ALGEBRAIC K-THEORY OF TOPOLOGICAL K-THEORY

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Abstract. Let ℓ_p be the p-complete connective Adams summand of topological K-theory, with coefficient ring $(\ell_p)_* = \mathbb{Z}_p[v_1]$, and let $V(1)$ be the Smith–Toda complex, with $BP_*(V(1)) = BP_*/(p, v_1)$. For $p \geq 5$ we explicitly compute the $V(1)$-homotopy of the algebraic K-theory spectrum of ℓ_p, denoted $V(1)_*K(\ell_p)$. In particular we find that it is a free finitely generated module over the polynomial algebra $P(v_2)$, except for a sporadic class in degree $2p-3$. Thus also in this case algebraic K-theory increases chromatic complexity by one. The proof uses the cyclotomic trace map from algebraic K-theory to topological cyclic homology, and the calculation is actually made in the $V(1)$-homotopy of the topological cyclic homology of ℓ_p.

Contents

Introduction

We are interested in the arithmetic of ring spectra.

To make sense of this we must work with structured ring spectra, such as S-algebras [EKMM], symmetric ring spectra [HSS] or Γ-rings [Ly]. We will refer to these as $\mathbb{S}$-algebras. The commutative objects are then commutative $\mathbb{S}$-algebras.

The category of rings is embedded in the category of $\mathbb{S}$-algebras by the Eilenberg–Mac Lane functor $R \mapsto HR$. We may therefore view an $\mathbb{S}$-algebra as a generalization of a ring in the algebraic sense. The added flexibility of $\mathbb{S}$-algebras provides room for new examples and constructions, which may eventually also shed light upon the category of rings itself.

In algebraic number theory the arithmetic of the ring of integers in a number field is largely captured by its Picard group, its unit group and its Brauer group. These are in turn reflected in the algebraic K-theory of the ring of integers. Algebraic K-theory is defined also in the generality of $\mathbb{S}$-algebras. We can thus view the algebraic K-theory of an $\mathbb{S}$-algebra as a carrier of some of its arithmetic properties.

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The algebraic K-theory of (connective) $\mathbb{S}$-algebras can be closely approximated by diagrams built from the algebraic K-theory of rings [Du]. Hence we expect that global structural properties enjoyed by algebraic K-theory as a functor of rings should also have an analogue for algebraic K-theory as a functor of $\mathbb{S}$-algebras.

We have in mind, in particular, the étale descent property of algebraic K-theory conjectured by Lichtenbaum [Li] and Quillen [Qu], which has been established for several classes of commutative rings [V], [RW], [HM2]. We are thus led to ask when a map of commutative $\mathbb{S}$-algebras $A \to B$ should be considered as an étale covering with Galois group G. In such a situation we may further ask whether the natural map $K(A) \to K(B)^{hG}$ to the homotopy fixed point spectrum for G acting on $K(B)$ induces an isomorphism on homotopy in sufficiently high degrees. These questions will be considered in more detail in [Ro2].

One aim of this line of inquiry is to find a conceptual description of the algebraic K-theory of the sphere spectrum, $K(S^0) = A(*)$, which coincides with Waldhausen's algebraic K-theory of the one-point space $*$. In [Ro1] the second author computed the mod 2 spectrum cohomology of $A(*)$ as a module over the Steenrod algebra, providing a very explicit description of this homotopy type. However, this result is achieved by indirect computation and comparison with topological cyclic homology, rather than by a structural property of the algebraic K-theory functor. What we are searching for here is a more memorable intrinsic explanation for the homotopy type appearing as the algebraic K-theory of an $\mathbb{S}$-algebra.

More generally, for a simplicial group G with classifying space $X = BG$ there is an $\mathbb{S}$-algebra $S^0[G]$, which can be thought of as a group ring over the sphere spectrum, and $K(S^0[G]) = A(X)$ is Waldhausen's algebraic K-theory of the space X. When X has the homotopy type of a manifold, $A(X)$ carries information about the geometric topology of that manifold. Hence an étale descent description of $K(S^0[G])$ will be of significant interest in geometric topology, reaching beyond algebraic K-theory itself.

In the present paper we initiate a computational exploration of this 'brave new world' of ring spectra and their arithmetic.

We begin by considering some interesting examples of (pro-)étale coverings in the category of commutative $\mathbb{S}$-algebras. For convenience we will choose to work locally, with $\mathbb{S}$-algebras that are complete at a prime p. For the purpose of algebraic K-theory this is less of a restriction than it may seem at first. What we have in mind here is that the square diagram

$$\begin{tikzcd}
K(A) \arrow[r] \arrow[d] & K(A_p) \arrow[d] \\
K(\pi_0 A) \arrow[r] & K(\pi_0 A_p)
\end{tikzcd}$$

is homotopy cartesian after p-adic completion [Du], when A is a connective $\mathbb{S}$-algebra, A_p its p-completion, $\pi_0 A$ its ring of path components and $\pi_0(A_p) \cong (\pi_0 A)_p$. This reduces the p-adic comparison of $K(A)$ and $K(A_p)$ to the p-adic comparison of $K(\pi_0 A)$ and $K(\pi_0 A_p)$, i.e., to a question about ordinary rings, which we view as a simpler question, or at least as one lying in better explored territory.

This leads us to study p-complete $\mathbb{S}$-algebras, or algebras over the p-complete sphere spectrum S^0_p. This spectrum is approximated in the category of commutative

$\mathbb{S}$-algebras (or E_∞ ring spectra) by a tower of chromatic localizations [Ra1]

$$S^0_p \to \cdots \to L_n S^0_p \to \cdots \to L_1 S^0_p \to L_0 S^0_p = H\mathbb{Q}_p\,.$$

Here $L_n = L_{E(n)}$ is Bousfield's localization functor [Bo], [EKMM] with respect to the nth Johnson–Wilson theory with coefficient ring $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}]$. By the Hopkins–Ravenel chromatic convergence theorem [Ra3, §8], the natural map $S^0_p \to \operatorname{holim}_n L_n S^0_p$ is a homotopy equivalence. For each $n \ge 1$ there is a further map of commutative $\mathbb{S}$-algebras $L_n S^0_p \to L_{K(n)} S^0_p$ to the Bousfield localization with respect to the nth Morava K-theory with coefficient ring $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$. This is an equivalence for $n = 0$, and $L_{K(1)} S^0_p \simeq J_p$ is the non-connective p-complete image-of-J spectrum.

There is a highly interesting sequence of commutative $\mathbb{S}$-algebras E_n constructed by Morava as spectra [Mo], by Hopkins and Miller [Re] as $\mathbb{S}$-algebras (or A_∞ ring spectra) and by Goerss and Hopkins [GH] as commutative $\mathbb{S}$-algebras (or E_∞ ring spectra). The coefficient ring of E_n is $(E_n)_* \cong W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]][u, u^{-1}]$. As a special case $E_1 \simeq KU_p$ is the p-complete complex topological K-theory spectrum.

The cited authors also construct a group action on E_n through commutative $\mathbb{S}$-algebra maps, by a semidirect product $G_n = S_n \rtimes C_n$ where S_n is the nth (profinite) Morava stabilizer group [Mo] and $C_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is the cyclic group of order n. There is a homotopy equivalence $L_{K(n)} S^0_p \simeq E_n^{hG_n}$, where the homotopy fixed point spectrum is formed in a continuous sense [DH], which reflects the Morava change of rings theorem [Mo].

Furthermore, the space of self equivalences of E_n in the category of commutative $\mathbb{S}$-algebras is weakly equivalent to its group of path components, which is precisely G_n. In fact the extension $L_{K(n)} S^0_p \to E_n$ qualifies as a pro-étale covering in the category of commutative $\mathbb{S}$-algebras, with Galois group weakly equivalent to G_n. The weak contractibility of each path component of the space of self equivalences of E_n (over either S^0_p or $L_{K(n)} S^0_p$) serves as the commutative $\mathbb{S}$-algebra version of the unique lifting property for étale coverings. Also the natural map $\zeta\colon E_n \to THH(E_n)$ is a $K(n)$-equivalence, cf. [MS1, 5.1], implying that the space of relative Kähler differentials of E_n over $L_{K(n)} S^0_p$ is contractible. See [Ro2] for further discussion.

There are further étale coverings of E_n. For example there is one with coefficient ring $W\mathbb{F}_{p^m}[[u_1, \dots, u_{n-1}]][u, u^{-1}]$ for each multiple m of n. Let E_n^{nr} be the colimit of these, with $E^{nr}_{n*} = W\bar{\mathbb{F}}_p[[u_1, \dots, u_{n-1}]][u, u^{-1}]$. Then $\mathrm{Gal}(E_n^{nr}/L_{K(n)} S^0_p)$ is weakly equivalent to an extension of S_n by the profinite integers $\hat{\mathbb{Z}} = \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. Let $\bar{E}_n$ be a maximal pro-étale covering of E_n, and thus of $L_{K(n)} S^0_p$. What is the absolute Galois group $\mathrm{Gal}(\bar{E}_n/L_{K(n)} S^0_p)$ of $L_{K(n)} S^0_p$?

The tower of commutative $\mathbb{S}$-algebras induces a tower of algebraic K-theory spectra

$$K(S^0_p) \to \cdots \to K(L_n S^0_p) \to \cdots \to K(J_p) \to K(\mathbb{Q}_p)$$

studied in the p-local case by Waldhausen [Wa2]. The natural map $K(S^0_p) \to \operatorname{holim}_n K(L_n S^0_p)$ may well be an equivalence, see [MS2]. We are thus led to study the spectra $K(L_n S^0_p)$, and their relatives $K(L_{K(n)} S^0_p)$. (More precisely, Waldhausen studied finite localization functors L^f_n characterized by their behavior on finite CW-spectra. However, for $n = 1$ the localization functors L_1 and L^f_1 agree, and this is the case that we will explore in the body of this paper. Hence we will suppress this distinction in the present discussion.)

Granting that $L_{K(n)}S^0_p \to E_n$ qualifies as an étale covering in the category of commutative $\mathbb{S}$-algebras, the descent question concerns whether the natural map

$$K(L_{K(n)}S^0_p) \to K(E_n)^{hG_n} \tag{0.1}$$

is a p-adic homotopy equivalence in sufficiently high dimensions. We conjecture that it does.

To analyze $K(E_n)$ we expect to use a localization sequence in algebraic K-theory to reduce to the algebraic K-theory of connective commutative $\mathbb{S}$-algebras, and to use the Bökstedt–Hsiang–Madsen cyclotomic trace map to topological cyclic homology to compute these [BHM]. The ring spectra E_n and $E(n)_p$ are closely related, and for $n \geq 1$ we expect that there is a cofiber sequence of spectra

$$K(BP\langle n-1\rangle_p) \to K(BP\langle n\rangle_p) \to K(E(n)_p) \tag{0.2}$$

analogous to the localization sequence $K(\mathbb{F}_p) \to K(\mathbb{Z}_p) \to K(\mathbb{Q}_p)$ in the case $n = 0$. Something similar should work for E_n.

The cyclotomic trace map

$$trc\colon K(BP\langle n\rangle_p) \to TC(BP\langle n\rangle_p; p) \simeq TC(BP\langle n\rangle; p)$$

induces a p-adic homotopy equivalence upon replacing the target with its connective cover [HM1]. Hence a calculation of $TC(BP\langle n\rangle; p)$ is as good as a calculation of $K(BP\langle n\rangle_p)$, after p-adic completion. In this paper we present computational techniques which are well suited for calculating $TC(BP\langle n\rangle; p)$, at least when $BP\langle n\rangle_p$ is a commutative $\mathbb{S}$-algebra and the Smith–Toda complex $V(n)$ exists as a ring spectrum. In the algebraic case $n = 0$, with $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$, these techniques simultaneously provide a simplification of the argument in [BM1], [BM2] computing $TC(\mathbb{Z}; p)$ and $K(\mathbb{Z}_p)$ for $p \geq 3$. Presumably the simplification is related to that appearing in different generality in [HM2].

It is also plausible that variations on these techniques apply when replacing $V(n)$ by another finite type $n+1$ ring spectrum, and the desired commutative $\mathbb{S}$-algebra structure on $BP\langle n\rangle_p$ is weakened to the existence of an $\mathbb{S}$-algebra map from a related commutative $\mathbb{S}$-algebra, such as MU or BP.

The first non-algebraic case occurs for $n = 1$. Then $E_1 \simeq KU_p$ has an action by $G_1 = \mathbb{Z}_p^\times \cong \Gamma \times \Delta$. Here $\mathbb{Z}_p \cong \Gamma = 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, $\mathbb{Z}/(p-1) \cong \Delta \subset \mathbb{Z}_p^\times$ and $k \in \mathbb{Z}_p^\times$ acts on E_1 like the p-adic Adams operation ψ^k on KU_p.

Let $L_p = E_1^{h\Delta}$ be the p-complete Adams summand with coefficient ring $(L_p)_* = \mathbb{Z}_p[v_1, v_1^{-1}]$, so $L_p \simeq E(1)_p$. Then Γ acts continuously on L_p with $J_p \simeq L_p^{h\Gamma}$. Let ℓ_p be the p-complete connective Adams summand with coefficient ring $(\ell_p)_* = \mathbb{Z}_p[v_1]$, so $\ell_p \simeq BP\langle 1\rangle_p$. We expect that there is a cofiber sequence of spectra

$$K(\mathbb{Z}_p) \to K(\ell_p) \to K(L_p)\,.$$

The previous calculation of $TC(\mathbb{Z}; p)$ [BM1], [BM2], and the calculation of $TC(\ell; p)$ presented in this paper, identify the p-adic completions of $K(\mathbb{Z}_p)$ and $K(\ell_p)$, respectively. Given an evaluation of the transfer map between them, this identifies

$K(L_p)$. The homotopy fixed points for the Γ-action on $K(L_p)$ induced by the Adams operations ψ^k for $k \in 1 + p\mathbb{Z}_p$ should then model $K(J_p) = K(L_{K(1)}S^0_p)$.

