TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THE PRIME FIELDS AND THE INTEGERS

JOHN ROGNES

1. Hochschild homology

Let k be a commutative ring. The category of k-modules is tensored over sets, meaning that for each k-module M and set T we can form

$$M \odot T = \bigoplus_{t \in T} M$$

so that

$$\mathrm{Mod}_k(M \odot T, N) \cong \{T \to \mathrm{Mod}_k(M, N)\}\,.$$

Likewise, the category of commutative k-algebras is tensored over sets, meaning that for each commutative k-algebra A and set T we can form

$$A \odot T = \bigotimes_{t \in T} A$$

so that

$$\mathrm{CAlg}_k(A \odot T, B) \cong \{T \to \mathrm{CAlg}_k(A, B)\}\,.$$

If $T_\bullet \colon [q] \mapsto T_q$ is a simplicial set, we can prolong the tensored structure, so that

$$M \odot T_\bullet \colon [q] \mapsto M \odot T_q$$

is a simplicial k-module, and

$$A \odot T_\bullet \colon [q] \mapsto A \odot T_q$$

is a simplicial commutative k-algebra. We can then form the associated normalized chain complexes, and consider their homology. In the k-module case we obtain $H_*^{cell}(|T_\bullet|; M)$. In the commutative k-algebra case we focus on the case $T_\bullet = S^1 = \Delta^1/\partial\Delta^1$.

Let Δ^1 be the simplicial 1-simplex, with q-simplices

$$\Delta^1_q = \{\alpha \colon [q] \to [1]\}$$

the order-preserving functions $\alpha \colon \{0, \dots, q\} \to \{0, 1\}$. There are $q+2$ of these. Let $0 \le s \le q+1$ correspond to α_s given by

$$\alpha_s(i) = \begin{cases} 0 & \text{for } 0 \le i < s, \\ 1 & \text{for } s \le i \le q. \end{cases}$$

Then

$$A \odot \Delta^1_q \cong A \odot \{0, 1, \dots, q, q+1\} \cong A \otimes A^{\otimes q} \otimes A\,.$$

The i-th face map $d_i \colon \Delta^1_q \to \Delta^1_{q-1}$ induces

$$d_i(x_0 \otimes x_1 \otimes \cdots \otimes x_q \otimes x_{q+1}) = x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{q+1}$$

for $0 \le i \le q$. The j-th degeneracy map $s_j \colon \Delta^1_q \to \Delta^1_{q+1}$ induces

$$s_j(x_0 \otimes x_1 \otimes \cdots \otimes x_q \otimes x_{q+1}) = x_0 \otimes \cdots \otimes x_j \otimes 1 \otimes x_{j+1} \otimes \cdots \otimes x_{q+1}\,.$$

Let $\partial\Delta^1$ be the simplicial subset with q-simplices

$$\partial\Delta^1 = \{\alpha_0, \alpha_{q+1}\}$$

the constant functions $\alpha \colon \{0, \dots q\} \to \{0, 1\}$. Then

$$A \odot \partial\Delta^1_q \cong A \odot \{0, q+1\} \cong A \otimes A\,.$$

Date: February 21st 2018.

The pushout square

$$\begin{array}{ccc} \partial\Delta^1 & \longrightarrow & \Delta^1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^1 \end{array}$$

of simplicial sets induces a pushout square

$$\begin{array}{ccc} A \otimes A & \longrightarrow & A \odot \Delta^1 \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \odot S^1 \end{array}$$

of commutative k-algebras. Hence

$$HH(A)_\bullet := A \odot S^1 \cong A \otimes_{A\otimes A} (A \odot \Delta^1)$$

is a simplicial commutative k-algebra under A, i.e., a simplicial commutative A-algebra, with

$$HH(A)_\bullet \colon [q] \mapsto A \otimes_{A\otimes A} (A \odot \Delta^1_q) \cong A \otimes A^{\otimes q} \,.$$

The i-th face map is given by

$$d_i(x_0 \otimes x_1 \otimes \cdots \otimes x_q) = \begin{cases} x_0x_1 \otimes x_2 \otimes \cdots \otimes x_q & \text{for } i = 0, \\ x_0 \otimes \cdots \otimes x_ix_{i+1} \otimes \cdots \otimes x_q & \text{for } 0 < i < q, \\ x_qx_0 \otimes x_1 \otimes \cdots \otimes x_{q-1} & \text{for } i = q. \end{cases}$$

The j-th degeneracy map is given by

$$s_j(x_0 \otimes x_1 \otimes \cdots \otimes x_q) = x_0 \otimes \cdots \otimes x_j \otimes 1 \otimes x_{j+1} \otimes \cdots \otimes x_q \,.$$

