FIXED POINTS OF TOPOLOGICAL HOCHSCHILD HOMOLOGY AND K-THEORY OF THE TWO-ADIC INTEGERS

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I will discuss a program to determine the algebraic K-theory of the ring of two-adic integers which originated with Marcel Bökstedt's ideas regarding the *trace map*

$$tr\colon K(\mathbb{Z}) \to T(\mathbb{Z}) = THH(\mathbb{Z})$$

In degree zero, this map takes an idempotent matrix (whose image is a projective module) to its trace (which is the rank of the module). Hochschild homology is the global object designed to receive higher–dimensional analogs of such traces, somewhat like K-theory globalizes the category of projective modules. Topological Hochschild homology is a refinement of this idea based on working with rings up to homotopy (algebras over the sphere spectrum) instead of ordinary rings (algebras over the integers). It turns out that $T(\mathbb{Z})$ is a product of Eilenberg–Mac Lane spaces, and so is determined by its homotopy groups which are

$$\pi_* T(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } * = 0, \\ \mathbb{Z}/i & \text{for } * = 2i-1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular $\pi_3 T(\mathbb{Z}) = \mathbb{Z}/2$ and the trace map $K_3(\mathbb{Z}) \to \pi_3 T(\mathbb{Z})$ is the surjection $\mathbb{Z}/48 \to \mathbb{Z}/2$ detecting the interesting part of $K_3(\mathbb{Z})$ not coming from stable homotopy.

A further refinement of this approach notes that the circle S^1 acts on $T(\mathbb{Z})$, and the trace map factors over the corresponding homotopy fixed points giving a *circle trace map*

$$tr_{S^1}\colon K(\mathbb{Z}) \to T(\mathbb{Z})^{hS^1}.$$

This will be a map of ring spectra, and if the Lichtenbaum–Quillen conjecture is true there should be a ring map from the model $JK(\mathbb{Z})$ for $K(\mathbb{Z})$ based on étale K-theory to $T(\mathbb{Z})^{hS^1}$. In particular there should be an algebra map linking the mod two spectrum homology algebras $H_*^{spec}(JK(\mathbb{Z}); \mathbb{Z}/2)$ and $H_*^{spec}(T(\mathbb{Z})^{hS^1}; \mathbb{Z}/2)$ compatible with the (co-)action of the dual Steenrod algebra A_*.

Here the source is relatively well known, and in the eighties Bökstedt began to make calculations of the target. He did not get complete answers, but thought for a while that although there were unresolved extension questions, none of the possible answers were compatible with the existence of such a ring map. This would contradict the Lichtenbaum–Quillen conjecture at two (in the form of Dwyer and Friedlander), but so far the details of such an argument remain unfinished.

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My subjective impression is that there is no contradiction, and that at two $K(\mathbb{Z})$ and $JK(\mathbb{Z})$ are the same, but I cannot prove this.

For the record, the Lichtenbaum–Quillen conjecture at two for the integers amounts to the assertion that the square of rings

$$\begin{array}{ccc} \mathbb{Z}[\frac{1}{2}] & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow c \\ \mathbb{Z}_3 & \longrightarrow & \mathbb{C} \end{array}$$

(for a suitable imbedding of $\mathbb{Z}_3$ into $\mathbb{C}$) induces a homotopy cartesian square in K-theory after completion at two, so that $K(\mathbb{Z}[\frac{1}{2}])$ would be the homotopy pullback of the maps

$$K(\mathbb{Z}_3) \to K(\mathbb{C}) \xleftarrow{c} K(\mathbb{R}).$$

Here we may identify $K(\mathbb{Z}_3) \simeq K(\mathbb{Z}/3) \simeq \operatorname{Im} J_{\mathbb{C}}$, $K(\mathbb{R}) \simeq \mathbb{Z} \times BO$ and $K(\mathbb{C}) \simeq \mathbb{Z} \times BU$ after completion. So the candidate for $K(\mathbb{Z}[\frac{1}{2}])$ is also the homotopy pullback of certain maps

$$\operatorname{Im} J_{\mathbb{C}} \to \mathbb{Z} \times BU \xleftarrow{c} \mathbb{Z} \times BO.$$