This brings us to the contents of the present paper. In §1 we produce two useful classes λ_1^K and λ_2^K in the algebraic K-theory of ℓ_p. In §2 we compute the $V(1)$-homotopy of the topological Hochschild homology of ℓ, simplifying the argument of [MS1]. In §3 we present notation concerning topological cyclic homology and the cyclotomic trace map of [BHM]. In §4 we make preparatory calculations in the spectrum homology of the S^1-homotopy fixed points of $THH(\ell)$. These are applied in §5 to prove that the canonical map from the C_{p^n} fixed points to the C_{p^n} homotopy fixed points of $THH(\ell)$ induces an equivalence on $V(1)$-homotopy above dimension $(2p-2)$, using [Ts]. In §6 we inductively compute the $V(1)$-homotopy of all these (homotopy) fixed point spectra, and their homotopy limit $TF(\ell; p)$. The action of the restriction map on this limit is then identified in §7. The pieces of the calculation are brought together in Theorem 8.4 of §8, yielding an explicit computation of the $V(1)$-homotopy of $TC(\ell; p)$:

Theorem 0.3. *Let $p \geq 5$. There is an isomorphism of $E(\lambda_1, \lambda_2) \otimes P(v_2)$-modules*

$$\begin{aligned} V(1)_* TC(\ell; p) \cong E(\lambda_1, \lambda_2, \partial) \otimes P(v_2) \\ \oplus E(\lambda_2) \otimes P(v_2) \otimes \mathbb{F}_p\{\lambda_1 t^e \mid 0 < e < p\} \\ \oplus E(\lambda_1) \otimes P(v_2) \otimes \mathbb{F}_p\{\lambda_2 t^{ep} \mid 0 < e < p\} \end{aligned}$$

with $|\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$, $|v_2| = 2p^2-2$, $|\partial| = -1$ and $|t| = -2$.

The p-completed cyclotomic trace map $K(\ell_p)_p \to TC(\ell_p; p) \simeq TC(\ell; p)$ identifies $K(\ell_p)_p$ with the connective cover of $TC(\ell; p)$. This yields the following expression for the $V(1)$-homotopy of $K(\ell_p)$, given in Theorem 9.1 of §9:

Theorem 0.4. *Let $p \geq 5$. There is an exact sequence of $E(\lambda_1, \lambda_2) \otimes P(v_2)$-modules*

$$0 \to \Sigma^{2p-3}\mathbb{F}_p \to V(1)_* K(\ell_p) \xrightarrow{trc} V(1)_* TC(\ell; p) \to \Sigma^{-1}\mathbb{F}_p \to 0$$

taking the degree $2p-3$ generator in $\Sigma^{2p-3}H\mathbb{F}_p$ to a class $a \in V(1)_{2p-3}K(\ell_p)$, and taking the class ∂ in $V(1)_{-1}TC(\ell; p)$ to the degree -1 generator in $\Sigma^{-1}H\mathbb{F}_p$.

The $V(1)$-homotopy of any spectrum is a $P(v_2)$-module, but we emphasize that $V(1)_* TC(\ell; p)$ is a free finitely generated $P(v_2)$-module, and $V(1)_* K(\ell_p)$ is free and finitely generated except for the summand $\mathbb{F}_p\{a\}$ in degree $2p-3$. Hence both $K(\ell_p)_p$ and $TC(\ell; p)$ are fp-spectra in the sense of [MR], with finitely presented mod p cohomology as a module over the Steenrod algebra. They both have fp-type 2, because $V(1)_* K(\ell_p)$ is infinite while $V(2)_* K(\ell_p)$ is finite, and similarly for $TC(\ell; p)$. In particular, $K(\ell_p)$ is closely related to elliptic cohomology.

More generally, at least if $BP\langle n\rangle_p$ is a commutative $\mathbb{S}$-algebra and $V(n)$ exists as a ring spectrum, similar calculations to those presented in this paper show that $V(n)_* TC(BP\langle n\rangle; p)$ is a free $P(v_{n+1})$-module on $2^{n+2} + 2^n(n+1)(p-1)$ generators. So algebraic K-theory takes such fp-type n commutative $\mathbb{S}$-algebras to fp-type $(n+1)$ commutative $\mathbb{S}$-algebras. If our ideas about localization sequences are correct then also $K(E_n)_p$ will be of fp-type $(n+1)$, and if étale descent holds in algebraic K-theory for $L_{K(n)}S^0_p \to E_n$ with $cd_p(G_n) < \infty$ then also $K(L_{K(n)}S^0_p)_p$ will be of fp-type $(n+1)$. The moral is that algebraic K-theory in many cases increments chromatic complexity by one, i.e., it produces a constant red-shift in stable homotopy theory.

Notations and conventions. For an $\mathbb{F}_p$ vector space V, let $E(V)$, $P(V)$ and $\Gamma(V)$ be the exterior algebra, polynomial algebra and divided power algebra on V, respectively. When V has a basis $\{x_1, \dots, x_n\}$, we write $E(x_1, \dots, x_n)$, $P(x_1, \dots, x_n)$ and $\Gamma(x_1, \dots, x_n)$ for these algebras. So $\Gamma(x) = \mathbb{F}_p\{\gamma_j(x) \mid j \geq 0\}$ with $\gamma_i(x) \cdot \gamma_j(x) = (i,j)\gamma_{i+j}(x)$. Let $P_h(x) = P(x)/(x^h = 0)$ be the truncated polynomial algebra of height h. For $a \leq b \leq \infty$ let $P_a^b(x) = \mathbb{F}_p\{x^k \mid a \leq k \leq b\}$ as a $P(x)$-module.

By an infinite cycle in a spectral sequence we mean a class x such that $d^r(x) = 0$ for all r. By a permanent cycle we mean an infinite cycle which is not a boundary, i.e., a class that survives to represent a nonzero class at E^∞. Differentials are often only given up to multiplication by a unit.

1. Classes in algebraic K-theory

1.1. E_∞ ring spectrum models. Let p be an odd prime. Let $\ell = BP\langle 1\rangle$ be the Adams summand of p-local connective topological K-theory. Its homotopy groups are $\ell_* \cong \mathbb{Z}_{(p)}[v_1]$, with $|v_1| = q = 2p-2$.

Its p-completion ℓ_p with $\ell_{p*} \cong \mathbb{Z}_p[v_1]$ admits a model as an E_∞ ring spectrum, which can be constructed as the algebraic K-theory spectrum of a perfect field k'. Let g be a prime power topologically generating the p-adic units and let $k' = \operatorname{colim}_{n\geq 0} \mathbb{F}_{g^{p^n}} \subset \bar{k}$ be a $\mathbb{Z}_p$-extension of $k = \mathbb{F}_g$. Then $\ell_p = K(k')_p$ is an E_∞ ring spectrum model for the p-completed Adams summand.

Likewise $j_p = K(k)_p$ and $ku_p = K(\bar{k})_p$ are E_∞ ring spectrum models for the p-completed image-of-J spectrum and the p-completed connective topological K-theory spectrum, respectively. The Frobenius automorphism $\sigma_g(x) = x^g$ induces the Adams operation ψ^g on both ℓ_p and ku_p. Then k is the fixed field of σ_g, and j_p is the connective cover of the homotopy fixed point spectrum for ψ^g acting on either one of ℓ_p or ku_p.

The E_∞ ring spectrum maps $S_p^0 \to j_p \to \ell_p \to ku_p \to H\mathbb{Z}_p$ induce E_∞ ring spectrum maps on algebraic K-theory:

$$K(S_p^0) \to K(j_p) \to K(\ell_p) \to K(ku_p) \to K(\mathbb{Z}_p)\,.$$

In particular these are H_∞ ring spectrum maps.

1.2. A first class in algebraic K-theory. The Bökstedt trace map $tr\colon K(\mathbb{Z}_p) \to THH(\mathbb{Z}_p)$ maps onto the first p-torsion in the image, which is $THH_{2p-1}(\mathbb{Z}_p) \cong \mathbb{Z}/p\{e\}$ [BM1]. Let $e^K \in K_{2p-1}(\mathbb{Z}_p)$ be a class with $tr(e^K) = e$.

There is a $(2p-2)$-connected linearization map $\ell_p \to H\mathbb{Z}_p$ of E_∞ ring spectra, which induces a $(2p-1)$-connected map $K(\ell_p) \to K(\mathbb{Z}_p)$.

Definition 1.3. Let $\lambda_1^K \in K_{2p-1}(\ell_p)$ be a chosen class mapping to $e^K \in K_{2p-1}(\mathbb{Z}_p)$ under the map induced by linearization $\ell_p \to H\mathbb{Z}_p$.

The image $tr(\lambda_1^K) \in THH_{2p-1}(\ell_p)$ of this class under the trace map $tr\colon K(\ell_p) \to THH(\ell_p)$ will map under linearization to $e \in THH_{2p-1}(\mathbb{Z}_p)$.

Remark 1.4. The class $\lambda_1^K \in K_{2p-1}(\ell_p)$ does not lift further back to $K_{2p-1}(S_p^0)$, since e^K has a nonzero image in π_{2p-2} of the homotopy fiber of $K(S_p^0) \to K(\mathbb{Z}_p)$ [Wa1]. Thus λ_1^K does also not lift to $K_{2p-1}(j_p)$, because the map $S_p^0 \to j_p$ is $(pq-2)$-connected. It is not clear if the induced action of ψ^g on $K(\ell_p)$ leaves λ_1^K invariant.

1.5. Homotopy and homology operations. For a spectrum X, let $D_pX = E\Sigma_p \ltimes_{\Sigma_p} X^{\wedge p}$ be its pth extended power. Part of the structure defining an H_∞ ring spectrum E is a map $\xi\colon D_pE \to E$. Then a mod p homotopy class $\theta \in \pi_m(D_pS^n; \mathbb{F}_p)$ determines a mod p homotopy operation

$$\theta^*\colon \pi_n(E) \to \pi_m(E; \mathbb{F}_p)$$

natural for maps of H_∞ ring spectra E. Its value on the homotopy class represented by a map $a\colon S^n \to E$ is the image of θ under the composite map

$$\pi_m(D_pS^n; \mathbb{F}_p) \xrightarrow{D_p(a)} \pi_m(D_pE; \mathbb{F}_p) \xrightarrow{\xi} \pi_m(E; \mathbb{F}_p)\,.$$

Likewise the Hurewicz image $h(\theta) \in H_m(D_pS^n; \mathbb{F}_p)$ induces a homology operation

$$h(\theta)^*\colon H_n(E; \mathbb{F}_p) \to H_m(E; \mathbb{F}_p)\,,$$

and the two operations are compatible under the Hurewicz homomorphisms.

For S^n with $n = 2k-1$ an odd dimensional sphere the bottom two cells of D_pS^n are in dimensions $pn + (p-2)$ and $pn + (p-1)$, and are connected by a mod p Bockstein. Hence the bottom two mod p homotopy classes of D_pS^n are in these two dimensions, and are called βP^k and P^k, respectively. Their Hurewicz images induce the Dyer–Lashof operations denoted βQ^k and Q^k in homology, cf. [Br].

1.6. A second class in algebraic K-theory. We use the H_∞ ring spectrum structure on $K(\ell_p)$ to produce a further element in its mod p homotopy.

Definition 1.7. Let $\lambda_2^K = (P^p)^*(\lambda_1^K) \in K_{2p^2-1}(\ell_p; \mathbb{F}_p)$ be the image under the mod p homotopy operation

$$(P^p)^*\colon K_{2p-1}(\ell_p) \to K_{2p^2-1}(\ell_p; \mathbb{F}_p)$$

of $\lambda_1^K \in K_{2p-1}(\ell_p)$.

Since the trace map $tr\colon K(\ell_p) \to THH(\ell_p)$ is an E_∞ ring spectrum map, it follows that $tr(\lambda_2^K) \in THH_{2p^2-1}(\ell_p; \mathbb{F}_p)$ equals the image of $tr(\lambda_1^K) \in THH_{2p-1}(\ell_p)$ under the mod p homotopy operation $(P^p)^*$. We shall identify this image in the next section.

Remark 1.8. It is not clear whether λ_2^K lifts to an integral homotopy class in $K_{2p^2-1}(\ell_p)$. The image of $e^K \in K_{2p-1}(\mathbb{Z}_p)$ in $K_{2p-1}(\mathbb{Q}_p; \mathbb{F}_p)$ is $v_1 d\log p$ for a class $d\log p \in K_1(\mathbb{Q}_p; \mathbb{F}_p)$ mapping to the generator of $K_0(\mathbb{F}_p; \mathbb{F}_p)$ in the K-theory localization sequence for $\mathbb{Z}_p$, [HM2]. It appears that the image of λ_2^K in $V(1)_{2p^2-1}(L_p)$ is $v_2 d\log v_1$ for a class $d\log v_1 \in V(1)_1K(L_p)$ mapping to the generator of $V(0)_0K(\mathbb{Z}_p)$ in the expected K-theory localization sequence for ℓ_p. The classes λ_1^K and λ_2^K are therefore related to logarithmic differentials for poles at p and v_1, respectively, which partially motivates the choice of the letter 'λ'.