The associated chain complex is the Hochschild complex $(C(A), b)$ with

$$C(A)_q = HH(A)_q = A \otimes A^{\otimes q}$$

and $b = \sum_{i=0}^q (-1)^i d_i \colon HH(A)_q \to HH(A)_{q-1}$. Its homology groups

$$HH_*(A) = H_*(C(A), b)$$

are the Hochschild homology groups of A. This is a graded commutative A-algebra, with multiplication given by the shuffle product. If A is flat over k, then $A \odot \Delta^1$ is a flat $A \odot \partial\Delta^1 \cong A \otimes A$-module resolution of $A \odot * \cong A$, and

$$HH_*(A) = \mathrm{Tor}_*^{A\otimes A}(A, A) \,.$$

Example 1.1. Let $A = E(x) = k\{1, x\}$ be the exterior algebra on one generator, with $x^2 = 0$. The normalized Hochschild complex $(NC(A)_*, b)$ is given in degree q bx

$$NC(A)_q = A \otimes \bar{A}^{\otimes q} \cong A\{1 \otimes x^{\otimes q}\}$$

where $\bar{A} = \mathrm{cok}(k \to A) = k\{x\}$. Suppose that A is graded and x is in an odd degree. The face maps d_i for $0 < i < q$ are zero because $x^2 = 0$, and $d_0 + (-1)^q d_q = 0$ because the cyclic permutation of $x^{\otimes q}$ has sign $(-1)^{q-1}$. Hence $b = 0$ and

$$HH_q(E(x)) \cong E(x)\{1 \otimes x^{\otimes q}\}$$

for each $q \geq 0$. We write $\sigma x = 1 \otimes x$ for the $E(x)$-module generator of $HH_1(E(x))$, and more generally write

$$\gamma_q(\sigma x) = 1 \otimes x \otimes \cdots \otimes x \,,$$

with q copies of x, for the $E(x)$-module generator of $HH_q(E(x))$. The shuffle product satisfies $\gamma_i(\sigma x) \cdot \gamma_j(\sigma x) = (i, j)\gamma_{i+j}(\sigma x)$, so we have a divided power algebra:

$$HH_*(E(x)) \cong E(x) \otimes \Gamma(\sigma x) \,.$$

2. De Rham forms

In low dimensions,

$$\dots \xrightarrow{b} C(A)_2 = A \otimes A^{\otimes 2} \xrightarrow{b} C(A)_1 = A \otimes A \xrightarrow{b} C(A)_0 = A \to 0 \,.$$

with $b(x_0 \otimes x_1) = x_0x_1 - x_1x_0$, and $b(x_0 \otimes x_1 \otimes x_2) = x_0x_1 \otimes x_2 - x_0 \otimes x_1x_2 + x_2x_0 \otimes x_1$.

The Kähler differentials Ω_A of A over k is the A-module generated by symbols dx for $x \in A$, subject to the relations

$$d(xy) = xdy + ydx$$

for $x, y \in A$. In other words, it is the cokernel of

$$b\colon A \otimes A^{\otimes 2} \longrightarrow A \otimes A \,,$$

with $x_0 \otimes x_1$ mapping to x_0dx_1. Hence $HH_0(A) \cong A$ and $HH_1(A) \cong \Omega_A$ for commutative k-algebras A.

The Kähler q-forms of A is the exterior power

$$\Omega^q_A = \Omega_A \wedge_A \cdots \wedge_A \Omega_A = \bigwedge_A^q \Omega_A \,,$$

i.e., an A-module generated by symbols

$$dx_1 \wedge \cdots \wedge dx_q$$

for $x_1, \dots, x_q \in A$, subject to suitable relations.

Example 2.1. If $k = \mathbb{R}$ and $A = C^\infty(M)$ is the algebra of smooth functions on a differentiable n-manifold M, then Ω_A is the $C^\infty(M)$-module of differential 1-forms on M, locally of the form $\sum_{i=1}^n f_i dx_i$, and Ω^q_A is the $C^\infty(M)$-module of differential q-forms on M, locally of the form $\sum_I f_I dx_I$, where $I = (i_1, \dots, i_q)$ with $1 \le i_1 < \cdots < i_q \le n$.

