THH AND TAQ

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I. Symmetric ring spectra (September 19th 2006)

We follow Hovey, Shipley and Smith (Symmetric spectra, JAMS, 1999).

I.1. Symmetric sequences.

Let Σ be the skeleton category of finite sets and bijections, with $ob\Sigma = \mathbb{N}_0$ (the non-negative integers), $\Sigma(n,n) = \Sigma_n$ (the symmetric group on n letters) for $n \geq 0$, and $\Sigma(m,n) = \emptyset$ (the empty set) for $m \neq n$.

Disjoint union of finite sets defines a functor $+\colon \Sigma \times \Sigma \to \Sigma$ that takes (m,n) to $m+n$ and maps $\Sigma(m,m) \times \Sigma(n,n) = \Sigma_m \times \Sigma_n$ to $\Sigma(m+n, m+n) = \Sigma_{m+n}$ by the standard inclusion $\Sigma_m \times \Sigma_n \to \Sigma_{m+n}$. It makes $(\Sigma, +, 0)$ a permutative (= strict symmetric monoidal) category.

A *symmetric sequence* of pointed simplicial sets, is a functor $X\colon \Sigma \to \mathcal{S}_*$, where $\mathcal{S}_*$ is the category of pointed simplicial sets. Equivalently, it is a sequence of pointed simplicial sets X_n, with a pointed left Σ_n-action on X_n, for each $n \geq 0$. Here $X_n = X(n)$ and $\pi \in \Sigma_n$ acts on X_n by $X(\pi)\colon X(n) \to X(n)$. We refer to X_n as the space at the n-th level of X.

A map $X \to Y$ of symmetric sequences is the same as a natural transformation of functors from Σ, or equivalently, a sequence of Σ_n-equivariant maps $X_n \to Y_n$, for $n \geq 0$. We write $\mathcal{S}_*^\Sigma$ for the category of symmetric sequences. Similarly, we can consider symmetric sequences of pointed topological spaces, or in any other category.

I.2. The tensor product.

Given two symmetric sequences X and Y, we can form their external smash product $X \bar{\otimes} Y$, which is the functor $\Sigma \times \Sigma \to \mathcal{S}_*$ that takes (p,q) to the smash product $X_p \wedge Y_q$ of pointed simplicial sets, with the obvious $\Sigma_p \times \Sigma_q$-action.

The left Kan extension of this functor along $+\colon \Sigma \times \Sigma \to \Sigma$ is, by definition, the *tensor product* $X \otimes Y$ of the two symmetric sequences.

$$\begin{tikzcd} \Sigma \times \Sigma \arrow[r, "X \bar{\otimes} Y"] \arrow[d, "+"'] & \mathcal{S}_* \\ \Sigma \arrow[ur, "X \otimes Y"'] & \end{tikzcd}$$

Its value at $n \geq 0$ is given by the following colimit formed in $\mathcal{S}_*$, over the left fiber

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX category $+/n$:

$$(X \otimes Y)_n = \operatorname*{colim}_{p+q\to n} X_p \wedge Y_q$$
$$= \bigvee_{p+q=n} \Sigma_{n+} \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \,.$$

We can present a k-simplex in $(X \otimes Y)_n$ as $(\alpha, x \wedge y)$, where $x \in X_p$ and $y \in Y_q$ are k-simplices, and $\alpha \in \Sigma_n$ is a morphism $p + q \to n$ in Σ. For $(\sigma, \tau) \in \Sigma_p \times \Sigma_q$ we have $(\alpha(\sigma \times \tau), x \wedge y) = (\alpha, \sigma(x) \wedge \tau(y))$.

The tensor product is coherently associative and unital, having the symmetric sequence I with $I_0 = S^0$ and $I_n = *$ (a point) for $n > 0$ as a unit. The role of the symmetric groups is to also make the tensor product commutative, up to the following coherent isomorphism.

The *twist isomorphism* $\tau \colon X \otimes Y \xrightarrow{\cong} Y \otimes X$ takes $(\alpha, x \wedge y)$ to $(\alpha\rho, y \wedge x)$, where $\rho \colon q + p \to p + q$ is the (q, p)-shuffle bijection.

$$\begin{tikzcd} x \wedge y & p+q \arrow[r, "\alpha"] & n \\ y \wedge x & q+p \arrow[u, "\rho"] \arrow[ur, "\alpha\rho"'] & \end{tikzcd}$$

It makes $(\mathcal{S}^{\Sigma}_*, \otimes, I)$ a symmetric monoidal category.

I.3. The sphere symmetric sequence.

