TOPOLOGICAL CYCLIC HOMOLOGY OF $\mathbb{S}$-ALGEBRAS

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1. Γ-spaces and $\mathbb{S}$-algebras

Let Γ^{op} be the category of finite pointed sets $k_+ = \{0, 1, \dots, k\}$ for $k \ge 0$, and base-point preserving functions. We often write $* = 0_+$ and $S^0 = 1_+$.

Γ^{op} admits a wedge sum $\vee \colon \Gamma^{op} \times \Gamma^{op} \to \Gamma^{op}$ taking (k_+, l_+) to the concatenation $(k+l)_+ \cong k_+ \vee l_+$, and a smash product $\wedge \colon \Gamma^{op} \times \Gamma^{op} \to \Gamma^{op}$ taking (k_+, l_+) to the lexicographically ordered product $(kl)_+ \cong k_+ \wedge l_+$.

Let $\mathcal{S}_*$ be the category of spaces, i.e., pointed simplicial sets. A Γ-space X is a functor $X \colon \Gamma^{op} \to \mathcal{S}_*$ with $X(0_+) = *$, i.e., a pointed functor. Let $\Gamma\mathcal{S}_*$ be the category of Γ-spaces. The morphisms are the natural transformations of functors.

As an example, let A be an abelian group. A functor $HA \colon k_+ \mapsto A \oplus \dots \oplus A$ (k summands A) is given on a morphism $f \colon k_+ \to l_+$ by $HA(f)(a_1, \dots, a_k) = (b_1, \dots, b_l)$, where $b_j = \sum_{f(i)=j} a_i$. We call HA the Eilenberg–Mac Lane Γ-space of A. This yields an embedding $H \colon \mathcal{A}b \to \Gamma\mathcal{S}_*$, where $\mathcal{A}b$ is the category of abelian groups.

We call $X(1_+) = X(S^0)$ the underlying space of X. A Γ-space X is special if the canonical map $X(k_+ \vee l_+) \to X(k_+) \times X(l_+)$ is a weak equivalence for all k and l. Equivalently the canonical map $X(k_+) \to X(1_+)^k$ is a weak equivalence for all k, so for X special $X(k_+)$ has the homotopy type of the product of k copies of the underlying space of X. In this case $\pi_0 X(1_+)$ naturally becomes a commutative monoid. If this monoid has inverses, i.e., is a commutative group, then we say that X is very special.

A Γ-space X extends to a functor $\mathcal{E}ns_* \to \mathcal{S}_*$ taking a pointed set T to the colimit $\operatorname{colim}_{k_+ \to T} X(k_+)$. It extends further to an endofunctor $\mathcal{S}_* \to \mathcal{S}_*$ taking a pointed simplicial set K to the diagonal of the simplicial space $[q] \mapsto X(K_q)$, i.e., the simplicial set $[q] \mapsto X(K_q)_q$. We also denote these extensions by X.

There is a natural map $X(K) \wedge L \to X(K \wedge L)$ for $K, L \in \mathcal{S}_*$. Let $S^1 = \Delta^1/\partial\Delta^1$. Taking $K = S^n = S^1 \wedge \dots \wedge S^1$ (n factors S^1) and $L = S^1$ we obtain the structure maps of a (pre-)spectrum $n \mapsto X(S^n)$, briefly denoted $X(S)$. The homotopy groups of the Γ-space X are defined as the homotopy groups of this spectrum, i.e., as

$$\pi_k(X) = \pi_k(X(S)) = \operatorname*{colim}_n \pi_{k+n} X(S^n)\,.$$

A map $X \to Y$ is called a stable equivalence if the induced map $\pi_k(X) \to \pi_k(Y)$ is an isomorphism for all $k \in \mathbb{Z}$.

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If X is special then the adjoint structure map $X(S^n) \to \Omega X(S^{n+1})$ is a weak equivalence for all $n \geq 1$ (X is a semi-Ω-spectrum), and if X is very special then this map is a weak equivalence for all $n \geq 0$ (X is an Ω-spectrum).

In the Eilenberg–Mac Lane example, $HA(S^n)$ is a $K(A, n)$-space and $HA(S)$ is the Eilenberg–Mac Lane spectrum of A. Its homotopy is $\pi_k(HA) = A$ for $k = 0$ and zero otherwise.