Then there should be a localization fiber sequence

$$K(\mathbb{Z}/2) \to K(\mathbb{Z}) \to K(\mathbb{Z}[\frac{1}{2}])$$

with $K(\mathbb{Z}/2) \simeq H\mathbb{Z}$ at two, so in the end the candidate $JK(\mathbb{Z})$ for $K(\mathbb{Z})$ at two fits into several different fiber sequences

$$\begin{aligned} \operatorname{Im} J_{\mathbb{R}} &\to JK(\mathbb{Z}) \to BBSO \\ SU &\to JK(\mathbb{Z}) \to \mathbb{Z} \times BO \\ BBO &\to JK(\mathbb{Z}) \to \operatorname{Im} J_{\mathbb{C}}. \end{aligned}$$

Here $\operatorname{Im} J_{\mathbb{R}}$ is the real image of J, defined by a fibration

$$\operatorname{Im} J_{\mathbb{R}} \to \mathbb{Z} \times BO \xrightarrow{\psi^3 - 1} BSpin$$

while $\operatorname{Im} J_{\mathbb{C}}$ is the complex image of J, defined by the fibration

$$\operatorname{Im} J_{\mathbb{C}} \to \mathbb{Z} \times BU \xrightarrow{\psi^3 - 1} BU.$$

The homotopy groups of these spaces are well known, as are the homotopy groups of the model $JK(\mathbb{Z})$ for $K(\mathbb{Z})$ at two.

Returning to Bökstedt's idea, there is now a refined version of the circle trace map due to Bökstedt, Hsiang and Madsen, called the *cyclotomic trace map*. Its construction runs as follows.

The circle action on $T(\mathbb{Z})$ determines spaces of fixed points $T(\mathbb{Z})^{C_{p^n}}$ for each subgroup $C_{p^n} \subset S^1$. Here p can be any prime; we will specialize to the case $p = 2$ a little later. There are natural inclusions

$$F\colon T(\mathbb{Z})^{C_{p^n}} \to T(\mathbb{Z})^{C_{p^{n-1}}}$$

which we will call *Frobenius maps*. These were called D in the old notation. In addition topological Hochschild homology admit other maps

$$R\colon T(\mathbb{Z})^{C_{p^n}} \to T(\mathbb{Z})^{C_{p^{n-1}}}$$

called *restriction maps*. These were called Φ in the old notation, and arise by restricting a C_{p^n}-equivariant map to its C_p-fixed points, giving a $C_{p^{n-1}}$-equivariant map. Both maps are interesting, and together give $T(\mathbb{Z})$ the structure of a *cyclotomic spectrum*.

The trace map tr admits lifts

$$tr_{p^n}\colon K(\mathbb{Z}) \to T(\mathbb{Z})^{C_{p^n}}$$

compatible with both F and R. In fact

$$F \circ tr_{p^n} \simeq R \circ tr_{p^n} \simeq tr_{p^{n-1}}.$$

The cyclotomic trace map combines these lifts into a map to the *topological cyclic homology* $TC(\mathbb{Z})$ at p.

$$\begin{array}{ccccccccc}
 & & K(\mathbb{Z}) & & & & & & \\
 & \swarrow^{trc} & \downarrow^{tr_{p^n}} & \searrow^{tr_{p^{n-1}}} & & & & & \\
TC(\mathbb{Z}) & \longrightarrow & T(\mathbb{Z})^{C_{p^n}} & \overset{F}{\underset{R}{\rightrightarrows}} & T(\mathbb{Z})^{C_{p^{n-1}}} & \rightrightarrows & \cdots & \rightrightarrows & T(\mathbb{Z}).
\end{array}$$

So by definition

$$TC(\mathbb{Z}) = \operatorname{holim}_{F,R} T(\mathbb{Z})^{C_{p^n}}$$

and $trc\colon K(\mathbb{Z}) \to TC(\mathbb{Z})$ is the cyclotomic trace map. Here implicitly all spaces are completed at our chosen prime p. We could also consider the homotopy limit over only the Frobenius maps

$$TF(\mathbb{Z}) = \operatorname{holim}_F T(\mathbb{Z})^{C_{p^n}}$$

and since F and R commute there is a fiber sequence

$$\dots \xrightarrow{\partial} TC(\mathbb{Z}) \xrightarrow{\pi} TF(\mathbb{Z}) \xrightarrow{R-1} TF(\mathbb{Z}) \to \dots$$

The ring map $\mathbb{Z} \to \mathbb{Z}_p$ to the ring of p-adic integers induces a nontrivial map $K(\mathbb{Z}) \to K(\mathbb{Z}_p)$, but after p-adic completion the natural map $TC(\mathbb{Z}) \to TC(\mathbb{Z}_p)$ is a homotopy equivalence. Hence the cyclotomic trace map $K(\mathbb{Z}) \to TC(\mathbb{Z}) \simeq TC(\mathbb{Z}_p)$ naturally factors through $K(\mathbb{Z}_p)$, and it is in fact as an invariant of the K-theory of the p-adic integers, rather than of the rational integers, that $TC(\mathbb{Z})$ is an interesting invariant.