Let $S^1 = \Delta^1/\partial\Delta^1$ be the simplicial circle, and let $S^n = S^1 \wedge \cdots \wedge S^1$ (n times) be the simplicial n-sphere, for $n \geq 0$. The symmetric group Σ_n acts on S^n by permuting the smash factors. This defines the *sphere* symmetric sequence S, with $S_n = S^n$.

There is a unit map $e \colon I \to S$ that is the identity at level 0.

We now define a multiplication map $m \colon S \otimes S \to S$ in $\mathcal{S}^{\Sigma}_*$. At the level of the n-th spaces, it is the Σ_n-equivariant map

$$\bigvee_{p+q=n} \Sigma_{n+} \wedge_{\Sigma_p \times \Sigma_q} S^p \wedge S^q \to S^n$$

whose restriction to the (p, q)-summand is left adjoint to the identity map $S^p \wedge S^q \to S^n$, viewed as a $(\Sigma_p \times \Sigma_q \to \Sigma_n)$-equivariant map.

Key Lemma. *(S, m, e) is a commutative monoid in $\mathcal{S}^{\Sigma}_*$.*

Commutativity is the assertion that $m = m\tau \colon S \otimes S \to S$.

((Also discuss function objects.))

I.4. Symmetric spectra.

By definition, a *symmetric spectrum* is a left S-module in the category of symmetric sequences. In other words, it is a symmetric sequence X together with a map $S \otimes X \to X$ that is associative and unital. We write $Sp^{\Sigma} = S$-Mod for the category of symmetric spectra.

Equivalently, a symmetric spectrum is a symmetric sequence X together with structure maps

$$\sigma \colon S^1 \wedge X_n \to X_{1+n}$$

for each $n \geq 0$, such that the m-fold composite map

$$\sigma^m \colon S^m \wedge X_n \to X_{m+n}$$

is $(\Sigma_m \times \Sigma_n \to \Sigma_{m+n})$-equivariant, for each $m, n \geq 0$. Here Σ_m acts on $S_m = S^m$, Σ_n acts on X_n, Σ_{m+n} acts on X_{m+n}, and $\Sigma_m \times \Sigma_n \to \Sigma_{m+n}$ is the standard inclusion.

To see this, note that the map $S \otimes X \to X$ is a sequence of Σ_n-equivariant maps

$$\bigvee_{p+q=n} \Sigma_{n+} \wedge_{\Sigma_p \wedge \Sigma_q} S^p \wedge X_q \to X_n$$

for $n \geq 0$, which is equivalent to a sequence of $(\Sigma_p \times \Sigma_q \to \Sigma_{p+q})$-equivariant maps $S^p \wedge X_q \to X_{p+q}$, for $p, q \geq 0$. The associativity condition satisfied by the module action ensures that it is enough to specify these maps for $p = 1$, i.e., to give the structure maps σ.

I.5. The smash product.

Recall that for a commutative ring R, the tensor product $\otimes$ of abelian groups gives rise to an internal tensor product $\otimes_R$ of left R-modules, given by the coequalizer diagram

$$M \otimes R \otimes N \rightrightarrows M \otimes N \longrightarrow M \otimes_R N\,.$$

Here M and N are left R-modules, but by commutativity M may also be regarded as a right R-module, and the two maps on the left are derived from the module actions $M \otimes R \to M$ and $R \otimes N \to N$, respectively. By commutativity, again, the left R-module action on M also induces one on the coequalizer $M \otimes_R N$.

Since S is a commutative monoid in symmetric sequences, the category $Sp^\Sigma = S$-Mod can be given an internal pairing $\otimes_S$ in the same way, which we prefer to denote by $\wedge$. So for two symmetric spectra X and Y, we define their *smash product* $X \wedge Y$ by the coequalizer diagram

$$X \otimes S \otimes Y \rightrightarrows X \otimes Y \longrightarrow X \wedge Y\,.$$

In other words, $X \wedge Y$ is the colimit in symmetric sequences of the left hand part of the diagram, and this limit can be formed level-wise as the colimit of the diagram

$$\bigvee\nolimits_{i+j+k=n} \Sigma_{n+} \wedge_{\Sigma_i \times \Sigma_j \times \Sigma_k} X_i \wedge S^j \wedge Y_k \rightrightarrows \bigvee\nolimits_{p+q=n} \Sigma_{n+} \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q$$

of pointed Σ_n-spaces. As in the algebraic case, the resulting symmetric sequence $X \wedge Y$ is naturally a symmetric spectrum.

The smash product of symmetric spectra is coherently associative and unital, having the sphere spectrum S as a unit. The twist isomorphism of symmetric sequences also induces a twist isomorphism $\tau \colon X \wedge Y \xrightarrow{\cong} Y \wedge X$ of symmetric spectra.

Proposition. *$(Sp^\Sigma, \wedge, S)$ is a symmetric monoidal category.*

((Also discuss function spectra.))