Given two Γ-spaces X and Y, their smash product $X \wedge Y$ is the Γ-space

$$k_+ \mapsto \operatorname*{colim}_{m_+ \wedge n_+ \to k_+} X(m_+) \wedge Y(n_+) .$$

This is the left Kan extension of the external smash product $X \bar{\wedge} Y \colon \Gamma^{op} \times \Gamma^{op} \to \Gamma\mathcal{S}_*$ taking (m_+, n_+) to $X(m_+) \wedge Y(n_+)$, over the smash product $\wedge\colon \Gamma^{op} \times \Gamma^{op} \to \Gamma^{op}$.

There is a stable homotopy equivalence $X(S) \wedge Y(S) \simeq (X \wedge Y)(S)$, so the smash product of Γ-spaces models the smash product of spectra in the stable homotopy category.

Given Γ-spaces X, Y and Z, and a morphism $f\colon X \wedge Y \to Z$, the composite

$$X(m_+) \wedge Y(n_+) \to (X \wedge Y)(m_+ \wedge n_+) \xrightarrow{f} Z(m_+ \wedge n_+)$$

is a natural transformation of functors $\Gamma^{op} \times \Gamma^{op} \to \mathcal{S}_*$, i.e., a morphism of bi-Γ-spaces. This correspondence is a bijection, so each such morphism $X \bar{\wedge} Y \to Z \circ \wedge$ of bi-Γ-spaces comes from a unique morphism $X \wedge Y \to Z$ of Γ-spaces.

The morphism of bi-Γ-spaces above also extends to a natural transformation

$$X(K) \wedge Y(L) \to Z(K \wedge L)$$

of functors $\mathcal{S}_* \times \mathcal{S}_* \to \mathcal{S}_*$.

The inclusion $\Gamma^{op} \to \mathcal{S}_*$ interpreting k_+ as a constant pointed simplicial set defines a Γ-space $\mathbb{S}$ called the sphere Γ-space. The extended endo-functor $\mathbb{S}\colon \mathcal{S}_* \to \mathcal{S}_*$ is the identity, and the associated spectrum is the sphere spectrum $n \mapsto S^n$. Thus its homotopy $\pi_k(\mathbb{S}) = \operatorname{colim}_n \pi_{k+n}(S^n)$ equals the stable homotopy groups of spheres.

The category $\Gamma\mathcal{S}_*$ of Γ-spaces equipped with the smash product pairing $\wedge\colon \Gamma\mathcal{S}_* \times \Gamma\mathcal{S}_* \to \Gamma\mathcal{S}_*$ and the unit object $\mathbb{S}$ is a symmetric monoidal category $(\Gamma\mathcal{S}_*, \wedge, \mathbb{S})$. This thus has similar formal properties to the category $\mathcal{A}b$ of abelian groups, with the tensor product pairing $\otimes\colon \mathcal{A}b \times \mathcal{A}b \to \mathcal{A}b$ and the unit object $\mathbb{Z}$.

A monoid (R, μ, η) in $(\mathcal{A}b, \otimes, \mathbb{Z})$ is an abelian group R equipped with a product $\mu\colon R \otimes R \to R$ and a unit map $\eta\colon \mathbb{Z} \to R$ satisfying associativity and unit conditions. This is precisely an associative ring with unit, or a $\mathbb{Z}$-algebra. It is a commutative ring, or a commutative $\mathbb{Z}$-algebra if $\mu \circ T = \mu$, where $T\colon R \otimes R \to R \otimes R$ is the twist isomorphism.

Likewise, an $\mathbb{S}$-algebra A is by definition a monoid (A, μ, η) in $(\Gamma\mathcal{S}_*, \wedge, \mathbb{S})$. It is thus a Γ-space A, equipped with a product $\mu\colon A \wedge A \to A$ and a unit $\eta\colon \mathbb{S} \to A$, satisfying associativity and unit conditions. If $\mu \circ T = \mu$, then A is a commutative $\mathbb{S}$-algebra.

The extended endofunctor $A\colon \mathcal{S}_* \to \mathcal{S}_*$ is now a functor with smash product (FSP). It comes equipped with a product map

$$A(K) \wedge A(L) \to A(K \wedge L)$$

and a unit map $K \to A(K)$, which are natural in $K, L \in \mathcal{S}_*$. These satisfy strict associativity and unit conditions.