2. Topological Hochschild homology

The topological Hochschild homology functor $THH(-)$, as well as its refined versions $THH(-)^{hS^1}$, $THH(-)^{C_{p^n}}$, $TF(-;p)$ and $TC(-;p)$, preserve p-adic equivalences. Hence we will tend to write $THH(\mathbb{Z})$ and $THH(\ell)$ in place of $THH(\mathbb{Z}_p)$ and $THH(\ell_p)$, and similarly for the other related functors.

2.1. Homology of $THH(\ell)$. The ring spectrum map $\ell \to H\mathbb{F}_p$ induces an injection on mod p homology, identifying $H_*(\ell;\mathbb{F}_p)$ with the subalgebra

$$H_*(\ell;\mathbb{F}_p) = E(\bar\tau_k \mid k \geq 2) \otimes P(\bar\xi_k \mid k \geq 1)$$

of the dual Steenrod algebra A_*.

There is a Bökstedt spectral sequence

$$E^2_{**} = HH_*(H_*(\ell;\mathbb{F}_p)) \Longrightarrow H_*(THH(\ell);\mathbb{F}_p) \tag{2.2}$$

with

$$E^2_{**} = H_*(\ell;\mathbb{F}_p) \otimes \Gamma(\sigma\bar\tau_k \mid k \geq 2) \otimes E(\sigma\bar\xi_k \mid k \geq 1)\,.$$

Here $\sigma x \in HH_1(-)$ is represented by $1 \otimes x$ in degree 1 of the Hochschild complex. It corresponds to the map $\sigma\colon \Sigma\ell \to THH(\ell)$ induced by the S^1 action on $THH(\ell)$ and the inclusion of 0-simplices $\ell \to THH(\ell)$.

By naturality with respect to the map $\ell \to H\mathbb{F}_p$, the differentials

$$d^{p-1}(\gamma_j(\sigma\bar\tau_k)) = \sigma\bar\xi_{k+1} \cdot \gamma_{j-p}(\sigma\bar\tau_k)$$

for $j \geq p$, found in the Bökstedt spectral sequence for $THH(\mathbb{F}_p)$, lift to the spectral sequence (2.2) above. Hence

$$E^p_{**} = H_*(\ell;\mathbb{F}_p) \otimes P_p(\sigma\bar\tau_k \mid k \geq 2) \otimes E(\sigma\bar\xi_1, \sigma\bar\xi_2)$$

and this equals the E^∞-term for bidegree reasons.

In $H_*(THH(\ell);\mathbb{F}_p)$ there are Dyer–Lashof operations acting, and $(\sigma\bar\tau_k)^p = Q^{p^k}(\sigma\bar\tau_k) = \sigma(Q^{p^k}(\bar\tau_k)) = \sigma\bar\tau_{k+1}$ for all $k \geq 2$ [St]. Thus as an algebra

$$H_*(THH(\ell);\mathbb{F}_p) \cong H_*(\ell;\mathbb{F}_p) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu) \tag{2.3}$$

where λ_1, λ_2 and μ are represented by $\sigma\bar\xi_1$, $\sigma\bar\xi_2$ and $\sigma\bar\tau_2$, respectively, in the Bökstedt spectral sequence. Here $|\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$ and $|\mu| = 2p^2$. Furthermore $Q^p(\lambda_1) = Q^p(\sigma\bar\xi_1) = \sigma(Q^p(\bar\xi_1)) = \sigma\bar\xi_2$, so we may assume that we have chosen $\lambda_2 = Q^p(\lambda_1)$.

2.4. $V(1)$-homotopy of $THH(\ell)$. Let $V(n)$ be the nth Smith–Toda complex, with homology $H_*(V(n);\mathbb{F}_p) \cong E(\bar\tau_0, \dots, \bar\tau_n)$. Thus $V(-1) = S^0$, $V(0)$ is the mod p Moore spectrum, and $V(1)$ is the cofiber of the multiplication by v_1-map $\Sigma^q V(0) \to V(0)$. There are cofiber sequences

$$S^0 \xrightarrow{p} S^0 \xrightarrow{i_0} V(0) \xrightarrow{j_0} S^1$$

and

$$\Sigma^q V(0) \xrightarrow{v_1} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1} V(0)$$

defining the maps labeled i_0, j_0, i_1 and j_1. When $p \geq 5$, $V(1)$ is a commutative ring spectrum [Ok].

For a spectrum X the rth (partially defined) v_1-Bockstein homomorphism $\beta_{1,r}$ is defined on the classes $x \in V(1)_*(X)$ with $j_1(x) \in V(0)_*(X)$ divisible by v_1^{r-1}. Then for $y \in V(0)_*(X)$ with $v_1^{r-1} \cdot y = j_1(x)$ let $\beta_{1,r}(x) = i_1(y) \in V(1)_*(X)$. So $\beta_{1,r}$ decreases degrees by $rq+1$.

Definition 2.5. Let $r(n) = 0$ for $n \le 0$, and let $r(n) = p^n + r(n-2)$ for all $n \ge 1$. Thus $r(2n-1) = p^{2n-1} + \cdots + p$ (n odd powers of p) and $r(2n) = p^{2n} + \cdots + p^2$ (n even powers of p). Note that $(p^2-1)r(2n-1) = p^{2n+1} - p$, while $(p^2-1)r(2n) = p^{2n+2} - p^2$.

Proposition 2.6 (McClure–Staffeldt). *There is an algebra isomorphism*

$$V(1)_* THH(\ell) \cong E(\lambda_1, \lambda_2) \otimes P(\mu)$$

with $|\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$ *and* $|\mu| = 2p^2$. *There are* v_1*-Bocksteins* $\beta_{1,p}(\mu) = \lambda_1$, $\beta_{1,p^2}(\mu^p) = \lambda_2$ *and generally* $\beta_{1,r(n)}(\mu^{p^{n-1}}) \neq 0$ *for* $n \ge 1$.

Proof. One proof proceeds as follows, leaving the v_1-Bockstein structure to the more detailed work of [MS1].

$H_*(THH(\ell); \mathbb{F}_p)$ is an A_*-comodule algebra. The coaction

$$\nu \colon H_*(THH(\ell); \mathbb{F}_p) \to A_* \otimes H_*(THH(\ell); \mathbb{F}_p)$$

agrees with the coproduct $\psi \colon A_* \to A_* \otimes A_*$ when both are restricted to the subalgebra $H_*(\ell; \mathbb{F}_p) \subset A_*$. Furthermore $\nu(\sigma x) = (1 \otimes \sigma)\psi(x)$, so $\nu(\lambda_1) = 1 \otimes \lambda_1$, $\nu(\lambda_2) = 1 \otimes \lambda_2$ and $\nu(\mu) = 1 \otimes \mu + \bar\tau_0 \otimes \lambda_2$.

There are change-of-rings isomorphisms of A_*-comodule algebras

$$\begin{aligned} H_*(V(1) \wedge THH(\ell); \mathbb{F}_p) &\cong (A_* \Box_{H_*(\ell;\mathbb{F}_p)} \mathbb{F}_p) \otimes (H_*(\ell; \mathbb{F}_p) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu)) \\ &\cong A_* \Box_{H_*(\ell;\mathbb{F}_p)} (H_*(\ell; \mathbb{F}_p) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu)) \\ &\cong A_* \otimes E(\lambda_1, \lambda_2) \otimes P(\mu) . \end{aligned}$$

The homology classes $1 \wedge \lambda_1$, $1 \wedge \lambda_2$ and $1 \wedge \mu - \bar\tau_0 \wedge \lambda_2$ in $H_*(V(1) \wedge THH(\ell); \mathbb{F}_p)$ are primitive, and correspond to the classes λ_1, λ_2 and μ under the isomorphisms above. Hence these three homology classes are in the Hurewicz image from spherical classes $\lambda_1 \in V(1)_{2p-1} THH(\ell)$, $\lambda_2 \in V(1)_{2p^2-1} THH(\ell)$ and $\mu \in V(1)_{2p^2} THH(\ell)$, respectively. $\square$

Proposition 2.7. *The classes* $\lambda_1^K \in K_{2p-1}(\ell_p)$ *and* $\lambda_2^K \in K_{2p^2-1}(\ell_p; \mathbb{F}_p)$ *map under the trace map to integral and mod* p *lifts of* $\lambda_1 \in V(1)_{2p-1} THH(\ell)$ *and* $\lambda_2 \in V(1)_{2p^2-1} THH(\ell)$, *respectively.*

Proof. The linearized image in $V(1)_{2p-1} THH(\mathbb{Z})$ of $\lambda_1 \in V(1)_{2p-1} THH(\ell)$ equals the mod p and v_1 reduction $i_1 i_0(e)$ of the class $e \in THH_{2p-1}(\mathbb{Z})$, since both classes have the same Hurewicz image $1 \wedge \sigma\bar\xi_1$ in $H_{2p-1}(V(1) \wedge THH(\mathbb{Z}); \mathbb{F}_p)$. Thus the mod p and v_1 reduction of the trace image $tr(\lambda_1^K)$ equals λ_1.

The Hurewicz image in $H_{2p^2-1}(THH(\ell); \mathbb{F}_p)$ of $tr(\lambda_2^K) = (P^p)^*(tr(\lambda_1^K))$ equals the image of the homology operation Q^p on the Hurewicz image $\lambda_1 = \sigma\bar\xi_1$ in $H_{2p-1}(THH(\ell); \mathbb{F}_p)$ of $tr(\lambda_1^K)$. This is $Q^p(\sigma\bar\xi_1) = \sigma Q^p(\bar\xi_1) = \sigma\bar\xi_2 = \lambda_2$. So the mod v_1 image in $V(1)_{2p^2-1} THH(\ell)$ of $tr(\lambda_2^K)$ equals λ_2, since both classes have the same Hurewicz image $1 \wedge \lambda_2$ in $H_{2p^2-1}(V(1) \wedge THH(\ell); \mathbb{F}_p)$. $\square$

3. Topological cyclotomy

We now review some terminology and notation concerning topological cyclic homology and the cyclotomic trace map.

3.1. Frobenius, restriction, Verschiebung. As already indicated, $THH(\ell)$ is an S^1-equivariant spectrum. Let $C_{p^n} \subset S^1$ be the cyclic group of order p^n. The Frobenius maps $F\colon THH(\ell)^{C_{p^n}} \to THH(\ell)^{C_{p^{n-1}}}$ are the usual inclusions of fixed point spectra that forget part of the invariance. Their homotopy limit defines

$$TF(\ell;p) = \operatorname*{holim}_{n,F} THH(\ell)^{C_{p^n}} .$$

There are also restriction maps $R\colon THH(\ell)^{C_{p^n}} \to THH(\ell)^{C_{p^{n-1}}}$, defined using the cyclotomic structure of $THH(\ell)$, cf. [HM1]. They commute with the Frobenius maps, and thus induce a self map $R\colon TF(\ell;p) \to TF(\ell;p)$. Its homotopy equalizer with the identity map defines the topological cyclic homology of ℓ, which was introduced in [BHM]:

$$TC(\ell;p) \xrightarrow{\pi} TF(\ell;p) \underset{1}{\overset{R}{\rightrightarrows}} TF(\ell;p) .$$

Hence there is a cofiber sequence

$$\Sigma^{-1}TF(\ell;p) \xrightarrow{\partial} TC(\ell;p) \xrightarrow{\pi} TF(\ell;p) \xrightarrow{1-R} TF(\ell;p) ,$$

which we shall use to compute $V(1)_*TC(\ell;p)$. There are also Verschiebung maps $V\colon THH(\ell)^{C_{p^{n-1}}} \to THH(\ell)^{C_{p^n}}$, defined up to homotopy in terms of the S^1-equivariant transfer.

3.2. The cyclotomic trace map. The Bökstedt trace map admits lifts

$$tr_n\colon K(\ell_p) \to THH(\ell)^{C_{p^n}}$$

for all $n \geq 0$, with $tr = tr_0$, which commute with the Frobenius maps and homotopy commute with the restriction maps up to preferred homotopy. Hence the limiting map $tr_F\colon K(\ell_p) \to TF(\ell;p)$ homotopy equalizes R and the identity map, and the resulting lift

$$trc\colon K(\ell_p) \to TC(\ell;p)$$

is the Bökstedt–Hsiang–Madsen cyclotomic trace map [BHM].