I.6. Algebraic structures.

A *symmetric ring spectrum* is a symmetric spectrum R with a unit map $e\colon S \to R$ and a multiplication map $m\colon R \wedge R \to R$, making the diagrams

$$\begin{tikzcd}
(R \wedge R) \wedge R \arrow[rr, "\cong"] \arrow[d, "m\wedge id"'] & & R \wedge (R \wedge R) \arrow[d, "id\wedge m"] \\
R \wedge R \arrow[r, "m"] & R & R \wedge R \arrow[l, "m"']
\end{tikzcd}$$

(associativity) and

$$\begin{tikzcd}
S \wedge R \arrow[r, "e\wedge id"] \arrow[dr, "\cong"'] & R \wedge R \arrow[d, "m"] & R \wedge S \arrow[l, "id\wedge e"'] \arrow[dl, "\cong"] \\
& R &
\end{tikzcd}$$

(unitality) commute. If furthermore the diagram

$$\begin{tikzcd}
R \wedge R \arrow[rr, "\tau", "\cong"'] \arrow[dr, "m"'] & & R \wedge R \arrow[dl, "m"] \\
& R &
\end{tikzcd}$$

(commutativity) commutes, then R is a *commutative symmetric ring spectrum.*

Let R be a symmetric ring spectrum. A left *R-module* is a symmetric spectrum M with a map $R\wedge M \to M$, such that the usual associativity and unitality diagrams commute. Similarly for right modules. We sometimes refer to R-modules as R-module spectra, for emphasis. Let R-Mod denote the category of R-modules and R-module maps.

If M is a right R-module and N is a left R-module, the smash product $M \wedge_R N$ is defined as the coequalizer

$$M \wedge R \wedge N \rightrightarrows M \wedge N \longrightarrow M \wedge_R N$$

in symmetric spectra.

If R is commutative (so that the distinction between left and right R-modules disappears), then $M \wedge_R N$ is also naturally an R-module. In this case, $(R\text{-Mod}, \wedge_R, R)$ is a symmetric monoidal category. An *R-algebra* A is a monoid in R-Mod. This amounts to a map $R \to A$ of symmetric ring spectra, such that R is central in A. A *commutative R-algebra* A is a commutative monoid in R-Mod. This amounts to a map $R \to A$ of commutative symmetric ring spectra.

I.7. Homotopy and homology.

There is a *stable model structure* on symmetric spectra, whose associated homotopy category $Ho(Sp^\Sigma)$ is equivalent to Boardman's stable homotopy category. Furthermore, the smash product of symmetric spectra induces a symmetric monoidal pairing on $Ho(Sp^\Sigma)$ that agrees with the smash product in Boardman's category, under the claimed equivalence.

The k-th homotopy group of a symmetric spectrum X is defined as the sequential colimit

$$\pi_k(X) = \operatorname*{colim}_n \pi_{n+k}(X_n)$$

over the composite maps

$$\pi_{n+k}(X_n) \xrightarrow{E} \pi_{1+n+k}(S^1 \wedge X_n) \xrightarrow{\sigma_*} \pi_{1+n+k}(X_{n+1})\,.$$

Taken together, these form a graded abelian group $\pi_*(X)$. The action by the symmetric groups plays no role here, so these are the homotopy groups of the underlying sequential spectrum of X. For a symmetric ring spectrum R, $\pi_*(R)$ is a graded ring. For a commutative symmetric ring spectrum R, $\pi_*(R)$ is a graded commutative ring.

A map of symmetric spectra $X \to Y$ that induces an isomorphism $\pi_*(X) \to \pi_*(Y)$ is called a π_*-equivalence. Each π_*-equivalence is a weak equivalence in the stable model structure on Sp^Σ, i.e., a stable equivalence, but the converse does **not** generally hold.

The stably fibrant symmetric spectra are the symmetric Ω-spectra, i.e., those X for which each X_n is a Kan (= fibrant) simplicial set, and each adjoint structure map $\tilde\sigma\colon X_n \to \Omega X_{n+1} = \underline{\mathcal{S}_*}(S^1, X_{n+1})$ is a weak equivalence. For maps $X \to Y$ between symmetric Ω-spectra, the stable equivalences are the same as the π_*-equivalences.

((Discuss orthogonal spectra. For symmetric spectra coming from orthogonal spectra, stable equivalences and π_*-equivalences agree. So for these, the homotopy groups are the homotopy invariant ones.))

I.8. The sphere spectrum.

The sphere spectrum S is a commutative symmetric ring spectrum, with the unit map $e\colon S \to S$ being the identity and the multiplication map $m\colon S \wedge S \to S$ being the natural isomorphism.