The associated spectrum $A(S)$ of an $\mathbb{S}$-algebra A becomes a ring spectrum, with product $A(S) \wedge A(S) \simeq (A \wedge A)(S) \to A(S)$ and unit $S = \mathbb{S}(S) \to A(S)$, but an $\mathbb{S}$-algebra is a stricter structure, defined in the category of Γ-spaces and strict maps, not just in the stable homotopy category.

When R is a (commutative) ring, the Eilenberg–Mac Lane Γ-space HR becomes a (commutative) $\mathbb{S}$-algebra. The sphere Γ-space $\mathbb{S}$ is the initial (commutative) $\mathbb{S}$-algebra.

2. Cyclic objects

Let Λ be Connes' cyclic category, with objects $\{[q] \mid q \geq 0\}$ and morphism sets

$$\Lambda([p], [q]) = \Delta([p], [q]) \times C_{p+1} \,.$$

By restriction to $\Delta^{op} \subset \Lambda^{op}$, a cyclic object X determines an underlying simplicial object, whose geometric realization $|X|$ admits a natural circle action (S^1-action).

3. Topological Hochschild homology

Let $I \subset \Gamma^{op}$ be the subcategory of injective functions $k_+ \to l_+$. The wedge sum and smash product functors restrict to the subcategory I.

For $x = k_+$ in I we write

$$S^x = S^1 \wedge \cdots \wedge S^1$$

(k factors S^1). Let $\mathrm{Map}(S^x, Y) \cong \Omega^k Y$ be the (based) simplicial mapping space. Its p-simplices is the set of simplicial maps $S^x \wedge \Delta^p_+ \to Y$.

Let A be an $\mathbb{S}$-algebra. For any $(q+1)$-tuple $x = (x_0, \dots, x_q)$ in I^{q+1} we define a Γ-space $k_+ \mapsto G(A, x)(k_+)$ by

$$G(A, x)(k_+) = \mathrm{Map}(S^{x_0} \wedge \cdots \wedge S^{x_q}, A(S^{x_0}) \wedge \cdots \wedge A(S^{x_q}) \wedge k_+) \,.$$

The association $x \mapsto G(A, x)$ is a functor $I^{q+1} \to \Gamma\mathcal{S}_*$, using in part the stabilization maps $A(S^n) \wedge S^1 \to A(S^{n+1})$. Its homotopy colimit defines the Γ-space

$$THH(A)_q = \operatorname*{hocolim}_{x \in I^{q+1}} G(A, x) \,.$$

There is a stable homotopy equivalence $THH(A)_q \simeq A \wedge \cdots \wedge A$ ($q+1$ factors A).

There are cyclic structure maps making $[q] \mapsto THH(A)_q$ a cyclic Γ-space, denoted by $THH(A)$. These are analogous to the cyclic structure maps defining the Hochschild complex. In particular the face maps use the product μ on A, and the degeneracies use the unit map η.

For example, the face map $d_0 \colon THH(A)_1 \to THH(A)_0$ takes a map $f \colon S^{x_0} \wedge S^{x_1} \wedge \Delta^p_+ \to A(S^{x_0}) \wedge A(S^{x_1}) \wedge k_+$ to $d_0(f) \colon S^{x_0 \vee x_1} \wedge \Delta^p_+ \to A(S^{x_0 \vee x_1}) \wedge k_+$, by means of the isomorphism $S^{x_0} \wedge S^{x_1} \cong S^{x_0 \vee x_1}$ and the product $A(S^{x_0}) \wedge A(S^{x_1}) \to A(S^{x_0 \vee x_1})$. The face map d_1 yields $d_1(f) \colon S^{x_1 \vee x_0} \wedge \Delta^p_+ \to A(S^{x_1 \vee x_0}) \wedge k_+$, and involves the twist isomorphism $x_0 \vee x_1 \cong x_1 \vee x_0$ both in the source and the target.

The cyclic structure gives each space $THH(A)(k_+)$ a natural S^1-action. The associated spectrum $THH(A)(S)$ is thus a spectrum with S^1-action. If A is a commutative $\mathbb{S}$-algebra, then there is a product $THH(A) \wedge THH(A) \to THH(A \wedge A) \to THH(A)$ and a unit $\mathbb{S} \to A \to THH(A)$, making $THH(A)$ a commutative $\mathbb{S}$-algebra. The composite $A \to THH(A)_0 \to THH(A)$ is then a map of commutative $\mathbb{S}$-algebras, making $THH(A)$ a commutative A-algebra.