3.3. The norm–restriction sequences. For each $n \geq 1$ there is a homotopy commutative diagram

$$\begin{tikzcd}
 & K(\ell_p) \arrow[d, "tr_n"'] \arrow[dr, "tr_{n-1}"] & \\
THH(\ell)_{hC_{p^n}} \arrow[r, "N"] \arrow[d, equal] & THH(\ell)^{C_{p^n}} \arrow[r, "R"] \arrow[d, "\Gamma_n"] & THH(\ell)^{C_{p^{n-1}}} \arrow[d, "\hat\Gamma_n"] \\
THH(\ell)_{hC_{p^n}} \arrow[r, "N^h"] & THH(\ell)^{hC_{p^n}} \arrow[r, "R^h"] & \hat{\mathbb{H}}(C_{p^n}, THH(\ell)) .
\end{tikzcd} \tag{3.4}$$

The lower part is the map of cofiber sequences that arises by smashing the S^1-equivariant cofiber sequence $ES^1_+ \to S^0 \to \widetilde{E}S^1$ with the S^1-equivariant map

$THH(\ell) \to F(ES^1_+, THH(\ell))$ and taking C_{p^n} fixed point spectra. For closed subgroups $G \subseteq S^1$ recall that $THH(\ell)^{hG} = F(ES^1_+, THH(\ell))^G$ is the G homotopy fixed point spectrum of $THH(\ell)$, and $\hat{\mathbb{H}}(G, THH(\ell)) = [\widetilde{E}S^1 \wedge F(ES^1_+, THH(\ell)]^G$ is the G Tate construction on $THH(\ell)$. The remaining terms of the diagram are then identified by the homotopy equivalences

$$THH(\ell)_{hC_{p^n}} \simeq [ES^1_+ \wedge THH(\ell)]^{C_{p^n}} \simeq [ES^1_+ \wedge F(ES^1_+, THH(\ell))]^{C_{p^n}}$$

and

$$THH(\ell)^{C_{p^{n-1}}} \simeq [\widetilde{E}S^1 \wedge THH(\ell)]^{C_{p^n}} .$$

We call N, R, N^h and R^h the norm, restriction, homotopy norm and homotopy restriction maps, respectively. We call Γ_n and $\hat{\Gamma}_n$ the canonical maps. The middle and lower cofiber sequences are the norm–restriction and homotopy norm–restriction sequences, respectively.

By passage to homotopy limits over Frobenius maps, there is also a limiting diagram

$$\begin{tikzcd}
 & K(\ell_p) \arrow[d, "tr_F"'] \arrow[dr, "tr_F"] & \\
\Sigma THH(\ell)_{hS^1} \arrow[r, "N"] \arrow[d, equal] & TF(\ell; p) \arrow[r, "R"] \arrow[d, "\Gamma"] & TF(\ell; p) \arrow[d, "\hat{\Gamma}"] \\
\Sigma THH(\ell)_{hS^1} \arrow[r, "N^h"] & THH(\ell)^{hS^1} \arrow[r, "R^h"] & \hat{\mathbb{H}}(S^1, THH(\ell)) .
\end{tikzcd} \tag{3.5}$$

Implicit here are the p-adic homotopy equivalences

$$\Sigma THH(\ell)_{hS^1} \simeq \operatorname*{holim}_{n,F} THH(\ell)_{hC_{p^n}}$$
$$THH(\ell)^{hS^1} \simeq \operatorname*{holim}_{n,F} THH(\ell)^{hC_{p^n}}$$
$$\hat{\mathbb{H}}(S^1, THH(\ell)) \simeq \operatorname*{holim}_{n,F} \hat{\mathbb{H}}(C_{p^n}, THH(\ell)) .$$

4. Circle homotopy fixed points

4.1. The circle trace map. The circle trace map

$$tr_{S^1} = \Gamma \circ tr_F \colon K(\ell_p) \to THH(\ell)^{hS^1} = F(ES^1_+, THH(\ell))^{S^1}$$

is a preferred lift of the trace map $tr\colon K(\ell_p) \to THH(\ell)$. We take S^∞ as our model for ES^1. Let

$$T^n = F(S^\infty/S^{2n-1}, THH(\ell))^{S^1}$$

for $n \geq 0$, so that there is a descending filtration $\{T^n\}_n$ on $T^0 = THH(\ell)^{hS^1}$, with layers $T^n/T^{n+1} \cong F(S^{2n+1}/S^{2n-1}, THH(\ell))^{S^1} \cong \Sigma^{-2n} THH(\ell)$.

4.2. The homology spectral sequence. Placing T^n in filtration $s = -2n$ and applying homology, we obtain a (not necessarily convergent) homology spectral sequence

$$E^2_{s,t} = H^{-s}(S^1; H_t(THH(\ell); \mathbb{F}_p)) \Longrightarrow H_{s+t}(THH(\ell)^{hS^1}; \mathbb{F}_p) \tag{4.3}$$

with

$$E^2_{**} = P(t) \otimes H_*(\ell; \mathbb{F}_p) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu)\,.$$

Here t has bidegree $(-2, 0)$ while the other generators are located on the vertical axis. (No confusion should arise from the double usage of t as a polynomial cohomology class and the vertical degree in this or other spectral sequences.)

Proposition 4.4. *There are differentials $d^2(\bar\xi_1) = t\lambda_1$, $d^2(\bar\xi_2) = t\lambda_2$, $d^2(\bar\tau_2) = t\mu$ and $d^{2p}(\bar\xi_1^p) = t^p\lambda_2$ in the spectral sequence (4.3).*

Proof. The d^2-differential

$$d^2_{0,t} \colon E^2_{0,t} \cong H_t(THH(\ell); \mathbb{F}_p)\{1\} \to E^2_{-2,t+1} \cong H_{t+1}(THH(\ell); \mathbb{F}_p)\{t\}$$

is adjoint to the S^1-action on $THH(\ell)$, hence restricts to σ on $H_t(\ell; \mathbb{F}_p)$. Thus $d^2(\bar\xi_i) = t\sigma\bar\xi_i = t\lambda_i$ for $i = 1$ and 2, and $d^2(\bar\tau_2) = t\sigma\bar\tau_2 = t\mu$.

Write $q = 2p-2$ and let $x \in C_q(T^0, T^1; \mathbb{F}_p)$ be a chain representing the differential $d^2(\bar\xi_1) = t\lambda_1$, i.e., x maps to a cycle representing $\bar\xi_1$ in $H_q(THH(\ell); \mathbb{F}_p) = E^2_{0,2p-2}$, and has boundary $dx \in C_{q-1}(T^1; \mathbb{F}_p)$ mapping to a cycle representing $t\lambda_1$ in $H_{q-1}(\Sigma^{-2}THH(\ell); \mathbb{F}_p) = E^2_{-2,2p-1}$. Let $\xi \colon D_pT^0 \to T^0$ be the pth H_∞ structure map for $T^0 = THH(\ell)^{hS^1}$, and form the chain $\xi_*(e_0 \otimes x^{\otimes p}) \in C_{pq}(T^0, T^p; \mathbb{F}_p)$. It maps to a cycle representing $\bar\xi_1^p$ in $H_{pq}(THH(\ell); \mathbb{F}_p) = E^2_{0,2p^2-2p}$, and has boundary $\xi_*(e_0 \otimes d(x^{\otimes p})) \in C_{pq-1}(T^p; \mathbb{F}_p)$. By a chain level calculation [Br, 3.4] in the extended pth power of the pair (D^{2p-2}, S^{2p-3}), this boundary is homologous to a unit multiple of $\xi_*(e_{p-1} \otimes (dx)^{\otimes p})$, representing $Q^{p-1}(t\lambda_1) = t^pQ^p(\lambda_1) = t^p\lambda_2$ in $H_{pq-1}(\Sigma^{-2p}THH(\ell); \mathbb{F}_p) = E^2_{-2p,2p^2-1}$. Hence there is a nontrivial differential $d^{2p}(\bar\xi_1^p) = t^p\lambda_2$ in the spectral sequence above. $\square$

4.5. The $V(1)$-homotopy spectral sequence. Applying $V(1)$-homotopy to the filtration $\{T^n\}_n$, in place of homology, we obtain a conditionally convergent $V(1)$-homotopy spectral sequence

$$E^2_{s,t}(S^1) = H^{-s}(S^1; V(1)_tTHH(\ell)) \Longrightarrow V(1)_{s+t}THH(\ell)^{hS^1} \tag{4.6}$$

with

$$E^2_{**}(S^1) = P(t) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu)\,.$$

Again t has bidegree $(-2, 0)$ while the other generators are located on the vertical axis.

Definition 4.7. Let

$$\alpha_1 \in \pi_{2p-3}(S^0),\ \beta_1' \in \pi_{2p^2-2p-1}V(0) \text{ and } v_2 \in \pi_{2p^2-2}V(1)$$

be the classes represented in their respective Adams spectral sequences by the cobar cycles $h_{10} = [\bar\xi_1]$, $h_{11} = [\bar\xi_1^p]$ and $[\bar\tau_2]$. So $j_1(v_2) = \beta_1'$ and $j_0(\beta_1') = \beta_1 \in \pi_{2p^2-2p-2}(S^0)$.

Consider the unit map $S^0 \to K(\ell_p) \to THH(\ell)^{hS^1}$, which is well defined after p-adic completion.

Proposition 4.8. *The classes $i_1 i_0(\alpha_1) \in \pi_{2p-3}V(1)$, $i_1(\beta'_1) \in \pi_{2p^2-2p-1}V(1)$ and $v_2 \in \pi_{2p^2-2}V(1)$ map under the unit map $V(1)_*S^0 \to V(1)_*THH(\ell)^{hS^1}$ to classes represented in $E^\infty(S^1)$ by $t\lambda_1$, $t^p\lambda_2$ and $t\mu$, respectively.*

Proof. Consider first the filtration subquotient $T^0/T^2 = F(S^3_+, THH(\ell))^{S^1}$. The unit map $V(1) \to V(1) \wedge (T^0/T^2)$ induces a map of Adams spectral sequences, taking the permanent cycles $[\bar\xi_1]$ and $[\bar\tau_2]$ in the source Adams spectral sequence to infinite cycles with the same cobar names in the target Adams spectral sequence. These are not boundaries in the cobar complex for the A_*-comodule $H_*(T^0/T^2; \mathbb{F}_p)$, because of the differentials $d^2(\bar\xi_1) = t\lambda_1$ and $d^2(\bar\tau_2) = t\mu$ that are present in the two-column spectral sequence converging to $H_*(T^0/T^2; \mathbb{F}_p)$. In detail, $H_{2p-2}(T^0/T^2; \mathbb{F}_p) = 0$ and $H_{2p^2-1}(T^0/T^2; \mathbb{F}_p)$ is spanned by the primitives λ_2 and $\lambda_1\bar\xi_1^p$.

Thus $[\bar\xi_1]$ and $[\bar\tau_2]$ are nonzero infinite cycles in the target Adams E_2-term. They have Adams filtration one, hence cannot be boundaries. Thus they are permanent cycles, and the classes $i_1 i_0(\alpha_1)$ and v_2 have nonzero images under the composite $V(1)_* \to V(1)_*(T^0) \to V(1)_*(T^0/T^2)$. Thus they are also detected in $V(1)_*(T^0)$, in filtration $s \geq -2$. For bidegree reasons the only possibility is that $i_1 i_0(\alpha_1)$ is detected in the $V(1)$-homotopy spectral sequence $E^\infty(S^1)$ as $t\lambda_1$, and v_2 is detected as $t\mu$.

Next consider the filtration subquotient $T^0/T^{p+1} = F(S^{2p+1}_+, THH(\ell))^{S^1}$. The unit map $V(1) \to V(1) \wedge (T^0/T^{p+1})$ again induces a map of Adams spectral sequences, taking the permanent cycle $[\bar\xi_1^p]$ in the source Adams spectral sequence to an infinite cycle with the same name. Again this is not a boundary in the cobar complex for the A_*-comodule $H_*(T^0/T^{p+1}; \mathbb{F}_p)$, because of the differential $d^{2p}(\bar\xi_1^p) = t^p\lambda_2$ that is present in the $(p+1)$ column spectral sequence converging to $H_*(T^0/T^{p+1}; \mathbb{F}_p)$.

Thus $[\bar\xi_1^p]$ is a nonzero infinite cycle in the target Adams E_2-term, of Adams filtration one. Hence the class $i_1(\beta'_1)$ has a nonzero image under the composite $V(1)_* \to V(1)_*(T^0) \to V(1)_*(T^0/T^{p+1})$. Thus it is also detected in $V(1)_*(T^0)$ in filtration $s \geq -2p$. Again for bidegree reasons the only possibility is that $i_1(\beta'_1)$ is detected in the $V(1)$-homotopy spectral sequence $E^\infty(S^1)$ as $t^p\lambda_2$. □

5. The homotopy limit property

5.1. Homotopy fixed point and Tate spectral sequences. For closed subgroups $G \subseteq S^1$ we will consider the (second quadrant) G homotopy fixed point spectral sequence

$$\begin{aligned} E^2_{s,t}(G) &= H^{-s}(G, V(1)_t THH(\ell)) \\ &\Longrightarrow V(1)_{s+t} THH(\ell)^{hG} . \end{aligned}$$

We also consider the (upper half plane) G Tate spectral sequence

$$\begin{aligned} \hat{E}^2_{s,t}(G) &= \hat{H}^{-s}(G, V(1)_t THH(\ell)) \\ &\Longrightarrow V(1)_{s+t} \hat{\mathbb{H}}(G, THH(\ell)) . \end{aligned}$$

When $G = S^1$ we have

$$E^2_{**}(S^1) = E(\lambda_1, \lambda_2) \otimes P(t, \mu)$$

since $H^*(S^1;\mathbb{F}_p) = P(t)$, and

$$\hat{E}^2_{**}(S^1) = E(\lambda_1,\lambda_2)\otimes P(t,t^{-1},\mu)$$

since $\hat{H}^*(S^1;\mathbb{F}_p) = P(t,t^{-1})$. When $G = C_{p^n}$ we have

$$E^2_{**}(C_{p^n}) = E(u_n,\lambda_1,\lambda_2)\otimes P(t,\mu)$$

since $H^*(C_{p^n};\mathbb{F}_p) = E(u_n)\otimes P(t)$, while

$$\hat{E}^2_{**}(C_{p^n}) = E(u_n,\lambda_1,\lambda_2)\otimes P(t,t^{-1},\mu)$$

since $\hat{H}^*(C_{p^n};\mathbb{F}_p) = E(u_n)\otimes P(t,t^{-1})$.