Theorem (McCarthy, Hesselholt–Madsen). *The cyclotomic trace map*

$$trc\colon K(\mathbb{Z}_p) \to TC(\mathbb{Z})$$

induces a homotopy equivalence on p-adically completed connective covers.

We say that *trc* is a *connective p-adic equivalence*. When p is an odd prime, we have the following calculation:

Theorem (Bökstedt–Madsen). *There is a homotopy equivalence*

$$TC(\mathbb{Z}) \simeq \operatorname{Im} J \times B \operatorname{Im} J \times BBU$$

on p-adically completed connective covers.

Here $\operatorname{Im} J$ is defined by the fiber sequence

$$\operatorname{Im} J \to \mathbb{Z} \times BU \xrightarrow{\psi^k - 1} BU$$

where k is a topological generator of the p-adic units. Such a complete description of the K-theory of a ring was previously essentially only known in the cases of finite fields (Quillen) and algebraically closed fields (Suslin).

What about the case $p = 2$? For multiplicative reasons the formula above cannot hold, neither with $\operatorname{Im} J$ replaced by $\operatorname{Im} J_{\mathbb{R}}$ or by $\operatorname{Im} J_{\mathbb{C}}$. Instead we have the following chain of results. Hereafter suppose all spaces are implicitly completed at two.

The main technical work goes into the following calculation.

Claim 1. *The nontrivial mod two homotopy groups of* $TC(\mathbb{Z})$ *have orders*

$$\#\pi_*(TC(\mathbb{Z}); \mathbb{Z}/2) = \begin{cases} 2 & \text{for } * = -1 \text{ or } 0, \\ 4 & \text{for } * = 1 \text{ or } * \geq 2 \text{ even}, \\ 8 & \text{for } * \geq 3 \text{ odd}. \end{cases}$$

Hence these are also the orders of $\pi_*(K(\mathbb{Z}_2); \mathbb{Z}/2)$ *for* $* \geq 0$.

The degree eight Adams map $A = v_1^4$ *acts injectively on the mod two homotopy groups above. Inverting* A *only changes the homotopy groups below degree* 1*, so* $K(\mathbb{Z}_2)$ *is essentially* K*-local, in the sense that the localization map* $K(\mathbb{Z}_2) \to L_K K(\mathbb{Z}_2)$ *is an equivalence above degree* 1*.*

Let us review the construction of the *Galois reduction map*,

$$red\colon K(\mathbb{Z}_2) \to \operatorname{Im} J_{\mathbb{C}}$$

by adapting ideas of Dwyer and Friedlander to the two-primary case.

We have inclusions

$$\mathbb{Z}_2 \subset \mathbb{Q}_2 \subset \mathbb{Q}_2(\mu_{2^\infty}) \subset \bar{Q}_2$$

and choose an imbedding $\bar{Q}_2 \to \mathbb{C}$. Here $\mathbb{Q}_2(\mu_{2^\infty})$ is the field obtained by adjoining all 2^nth roots of unity to $\mathbb{Q}_2$, for all n, while $\bar{Q}_2$ is the algebraic closure. There is a

Galois automorphism θ^3 of $\mathbb{Q}_2(\mu_{2^\infty})$ fixing $\mathbb{Q}_2$ and given by $\theta^3(\zeta) = \zeta^3$ for each 2^nth root of unity ζ. Next we may choose an extension of θ^3 to a Galois automorphism ϕ^3 of $\bar{Q}_2$. This involves a choice, but we will only require that $\phi^3(\sqrt{3}) = +\sqrt{3}$, rather than $-\sqrt{3}$. We have this choice because $\sqrt{3} \notin \mathbb{Q}_2(\mu_{2^\infty})$.