For example, when $A = \mathbb{S}$ each iterated degeneracy map $\mathbb{S} \simeq THH(\mathbb{S})_0 \to THH(\mathbb{S})_q$ is a stable equivalence. It follows that the inclusion of zero-simplices $THH(\mathbb{S})_0 \to THH(\mathbb{S})$ is a stable equivalence, so $THH(\mathbb{S}) \simeq \mathbb{S}$.

When $A = HR$ with R a ring, we usually write $THH(R)$ for $THH(HR)$. Rationally, we have $H_*(THH(A)_q; \mathbb{Q}) \cong H_*(A; \mathbb{Q}) \otimes \cdots \otimes H_*(A; \mathbb{Q})$ ($(q+1)$ factors $H_*(A; \mathbb{Q})$), and there is an isomorphism $H_*(THH(A); \mathbb{Q}) \cong HH_*(H_*(A; \mathbb{Q}))$. When $A = HR$, $H_*(HR; \mathbb{Q}) \cong R \otimes \mathbb{Q}$, so rationally the topological Hochschild homology of HR agrees with the Hochschild homology of R. The difference consists of torsion groups.

4. Frobenius maps and TF

Let $C_r \subset S^1$ be the cyclic subgroup of order r. Let p be a prime. The S^1-action on $THH(A)$ arising from the cyclic structure restricts to a C_{p^n}-action for each $n \geq 0$. The fixed points for this action is the Γ-space $THH(A)^{C_{p^n}}$, taking k_+ to the fixed point space $THH(A)(k_+)^{C_{p^n}}$.

We think of $C_{p^{n-1}}$ as a subgroup of C_{p^n} of index p. Then the C_{p^n}-fixed points for the circle action on $THH(A)$ are contained in the $C_{p^{n-1}}$-fixed points, by neglecting part of the invariance.

The *Frobenius map*

$$F = F_p \colon THH(A)^{C_{p^n}} \to THH(A)^{C_{p^{n-1}}}$$

is defined as the inclusion between these fixed point sets, interpreting a point in $THH(A)(k_+)$ that is fixed by C_{p^n} as in particular being fixed by $C_{p^{n-1}}$.

These assemble to a sequential limit diagram

$$\dots \xrightarrow{F} THH(A)^{C_{p^n}} \xrightarrow{F} THH(A)^{C_{p^{n-1}}} \xrightarrow{F} \dots \xrightarrow{F} THH(A)^{C_p} \xrightarrow{F} THH(A) \,.$$

Replacing each map by a fibration and taking the limit, or more precisely taking the homotopy limit of this diagram, defines the functor TF:

$$TF(A;p) = \operatorname*{holim}_{n,F} THH(A)^{C_{p^n}}$$

where the maps in the limit are the Frobenius maps $F = F_p$.

There are canonical map

$$\Gamma_n \colon THH(A)^{C_{p^n}} \to THH(A)^{hC_{p^n}} = \mathrm{Map}(EC_{p^n +}, THH(A))^{C_{p^n}}$$

from the fixed points to the homotopy fixed points for the C_{p^n}-action on $THH(A)$. We obtain maps

$$TF(A;p) = \operatorname*{holim}_{n} THH(A)^{C_{p^n}} \to \operatorname*{holim}_{n} THH(A)^{hC_{p^n}} \,.$$

After p-adic completion, the natural map

$$THH(A)^{hS^1} \to \operatorname*{holim}_{n} THH(A)^{hC_{p^n}}$$

is a homotopy equivalence. Hence there is a canonical map

$$\Gamma\colon TF(A;p)^{\wedge}_p \to THH(A)^{hS^1}{}^{\wedge}_p$$

which in some cases is a homotopy equivalence, or a homotopy equivalence on suitable connective covers. Hence TF may be thought of as close to the S^1-homotopy fixed points of THH. In this sense, TF is close to a topological negative cyclic homology.

Even if $THH(A)$ is of finite type, i.e., each homotopy group is finitely generated, it is usually not the case that $TF(A;p)$ and $THH(A)^{hS^1}$ are of finite type. Hence we shall seek to reduce the size of $TF(A;p)$ further, by taking into account more structure available in $THH(A)$.