The homotopy restriction map R^h induces a map of spectral sequences

$$E^*(R^h)\colon E^*(G)\to \hat{E}^*(G)\,,$$

which on E^2-terms identifies $E^2(G)$ with the restriction of $\hat{E}^2(G)$ to the second quadrant.

The Frobenius and Verschiebung maps F and V are compatible under Γ_n and Γ_{n-1} with homotopy Frobenius and Verschiebung maps F^h and V^h that induce maps of homotopy fixed point spectral sequences

$$E^*(F^h)\colon E^*(C_{p^n})\to E^*(C_{p^{n-1}})$$

and

$$E^*(V^h)\colon E^*(C_{p^{n-1}})\to E^*(C_{p^n})\,.$$

Here $E^2(F^h)$ maps the even columns isomorphically and the odd columns trivially. On the other hand, $E^2(V^h)$ maps the odd columns isomorphically and the even columns trivially. This pattern persists to higher E^r-terms, until a differential of odd length appears in either spectral sequence. Thus the spectral sequences $E^*(C_{p^n})$ and $E^*(C_{p^{n-1}})$ are abstractly isomorphic up to and including the E^r-term, where r is the length of the first odd differential in either spectral sequence. The same remarks apply for the Tate spectral sequences.

5.2. Input for Tsalidis' theorem.

Definition 5.3. A map $A_*\to B_*$ of graded groups is k-coconnected if it is an isomorphism in all dimensions $> k$ and injective in dimension k.

Theorem 5.4. *The canonical map*

$$\hat{\Gamma}_1\colon THH(\ell)\to \hat{\mathbb{H}}(C_p, THH(\ell))$$

induces a $(2p-2)$-coconnected map on $V(1)$-homotopy.

Proof. Consider diagram (3.4) in the case $n=1$. The classes $i_1 i_0(\alpha_1)$, $i_1(\beta_1')$ and v_2 in $V(1)_*$ map through $V(1)_*K(\ell_p)$ and $\Gamma_1\circ tr_1$ to classes in $V(1)_*THH(\ell)^{hC_p}$ that are detected by $t\lambda_1$, $t^p\lambda_2$ and $t\mu$ in $E^\infty(C_p)$, respectively. Continuing by R^h to $V(1)_*\hat{\mathbb{H}}(C_p, THH(\ell))$ these classes factor through $V(1)_*THH(\ell)$, where they pass through zero groups. Hence the images of $t\lambda_1$, $t^p\lambda_2$ and $t\mu$ in $\hat{E}^\infty(C_p)$ must be zero, i.e., these infinite cycles in $\hat{E}^2(C_p)$ are boundaries. For dimension reasons the only possibilities are

$$d^{2p}(t^{1-p}) = t\lambda_1$$
$$d^{2p^2}(t^{p-p^2}) = t^p\lambda_2$$
$$d^{2p^2+1}(u_1 t^{-p^2}) = t\mu\,.$$

The classes $i_1 i_0(\lambda_1^K)$ and $i_1(\lambda_2^K)$ in $V(1)_*K(\ell_p)$ map by $\Gamma_1 \circ tr_1$ to classes in $V(1)_*THH(\ell)^{hC_p}$ that have Frobenius images λ_1 and λ_2 in $V(1)_*THH(\ell)$, and hence survive as permanent cycles in $E^\infty_{0,*}(C_p)$. Thus their images λ_1 and λ_2 in $\hat{E}^*(C_p)$ are infinite cycles.

Hence the various E^r-terms of the C_p Tate spectral sequence are:

$$\hat{E}^2(C_p) = E(u_1, \lambda_1, \lambda_2) \otimes P(t, t^{-1}, t\mu)$$
$$\hat{E}^{2p+1}(C_p) = E(u_1, \lambda_1, \lambda_2) \otimes P(t^p, t^{-p}, t\mu)$$
$$\hat{E}^{2p^2+1}(C_p) = E(u_1, \lambda_1, \lambda_2) \otimes P(t^{p^2}, t^{-p^2}, t\mu)$$
$$\hat{E}^{2p^2+2}(C_p) = E(\lambda_1, \lambda_2) \otimes P(t^{p^2}, t^{-p^2})\,.$$

For bidegree reasons there are no further differentials, so $\hat{E}^{2p^2+2}(C_p) = \hat{E}^\infty(C_p)$ and the classes λ_1, λ_2 and $t^{\pm p^2}$ are permanent cycles.

The map $\hat{\Gamma}_1 : THH(\ell) \to \hat{\mathbb{H}}(C_p, THH(\ell))$ induces on $V(1)$-homotopy the homomorphism

$$E(\lambda_1, \lambda_2) \otimes P(\mu) \to E(\lambda_1, \lambda_2) \otimes P(t^{p^2}, t^{-p^2})$$

that maps $\lambda_1 \mapsto \lambda_1$, $\lambda_2 \mapsto \lambda_2$ and $\mu \mapsto t^{-p^2}$. For the classes $i_1 i_0(\lambda_1^K)$ and $i_1(\lambda_2^K)$ in $V(1)_*K(\ell_p)$ map by tr to λ_1 and λ_2 in $V(1)_*THH(\ell)$, and by $R^h \circ \Gamma_1 \circ tr_1$ to the classes in $V(1)_*\hat{\mathbb{H}}(C_p, THH(\ell))$ represented by λ_1 and λ_2. The class μ in $V(1)_*THH(\ell)$ must have nonzero image in $V(1)_*\hat{\mathbb{H}}(C_p, THH(\ell))$, since its pth v_1-Bockstein $\beta_{1,p}(\mu) = \lambda_1$ has nonzero image there. Thus μ maps to the class represented by t^{-p^2}, up to a unit multiple which we ignore. So $V(1)_*\hat{\Gamma}_1$ is an isomorphism in dimensions greater than $|\lambda_1\lambda_2 t^{p^2}| = 2p-2$, and is injective in dimension $2p-2$. $\square$

5.5. The homotopy limit property.

Theorem 5.6. *The canonical maps*

$$\Gamma_n \colon THH(\ell)^{C_{p^n}} \to THH(\ell)^{hC_{p^n}}$$
$$\hat{\Gamma}_n \colon THH(\ell)^{C_{p^{n-1}}} \to \hat{\mathbb{H}}(C_{p^n}, THH(\ell))$$

and

$$\Gamma \colon TF(\ell; p) \to THH(\ell)^{hS^1}$$
$$\hat{\Gamma} \colon TF(\ell; p) \to \hat{\mathbb{H}}(S^1, THH(\ell))$$

all induce $(2p-2)$-coconnected maps on $V(1)$-homotopy.

Proof. The claims for Γ_n and $\hat{\Gamma}_n$ follows from 5.4 and a theorem of Tsalidis [Ts]. The claims for Γ and $\hat{\Gamma}$ follow by passage to homotopy limits, using the p-adic homotopy equivalence $THH(\ell)^{hS^1} \simeq \operatorname{holim}_{n,F} THH(\ell)^{hC_{p^n}}$ and its analogue for the Tate constructions. $\square$

6. Higher fixed points

Let $[k] = 1$ when k is odd, and $[k] = 2$ when k is even. Let $\lambda'_{[k]} = \lambda_{[k+1]}$, so that $\{\lambda_{[k]}, \lambda'_{[k]}\} = \{\lambda_1, \lambda_2\}$ for all k. We write $v_p(k)$ for the p-valuation of k, i.e., the exponent of the greatest power of p that divides k. By convention, $v_p(0) = +\infty$. Recall the integers $r(n)$ from 2.5.

Theorem 6.1. *In the C_{p^n} Tate spectral sequence $\hat{E}^*(C_{p^n})$ there are differentials*

$$d^{2r(k)}(t^{p^{k-1}-p^k}) = \lambda_{[k]} t^{p^{k-1}} (t\mu)^{r(k-2)}$$

for all $1 \le k \le 2n$, and

$$d^{2r(2n)+1}(u_n t^{-p^{2n}}) = (t\mu)^{r(2n-2)+1}\,.$$

The classes λ_1, λ_2 and $t\mu$ are infinite cycles.

We shall prove this by induction on n, the case $n = 1$ being settled in the previous section. Hence we assume the theorem holds for one $n \ge 1$ and we will establish its assertions for $n + 1$.

The terms of the Tate spectral sequence are

$$\begin{aligned}\hat{E}^{2r(m)+1}(C_{p^n}) = {} & E(u_n, \lambda_1, \lambda_2) \otimes P(t^{p^m}, t^{-p^m}, t\mu) \\ & \oplus \bigoplus_{k=3}^{m} E(u_n, \lambda'_{[k]}) \otimes P_{r(k-2)}(t\mu) \otimes \mathbb{F}_p\{\lambda_{[k]} t^i \mid v_p(i) = k-1\}\end{aligned}$$

for $1 \le m \le 2n$. Next

$$\begin{aligned}\hat{E}^{2r(2n)+2}(C_{p^n}) = {} & E(\lambda_1, \lambda_2) \otimes P_{r(2n-2)+1}(t\mu) \otimes P(t^{p^{2n}}, t^{-p^{2n}}) \\ & \oplus \bigoplus_{k=3}^{2n} E(u_n, \lambda'_{[k]}) \otimes P_{r(k-2)}(t\mu) \otimes \mathbb{F}_p\{\lambda_{[k]} t^i \mid v_p(i) = k-1\}\,.\end{aligned}$$

For bidegree reasons the remaining differentials are zero, so $\hat{E}^{2r(2n)+2}(C_{p^n}) = \hat{E}^\infty(C_{p^n})$, and the classes $t^{\pm p^{2n}}$ are permanent cycles.

Proposition 6.2. *The associated graded of $V(1)_* \hat{\mathbb{H}}(C_{p^n}, THH(\ell))$ is*

$$\begin{aligned}\hat{E}^\infty(C_{p^n}) = {} & E(\lambda_1, \lambda_2) \otimes P_{r(2n-2)+1}(t\mu) \otimes P(t^{p^{2n}}, t^{-p^{2n}}) \\ & \oplus \bigoplus_{k=3}^{2n} E(u_n, \lambda'_{[k]}) \otimes P_{r(k-2)}(t\mu) \otimes \mathbb{F}_p\{\lambda_{[k]} t^i \mid v_p(i) = k-1\}\,.\end{aligned}$$

Comparing $E^*(C_{p^n})$ with $\hat{E}^*(C_{p^n})$ via the homotopy restriction map R^h, we obtain the following:

Proposition 6.3. *In the C_{p^n} homotopy fixed point spectral sequence $E^*(C_{p^n})$ there are differentials*

$$d^{2r(k)}(t^{p^{k-1}}) = \lambda_{[k]} t^{p^k + p^{k-1}} (t\mu)^{r(k-2)}$$

for all $1 \le k \le 2n$*, and*

$$d^{2r(2n)+1}(u_n) = t^{p^{2n}}(t\mu)^{r(2n-2)+1}\,.$$

The classes λ_1*,* λ_2 *and* $t\mu$ *are infinite cycles.*

We also consider the μ-inverted spectral sequences $\mu^{-1}E^*(G)$ for G closed in S^1, obtained by inverting $t\mu$ in $\hat{E}^*(G)$ and restricting to the left half plane. The E^2-term $\mu^{-1}E^2(G)$ is obtained from $E^2(G)$ by inverting μ. At each term, the natural map $E^*(G) \to \mu^{-1}E^*(G)$ is an isomorphism in total degrees greater than $2p-2$, and an injection in total degree $2p-2$.

Proposition 6.4. *In the* μ*-inverted spectral sequence* $\mu^{-1}E^*(C_{p^n})$ *there are differentials*

$$d^{2r(k)}(\mu^{p^k-p^{k-1}}) = \lambda_{[k]}(t\mu)^{r(k)}\mu^{-p^{k-1}}$$

for all $1 \le k \le 2n$*, and*

$$d^{2r(2n)+1}(u_n\mu^{p^{2n}}) = (t\mu)^{r(2n)+1}\,.$$

The classes λ_1*,* λ_2 *and* $t\mu$ *are infinite cycles.*

The terms of the μ-inverted spectral sequence are

$$\begin{aligned}\mu^{-1}E^{2r(m)+1}(C_{p^n}) = {} & E(u_n,\lambda_1,\lambda_2)\otimes P(\mu^{p^m},\mu^{-p^m},t\mu)\\ & \oplus \bigoplus_{k=1}^{m} E(u_n,\lambda'_{[k]})\otimes P_{r(k)}(t\mu)\otimes \mathbb{F}_p\{\lambda_{[k]}\mu^j \mid v_p(j)=k-1\}\end{aligned}$$

for $1 \le m \le 2n$. Next

$$\begin{aligned}\mu^{-1}E^{2r(2n)+2}(C_{p^n}) = {} & E(\lambda_1,\lambda_2)\otimes P_{r(2n)+1}(t\mu)\otimes P(\mu^{p^{2n}},\mu^{-p^{2n}})\\ & \oplus \bigoplus_{k=1}^{2n} E(u_n,\lambda'_{[k]})\otimes P_{r(k)}(t\mu)\otimes \mathbb{F}_p\{\lambda_{[k]}\mu^j \mid v_p(j)=k-1\}\,.\end{aligned}$$

Again $\mu^{-1}E^{2r(2n)+2}(C_{p^n}) = \mu^{-1}E^\infty(C_{p^n})$ for bidegree reasons, and the classes $\mu^{\pm p^{2n}}$ are permanent cycles.