An S-algebra is the same as a symmetric ring spectrum, and a commutative S-algebra is the same as a commutative symmetric ring spectrum.

The homotopy groups $\pi_*(S) = \pi_*^S$ are the stable homotopy groups of spheres.

I.9 Eilenberg–Mac Lane spectra.

Let A be an abelian group. For each set U let $A[U] = \bigoplus_{u\in U} A$, and for each pointed set U let $A(U) = A[U]/A[*]$, where $*$ is the base point in U. This construction is natural in U, so that each pointed function $f\colon U \to V$ induces an abelian group homomorphism $A(f)\colon A(U) \to A(V)$. By forgetting structure, we can view $A(f)$ as a function of pointed sets.

For each simplicial set $W_\bullet$, let $A(W_\bullet)$ be the pointed simplicial set $[q] \mapsto A(W_q)$. This makes sense by the naturality property just stated.

For example, when $W_\bullet = S^1 = \Delta^1/\partial\Delta^1$, Δ^1_q consists of the $(q+2)$ morphisms $[q] \to [1]$ in Δ, so $W_q = S^1_q$ consists of the q surjective morphisms $[q] \to [1]$, together with the base point. Hence $A(S^1_q) = A \oplus \cdots \oplus A = A^q$, and $A(S^1) = BA$ is the *bar construction* on A. We usually write $[a_1|\dots|a_q]$ for an element of $A(S^1_q) = BA_q$.

Iterating n times, the *Eilenberg–Mac Lane spectrum* HA is defined as the symmetric spectrum with n-th space $HA_n = A(S^n) = B^nA$, where Σ_n permutes the smash factors in $S^n = S^1 \wedge \cdots \wedge S^1$, or equivalently, the ordering of the n bar constructions. The structure map $\sigma\colon S^1 \wedge HA_n \to HA_{1+n}$ is given in degree q by the inclusion $S^1_q \wedge A(S^n_q) \to A(S^1_q \wedge S^n_q)$.

The n-fold bar construction $HA_n = B^nA$ is a $K(A,n)$-complex, with homotopy groups $\pi_{n+k}B^nA = A$ for $k = 0$ and $= 0$ for $k \neq 0$. Being a simplicial (abelian) group, it is a Kan simplicial set. Furthermore the adjoint structure map $\tilde\sigma\colon HA_n \to \Omega HA_{1+n}$ equals the weak homotopy equivalence $B^nA \to \Omega B^{1+n}A$, so HA is a symmetric Ω-spectrum (= stably fibrant). Hence the homotopy groups $\pi_k HA = A$ for $k = 0$ and $= 0$ for $k \neq 0$ are homotopically significant.

Let R be an associative ring. There is a natural multiplication map $m\colon HR \wedge HR \to HR$ and unit map $e\colon S \to HR$ that make HR a symmetric ring spectrum, alias an S-algebra. The multiplication map is derived from a map $HR \otimes HR \to HR$ of symmetric sequences, which at the n-th level is a wedge sum over (p, q) with $p + q = n$ of Σ_n-equivariant maps. These are left adjoint to the $(\Sigma_p \times \Sigma_q \to \Sigma_n)$-equivariant maps

$$R(S^p) \wedge R(S^q) \to R(S^p \wedge S^q) = R(S^n)$$

given by

$$\sum_i r_i x_i \wedge \sum_j r'_j y_j \mapsto \sum_{i,j} r_i r'_j (x_i \wedge y_j)$$

where $r_i, r'_j \in R$, the x_i are simplices in S^p and the y_j are simplices in S^q. The product $r_i r'_j$ is formed in the ring R.

The opposite multiplication, $m\tau\colon HR \wedge HR \to HR$ is likewise derived from the map given by

$$\sum_i r_i x_i \wedge \sum_j r'_j y_j \mapsto \rho(\sum_{i,j} r'_j r_i (y_j \wedge x_i)) = \sum_{i,j} r'_j r_i (x_i \wedge y_j)\,.$$

Hence HR is a commutative symmetric ring spectrum if and only if R is a commutative ring.

The Eilenberg–Mac Lane functor $A \mapsto HA$ embeds abelian groups into symmetric spectra, rings into symmetric ring spectra, and more generally embeds algebra into topology in the form of stable homotopy theory.

Proposition. *Let R be a ring, M a right R-module and N a left R-module. Then HM is a left HR-module, HN is a left HR-module, and*

$$\pi_i(HM \wedge_{HR} HN) \cong \operatorname{Tor}_i^R(M, N)\,.$$

((Similarly, $\pi_i F_{HR}(HM, HN) \cong \operatorname{Ext}_R^{-i}(M, N)$ for two left R-modules M and N, where F_{HR} denotes the HR-module function spectrum.))

I.10 Thom spectra.