6. Edgewise subdivision

The S^1-action on $THH(A)$ is not simplicial. Edgewise subdivision is a method to replace $THH(A)$ by another simplicial space $sd_rTHH(A)$, which admits a simplicial C_r-action, such that there is a natural homeomorphism of geometric realizations

$$D\colon |sd_rTHH(A)| \xrightarrow{\cong} |THH(A)|$$

identifying the simplicial C_r-action on the left with the C_r-action on the right that comes from restricting the S^1-action to the subgroup $C_r \subset S^1$. Hence there is a homeomorphism

$$D^{C_r}\colon |(sd_rTHH(A))^{C_r}| \xrightarrow{\cong} |THH(A)|^{C_r}$$

and $(sd_rTHH(A))^{C_r}$ provides a simplicial model for the C_r-fixed points.

7. Restriction maps and TR

Now consider C_p as a subgroup of C_{p^n}, with quotient group $C_{p^{n-1}}$. The *restriction map*

$$R = R_p\colon THH(A)^{C_{p^n}} \to THH(A)^{C_{p^{n-1}}}$$

is defined by applying $C_{p^{n-1}}$-fixed points to the geometric realization of a simplicial S^1-equivariant map

$$R_p\colon sd_pTHH(A)^{C_p} \to THH(A)\,.$$

On q-simplices, this is a map of Γ-spaces

$$(R_p)_q\colon (sd_pTHH(A)_q)^{C_p} = (THH(A)_{p(q+1)-1})^{C_p} \to THH(A)_q\,.$$

An r-simplex in the homotopy colimit defining $THH(A)_{p(q+1)-1}(k_+)$ is a chain of maps $x^0 \leftarrow \dots \leftarrow x^r = x = (x_0, \dots, x_{p(q+1)-1})$ in $I^{p(q+1)}$, together with a map

$$f\colon S^{x_0} \wedge \dots \wedge S^{x_{p(q+1)-1}} \wedge \Delta^r_+ \to A(S^{x_0}) \wedge \dots \wedge A(S^{x_{p(q+1)-1}}) \wedge k_+\,.$$

The generator of the C_p-action permutes the factors in $I^{p(q+1)}$ by cyclically shifting them $(q+1)$ positions to the right, and similarly for the $p(q+1)$ smash product factors in the source and target of the map f. (The final factors Δ^r_+ and k_+ are fixed.)

The source of $(R_p)_q$ consists of the C_p-invariant chains $x^0 \leftarrow \dots \leftarrow x^r = x$, together with the C_p-equivariant maps f as above.

A $p(q+1)$-tuple $x \in I^{p(q+1)}$ is C_p-invariant precisely when it has the form $\Delta_p(y) = (y, \dots, y)$ for $y \in I^{q+1}$. Here $\Delta_p \colon I^{q+1} \to I^{p(q+1)}$ is the p-fold diagonal embedding. Thus we may assume that the C_p-invariant chain $x^0 \leftarrow \dots \leftarrow x^r = x$ arises by applying Δ_p to a chain $y^0 \leftarrow \dots \leftarrow y^r = y = (y_0, \dots, y_q)$ in I^{q+1}. So

$$x = \Delta(y) = (y_0, \dots, y_q, \dots, y_0, \dots, y_q)$$

is y repeated p times.

A C_p-equivariant map $f \colon X \to Y$ induces a map $f^{C_p} \colon X^{C_p} \to Y^{C_p}$ by restriction to the C_p-fixed point spaces. This is the core of the construction of the restriction maps. We apply this to the source of f:

$$X = (S^{y_0} \wedge \dots \wedge S^{y_q})^{\wedge p} \wedge \Delta^r_+ \,.$$

Here the generator of C_p cyclically permutes the p wedge factors, so the C_p-fixed points are the diagonal copy

$$X^{C_p} \cong S^{y_0} \wedge \dots \wedge S^{y_q} \wedge \Delta^r_+ \,.$$

We also apply this to the target of f:

$$Y = (A(S^{y_0}) \wedge \dots \wedge A(S^{y_q}))^{\wedge p} \wedge k_+ \,.$$

The C_p-fixed points are the diagonal copy

$$Y^{C_p} \cong A(S^{y_0}) \wedge \dots \wedge A(S^{y_q}) \wedge k_+ \,.$$

Via these identifications, R_p takes a C_p-invariant (r,q)-simplex in $THH(A)(k_+)$ determined by $\Delta_p(y^0) \leftarrow \dots \leftarrow \Delta_p(y^r) = \Delta_p(y)$ and a C_p-equivariant map $f \colon X \to Y$, to the (r,q)-simplex in $THH(A)(k_+)$ determined by $y^0 \leftarrow \dots \leftarrow y^r = y$ and the restricted map $f^{C_p} \colon X^{C_p} \to Y^{C_p}$, identified as a map $f^{C_p} \colon S^{y_0} \wedge \dots \wedge S^{y_q} \wedge \Delta^r_+ \to A(S^{y_0}) \wedge \dots \wedge A(S^{y_q}) \wedge k_+$.