Then ϕ^3 induces a self–map of $K(\bar{Q}_2)$ compatible under Suslin's equivalence $K(\bar{Q}_2) \simeq K(\mathbb{C}) \simeq \mathbb{Z} \times BU$ with the Adams operation ψ^3. Hence we can arrange to have maps of ring spectra

$$K(\mathbb{Z}_2) \to K(\mathbb{Q}_2) \to K(\bar{Q}_2)^{h\phi^3} \xrightarrow{\simeq} (\mathbb{Z} \times BU)^{h\psi^3}$$

where the superscript $(\ )^{hf}$ means the homotopy fiber of $f-1$. Passing to connective covers we get the reduction map

$$red\colon K(\mathbb{Z}_2) \to \operatorname{Im} J_{\mathbb{C}}.$$

We define the *reduced K-theory* $K^{red}(\mathbb{Z}_2)$ as the homotopy fiber of *red*.

Claim 2. *The reduction map induces surjections on mod two homotopy in all degrees, and split surjections on integral homotopy in all degrees. The nontrivial mod two homotopy groups of $K^{red}(\mathbb{Z}_2)$ have orders*

$$\#\pi_*(K^{red}(\mathbb{Z}_2); \mathbb{Z}/2) = \begin{cases} 2 & \text{for } * = 1 \text{ or } * \geq 2 \text{ even,} \\ 4 & \text{for } * \geq 3 \text{ odd.} \end{cases}$$

Note that the two-adic unit 3 in $\bar{Q}_2$ has a square root which is invariant under ϕ^3. Hence its symbol $\{3\}$ in K_1 is divisible by two, and maps to zero in $\pi_1 \operatorname{Im} J_{\mathbb{C}} \cong \mathbb{Z}/2$. Thus $\pi_1 K^{red}(\mathbb{Z}_2) \cong \mathbb{Z}_2$ is free on one generator, namely $\{3\}$. This class is represented by an infinite loop map $\{3\}\colon Q(S^1) \to K^{red}(\mathbb{Z}_2)$, and since $K(\mathbb{Z}_2)$ and thus $K^{red}(\mathbb{Z}_2)$ is essentially K-local, there is a factorization in the diagram

$$\begin{array}{ccccc} & & Q(S^1) & \xrightarrow{\{3\}} & K^{red}(\mathbb{Z}_2) \\ & & \downarrow & & \downarrow \\ B\operatorname{Im} J_{\mathbb{R}} & \longrightarrow & L_K Q(S^1) & \longrightarrow & L_K K^{red}(\mathbb{Z}_2) \end{array}$$

giving a map $B\operatorname{Im} J_{\mathbb{R}} \to K^{red}(\mathbb{Z}_2)$ inducing an isomorphism on π_1. There is a complexification map $Bc\colon B\operatorname{Im} J_{\mathbb{R}} \to B\operatorname{Im} J_{\mathbb{C}}$. Briefly let W denote its homotopy fiber, as in the diagram below:

$$\begin{array}{ccccc} W & \longrightarrow & B\operatorname{Im} J_{\mathbb{R}} & \xrightarrow{Bc} & B\operatorname{Im} J_{\mathbb{C}} \\ & & \downarrow & & \\ & & K^{red}(\mathbb{Z}_2) & & \end{array}$$

Claim 3. *The composite $W \to B\operatorname{Im} J_{\mathbb{R}} \to K^{red}(\mathbb{Z}_2)$ is null homotopic. Hence there is an extension $B\operatorname{Im} J_{\mathbb{C}} \to K^{red}(\mathbb{Z}_2)$ of the map $\{3\}$, which induces an isomorphism on π_1.*

The proof uses the description of $W = \mathrm{hofib}(Bc)$ as an invertible spectrum in the K-local category at two, representing an element of order 4 in the Picard group Pic_1 (chromatic type 1) of Hopkins, Mahowald and Sadofsky. These authors give an inductive procedure for how to construct W from $*$ by successively attaching mapping cones over $L_K(S^i/2)$ where $S^i/2$ is a mod two Moore spectrum. So to check that $W \to K^{red}(\mathbb{Z}_2)$ is null homotopic it suffices to check that certain maps $S^i/2 \to L_K S^i/2 \to K^{red}(\mathbb{Z}_2)$ can be arranged to be null homotopic, which amounts to questions about $\pi_{i+1}(K^{red}(\mathbb{Z}_2); \mathbb{Z}/2)$. This is precisely the kind of information the TC calculations supplied, and the result can be proved in this way.