To any Euclidean $\mathbb{R}^n$-bundle $\xi\colon E \to X$, with principal $O(n)$-bundle $P \to X$, we can associate the Thom complex $Th(\xi) = P_+ \wedge_{O(n)} S^n$. Now $S^n = S^{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ as a pointed topological space, still with $S^n \cong S^1 \wedge \cdots \wedge S^1$.

((Universal case γ^n with principal bundle $EO(n) \to BO(n)$.))

((See also Chapter V.))

II. THH: structural properties (October 3rd 2006)

Recall the Hochschild–Kostant–Rosenberg theorem: for a smooth algebra A over a commutative ring k, the Hochschild homology $HH_*(A)$ is isomorphic to the exterior algebra Ω^*_A of differential forms on A, generated by the Kähler differentials $HH_1(A) \cong \Omega^1_A$. Connes' B-operator $HH_n(A) \to HH_{n+1}(A)$ then corresponds to the exterior derivation $d\colon \Omega^n_A \to \Omega^{n+1}_A$. For commutative, non-smooth A there is still a map $\Omega^*_A \to HH_*(A)$, and the Hochschild homology can even be defined for non-commutative A. It is therefore common to interpret $HH_*(A)$, with Connes' operator, as a kind of non-commutative de Rham complex over A.

We will study the extension of this de Rham theory to brave new rings, i.e., to associative S-algebras, where S is the sphere spectrum. To be concrete, let B be a symmetric ring spectrum. Its topological Hochschild homology, denoted $THH(B)$, can be defined in several equivalent ways. We begin with one that is rather explicit.

II.1. The Hochschild complex.

Definition. Let $THH(B)_\bullet$ be the simplicial symmetric spectrum

$$[q] \mapsto THH(B)_q = B \wedge B \wedge \cdots \wedge B$$

($(q+1)$ copies of B), with face maps

$$d_i(b_0 \wedge b_1 \wedge \cdots \wedge b_q) = \begin{cases} b_q b_0 \wedge b_1 \wedge \cdots \wedge b_{q-1} & \text{for } i = 0, \\ b_0 \wedge \cdots \wedge b_i b_{i+1} \wedge \cdots \wedge b_q & \text{for } 0 < i \leq q, \end{cases}$$

and degeneracy maps

$$s_j(b_0 \wedge b_1 \wedge \cdots \wedge b_q) = b_0 \wedge \cdots \wedge b_j \wedge 1 \wedge b_{j+1} \wedge \cdots \wedge b_q$$

for $0 \leq j \leq q$. Let

$$THH(B) = |THH(B)_\bullet|$$

be the geometric realization.

These formulas must be interpreted in terms of the product and unit maps $\mu\colon B \wedge B \to B$ and $\eta\colon S \to B$, and the cyclic twist map $\tau\colon B^{\wedge(q+1)} \to B^{\wedge(q+1)}$. Note that $1 \wedge b_0 \wedge \cdots \wedge b_q$ is not degenerate, unless some $b_j = 1$.

The explicit Hochschild complex can also be interpreted as the homology of B as a B-bimodule (= left $B \wedge B^{op}$-module):

$$THH(B) = \mathrm{Tor}^{B \wedge B^{op}}(B, B) := B \wedge_{B \wedge B^{op}} B\,.$$

If B is resolved as a B-bimodule by the two-sided bar complex

$$\beta(B, B, B)_\bullet\colon [q] \mapsto B \wedge B \wedge \cdots \wedge B \wedge B$$

($(q+2)$ copies of B), then $THH(B)_\bullet = B \wedge_{B \wedge B^{op}} \beta(B, B, B)_\bullet$.

II.2. Cyclic structure.

The simplicial construction admits the refined structure of being a cyclic object.

Definition. $THH(B)_\bullet$ is a cyclic symmetric spectrum, with cyclic twist operators

$$t_q(b_0 \wedge b_1 \wedge \cdots \wedge b_q) = b_q \wedge b_0 \wedge \cdots \wedge b_{q-1}\,.$$

Hence $THH(B)$ naturally admits a left S^1-action

$$\alpha\colon S^1_+ \wedge THH(B) \to THH(B)\,.$$

For example, the S^1-action takes a 0-simplex $b \in THH(B)_0$ once around the geometric realization of the 1-simplex $t_1 s_0(b) = 1 \wedge b$.

The inclusion of 0-simplices defines a map $\eta\colon B \to THH(B)$. When combined with the S^1-action, we obtain a map

$$\omega\colon S^1_+ \wedge B \to THH(B)\,.$$

The retraction $S^1_+ \to 1_+$ in the cofiber sequence

$$1_+ \to S^1_+ \to S^1$$

defines a preferred stable section $\sigma\colon S^1 \to S^1_+$. Combined with the S^1-action on $THH(B)$, it defines a stable map $\alpha(\sigma \wedge id)$ that we denote

$$d\colon \Sigma THH(B) \to THH(B)\,.$$

((The author usually writes σ in place of d for this map, called the suspension operator.)) In terms of this preferred splitting of $S^1_+ \wedge THH(B)$, we can write α as (id, d).