Proposition 6.5. *The associated graded* $E^\infty(C_{p^n})$ *of* $V(1)_*THH(\ell)^{hC_{p^n}}$ *maps by a* $(2p-2)$*-coconnected map to*

$$\begin{aligned}\mu^{-1}E^\infty(C_{p^n}) = {} & E(\lambda_1,\lambda_2)\otimes P_{r(2n)+1}(t\mu)\otimes P(\mu^{p^{2n}},\mu^{-p^{2n}})\\ & \oplus \bigoplus_{k=1}^{2n} E(u_n,\lambda'_{[k]})\otimes P_{r(k)}(t\mu)\otimes \mathbb{F}_p\{\lambda_{[k]}\mu^j \mid v_p(j)=k-1\}\,.\end{aligned}$$

Proof of Theorem 6.1. By our inductive hypothesis, the abutment $\mu^{-1}E^\infty(C_{p^n})$ contains summands

$$P_{r(2n-1)}(t\mu)\{\lambda_1\mu^{p^{2n-2}}\},\ P_{r(2n)}(t\mu)\{\lambda_2\mu^{p^{2n-1}}\} \text{ and } P_{r(2n)+1}(t\mu)\{\mu^{p^{2n}}\}$$

representing elements in $V(1)_* THH(\ell)^{C_{p^n}}$. By inspection there are no permanent cycles in the same total degree and of lower s-filtration in $\mu^{-1}E^\infty(C_{p^n})$ than $(t\mu)^{r(2n-1)} \cdot \lambda_1 \mu^{p^{2n-2}}$, $(t\mu)^{r(2n)} \cdot \lambda_2 \mu^{p^{2n-1}}$ and $(t\mu)^{r(2n)+1} \cdot \mu^{p^{2n}}$, respectively. So the three homotopy classes represented by $\lambda_1 \mu^{p^{2n-2}}$, $\lambda_2 \mu^{p^{2n-1}}$ and $\mu^{p^{2n}}$ are v_2-torsion classes of orders precisely $r(2n-1)$, $r(2n)$ and $r(2n)+1$, respectively.

Consider the commutative diagram

$$\begin{tikzcd}
THH(\ell)^{hC_{p^n}} \arrow[d, "F^n"] & THH(\ell)^{C_{p^n}} \arrow[l, "\Gamma_n"'] \arrow[r, "\hat\Gamma_{n+1}"] \arrow[d, "F^n"] & \hat{\mathbb{H}}(C_{p^{n+1}}, THH(\ell)) \arrow[d, "F^n"] \\
THH(\ell) & THH(\ell) \arrow[l, "\Gamma_0"', "="] \arrow[r, "\hat\Gamma_1"] & \hat{\mathbb{H}}(C_p, THH(\ell)) \,.
\end{tikzcd}$$

Here F^n is the n-fold Frobenius map forgetting C_{p^n}-invariance. The right hand diagram commutes because $\hat\Gamma_{n+1}$ is constructed as the C_{p^n}-invariant part of an S^1-equivariant model for $\hat\Gamma_1$.

The above three homotopy classes in $V(1)_* THH(\ell)^{C_{p^n}}$ map by the middle F^n to homotopy classes in $V(1)_* THH(\ell)$ with the same names, and by $\hat\Gamma_1$ to homotopy classes in $V(1)_* \hat{\mathbb{H}}(C_p, THH(\ell))$ represented by $\lambda_1 t^{-p^{2n}}$, $\lambda_2 t^{-p^{2n+1}}$ and $t^{-p^{2n+2}}$ in $\hat E^\infty(C_p)$, respectively. Hence they map by $\hat\Gamma_{n+1}$ to permanent cycles in $\hat E^*(C_{p^{n+1}})$ with these images under the right hand F^n.

By comparison over homotopy Frobenius and Verschiebung maps, there are isomorphisms $\hat E^r(C_{p^n}) \cong \hat E^r(C_{p^{n+1}})$ for all $r \le 2r(2n)+1$, taking u_n to u_{n+1}. This determines the d^r-differentials and E^r-terms of $\hat E^*(C_{p^{n+1}})$ up to and including the E^r-term with $r = 2r(2n)+1$:

$$\begin{aligned}
\hat E^{2r(2n)+1}(C_{p^{n+1}}) &= E(u_{n+1}, \lambda_1, \lambda_2) \otimes P(t^{p^{2n}}, t^{-p^{2n}}, t\mu) \\
&\oplus \bigoplus_{k=3}^{2n} E(u_{n+1}, \lambda'_{[k]}) \otimes P_{r(k-2)}(t\mu) \otimes \mathbb{F}_p\{\lambda_{[k]} t^i \mid v_p(i) = k-1\} \,.
\end{aligned}$$

By inspection there are no permanent cycles in the same total degree and of higher s-filtration in $\hat E^*(C_{p^{n+1}})$ than $\lambda_1 t^{-p^{2n}}$, $\lambda_2 t^{-p^{2n+1}}$ and $t^{-p^{2n+2}}$, respectively. So also the equivalence $\hat\Gamma_{n+1}\Gamma_n^{-1}$ takes the homotopy classes represented by $\lambda_1 \mu^{p^{2n-2}}$, $\lambda_2 \mu^{p^{2n-1}}$ and $\mu^{p^{2n}}$ to homotopy classes represented by $\lambda_1 t^{-p^{2n}}$, $\lambda_2 t^{-p^{2n+1}}$ and $t^{-p^{2n+2}}$, respectively.

Since $\hat\Gamma_{n+1}\Gamma_n^{-1}$ induces an isomorphism on $V(1)$-homotopy in dimensions $> (2p-2)$, it preserves the v_2-torsion order of these classes. Thus the infinite cycles

$$(t\mu)^{r(2n-1)} \cdot \lambda_1 t^{-p^{2n}}, \ (t\mu)^{r(2n)} \cdot \lambda_2 t^{-p^{2n+1}} \text{ and } (t\mu)^{r(2n)+1} \cdot t^{-p^{2n+2}}$$

are all boundaries in $\hat E^*(C_{p^{n+1}})$. These are all $t\mu$-periodic classes in $\hat E^r(C_{p^{n+1}})$ for $r = 2r(2n)+1$, hence cannot be hit by differentials on the $t\mu$-torsion classes in this E^r-term.

This leaves the $t\mu$-periodic part $E(u_{n+1}, \lambda_1, \lambda_2) \otimes P(t^{p^{2n}}, t^{-p^{2n}}, t\mu)$, where all the generators above the horizontal axis are infinite cycles. Hence the differentials hitting $(t\mu)^{r(2n-1)} \cdot \lambda_1 t^{-p^{2n}}$, $(t\mu)^{r(2n)} \cdot \lambda_2 t^{-p^{2n+1}}$ and $(t\mu)^{r(2n)+1} \cdot t^{-p^{2n+2}}$ must originate on the horizontal axis, and the only possibilities are

$$\begin{aligned} d^{2r(2n+1)}(t^{-p^{2n}-p^{2n+1}}) &= (t\mu)^{r(2n-1)} \cdot \lambda_1 t^{-p^{2n}} \\ d^{2r(2n+2)}(t^{-p^{2n+1}-p^{2n+2}}) &= (t\mu)^{r(2n)} \cdot \lambda_2 t^{-p^{2n+1}} \\ d^{2r(2n+2)+1}(u_{n+1} t^{-2p^{2n+2}}) &= (t\mu)^{r(2n)+1} \cdot t^{-p^{2n+2}} . \end{aligned}$$

The algebra structure on $\hat{E}^*(C_{p^{n+1}})$ allows us to rewrite these differentials as the remaining differentials asserted by case $n+1$ of 6.1. $\square$

Passing to the limit over the Frobenius maps, we obtain:

Theorem 6.6. *The associated graded of $V(1)_* \hat{\mathbb{H}}(S^1, THH(\ell))$ is*

$$\begin{aligned} \hat{E}^\infty(S^1) = E(\lambda_1, \lambda_2) &\otimes P(t\mu) \\ &\oplus \bigoplus_{k \geq 3} E(\lambda'_{[k]}) \otimes P_{r(k-2)}(t\mu) \otimes \mathbb{F}_p\{\lambda_{[k]} t^i \mid v_p(i) = k-1\} . \end{aligned}$$

Theorem 6.7. *The associated graded $E^\infty(S^1)$ of $V(1)_* THH(\ell)^{hS^1}$ maps by a $(2p-2)$-coconnected map to*

$$\begin{aligned} \mu^{-1} E^\infty(S^1) = E(\lambda_1, \lambda_2) &\otimes P(t\mu) \\ &\oplus \bigoplus_{k \geq 1} E(\lambda'_{[k]}) \otimes P_{r(k)}(t\mu) \otimes \mathbb{F}_p\{\lambda_{[k]} \mu^j \mid v_p(j) = k-1\} . \end{aligned}$$

Each of these E^∞ terms compute $V(1)_* TF(\ell; p)$ in dimensions $> (2p-2)$, by way of the $(2p-2)$-coconnected maps Γ and $\hat{\Gamma}$, respectively.

7. The restriction map

We now evaluate the homomorphism

$$R_* \colon V(1)_* TF(\ell; p) \to V(1)_* TF(\ell; p)$$

induced on $V(1)$-homotopy by the restriction map R, in dimensions $> (2p-2)$. The source and target are both identified with $V(1)_* THH(\ell)^{hS^1}$ via Γ_*. Then R_* is identified with the composite homomorphism $(\Gamma \hat{\Gamma}^{-1})_* \circ R^h_*$, where R^h is the homotopy restriction map. The latter induces a map of spectral sequences

$$E^*(R^h) \colon E^*(S^1) \to \hat{E}^*(S^1) ,$$

where the E^∞ terms are given in 6.6 and 6.7.

Proposition 7.1. *In dimensions $> (2p-2)$ the homomorphism $E^\infty(R^h)$ maps:*

(a) $E(\lambda_1, \lambda_2) \otimes P(t\mu)$ in $E^\infty(S^1)$ isomorphically to $E(\lambda_1, \lambda_2) \otimes P(t\mu)$ in $\hat{E}^\infty(S^1)$.

(b) $E(\lambda'_{[k]}) \otimes P_{r(k)}(t\mu) \otimes \mathbb{F}_p\{\lambda_{[k]} \mu^{-ep^{k-1}}\}$ in $E^\infty(S^1)$ onto $E(\lambda'_{[k]}) \otimes P_{r(k-2)}(t\mu) \otimes \mathbb{F}_p\{\lambda_{[k]} t^{ep^{k-1}}\}$ in $\hat{E}^\infty(S^1)$, for $k \geq 3$ and $0 < e < p$.

(c) the remaining terms in $E^\infty(S^1)$ to zero.

Proof. Case (a) is clear. For (b) and (c) note that $E^\infty(R^h)$ maps the term

$$E(\lambda'_{[k]}) \otimes P_{r(k)}(t\mu) \otimes \mathbb{F}_p\{\lambda_{[k]}\mu^{-ep^{k-1}}\}$$

in $E^\infty(S^1)$ to the term

$$E(\lambda'_{[k]}) \otimes P_{r(k-2)}(t\mu) \otimes \mathbb{F}_p\{\lambda_{[k]}t^{ep^{k-1}}\}$$

in $\hat{E}^\infty(S^1)$. Here e is prime to p. For $e > p$ the source and target are in negative dimensions, while for $e < 0$ the source and target are concentrated in disjoint dimensions. The cases $0 < e < p$ remain, when the map is a surjection since $r(k) - ep^{k-1} > r(k-2)$. □

This identifies the image of R^h_*, by the following lemma from [BM1, §2].

Lemma 7.2. *The representatives in $E^\infty(S^1)$ of the kernel of R^h_* equal the kernel of $E^\infty(R^h)$. Hence the image of R^h_* is isomorphic to the image of $E^\infty(R^h)$.*

The composite equivalence $\Gamma\hat{\Gamma}^{-1}$ does not induce a map of spectral sequences. Nonetheless it induces an isomorphism of $E(\lambda_1, \lambda_2) \otimes P(v_2)$-modules on $V(1)$-homotopy in dimensions $> (2p-2)$. Here v_2 acts by multiplication in $V(1)_*$, while multiplications by λ_1 and λ_2 are realized by the images of λ_1^K and λ_2^K, since both Γ and $\hat{\Gamma}$ are ring spectrum maps.

Proposition 7.3. *In dimensions $> (2p-2)$ the composite equivalence $\Gamma\hat{\Gamma}^{-1}$ induces an isomorphism*

$$V(1)_*\hat{\mathbb{H}}(S^1, THH(\ell)) \cong V(1)_*THH(\ell)^{hS^1}$$

of $P(v_2)$-modules, taking all classes represented by $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}(t\mu)^m t^i$ in $\hat{E}^\infty(S^1)$ to classes represented by $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}(t\mu)^m \mu^j$ in $E^\infty(S^1)$ with $i + p^2 j = 0$. Here $0 \le \epsilon_1, \epsilon_2 \le 1$ and $m \ge 0$.