Claim 4. *The map $B\operatorname{Im} J_{\mathbb{C}} \to K^{red}(\mathbb{Z}_2)$ induces an injection on mod two homotopy groups in all degrees, and split injections on all integral homotopy groups. Hence its (infinite loop space) cofiber X has the mod two homotopy groups of BBU, and in fact $X \simeq BBU$.*

Hence we have the following theorem.

Theorem (Rognes). *The two-completed homotopy type of $K(\mathbb{Z}_2)$ as an infinite loop space is determined by the following diagram of infinite loop space fiber sequences*

$$\begin{array}{ccccc} B\operatorname{Im} J_{\mathbb{C}} & & & & \\ \downarrow & & & & \\ K^{red}(\mathbb{Z}_2) & \longrightarrow & K(\mathbb{Z}_2) & \xrightarrow{red} & \operatorname{Im} J_{\mathbb{C}} \\ \downarrow & & & & \\ BBU. & & & & \end{array}$$

The vertical connecting map $f\colon BU \to B\operatorname{Im} J_{\mathbb{C}}$ represents a generator of the group of such infinite loop maps which induce trivial homomorphisms on homotopy. This group is free of rank one as a module over the two-adic integers.

The horizontal connecting map $g\colon \operatorname{Im} J_{\mathbb{C}} \to BK^{red}(\mathbb{Z}_2)$ also represents a generator of the group of such infinite loop maps, which again is free of rank one as a module over the two-adic integers.

Both fibrations may thus be viewed as maximally twisted extensions. But looking at K-groups we have

$$K_*(\mathbb{Z}_2) \cong \pi_* \operatorname{Im} J_{\mathbb{C}} \times \pi_* B\operatorname{Im} J_{\mathbb{C}} \times \pi_* BBU.$$

How do we access $\pi_*(TC(Z); Z/2)$? Through the fiber sequence

$$TC(Z) \xrightarrow{\pi} TF(Z) \xrightarrow{R-1} TF(Z).$$

Here

$$TF(Z) = \operatorname{holim} T(Z)^{C_{2^n}} \to \operatorname{holim} T(Z)^{hC_{2^n}} \leftarrow T(Z)^{hS^1}.$$

The last map is a two-complete homotopy equivalence.

Theorem (Tsalidis). *If the natural map $\Gamma_1 \colon T(Z)^{C_p} \to T(Z)^{hC_p}$ is a connective p-adic equivalence, then so is $\Gamma_n \colon T(Z)^{C_{p^n}} \to T(Z)^{hC_{p^n}}$ for all $n \geq 1$.*

The proof is a reduction to the rank one elementary abelian p-group case, along the lines of Carlsson's reduction of the Segal conjecture.

Hence once we check that Γ_1 is a connective two-adic equivalence, then the composite map

$$K(Z_2) \xrightarrow{trc} TC(Z) \xrightarrow{\pi} TF(Z)$$

is equivalent to the circle trace map tr_{S^1} studied by Bökstedt. The cyclotomic trace map merely takes into account the additional coherence required by the R-maps. So we want to understand $\pi_*(TF(Z); Z/2)$, which equals $\pi_*(T(Z)^{hS^1}; Z/2)$ in nonnegative degrees, and the limiting case of the reduction map $R \colon TF(Z) \to TF(Z)$ compared to the identity.

We will do this in inductive steps, proceding from the mod two homotopy of $T(Z)$ to $T(Z)^{C_2}$, through to that of $T(Z)^{C_{2^n}}$ for all n, simultaneously keeping an eye on $R \colon T(Z)^{C_{2^n}} \to T(Z)^{C_{2^{n-1}}}$. The inductive approach is made possible by the following *norm–restriction fiber sequence*

$$T(Z)_{C_{2^n}} \xrightarrow{N} T(Z)^{C_{2^n}} \xrightarrow{R} T(Z)^{C_{2^{n-1}}}.$$

Here the norm map N followed by a forgetful map gives the C_{2^n}-equivariant transfer. Mapping $T(Z)$ to the function spectrum $F(ES^1_+, T(Z))$ takes the fiber sequence above to the following *homotopy norm–restriction sequence*

$$T(Z)_{C_{2^n}} \xrightarrow{N^h} T(Z)^{hC_{2^n}} \xrightarrow{R^h} \hat{\mathbb{H}}(C_{2^n}, T(Z)).$$

Here $\hat{\mathbb{H}}(C_{2^n}, T(Z))$ denotes the *Tate construction* for the action of C_{2^n} on $T(Z)$, which merges the homotopy orbit and homotopy fixed point constructions to its left, just like Tate cohomology merges group homology and cohomology.