As observed by Hesselholt (1996, 1.4.4), the iterated map $dd\colon \Sigma^2 THH(B) \to THH(B)$ satisfies

$$dd = d\eta = \eta d\,,$$

where now $\eta \in \pi_1(S)$ denotes the stable Hopf map. For the composite stable map

$$S^1 \wedge S^1 \xrightarrow{\sigma\wedge\sigma} S^1_+ \wedge S^1_+ \cong (S^1 \times S^1)_+ \xrightarrow{m_+} S^1_+$$

factors as the stable map

$$S^1 \wedge S^1 \xrightarrow{\eta} S^1 \xrightarrow{\sigma} S^1_+\,,$$

where η arises (after one suspension) as the Hopf construction on the multiplication $m\colon S^1 \times S^1 \to S^1$. See [Ha61] for details. Thus dd is null-homotopic whenever η acts trivially on $THH(B)$, up to homotopy.

II.3. Tensored structure.

When B is commutative, there is a more concise definition

$$THH(B)_\bullet = B \otimes S^1_\bullet\,,$$

where $S^1_\bullet = \Delta^1/\partial\Delta^1$ is the simplicial circle, with $(q+1)$ simplices in degree q.

The tensor product $B \otimes X_\bullet$ for simplicial sets $X_\bullet$ is defined by prolongation of the smash power

$$B \otimes Y = \bigwedge_{y\in Y} B\,,$$

defined for (finite) sets Y. The two-sided bar construction can then be written as $\beta(B,B,B)_\bullet = B \otimes \Delta^1$, where Δ^1 has $(q+2)$ simplices in degree q.

The topological formula $THH(B) = B \otimes S^1$ also exhibits the S^1-action α, by rotation in the S^1-term. Since these constructions take place in the category of commutative symmetric ring spectra, the unit map

$$\eta\colon B = B \otimes 1 \to B \otimes S^1 = THH(B)$$

makes $THH(B)$ a commutative B-algebra. The collapse map $S^1 \to 1$ induces an augmentation

$$\epsilon\colon THH(B) = B \otimes S^1 \to B \otimes 1 = B\,.$$

The pinch map $S^1 \to S^1 \vee S^1$, the fold map $S^1 \vee S^1 \to S^1$ and the flip map $S^1 \to S^1$ induce maps

$$\begin{aligned}
&\psi\colon THH(B) \to THH(B) \wedge_B THH(B)\\
&\phi\colon THH(B) \wedge_B THH(B) \to THH(B)\\
&\chi\colon THH(B) \to THH(B)
\end{aligned}$$

that make $THH(B)$ a commutative Hopf algebra over B.

Note that ψ takes values in the smash product over B, not in the smash product over S. Furthermore, it is only A_∞ coassociative and counital. Similarly, χ is only a homotopy inverse.

II.4. E_n-algebras.

Intermediate between strictly associative ring spectra and strictly commutative ring spectra, we have the notion of an E_n ring spectrum, which comes with an action of an operad weakly equivalent to the little n-cubes operad. The E_1 ring spectra are the same as A_∞ ring spectra, which can be rigidified to strictly associative ring spectra, while the E_∞ ring spectra can be rigidified to strictly commutative ring spectra. By restriction of the operad action, an E_{n+1} ring spectrum is in particular an E_n ring spectrum.

Fiedorowicz and Vogt show that an E_{n+1} ring spectrum B can be rigidified to an E_n algebra in strictly associative ring spectra, i.e., to a strictly associative ring spectrum with an action by an E_n operad, in that category. This E_n operad action carries over for topological functors from strictly associative ring spectra to other categories. Basterra and Mandell have announced similar results.

For example, THH is a functor from symmetric ring spectra to symmetric spectra with S^1-action. Hence THH of any E_{n+1} ring spectrum B produces an E_n algebra in symmetric spectra with S^1-action. In other words, $THH(B)$ is then an E_n ring spectrum with S^1-action. Similarly, the natural map $\eta\colon B \to THH(B)$ is then an E_n ring spectrum map.

For $n = \infty$, this recovers the fact that THH of a commutative symmetric ring spectrum B is again a commutative B-algebra. However, there are natural examples of spectra that admit an E_n ring spectrum structure for some finite $n \geq 2$, that are not known to admit an E_∞ structure.

II.5. Many objects.