The isomorphism $\Omega_A \cong HH_1(A)$ extends, using the graded commutative A-algebra structure on $HH_*(A)$, to an A-algebra homomorphism

$$\phi\colon \Omega^*_A \longrightarrow HH_*(A) \,.$$

Theorem 2.2 (Hochschild–Kostant–Rosenberg)**.** *If A is a smooth k-algebra, then ϕ is an isomorphism.*

Example 2.3. If $A = k[x]$, then $\Omega_A = A\{dx\}$ and

$$HH_*(k[x]) \cong k[x] \otimes E(dx)$$

where $E(dx) = k\{1, dx\}$ is the exterior algebra on dx. Here $dx \in \Omega_A$ corresponds to the class of $\sigma x = 1 \otimes x$ in $HH_1(A)$.

3. De Rham cohomology and the circle action

The differential $d\colon A \to \Omega_A$, mapping x to dx, extends to a k-linear derivation

$$d\colon \Omega^q_A \longrightarrow \Omega^{q+1}_A$$

with $d \circ d = 0$. In the case $A = C^\infty(M)$, this is the exterior (de Rham) differential. Connes defined an operator

$$B\colon C_q(A) \longrightarrow C_{q+1}(A)$$

that induces a derivation

$$B\colon HH_q(A) \longrightarrow HH_{q+1}(A) \,,$$

and the exterior differential d and Connes' B-operator are compatible under ϕ:

$$\begin{array}{ccc} \Omega^q_A & \xrightarrow{d} & \Omega^{q+1}_A \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ HH_q(A) & \xrightarrow{B} & HH_{q+1}(A) \end{array}$$

Hence there is a natural homomorphism

$$H_q^{dR}(A) = H_q(\Omega_A^*, d) \xrightarrow{\phi} H_q(HH_*(A), B)$$

from the de Rham homology of A to the homology of Hochschild homology with respect to the B-operator. (This is close to Connes' periodic and cyclic homology.)

The simplicial object $HH(A)_\bullet = A \odot S^1 \colon [q] \mapsto A \otimes A^{\otimes q}$ is a cyclic object, in the sense that the cyclic permutations

$$t_q \colon A \otimes A^{\otimes q} \longrightarrow A \otimes A^{\otimes q}$$

given by

$$t_q(x_0 \otimes x_1 \otimes \cdots \otimes x_q) = x_q \otimes x_0 \otimes \cdots \otimes x_{q-1}$$

satisfy $t_q^{q+1} = \mathrm{id}$, and are suitably compatible with the face and degeneracy maps.

For any cyclic set the topological realization of the underlying simplicial set has a natural circle action. In the case of

$$HH(A) := |HH(A)_\bullet| = |A \odot S^1|$$

the action

$$\rho \colon HH(A) \wedge S^1_+ \longrightarrow HH(A)$$

is base-point preserving. The S^1-orbit of a 0-simplex $x \in A = HH(A)_0$ is given by the 1-simplex

$$t_1 s_0(x) = 1 \otimes x \in HH(A)_1$$

which corresponds to a closed loop at x in $HH(A)$. The S^1-orbit of a q-simplex traces through $q+1$ different $q+1$-simplices, in a similar manner.

The topological realization $HH(A)$ of the simplicial abelian group underlying $HH(A)_\bullet$ has homotopy groups given by the homology of the (normalized) Hochschild complex $NC_*(A)$:

$$\pi_*(HH(A)) \cong HH_*(A)\,.$$

The circle action ρ induces a homomorphism

$$\sigma \colon \pi_q(HH(A)) \longrightarrow \pi_{q+1}(HH(A))\,,$$

i.e., an operator $\sigma \colon HH_q(A) \to HH_{q+1}(A)$. For example, if $x \in A = C(A)_0 \cong HH_0(A)$ we deduce that $\sigma x \in HH_1(A)$ is the class of $1 \otimes x$ in $C(A)_1$, corresponding to $dx \in \Omega_A$. In fact, σ is equal to Connes' B-operator.

4. Topological Hochschild homology

Let $(Sp, \wedge, S, \gamma, F)$ be one of the closed symmetric monoidal categories of structured ring spectra. Examples include symmetric spectra and orthogonal spectra. Let A be a strictly commutative ring spectrum, i.e., a commutative monoid in Sp, with multiplication $\mu \colon A \wedge A \to A$ and unit $\eta \colon S \to A$. We refer to A as a commutative S-algebra. The category CAlg_S is tensored over sets, by letting

$$A \odot T = \bigwedge_{t \in T} A$$

so that

$$\mathrm{CAlg}_S(A \odot T, B) \cong \{T \mapsto \mathrm{CAlg}_S(A, B)\}\,.$$

Let

$$THH(A)_\bullet := A \odot S^1 \colon [q] \mapsto A \odot S^1_q \cong A \wedge A \wedge \cdots \wedge A$$