Proof. Let $\hat{T} = \hat{\mathbb{H}}(C_p, THH(\ell))$. The S^1-equivariant map $\hat{\Gamma}_1 \colon THH(\ell) \to \hat{T}$ induces a map of S^1 homotopy fixed points, which realizes the localization homomorphism $E^*(S^1) \to \mu^{-1}E^*(S^1)$ on the level of spectral sequences. It follows that there is a homotopy equivalence $V(1) \wedge \hat{\mathbb{H}}(S^1, THH(\ell)) \simeq V(1) \wedge \hat{T}^{hS^1}$ which agrees with $\Gamma\hat{\Gamma}^{-1}$ on $(2p-2)$-connected covers.

The v_2-indivisible elements in $V(1)_*\hat{\mathbb{H}}(S^1, THH(\ell))$ are represented in $\hat{E}^\infty(S^1)$ by the classes

$$\begin{aligned}\hat{E}^\infty(S^1)/(t\mu) = {} & E(\lambda_1, \lambda_2) \\ & \oplus \bigoplus_{k \ge 3} E(\lambda'_{[k]}) \otimes \mathbb{F}_p\{\lambda_{[k]}t^i \mid v_p(i) = k-1\}.\end{aligned}$$

The v_2-indivisible elements in $V(1)_*\hat{T}^{hS^1}$ are represented in $\mu^{-1}E^\infty(S^1)$ by the classes

$$\begin{aligned}\mu^{-1}E^\infty(S^1)/(t\mu) = {} & E(\lambda_1, \lambda_2) \\ & \oplus \bigoplus_{k \ge 1} E(\lambda'_{[k]}) \otimes \mathbb{F}_p\{\lambda_{[k]}\mu^j \mid v_p(j) = k-1\}.\end{aligned}$$

The asserted homotopy equivalence induces an isomorphism between these two terms, which by a dimension count must be given by

$$\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}t^i \longmapsto \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^j$$

with $i + p^2 j = 0$. Hence the same formulas hold modulo multiples of v_2 on $V(1)$-homotopy. Taking the $P(v_2)$-module structure into account, the corresponding formulas including factors $(t\mu)^m$ also hold, and express the isomorphism

$$V(1)_*\hat{\mathbb{H}}(S^1, THH(\ell)) \cong V(1)_*\hat{T}^{hS^1}$$

which agrees with $\Gamma\hat{\Gamma}^{-1}$ in dimensions $> (2p-2)$. □

Definition 7.4. Let $A = E(\lambda_1, \lambda_2) \otimes P(t\mu)$,

$$B_k = E(\lambda'_{[k]}) \otimes P_{r(k)}(t\mu) \otimes \mathbb{F}_p\{\lambda_{[k]}\mu^{-ep^{k-1}} \mid 0 < e < p\}$$

and $B = \bigoplus_{k\geq 1} B_k$. Let C be the span of the remaining monomial terms in $\mu^{-1}E^\infty(S^1)$. Then $E^\infty(S^1) = A \oplus B \oplus C$ in dimensions $> (2p-2)$.

Theorem 7.5. *In dimensions $> (2p-2)$ there are subgroups $\widetilde{A} = E(\lambda_1, \lambda_2) \otimes P(v_2)$, $\widetilde{B}_k$ and $\widetilde{C}$ of $V(1)_*TF(\ell;p)$ represented by A, B_k and C in $E^\infty(S^1)$, respectively, such that*

(a) R_* is the identity on $\widetilde{A}$.
(b) R_* maps $\widetilde{B}_{k+2}$ onto $\widetilde{B}_k$ for all $k \geq 1$.
(c) R_* is zero on $\widetilde{B}_1$, $\widetilde{B}_2$ and $\widetilde{C}$.

In these dimensions $V(1)_*TF(\ell;p) = \widetilde{A} \oplus \widetilde{B} \oplus \widetilde{C}$, with $\widetilde{B} = \prod_{k\geq 1}\widetilde{B}_k$.

Proof. At the level of $E^\infty(S^1)$, the composite map $\Gamma\hat{\Gamma}^{-1} \circ E^\infty(R^h)$ is the identity on A, maps B_{k+2} onto B_k for all $k \geq 1$ and is zero on B_1, B_2 and C, by 7.1 and 7.3. The task is to find lifts of these groups to $V(1)_*TF(\ell;p)$ such that R_* has similar properties.

Let $\widetilde{A} = E(\lambda_1, \lambda_2) \otimes P(v_2) \subset V(1)_*TF(\ell;p)$ be the subalgebra generated by the images of the classes λ_1^K, λ_2^K and v_2 in $V(1)_*K(\ell_p)$. Then $\widetilde{A}$ lifts A and consists of classes in the image from $V(1)_*K(\ell_p)$. Hence R_* is the identity on $\widetilde{A}$.

By 7.1 we have $C \subset \ker E^\infty(R^h)$. Thus by 7.2 there is a subgroup $\widetilde{C}$ in $\ker(R_*) \cong \ker(R_*^h)$ represented by C. Then R_* is zero on $\widetilde{C}$.

Note that $\mathrm{im}(R_*)$ and $\ker(R_*)$ span $V(1)_*TF(\ell;p)$. For by 7.1 the representatives of $\mathrm{im}(R_*)$ span $A \oplus B$, and the representatives of the subgroup $\widetilde{C}$ in $\ker(R_*)$ span C. Thus the classes in $\mathrm{im}(R_*)$ and $\ker(R_*)$ have representatives spanning $E^\infty(S^1)$, and therefore span all of $V(1)_*TF(\ell;p)$. Hence the image of R_* on $V(1)_*TF(\ell;p)$ equals the image of its restriction to $\mathrm{im}(R_*)$.

Consider the subgroup

$$\begin{aligned} B_k^0 &= B_k \cap \ker E^\infty(R^h) \\ &= E(\lambda'_{[k]}) \otimes \bigoplus_{0<e<p} P^{r(k)-1}_{r(k-2)+ep^{k-1}}(t\mu) \otimes \mathbb{F}_p\{\lambda_{[k]}\mu^{-ep^{k-1}}\}. \end{aligned}$$

This can be lifted to $\mathrm{im}(R_*)$ by 7.1, and to $\ker(R_*)$ by 7.2. By an argument using the strong convergence of the spectral sequence $E^*(S^1)$ (exercise !), it can be simultaneously lifted to a subgroup of $\mathrm{im}(R_*) \cap \ker(R_*)$. Let $\widetilde{B}^0_k \subset \mathrm{im}(R_*) \cap \ker(R_*)$ be such a lift.

Inductively for $n \geq 1$ let $B^n_k \subset B_{k+2n} \subset E^\infty(S^1)$ be the subgroup generated by the monomials mapped by $E^\infty(R^h)$ and $\Gamma\hat{\Gamma}^{-1}$ to the monomials generating B^{n-1}_k. Then B_k is the span of all B^n_{k-2n} for $n \geq 0$.

Suppose inductively that we have chosen a lift $\widetilde{B}^n_k \subset \mathrm{im}(R_*)$ of B^n_k which maps by R_* to $\widetilde{B}^{n-1}_k$ for $n \geq 1$ and to zero for $n = 0$. Then choose classes in $\mathrm{im}(R_*)$ mapping by R_* to generators of $\widetilde{B}^n_k$ and let $\widetilde{B}^{n+1}_k$ be the subgroup they generate. Then $\widetilde{B}^{n+1}_k$ is a lift of B^{n+1}_k by 7.1 and 7.3.

Let $\widetilde{B}_k \subset V(1)_*TF(\ell;p)$ be the span of all $\widetilde{B}^n_{k-2n}$ for $n \geq 0$. Then $\widetilde{B}_k$ is represented by all of B_k, R_* maps $\widetilde{B}_{k+2}$ onto $\widetilde{B}_k$ for $k \geq 1$, and $\widetilde{B}_1$ and $\widetilde{B}_2$ lie in $\ker(R_*)$. $\square$

8. Topological cyclic homology

Recall from 3.1 the long exact sequence

$$\dots \xrightarrow{\partial} V(1)_*TC(\ell;p) \xrightarrow{\pi} V(1)_*TF(\ell;p) \xrightarrow{R_*-1} V(1)_*TF(\ell;p) \xrightarrow{\partial} \dots . \tag{8.1}$$

Proposition 8.2. *In dimensions $> (2p-2)$ there are isomorphisms*

$$\begin{aligned} \ker(R_*-1) \cong\ & E(\lambda_1,\lambda_2) \otimes P(v_2) \\ & \oplus E(\lambda_2) \otimes P(v_2) \otimes \mathbb{F}_p\{\lambda_1 t^e \mid 0 < e < p\} \\ & \oplus E(\lambda_1) \otimes P(v_2) \otimes \mathbb{F}_p\{\lambda_2 t^{ep} \mid 0 < e < p\} \end{aligned}$$

and

$$\mathrm{cok}(R_*-1) \cong E(\lambda_1,\lambda_2) \otimes P(v_2) .$$

Proof. By 7.5 the homomorphism R_*-1 is zero on $\widetilde{A} = E(\lambda_1,\lambda_2) \otimes P(v_2)$ and an isomorphism on $\widetilde{C}$. The remainder of $V(1)_*TF(\ell;p)$ decomposes as

$$\widetilde{B} = \prod_{k \text{ odd}} \widetilde{B}_k \oplus \prod_{k \text{ even}} \widetilde{B}_k$$

and R_* takes $\widetilde{B}_{k+2}$ to $\widetilde{B}_k$ for $k \geq 1$, forming two sequential limit systems. Hence there is an exact sequence

$$0 \to \lim_{k \text{ odd}} \widetilde{B}_k \to \prod_{k \text{ odd}} \widetilde{B}_k \xrightarrow{R_*-1} \prod_{k \text{ odd}} \widetilde{B}_k \to \lim^1_{k \text{ odd}} \widetilde{B}_k \to 0$$

and a corresponding one for k even. The right derived limit vanishes since each $\widetilde{B}_k$ has finite type. Hence it remains to prove that in dimensions $> (2p-2)$,

$$\lim_{k \text{ odd}} \widetilde{B}_k \cong E(\lambda_2) \otimes P(v_2) \otimes \mathbb{F}_p\{\lambda_1 t^e \mid 0 < e < p\}$$

and

$$\lim_{k \text{ even}} \widetilde{B}_k \cong E(\lambda_1) \otimes P(v_2) \otimes \mathbb{F}_p\{\lambda_2 t^{ep} \mid 0 < e < p\}\,.$$

Each $\widetilde{B}_k \cong B_k$ is a sum of $(2p-2)$ finite cyclic $P(v_2)$-modules. The restriction homomorphisms R_* respect this sum decomposition, and map each cyclic module surjectively onto the next. Hence their limit is a sum of $(2p-2)$ cyclic modules, and it remains to check that these are infinite cyclic, i.e., not bounded above.

For k odd the 'top' class $\lambda_1\lambda_2(t\mu)^{r(k)-1}\mu^{-ep^{k-1}}$ in B_k is in dimension $2p^{k+1}(p-e)$. For k even the corresponding class in B_k is in dimension $2p^{k+1}(p-e)+2p-2p^2$. In both cases the dimension grows to $+\infty$ for $0 < e < p$ as k grows.

For k odd each infinite cyclic $P(v_2)$-module contains a class in non-negative degree with nonzero image in $\widetilde{B}_1 \cong B_1$, namely the classes $\lambda_1 t^e$ and $\lambda_1\lambda_2 t^e$ for $0 < e < p$. Hence we take these as generators for $\lim_{k \text{ odd}} \widetilde{B}_k$. Likewise there are generators in non-negative degrees for $\lim_{k \text{ even}} \widetilde{B}_k$ with nonzero image in $\widetilde{B}_2 \cong B_2$, namely the classes $\lambda_2 t^{ep}$ and $\lambda_1\lambda_2 t^{ep}$ for $0 < e < p$. $\square$

Let $e \in \pi_{2p-1}TC(\mathbb{Z};p)$ be the image of $e^K \in K_{2p-1}(\mathbb{Z}_p)$, and let $\partial \in \pi_{-1}TC(\mathbb{Z};p)$ be the image of $1 \in \pi_0 TF(\mathbb{Z};p)$ under $\partial\colon \Sigma^{-1}TF(\mathbb{Z};p) \to TC(\mathbb{Z};p)$. We recall from [BM1], [BM2] the calculation of the mod p homotopy of $TC(\mathbb{Z};p)$.

Theorem 8.3 (Bökstedt–Madsen).

$$V(0)_*TC(\mathbb{Z};p) \cong E(e,\partial) \otimes P(v_1) \oplus P(v_1) \otimes \mathbb{F}_p\{et^i \mid 0 < i < p\}\,.$$

Hence

$$V(1)_*TC(\mathbb{Z};p) \cong E(e,\partial) \oplus \mathbb{F}_p\{et^i \mid 0 < i < p\}\,.$$

The $(2p-2)$-connected map $\ell_p \to H\mathbb{Z}_p$ induces a $(2p-1)$-connected map $K(\ell_p) \to K(\mathbb{Z}_p)$, and thus a $(2p-1)$-connected map $TC(\ell;p) \to TC(\mathbb{Z};p)$ after p-adic completion, by [Du]. This brings us to our main theorem.

Theorem 8.4. *There is an isomorphism of $E(\lambda_1,\lambda_2) \otimes P(v_2)$-modules*

$$\begin{aligned} V(1)_*TC(\ell;p) \cong E(\lambda_1,\lambda_2,\partial) &\otimes P(v_2) \\ &\oplus E(\lambda_2) \otimes P(v_2) \otimes \mathbb{F}_p\{\lambda_1 t^e \mid 0 < e < p\} \\ &\oplus E(\lambda_1) \otimes P(v_2) \otimes \mathbb{F}_p\{\lambda_2 t^{ep} \mid 0 < e < p\} \end{aligned}$$

with $|\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$, $|v_2| = 2p^2-2$, $|\partial| = -1$ *and* $|t| = -2$.