There is also a more flexible definition, in terms of the category of finite cell B-modules, enriched in symmetric spectra.

Definition. Let $\mathcal{C}$ be a small category enriched in symmetric spectra. Its topological Hochschild homology $THH(\mathcal{C})$ is the geometric realization of the cyclic symmetric spectrum

$$THH(\mathcal{C})_q = \bigvee_{c_0,\dots,c_q \in ob\mathcal{C}} \mathcal{C}(c_0, c_q) \wedge \mathcal{C}(c_1, c_0) \wedge \cdots \wedge \mathcal{C}(c_q, c_{q-1})\,,$$

with simplicial face and degeneracy operators, and cyclic operators, suggested by composition, identity and twist in the diagram

$$\begin{tikzcd} c_0 \arrow[rrr, bend right] & c_1 \arrow[l] & \cdots \arrow[l] & c_q \arrow[l] \end{tikzcd}$$

When $\mathcal{C}$ is the enriched category with only one object $*$, and $\mathcal{C}(*, *) = B$, then this definition simplifies to the original definition of $THH(B)$.

For another example, let $\mathcal{C} = \mathcal{F}_B$ be the category of finitely generated free B-modules, with objects $\bigvee^n B = B \wedge n_+$ for $n \geq 0$, and morphisms

$$\mathcal{F}_B(\bigvee^m B, \bigvee^n B) = \prod_m \bigvee^n B$$

as symmetric spectra. The previous example embeds into this one, taking $*$ to the free B-module of rank 1, and the induced map

$$THH(B) \to THH(\mathcal{F}_B)$$

is an equivalence. See Dundas et al, Lemma 2.5.17, for the proof when B is connective. We refer to this equivalence as Morita equivalence.

Presumably, letting $\mathcal{C} = \mathcal{C}_B$ be the category of finite cell B-modules and B-module maps, the inclusion $\mathcal{F}_B \to \mathcal{C}_B$ also induces an equivalence

$$THH(\mathcal{F}_B) \to THH(\mathcal{C}_B)\,.$$

((Find a reference.))

It may also be useful to work with $THH(\mathcal{C})$, where $\mathcal{C}$ is a category of coherent crystals over B.

II.6. Étale maps.

Let A be a commutative S-algebra, and let B be an A-algebra, commutative or not. We can define the relative topological Hochschild homology $THH^A(B)$ as the geometric realization of the simplicial spectrum

$$[q] \mapsto THH^A(B)_q = B \wedge_A B \wedge_A \cdots \wedge_A B\,.$$

There is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & THH(A) \arrow[r, "\epsilon"] \arrow[d] & A \arrow[d] \\ B \arrow[r] & THH(B) \arrow[r] & THH^A(B)\,. \end{tikzcd}$$

By assumption, A is central in B, so $THH(B)$ is a $THH(A)$-module. The induced map

$$THH(B) \wedge_{THH(A)} A \to THH^A(B)$$

is an equivalence, so the left hand side may also be taken as the definition of the relative topological Hochschild homology. ((Check this for non-commutative B!)) If B is commutative, this map is an equivalence of commutative B-algebras.

Definition. We say that $A \to B$ is formally thh-étale if the relative unit map $\eta\colon B \to THH^A(B)$ is an equivalence.

In that case, the space of associative A-algebra derivations of B with values in any symmetric bimodule M is contractible, and conversely. This uses the cofiber sequence

$$B \wedge_{B\wedge_A B^{op}} I_{B/A} \to B \to THH^A(B)$$

and the equivalences

$$\mathcal{A}\mathrm{Der}_A(B, M) \simeq \mathcal{M}_{B\wedge_A B^{op}}(I_{B/A}, M) \simeq \mathcal{M}_B(B \wedge_{B\wedge_A B^{op}} I_{B/A}, M)$$

for symmetric B-modules M. See Lazarev (2001) and Rognes (Galois, Proposition 9.2.5). Here $I_{B/A}$ is the homotopy fiber of $B \wedge_A B \to B$.

II.7. Quasi-coherence.

Lemma. *If the natural map*

$$B \wedge_A THH(A) \to THH(B)$$

is a weak equivalence, then $A \to B$ is formally thh-étale.

Conversely, if $A \to B$ is formally thh-étale and either (1) $\epsilon\colon THH(A) \to A$ is faithful, (2) A and B are connective, or (3) $A \to B$ is separable, then $B \wedge_A THH(A) \to THH(B)$ is a weak equivalence.

Proof. If the induced map

$$B \wedge_A THH(A) \to THH(B)$$

is an equivalence, then $A \to B$ is formally thh-étale, by base change along ϵ.

Conversely, if we assume that $\epsilon\colon THH(A) \to A$ is faithful, then $B\wedge_A THH(A) \to THH(B)$ is an equivalence for all formally étale $A \to B$, so that THH is a quasi-coherent étale sheaf over A.