be the simplicial commutative S-algebra obtained by prolongation, with $1+q$ copies of A on the right hand side. The topological realization

$$THH(A) := \coprod_{q \geq 0} THH(A)_q \wedge \Delta^q_+ /{\sim}$$

defines the topological Hochschild homology spectrum of A, which is a commutative S-algebra under $A = A \odot S^1_0$, i.e., a commutative A-algebra. The topological Hochschild homology groups of A are the (spectrum level) homotopy groups

$$THH_q(A) = \pi_q THH(A)\,.$$

Taken together, $\pi_* THH(A)$ is a graded commutative $\pi_*(A)$-algebra.

The simplicial object $THH(A)_\bullet$ is a cyclic object, with additional structure maps

$$t_q = \gamma \colon A^{\wedge q} \wedge A \longrightarrow A \wedge A^{\wedge q} \,.$$

Its geometric realization has a circle action

$$\rho \colon THH(A) \wedge S^1_+ \longrightarrow THH(A) \,.$$

Let $s \in \pi_1^S(S^1_+)$ be a generator with Hurewicz image the fundamental class $\sigma \in H_1(S^1_+)$, such that $s^2 = \eta s$. The group action induces a pairing

$$\rho_* \colon \pi_q THH(A) \otimes \pi_1^S(S^1_+) \longrightarrow \pi_{q+1} THH(A)$$

and we define the homotopical s-operator $s \colon THH_q(A) \to THH_{q+1}(A)$ by $sx = \rho_*(x \otimes s)$. Passing to homology, there is also a pairing

$$\rho_* \colon H_q THH(A) \otimes H_1(S^1_+) \longrightarrow H_{q+1} THH(A)$$

and a homological σ-operator $\sigma \colon H_q THH(A) \to H_{q+1} THH(A)$ given by $\sigma x = \rho_*(x \otimes \sigma)$.

5. Dyer–Lashof operations

The circle action has a right adjoint

$$\tilde{\rho} \colon THH(A) \longrightarrow F(S^1_+, THH(A)) \xleftarrow{\simeq} THH(A) \wedge DS^1_+ \,,$$

where $DS^1_+ = F(S^1_+, S)$ is the functional (Spanier–Whitehead) dual of S^1_+. These are maps of commutative S-algebras. Passing to homology, we obtain an algebra homomorphism

$$\tilde{\rho}_* \colon H_* THH(A) \longrightarrow H_* F(S^1_+, THH(A)) \cong H_* THH(A) \otimes H_* DS^1_+ \,.$$

Here $H_* DS^1_+ \cong \tilde{H}^{-*}(S^1_+) = E(\iota)$, with $\iota \in H^1(S^1_+)$ dual to $\sigma \in H_1(S^1_+)$, and

$$\tilde{\rho}_*(x) = x \otimes 1 + \sigma x \otimes \iota \,.$$

Proposition 5.1. $\sigma(xy) = x \cdot \sigma y + (-1)^{|y|} \sigma x \cdot y$.

Proof. From $\tilde{\rho}_*(xy) = \tilde{\rho}_*(x)\tilde{\rho}_*(y)$ and $\iota^2 = 0$ we obtain

$$\sigma(xy) \otimes \iota = x \cdot \sigma y \otimes \iota + (-1)^{|y|} \sigma x \cdot y \otimes \iota \,.$$

□

Let p be a prime and let $H_*(X) = H_*(X; \mathbb{Z}/p)$ denote mod p homology. For commutative S-algebras A there is a natural Bockstein homomorphism

$$\beta \colon H_q(A) \longrightarrow H_{q-1}(A) \,,$$

satisfying $\beta(xy) = \beta(x)y + (-1)^{|x|} x \beta(y)$. In particular, $\beta(x^p) = 0$. Furthermore, there are natural Dyer–Lashof operations

$$Q^k \colon H_q(A) \longrightarrow H_{q+(2p-2)k}(A)$$

satisfying the Cartan formula

$$Q^k(xy) = \sum_{i+j=k} Q^i(x) Q^j(y) \,.$$

These operations generalize the p-th power homomorphism, and in particular

$$Q^k(x) = x^p$$

if $|x| = 2k$.

Proposition 5.2 (Bökstedt/Angeltveit-R.). $Q^k(\sigma x) = \sigma Q^k(x)$.

