

Federer (1956): Spectral sequence

$$E^2_{s,t} = H^{-s}(M; \pi_t(M)) \Longrightarrow \pi_{s+t}(Map(M, M))$$

for M simple.



Surgery Theory

The structure space

$$G(M)/\widetilde{CAT}(M)$$

of *CAT* models for *M*, up to block automorphism, is determined by surgery exact sequence.

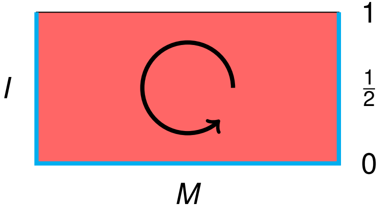



Pseudoisotopy Theory

Cerf (1966): Pseudoisotopy space

$$P(M) = CAT(M \times I \operatorname{rel} M \times 0 \cup \partial M \times I)$$

compares block automorphisms and automorphisms.



Reflection about $\frac{1}{2}$ induces an involution $P(M) \to P(M)$ (= $\mathbb{Z}/2$-action).

Unblocking Automorphisms

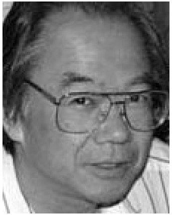
Hatcher (1978): Spectral sequence

$$E^1_{s,t} = \pi_t P(M \times I^s) \Longrightarrow \pi_{s+t+1}(\widetilde{CAT}(M)/CAT(M))$$
$$E^2_{s,t} = H_s(\mathbb{Z}/2; \pi_t P(M)) \qquad \text{in stable range}$$

Igusa (1988): Stability isomorphism

$$\pi_i P(M) \xrightarrow{\cong} \operatorname*{colim}_s \pi_i P(M \times I^s)$$

for $CAT = DIFF$ and $i \lesssim n/3$.



Algebraic K-Theory

Waldhausen–Jahren–R. (1979–2013, stated for $CAT = DIFF$):

Theorem (Stable Parametrized s-Cobordism Theorem)

There is a natural equivalence

$$\operatorname{colim}_s P(M \times I^s) \simeq \Omega^2 Wh(M)$$

where $A(M) := K(\mathbb{S}[\Omega M]) \simeq Q(M_+) \times Wh(M)$.

Corollary

If M is aspherical with $\pi = \pi_1(M)$, then

$$\pi_i P(M) \otimes \mathbb{Q} \cong \pi_{i+2} Wh(M) \otimes \mathbb{Q} \subset K_{i+2}(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$$

for $i \lesssim n/3$.

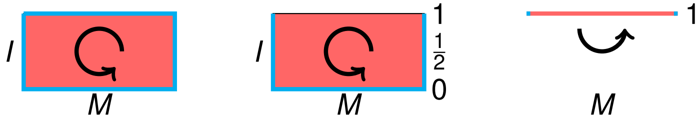
Diffeomorphisms of Discs, Revisited

Example

$$\pi_i P(D^n) \otimes \mathbb{Q} \cong K_{i+2}(\mathbb{Z}) \otimes \mathbb{Q} \overset{\text{Borel}}{\cong} \begin{cases} \mathbb{Q} & \text{for } i \equiv 3 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

for $i \lesssim n/3$.

Fiber sequence $DIFF(D^{n+1} \operatorname{rel} \partial) \longrightarrow P(D^n) \longrightarrow DIFF(D^n \operatorname{rel} \partial)$:

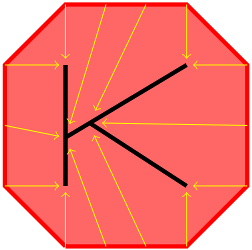


- -1-eigenspace for involution on $\pi_i P(D^n)$ comes from $\pi_i DIFF(D^{n+1} \operatorname{rel} \partial)$;
- $+1$-eigenspace maps to $\pi_i DIFF(D^n \operatorname{rel} \partial)$.

Manifold Part of the Proof

A CAT *n-thickening* of a space K is a map $u\colon M \to K$ where

- M is a stably framed compact n-manifold,
- each preimage $u^{-1}(k)$ is contractible, for $k \in K$, and
- each restricted preimage $u^{-1}(k) \cap \partial M$ is simply connected.



We prove that the space

$$T^n(K) := \{u\colon M \to K\}$$

of PL n-thickenings is at least $(n - 2\dim(K) - 6)$-connected.
Hence $\operatorname{colim}_n T^n(K)$ is contractible.

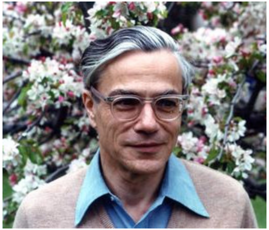
Strategy for an Alternative Proof

- Waldhausen's proof uses smoothing theory to use PL manifolds to prove a theorem about DIFF manifolds.
- Might instead try to prove directly that for DIFF n-thickenings

$$K \mapsto \operatorname*{colim}_n \pi_* T^n(K)$$

defines a homology theory.
- Its coefficient spectrum will then be $\Omega Wh(*)$.

Harmonic Analysis on Symmetric Spaces



- Rational $\pi_i DIFF(D^n \operatorname{rel} \partial)$ comes from $K_{i+2}(\mathbb{Z})$.
- Borel (1975): Calculated

$$H^*(SL_N(\mathbb{Z}); \mathbb{R})$$

  using harmonic analysis for square integrable differential forms on finite volume quotients X/Γ of the symmetric space $X = SL_N(\mathbb{R})/O(N)$.
- His calculation

$$K_*(\mathbb{Z}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & * = 4j+1 \text{ for } j \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

  for $* > 0$ follows.

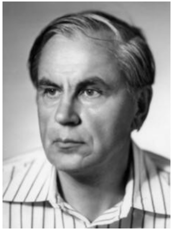
The Absolute Galois Group

- Alternative approach to rational $K_*(\mathbb{Z})$ or $K_*(\mathbb{Q})$ using Galois descent:
- Shafarevich conjecture (1964): The absolute Galois group of $\mathbb{Q}^{ab}$ is a free profinite group on countably many generators.
- If true, descent from $K(\bar{\mathbb{Q}})$ along

$$K(\mathbb{Q}) \longrightarrow K(\mathbb{Q}^{ab}) \longrightarrow K(\bar{\mathbb{Q}})$$

will recover Borel's result.

- The generator of weight $2j+1$, for $j \geq 1$, contributes via Galois/motivic H^1 to $K_*(\mathbb{Z})$ in degree $* = 2(2j+1) - 1 = 4j+1$.



The Abel Prize Laureates

In Memory of John and Alicia Nash