There is a map of fiber sequences, which is the identity on the left, the natural map Γ_n in the middle, and a similar map

$$\hat{\Gamma}_n \colon T(Z)^{C_{2^{n-1}}} \to \hat{\mathbb{H}}(C_{2^n}, T(Z))$$

on the right. Clearly Γ_n is a connective two-adic equivalence if and only if $\hat{\Gamma}_n$ is one. In fact the following theorem

Theorem (Rognes). Γ_1 *is a connective two-adic equivalence.*

is proved by showing that $\hat{\Gamma}_1 \colon T(Z) \to \hat{\mathbb{H}}(C_2, T(Z))$ induces an isomorphism on mod two homotopy in all nonnegative degrees. This is feasible because

$$\pi_*(T(Z); Z/2) \cong \mathbb{Z}/2[e_3, e_4]/(e_3^2 = 0)$$

is known, and there is a spectral sequence $\hat{E}^*(C_2)$:

$$E^2_{s,t} = \hat{H}^{-s}(C_2; \pi_t(T(Z); Z/2)) \Longrightarrow \pi_{s+t}(\hat{\mathbb{H}}(C_2, T(Z)); Z/2)$$

Here $\hat{H}^{-s}$ denotes Tate cohomology. The theorem is proved by a comparison between the spectral sequence above, which is *not* an algebra spectral sequence, and correponding spectral sequences for mod four homotopy and for the action of S^1 in place of C_2.

The whole technical problem at two is that the mod two Moore spectrum $M = S^0/2$ representing mod two homotopy is not a ring spectrum. The obstruction to finding a retraction to the (left) unital map $M \to M \wedge M$ factors through the Hopf map η, so because multiplication by η is null homotopic on $T(Z)$, there is an algebra structure on $\pi_*(T(Z); Z/2)$. But this null homotopy cannot be made even C_2-equivariant, and so there is no natural algebra structure on $\pi_*(T(Z)^{C_{2^n}}; Z/2)$ for $n \geq 1$. In particular there is no natural algebra structure on the mod two homotopy spectral sequences we are considering.

Hence all Γ_n are connective two-adic equivalences, and we may try to compute the mod two homotopy of $T(Z)^{C_{2^n}}$ from the spectral sequences $E^*(C_{2^n})$:

$$E^2_{s,t} = H^{-s}(C_{2^n}; \pi_t(T(Z); Z/2)) \Longrightarrow \pi_{s+t}(T(Z)^{hC_{2^n}}; Z/2)$$

Here H^{-s} is ordinary group cohomology, which can be found from Tate cohomology by truncation.

The inductive scheme goes as follows.

Suppose we know the mod two homotopy of $T(Z)^{C_{2^{n-1}}}$. This is the mod two homotopy of $\hat{\mathbb{H}}(C_{2^n}, T(Z))$ in nonnegative degrees. Try to backtrack in the spectral sequence $\hat{E}^*(C_{2^n})$ to determine the differential structure, given the E^2-term and the abutment. Then truncate the spectral sequence to give $E^*(C_{2^n})$. This converges to the mod two homotopy of $T(Z)^{C_{2^n}}$. And the map $R\colon T(Z)^{C_{2^n}} \to T(Z)^{C_{2^{n-1}}}$ is induced by the map of spectral sequences $E^*(C_{2^n}) \to \hat{E}^*(C_{2^n})$.

How about the Adams $A = v_1^4$-periodicity ? By an explicit symbol calculation, and a theorem of Calvin Moore, η^3 in $\pi_3 Q(S^0)$ maps to zero in $K_3(Z_2)$. Hence ν maps to a four-torsion class in $K_3(Z_2)$, which lifts over a mod four Bockstein to a class $\tilde{\nu}_4 \in K_4(\mathbb{Z}_2; Z/4)$. Mod four homotopy acts upon mod two homotopy, and multiplication by $\tilde{\nu}_4$ behaves as multiplication by v_1^2; as closely as possible. At least a detailed look at $\tilde{\nu}_4^2$ shows that it acts like A.