Proof. By naturality, $Q^k(\tilde{\rho}_*(x)) = \tilde{\rho}_*(Q^k(x))$, so
$$Q^k(x \otimes 1 + \sigma x \otimes \iota) = Q^k(x) \otimes 1 + \sigma Q^k(x) \otimes \iota\,.$$
Here $Q^j(1) = 0$ and $Q^j(\iota) = 0$ for $j \neq 0$, and $Q^0(1) = 1$ and $Q^0(\iota) = \iota$, all in $H_*(DS^1_+) \cong \tilde{H}^{-*}(S^1_+) \cong H^{-*}(S^1)$. Hence
$$Q^k(\sigma x) \otimes \iota = \sigma Q^k(x) \otimes \iota\,.$$
□

6. The dual Steenrod algebra

Let p be an odd prime, and let $H = H\mathbb{Z}/p$ be the mod p Eilenberg–Mac Lane (ring) spectrum, with n-th space a $K(\mathbb{Z}/p, n)$. Write $H_*(X) = \pi_*(H \wedge X)$ for mod p homology.

For $n = 1$,
$$H^*(K(\mathbb{Z}/p, 1)) = E(x) \otimes P(y)$$
with $|x| = 1$, $|y| = 2$, $\beta(x) = y$, and
$$H_*(K(\mathbb{Z}/p, 1)) = \mathbb{Z}/p\{\alpha_n \mid n \geq 0\}$$
with α_n dual to $x^\epsilon y^m$ for $n = 2m+\epsilon$, $m \geq 0$, $\epsilon \in \{0, 1\}$. Here $\alpha_1 \cdot \alpha_{2m} = \alpha_{2m+1}$, and $\alpha_{2i} \cdot \alpha_{2j} = (i, j)\alpha_{2(i+j)}$, so
$$H_*(K(\mathbb{Z}/p, 1)) = E(\alpha_1) \otimes \Gamma(\alpha_2)\,.$$

For $n = 2$ we use the map $K(\mathbb{Z}, 2) \to K(\mathbb{Z}/p, 2)$, where
$$H^*(K(\mathbb{Z}, 2)) = P(y)$$
with $|y| = 2$, and
$$H_*(K(\mathbb{Z}, 2)) = \mathbb{Z}/p\{\beta_m \mid m \geq 0\}$$
with β_m dual to y^m. Here $\beta_i \cdot \beta_j = (i, j)\beta_{i+j}$, so
$$H_*(K(\mathbb{Z}, 2)) = \Gamma(\beta_1)\,.$$

Definition 6.1. For $i \geq 0$ let $\tau_i \in H_*(H)$, with $|\tau_i| = 2p^i - 1$, be the image of $\alpha_{2p^i} = \gamma_{p^i}(\alpha_2)$ under the colimit structure map
$$H_{*+1}(K(\mathbb{Z}/p, 1)) \longrightarrow H_*(H)\,.$$

For $i \geq 1$ let $\xi_i \in H_*(H)$, with $|\xi_i| = 2p^i - 2$, be the image of $\beta_{p^i} = \gamma_{p^i}(\beta_1)$ under the colimit structure map
$$H_{*+2}(K(\mathbb{Z}, 2)) \longrightarrow H_{*+2}(K(\mathbb{Z}/p, 2)) \longrightarrow H_*(H)\,.$$

Theorem 6.2 (Milnor)**.**
$$\mathscr{A}_* = H_*(H) \cong E(\tau_i \mid i \geq 0) \otimes P(\xi_i \mid i \geq 1)$$
The coproduct $\psi\colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$*, induced by* $1 \wedge \eta \wedge 1\colon H \wedge H \to H \wedge H \wedge H$*, is given by*
$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_{i+j=k} \xi_i^{p^j} \otimes \tau_j$$
and
$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j\,,$$
where $\xi_0 = 1$.

The involution $\chi\colon \mathscr{A}_* \to \mathscr{A}_*$, induced by the symmetry $\gamma\colon H \wedge H \to H \wedge H$, can be recursively calculated using
$$\phi(1 \otimes \chi)\psi = \eta\epsilon\,.$$
Let $\bar{\tau}_i = \chi(\tau_i)$ and $\bar{\xi}_i = \chi(\xi_i)$ be the conjugate generators of the dual Steenrod algebra.
$$\mathscr{A}_* \cong E(\bar{\tau}_i \mid i \geq 0) \otimes P(\bar{\xi}_i \mid i \geq 1)\,.$$
with
$$\psi(\bar{\tau}_k) = 1 \otimes \bar{\tau}_k + \sum_{i+j=k} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i}$$

and

$$\psi(\bar\xi_k) = \sum_{i+j=k} \bar\xi_i \otimes \bar\xi_j^{p^i} \,.$$

Proposition 6.3 (Kristensen/Steinberger)**.** *The action of the Bockstein and Dyer–Lashof operations on $H_*(H) = \mathscr{A}_*$ is known. In particular, $\beta(\bar\tau_k) = (\pm)\bar\xi_k$ and*

$$Q^{p^k}(\bar\tau_k) = \bar\tau_{k+1}$$

for each $k \ge 0$.