Proof. This follows in dimensions $> (2p-2)$ from 8.2 and the exact sequence (8.1). It follows in dimensions $\le (2p-2)$ from 8.3 and the $(2p-1)$-connected map $V(1)_*TC(\ell;p) \to V(1)_*TC(\mathbb{Z};p)$. It remains to check that the module structures are compatible for multiplications crossing dimension $(2p-2)$.

The classes $E(\lambda_1) \oplus \mathbb{F}_p\{\lambda_1 t^e \mid 0 < e < p\}$ in $V(1)_*TC(\ell;p)$ map to $E(e) \oplus \mathbb{F}_p\{et^i \mid 0 < i < p\}$ in $V(1)_*TC(\mathbb{Z};p)$, and map by $\Gamma \circ \pi$ to classes with the same names in the S^1 homotopy fixed point spectral sequence for $THH(\mathbb{Z})$. By naturality, the given classes in $V(1)_*TC(\ell;p)$ map by $\Gamma \circ \pi$ to classes with the same names in $E^\infty(S^1)$. Here these classes generate free $E(\lambda_2) \otimes P(t\mu)$-modules. For degree reasons multiplication by λ_1 is zero on each $\lambda_1 t^e$. Hence the $E(\lambda_1,\lambda_2) \otimes P(v_2)$-module structure on the given classes is as claimed.

Finally the class ∂ in $V(1)_{-1}TC(\ell;p)$ is the image under the connecting homomorphism ∂ of the class 1 in $V(1)_*TF(\ell;p)$, which generates the free $E(\lambda_1,\lambda_2)\otimes P(v_2)$-module $\mathrm{cok}(R_*-1)$ of 8.2. Hence also the module structure on ∂ and $\lambda_1\partial$ is as claimed. $\square$

A very important feature of this calculational result is that $V(1)_*TC(\ell;p)$ is a finitely generated free $P(v_2)$-module. Thus $TC(\ell;p)$ is a fp-spectrum of fp-type 2 in the sense of [MR]. Notice that $V(1)_*TF(\ell;p)$ is not a free $P(v_2)$-module. On the other hand we have the following calculation for the companion functor $TR(\ell;p)=\mathrm{holim}_{n,R}\,THH(\ell)^{C_{p^n}}$, showing that $V(1)_*TR(\ell;p)$ is a free but not finitely generated $P(v_2)$-module.

Theorem 8.5. *There is an isomorphism of $E(\lambda_1,\lambda_2)\otimes P(v_2)$-modules*

$$\begin{aligned} V(1)_*TR(\ell;p) \cong{} & E(\lambda_1,\lambda_2)\otimes P(v_2) \\ & \oplus \bigoplus_{n\geq 1} E(u,\lambda_2)\otimes P(v_2)\otimes \mathbb{F}_p\{\lambda_1 t^e \mid 0<e<p\} \\ & \oplus \bigoplus_{n\geq 1} E(u,\lambda_1)\otimes P(v_2)\otimes \mathbb{F}_p\{\lambda_2 t^{ep} \mid 0<e<p\}\,. \end{aligned}$$

The nth summand classes $u^\delta\lambda_1 t^e$ and $u^\delta\lambda_2 t^{ep}$ for $0\leq\delta\leq 1$ and $0<e<p$ are detected in $V(1)_*THH(\ell)^{C_{p^n}}$ by the classes representing $u_n^\delta\lambda_1 t^e$ and $u_n^\delta\lambda_2 t^{ep}$ in $E^\infty(C_{p^n})$, respectively.

We omit the proof.

9. Algebraic K-theory

We are now in a position to describe the $V(1)$-homotopy of the algebraic K-theory spectrum of the p-completed Adams summand of connective topological K-theory, i.e., $V(1)_*K(\ell_p)$. We use the cyclotomic trace map to largely identify it with the corresponding topological cyclic homology. Hence we will identify the algebraic K-theory classes λ_1^K and λ_2^K with their cyclotomic trace images λ_1 and λ_2, in this section.

Theorem 9.1. *There is an exact sequence of $E(\lambda_1,\lambda_2)\otimes P(v_2)$-modules*

$$0\to\Sigma^{2p-3}\mathbb{F}_p\to V(1)_*K(\ell_p)\xrightarrow{trc} V(1)_*TC(\ell;p)\to\Sigma^{-1}\mathbb{F}_p\to 0$$

taking the degree $2p-3$ generator in $\Sigma^{2p-3}H\mathbb{F}_p$ to a class $a\in V(1)_{2p-3}K(\ell_p)$, and taking the class ∂ in $V(1)_{-1}TC(\ell;p)$ to the degree -1 generator in $\Sigma^{-1}H\mathbb{F}_p$. Hence

$$\begin{aligned} V(1)_*K(\ell_p) \cong{} & E(\lambda_1,\lambda_2)\otimes P(v_2) \\ & \oplus P(v_2)\otimes\mathbb{F}_p\{\partial\lambda_1,\partial v_2,\partial\lambda_2,\partial\lambda_1\lambda_2\} \\ & \oplus E(\lambda_2)\otimes P(v_2)\otimes\mathbb{F}_p\{\lambda_1 t^e \mid 0<e<p\} \\ & \oplus E(\lambda_1)\otimes P(v_2)\otimes\mathbb{F}_p\{\lambda_2 t^{pe} \mid 0<e<p\} \\ & \oplus \mathbb{F}_p\{a\}\,. \end{aligned}$$

Proof. By [HM1] the map $\ell_p \to H\mathbb{Z}_p$ induces a map of horizontal cofiber sequences of p-complete spectra:

$$\begin{tikzcd} K(\ell_p)_p \arrow[r, "trc"] \arrow[d] & TC(\ell;p) \arrow[r] \arrow[d] & \Sigma^{-1}H\mathbb{Z}_p \arrow[d, equal] \\ K(\mathbb{Z}_p)_p \arrow[r, "trc"] & TC(\mathbb{Z};p) \arrow[r] & \Sigma^{-1}H\mathbb{Z}_p . \end{tikzcd}$$

Here $V(1)_*\Sigma^{-1}H\mathbb{Z}_p$ is $\mathbb{F}_p$ in degrees -1 and $2p-2$, and 0 otherwise. Clearly ∂ in $V(1)_*TC(\ell;p)$ maps to the generator in degree -1, since $K(\ell_p)_p$ is a connective spectrum. The connecting map in $V(1)$-homotopy for the lower cofiber sequence takes the generator in degree $(2p-2)$ to the nonzero class $i_1(\partial v_1)$ in $V(1)_{2p-3}K(\mathbb{Z}_p)$. By naturality it factors through $V(1)_{2p-3}K(\ell_p)$, where we let a be its image. □

Hence also $K(\ell_p)_p$ is an fp-spectrum of fp-type 2. By [MR, 3.2] its mod p spectrum cohomology is finitely presented as a module over the Steenrod algebra, hence is induced up from a finite module over a finite subalgebra of the Steenrod algebra. In particular, $K(\ell_p)$ is closely related to elliptic cohomology.

We conclude with some comments on the v_1-Bockstein spectral sequence leading from the $V(1)$-homotopy of $K(\ell_p)$ to its $V(0)$-homotopy, i.e., its mod p homotopy. For any X, classes in the image of $i_1\colon V(0)_*X \to V(1)_*X$ are called mod p classes, while classes in the image of $i_1 i_0\colon \pi_*X_p \to V(1)_*X$ are called integral classes.

Lemma 9.2. *The classes* 1, $\partial\lambda_1$, λ_1 *and* $\lambda_1 t^e$ *for* $0 < e < p$ *are integral classes both in* $V(1)_*K(\ell_p)$ *and* $V(1)_*TC(\ell;p)$. *Also* ∂ *is integral in* $V(1)_*TC(\ell;p)$, *while* a *is integral in* $V(1)_*K(\ell_p)$.

The classes $\partial\lambda_2$, λ_2, $\partial\lambda_1\lambda_2$, $\lambda_1\lambda_2$, $\lambda_1\lambda_2 t^e$, $\lambda_2 t^{ep}$ *and* $\lambda_1\lambda_2 t^{ep}$ *for* $0 < e < p$ *are mod* p *classes in both* $V(1)_*K(\ell_p)$ *and* $V(1)_*TC(\ell;p)$.

We are not excluding the possibility that some of the mod p classes are actually integral classes.

Proof. Each v_1-Bockstein $\beta_{1,r}$ lands in a trivial group when applied to the classes ∂, 1, a and $\lambda_1 t^e$ for $0 < e < p$ in $V(1)_*K(\ell_p)$ or $V(1)_*TC(\ell;p)$. Hence these are at least mod p classes.

Since 1 maps to an element of infinite order in $\pi_0 TC(\mathbb{Z};p) \cong \mathbb{Z}_p$ and the other classes sit in odd degrees, all mod p^r Bocksteins on these classes are zero. Hence they are integral classes. The class λ_1 is integral by construction, hence so is the product $\partial\lambda_1$.

The mod p homotopy operation $(P^{p-e})^*$ takes $\lambda_1 t^e$ in integral homotopy to $\lambda_2 t^{ep}$ in mod p homotopy, for $0 < e < p$. Hence these are all mod p classes, as is λ_2 by construction. The remaining classes listed are then products of established integral and mod p classes, and are therefore mod p classes. □

The classes listed in this lemma generate $V(1)_*K(\ell_p)$ and $V(1)_*TC(\ell;p)$ as $P(v_2)$-modules. But v_2 itself is not a mod p class.

Lemma 9.3. *Let* x *be a mod* p *(or integral) class of* $V(1)_*K(\ell_p)$ *or* $V(1)_*TC(\ell;p)$ *and let* $t \geq 0$. *Then*

$$\beta_{1,1}(v_2^t \cdot x) = t v_2^{t-1} i_1(\beta_1') \cdot x \,.$$

In particular $i_1(\beta'_1) \cdot 1 = t^p \lambda_2$ *and* $i_1(\beta'_1) \cdot \lambda_1 = t^p \lambda_1 \lambda_2$.

We expect that $i_1(\beta'_1) \cdot t^{p^2-p}\lambda_2 = \partial\lambda_2$ and $i_1(\beta'_1) \cdot t^{p^2-p}\lambda_1\lambda_2 = \partial\lambda_1\lambda_2$, by symmetry considerations.

Proof. The v_1-Bockstein $\beta_{1,1} = i_1 j_1$ acts as a derivation by [Ok]. By definition $j_1(v_2) = \beta'_1 = [h_{11}]$, which is detected as $t^p\lambda_2$ by 4.8. Clearly $j_1(x) = 0$ for mod p classes x. □

In $V(1)_*$ the powers v_2^t support nonzero differentials $\beta_{1,1}(v_2^t) = t v_2^{t-1} i_1(\beta'_1)$ for $p \nmid t$. The first nonzero differential on v_2^p is $\beta_{1,p}$:

Lemma 9.4. $\beta_{1,p}(v_2^p) = [h_{12}] \neq 0$ *in* $V(1)_*$.

We refer to [Ra2, §4.4] for background for the following calculation.

Proof. In the BP-based Adams–Novikov spectral sequence for $V(0)$ the relation $j_1(v_2^p) = v_1^{p-1}\beta'_{p/p}$ holds, where $\beta'_{p/p}$ is the class represented by $h_{12} + v_1^{p^2-p}h_{11}$ in degree 1 of the cobar complex. Its integral image $\beta_{p/p} = j_0(\beta'_{p/p})$ is represented by b_{11}, and supports the Toda differential $d_{2p-1}(\beta_{p/p}) = \alpha_1\beta_1^p$. This differential lifts to $d_{2p-1}(\beta'_{p/p}) = v_1\beta_1^p$ in the Adams–Novikov spectral sequence for $V(0)$. Consider the image of $\beta'_{p/p}$ under i_1 in the Adams–Novikov spectral sequence for $V(1)$, which is represented by h_{12} in the cobar complex. Then $d_{2p-1}(i_1(\beta'_{p/p})) = i_1(v_1\beta_1^p) = 0$. By sparseness and the vanishing line there are no further differentials, and $i_1(\beta'_{p/p}) = [h_{12}]$ represents a nonzero element of $V(1)_*$. Hence $\beta_{1,p}(v_2^p) = [h_{12}]$, as claimed. □

Remark 9.5. We would like to obtain the mod p homotopy groups $V(0)_*TC(\ell;p)$ by means of the v_1-Bockstein spectral sequence. This requires, first, that we compute the product with $\beta_{1,1}(v_2) = i_1(\beta'_1)$ in the remaining $V(1)$-homotopy groups. Next we must identify the image of $\beta_{1,p}(v_2^p) = [h_{12}]$ in $V(1)_*TC(\ell;p)$. Imaginably there is a homology differential in (4.3) on $[\bar\xi^{p^2}]$ hitting $(t\mu)^p\lambda_1 t^{p^2}$, if $(P^{p^2})^*(\lambda_2) = (t\mu)^p\lambda_1$. Then most likely $[h_{12}]$ is detected by $(t\mu)^p\lambda_1 t^{p^2}$ in $V(1)_*TF(\ell;p)$, which is identified under R with the class $(t\mu)^p\lambda_1\mu^{-1}$ generating $V(1)_*TC(\ell;p)$ in this degree. The general picture appears to be complicated.

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