The resulting map $R_p \colon THH(A)^{C_p} \to THH(A)$ is a cyclic map, hence S^1-equivariant. Taking $C_{p^{n-1}}$-fixed points for $n \ge 1$ defines the various restriction maps, as displayed above. They assemble to a sequential limit diagram

$$\dots \xrightarrow{R} THH(A)^{C_{p^n}} \xrightarrow{R} THH(A)^{C_{p^{n-1}}} \xrightarrow{R} \dots \xrightarrow{R} THH(A)^{C_p} \xrightarrow{R} THH(A) \,.$$

Taking the homotopy limit of this diagram defines the functor TR:

$$TR(A; p) = \operatorname*{holim}_{n,R} THH(A)^{C_{p^n}}$$

where the maps in the limit are the restriction maps $R = R_p$.

Since the restriction maps arise by taking fixed points of an S^1-equivariant map, they commute with the forgetful Frobenius maps.

8. Topological cyclic homology TC

Let p be a prime, and let $\mathcal{RF}$ be the category with objects $1, p, \dots, p^n, \dots$ for $n \geq 0$, and commuting morphisms $r, f \colon p^n \to p^{n-1}$ for all $n \geq 1$. Thus there are $(k+1)$ distinct morphisms $p^{n+k} \to p^n$, given as the various composites $r^i f^j$ for $i + j = k$. Then

$$p^n \mapsto THH(A)^{C_{p^n}}$$

defines a functor $\mathcal{RF} \to \Gamma\mathcal{S}_*$, taking r to the restriction map $R = R_p$ and f to the Frobenius map $F = F_p$. We define the p-primary *topological cyclic homology* of the $\mathbb{S}$-algebra A to be

$$TC(A;p) = \operatorname*{holim}_{p^n \in \mathcal{RF}} THH(A)^{C_{p^n}} .$$

Alternatively, TC may be described as a homotopy equalizer for maps between sequential homotopy limits, thus avoiding the details of how this more complicated homotopy limit is defined.

Since the R- and F-maps commute, the R-maps induce a self-map R of $TF(A;p)$, and the F-maps induce a self map F of $TR(A;p)$. There are homotopy equalizer diagrams

$$TC(A;p) \xrightarrow{\pi} TF(A;p) \underset{1}{\overset{R}{\rightrightarrows}} TF(A;p)$$

and

$$TC(A;p) \xrightarrow{\pi} TR(A;p) \underset{1}{\overset{F}{\rightrightarrows}} TR(A;p) .$$

Thus TC is homotopy equivalent to the homotopy fiber of $R - 1$ acting on TF, or of $F - 1$ acting on TR.

9. The norm–restriction sequence

For each $n \geq 1$ there is a (homotopy) cofiber sequence of Γ-spaces

$$THH(A)_{hC_{p^n}} \xrightarrow{N} THH(A)^{C_{p^n}} \xrightarrow{R} THH(A)^{C_{p^{n-1}}} .$$

Here the homotopy orbit construction $THH(A)_{hC_{p^n}}$ is the Γ-space taking k_+ to $EC_{p^n +} \wedge_{C_{p^n}} THH(A)(k_+)$, where EC_{p^n} is a free contractible C_{p^n}-space. Its underlying spectrum is $m \mapsto EC_{p^n +} \wedge_{C_{p^n}} THH(A)(S^m)$.

A map $F \colon A \to B$ of $\mathbb{S}$-algebras inducing a stable equivalence on $THH(-)$, e.g. a stable equivalence $A \to B$, will induce a stable equivalence on all homotopy orbit spectra $THH(-)_{hC_{p^n}}$. By the norm–restriction sequence and induction, it also induces a stable equivalence on the C_{p^n}-fixed point subspectra of THH, for each $n \geq 0$. Hence it also induces a stable equivalence on $TF(-;p)$, $TR(-;p)$ and $TC(-;p)$.