Let $H\mathbb{Z}$ be the integral Eilenberg–Mac Lane (ring) spectrum. The map $H\mathbb{Z} \to H$ induces an injection in homology:

Theorem 6.4.

$$H_*(H\mathbb{Z}) \cong E(\bar\tau_i \mid i \ge 1) \otimes P(\bar\xi_i \mid i \ge 1)\,.$$

7. Bökstedt's calculations

For each commutative ring R we write $THH(R)$ for $THH(A)$, where $A = HR$ is the associated Eilenberg–Mac Lane ring spectrum.

Theorem 7.1 (Bökstedt)**.** *(a)*

$$H_*THH(\mathbb{Z}/p) \cong \mathscr{A}_* \otimes P(\sigma\bar\tau_0)\,.$$

(b)

$$\pi_*THH(\mathbb{Z}/p) = P(\mu_0) = \mathbb{Z}/p[\mu_0]$$

with $|\mu_0| = 2$.

(c)

$$\pi_i THH(\mathbb{Z}/p) \cong \begin{cases} \mathbb{Z}/p & \text{for } i = 2n \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 7.2 (Bökstedt)**.** *(a)*

$$H_*THH(\mathbb{Z}) \cong H_*(H\mathbb{Z}) \otimes E(\sigma\bar\xi_1) \otimes P(\sigma\bar\tau_1)\,.$$

(b)

$$\pi_*(THH(\mathbb{Z}); \mathbb{Z}/p) = E(\lambda_1) \otimes P(\mu_1) = \Lambda(\lambda_1) \otimes \mathbb{Z}/p[\mu_1]$$

with $|\lambda_1| = 2p-1$ and $|\mu_1| = 2p$.

(c)

$$\pi_i THH(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}/n & \text{for } i = 2n-1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We apply homology to the skeleton filtration

$$A = THH(A)^{(0)} \subset \cdots \subset THH(A)^{(s-1)} \subset THH(A)^{(s)} \subset \cdots \subset THH(A)$$

with quotients

$$THH(A)^{(s)}/THH(A)^{(s-1)} \cong A \wedge \bar A^{\wedge s} \wedge \Delta^s/\partial\Delta^s\,.$$

Here $\bar A$ denotes the cofiber of $\eta\colon S \to A$. The resulting Bökstedt spectral sequence has

$$E^1_{s,*} = H_{s+*}(A \wedge \bar A^{\wedge s} \wedge \Delta^s/\partial\Delta^s) \cong H_*(A) \otimes H_*(\bar A)^{\otimes s} = NC(H_*(A))_s$$

and $d^1_s\colon E^1_{s,*} \to E^1_{s-1,*}$ agrees with the normalized Hochschild boundary b. Hence

$$E^2_{s,*} \cong HH_s(H_*(A)) \Longrightarrow H_{s+*}THH(A)\,.$$

There is a coproduct

$$\psi\colon THH(A) \longrightarrow THH(A) \wedge_A THH(A)$$

that makes the spectral sequence an $H_*(A)$-Hopf algebra spectral sequence, as long as a flatness hypothesis is satisfied, see Angeltveit–R.

Proof of Theorem 7.1. In the case $R = \mathbb{Z}/p$ we have $A = H$ and $H_*(A) = \mathscr{A}_*$. Hence

$$E^2_{s,*} \cong HH_s(\mathscr{A}_*) \Longrightarrow H_{s+*}THH(\mathbb{Z}/p)\,.$$

Let p be odd. From

$$\mathscr{A}_* \cong E(\bar\tau_k \mid k \ge 0) \otimes P(\bar\xi_k \mid k \ge 1)$$

and the Künneth theorem we obtain

$$\begin{aligned} HH_*(\mathscr{A}_*) &\cong \bigotimes_{k\ge0} HH_*(E(\bar\tau_k)) \otimes \bigotimes_{k\ge1} HH_*(P(\bar\xi_k)) \\ &\cong \bigotimes_{k\ge0} E(\bar\tau_k)\otimes\Gamma(\sigma\bar\tau_k) \otimes \bigotimes_{k\ge1} P(\bar\xi_k)\otimes E(\sigma\bar\xi_k) \\ &\cong \mathscr{A}_* \otimes \Gamma(\sigma\bar\tau_k \mid k\ge0) \otimes E(\sigma\bar\xi_k \mid k \ge 1)\,. \end{aligned}$$

Here $\mathscr{A}_* = E^2_{0,*} = HH_0(\mathscr{A}_*)$. The E^2-term is generated as a graded commutative $\mathscr{A}_*$-algebra by the divided powers

$$\gamma_{p^i}(\sigma\bar\tau_k)$$

for $i \ge 0$ and $k \ge 0$, in bidegree $p^i(1, 2p^k - 1)$, and the exterior classes

$$\sigma\bar\xi_k$$

for $k \ge 1$, in bidegree $(1, 2p^k - 2)$.

We claim that there are differentials

$$d^{p-1}(\gamma_p(\sigma\bar\tau_k)) \doteq \sigma\bar\xi_{k+1} \tag{1}$$

for each $k \ge 0$. Here $\doteq$ means equality up to multiplication by a unit in $\mathbb{Z}/p$.

Using the Hopf algebra structure, differential (1) propagates to differentials

$$d^{p-1}(\gamma_m(\sigma\bar\tau_k)) \doteq \gamma_{m-p}(\sigma\bar\tau_k)\sigma\bar\xi_{k+1}$$

for each $m \ge p$ and $k \ge 0$. By the Künneth theorem, this leaves

$$E^p_{*,*} \cong \mathscr{A}_* \otimes P_p(\sigma\bar\tau_k \mid k \ge 0)\,.$$

Here

$$P_p(\sigma x) = \mathbb{Z}/p[\sigma x]/(\sigma x)^p \cong \mathbb{Z}/p\{\gamma_m(\sigma x) \mid 0 \le m < p\} \subset \Gamma(\sigma x)$$

is the truncated polynomial algebra on σx of height p. The E^p-term is generated over $\mathscr{A}_*$ by classes in filtration $s = 1$, hence there is no room for any further differentials, and $E^p = E^\infty$.

Consider the composite

$$H_*(A) \longrightarrow H_*THH(A) \xrightarrow{\sigma} H_{*+1}THH(A)\,.$$

From $Q^{p^k}(\bar\tau_k) = \bar\tau_{k+1}$, and $|\sigma\bar\tau_k| = 2p^k$, we deduce

$$\sigma\bar\tau_{k+1} = \sigma Q^{p^k}(\bar\tau_k) = Q^{p^k}(\sigma\bar\tau_k) = (\sigma\bar\tau_k)^p$$

in $H_*THH(\mathbb{Z}/p)$. Hence $P_p(\sigma\bar\tau_k \mid k \ge 0)$ is the associated graded of $P(\sigma\bar\tau_0)$, and

$$H_*THH(\mathbb{Z}/p) \cong \mathscr{A}_* \otimes P(\sigma\bar\tau_0)$$

as $\mathscr{A}_*$-algebra.

It follows that $THH(\mathbb{Z}/p) \simeq \bigvee_{m\ge0} \Sigma^{2m} H$, there is a unique class

$$\mu_0 \in \pi_2 THH(\mathbb{Z}/p)$$

with Hurewicz image $\sigma\bar\tau_0$, and

$$\pi_*THH(\mathbb{Z}/p) \cong P(\mu_0) = \mathbb{Z}/p[\mu_0]\,.$$

It remains to establish differential (1). Here is a quick argument due to Ausoni. From $\beta(\bar\tau_{k+1}) = (\pm)\bar\xi_{k+1}$ we deduce

$$\sigma\bar\xi_{k+1} = (\pm)\beta(\sigma\bar\tau_{k+1}) = \beta((\sigma\bar\tau_k)^p) = 0$$

in $H_*THH(\mathbb{Z}/p)$. Hence $\sigma\bar\xi_{k+1} \in E^2_{1,2p^{k+1}-2}$ must be a boundary, i.e., the target of a differential. By induction on k, the only possible source of a differential is $\gamma_p(\sigma\bar\tau_k) \in E^2_{p,p(2p^k-1)}$. □

Proof of Theorem 7.2. In the case $R = \mathbb{Z}$ we have $A = H\mathbb{Z}$. Hence

$$E^2_{s,*} \cong HH_s(H_*(H\mathbb{Z})) \Longrightarrow H_{s+*}THH(\mathbb{Z})\,.$$

Let p be odd. From

$$H_*(H\mathbb{Z}) \cong E(\bar\tau_k \mid k \geq 1) \otimes P(\bar\xi_k \mid k \geq 1)$$

we obtain

$$\begin{aligned} HH_*(H_*(H\mathbb{Z})) &\cong \bigotimes_{k\geq 1} HH_*(E(\bar\tau_k)) \otimes \bigotimes_{k \geq 1} HH_*(P(\bar\xi_k)) \\ &\cong \bigotimes_{k\geq 1} E(\bar\tau_k) \otimes \Gamma(\sigma\bar\tau_k) \otimes \bigotimes_{k\geq 1} P(\bar\xi_k) \otimes E(\sigma\bar\xi_k) \\ &\cong H_*(H\mathbb{Z}) \otimes \Gamma(\sigma\bar\tau_k \mid k \geq 1) \otimes E(\sigma\bar\xi_k \mid k \geq 1)\,. \end{aligned}$$

The E^2-term is generated as a graded commutative $\mathscr{A}_*$-algebra by the divided powers

$$\gamma_{p^i}(\sigma\bar\tau_k)$$

for $i \geq 0$ and $k \geq 1$, in bidegree $p^i(1, 2p^k - 1)$, and the exterior classes

$$\sigma\bar\xi_k$$

for $k \geq 1$, in bidegree $(1, 2p^k - 2)$.

By naturality with respect to $H\mathbb{Z} \to H$ there are differentials

$$d^{p-1}(\gamma_p(\sigma\bar\tau_k)) \doteq \sigma\bar\xi_{k+1}$$

for each $k \geq 1$. Using the Hopf algebra structure, they propagate to differentials

$$d^{p-1}(\gamma_m(\sigma\bar\tau_k)) \doteq \gamma_{m-p}(\sigma\bar\tau_k)\sigma\bar\xi_{k+1}$$

for each $m \geq p$ and $k \geq 1$. By the Künneth theorem, this leaves

$$E^p_{*,*} \cong H_*(H\mathbb{Z}) \otimes P_p(\sigma\bar\tau_k \mid k \geq 1) \otimes E(\sigma\bar\xi_1)\,.$$

The E^p-term is generated over $H_*(H\mathbb{Z})$ by classes in filtration $s = 1$, hence there is no room for any further differentials, and $E^p = E^\infty$.

From $Q^{p^k}(\bar\tau_k) = \bar\tau_{k+1}$, or by naturality, we deduce

$$\sigma\bar\tau_{k+1} = (\sigma\bar\tau_k)^p$$

in $H_*THH(\mathbb{Z})$. Hence $P_p(\sigma\bar\tau_k \mid k \geq 1)$ is the associated graded of $P(\sigma\bar\tau_1)$. Furthermore, $\sigma\bar\xi_1$ is in odd total degree, so $(\sigma\bar\xi_1)^2 = 0$ in $H_*THH(\mathbb{Z})$. Hence

$$H_*THH(\mathbb{Z}) \cong H_*(\mathbb{Z}) \otimes P(\sigma\bar\tau_1) \otimes E(\sigma\bar\xi_1)$$

as $H_*(H\mathbb{Z})$-algebra.

It follows that $THH(\mathbb{Z}) \wedge S/p$ is a generalized Eilenberg–Mac Lane spectrum, there are unique classes

$$\lambda_1 \in \pi_{2p-1}(THH(\mathbb{Z}); \mathbb{Z}/p) \qquad \text{and} \qquad \mu_1 \in \pi_{2p}(THH(\mathbb{Z}); \mathbb{Z}/p)$$

with Hurewicz images $\sigma\bar\xi_1 \wedge 1$ and $\sigma\bar\tau_1 \wedge 1 + \sigma\bar\xi_1 \wedge \epsilon_0$, respectively. Here $H_*(S/p) = E(\epsilon_0)$. Hence

$$\pi_*(THH(\mathbb{Z}); \mathbb{Z}/p) \cong E(\lambda_1) \otimes P(\mu_1) = \Lambda(\lambda_1) \otimes \mathbb{Z}/p[\mu_1]\,.$$

The integral structure of $\pi_*THH(\mathbb{Z})$ follows by Bockstein spectral sequence arguments. □

References

[1] Vigleik Angeltveit and John Rognes, *Hopf algebra structure on topological Hochschild homology*, Algebr. Geom. Topol. **5** (2005), 1223–1290. MR2171809

[2] Christian Ausoni, *Topological Hochschild homology of connective complex K-theory*, Amer. J. Math. **127** (2005), no. 6, 1261–1313. MR2183525

[3] Lars Hesselholt and Ib Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology **36** (1997), no. 1, 29–101. MR1410465