Similarly, if A and B are connective, and $A \to B$ is formally étale, then $B \wedge_A THH(A) \to THH(B)$ is an equivalence. For its cofiber C is then connective and satisfies $C \wedge_{THH(A)} A \simeq *$, which implies $C \simeq *$ since $\pi_0 THH(A) \cong \pi_0 A$. These hypotheses are most relevant when $A = HR$ and $B = HT$ are both ordinary rings, with $R \to T$ (formally) étale, when they show that THH is a quasi-coherent étale sheaf over $A = HR$.

Along the same lines, there are equivalences

$$\begin{aligned} B \wedge_A THH(A) &= B \wedge_A A \wedge_{A\wedge A} A \simeq B \wedge_{A\wedge A} A \\ &\cong A \wedge_{A\wedge A} B \simeq (B \wedge_A B^{op}) \wedge_{B\wedge B^{op}} B = THH(B, B \wedge_A B)\,, \end{aligned}$$

which is equivalent to $THH(B)$ if $THH(B, I_{B/A}) \simeq *$. This condition holds if $A \to B$ is separable, so that $\mu\colon B \wedge_A B \to B$ admits a bimodule section σ, and the resulting idempotent $\delta = \sigma\mu$ in $\pi_0(B\wedge_A B^{op}) = B_0^A(B)$ lifts to $\pi_0(B\wedge B^{op}) = B_0(B)$. For then $B \simeq B[\delta^{-1}]$ as a $B \wedge B^{op}$-module, while $I_{B/A}[\delta^{-1}] \simeq *$. Compare Geller and Weibel (1991). $\square$

((In what generality is $B_0^A(B) \to B_0(B)$ surjective on idempotents?))

II.8. Galois descent (October 24th 2006).

Lemma. *If $A \to B$ is a faithful G-Galois extension, with G finite, then*

$$i\colon THH(A) \to THH(B)^{hG}$$

is a weak equivalence.

Proof. It suffices to show that $1 \wedge i\colon B \wedge_A THH(A) \to B \wedge_A THH(B)^{hG}$ is a weak equivalence, since B is faithful over A. To see this, we factor $1 \wedge i$ as the following chain of weak equivalences:

$$\begin{aligned} B \wedge_A THH(B)^{hG} &\to (B \wedge_A THH(B))^{hG} \\ &\leftarrow (B \wedge_A B \wedge_A THH(A))^{hG} \\ &\to (F(G_+, B) \wedge_A THH(A))^{hG} \\ &\to F(G_+, B \wedge_A THH(A))^{hG} \simeq B \wedge_A THH(A) \end{aligned}$$

These use that B is dualizable over A, B is separable over A, B is Galois over A and G is finite, respectively. □

Are there similar quasi-coherence (resp. Galois descent) results for relative theories $THH(A|K)$ and $THH(B|L)$ (to be defined later), under weaker hypotheses (tame ramification, resp. something more general) on $A \to B$?

References

[Ha61] Hardie, K.A., *A generalization of the Hopf construction*, Q. J. Math., Oxf. II. Ser. **12** (1961), 196–204.

III. THH: calculations (October 10th 2006)

To make calculations, we compare the spectrum level structures with the induced structures at the level of homotopy and homology, usually with mod p coefficients.

III.1. Homotopy.

For spectra X and Y we write $[X, Y]$ for the abelian group of maps from X to Y in the stable homotopy category. More generally, $[X, Y]_n = [\Sigma^n X, Y]$ is the abelian group of degree n maps, and $[X, Y]_*$ is the resulting graded abelian group.

The homotopy groups of a spectrum X are defined by

$$\pi_*(X) = [S, X]_* .$$

The stable homotopy groups of spheres is the graded commutative ring

$$\pi_*(S) = [S, S]_* ,$$

which naturally acts on $\pi_*(X)$ by composition:

$$\pi_*(S) \otimes_{\mathbb{Z}} \pi_*(X) = [S, S]_* \otimes_{\mathbb{Z}} [S, X]_* \xrightarrow{\circ} [S, X]_* = \pi_*(X) .$$

However, the groups $\pi_*(S)$ are mostly unknown, and its homological properties are terrible, so it is usually not convenient to work with the $\pi_*(S)$-module $\pi_*(X)$ as an algebraic invariant of X.

III.2. Homology.

Let p be a prime, and write $H\mathbb{F}_p$ for the mod p Eilenberg–Mac Lane spectrum. It is a commutative S-algebra, with unit $\eta\colon S \to H\mathbb{F}_p$ and product $\mu\colon H\mathbb{F}_p \wedge H\mathbb{F}_p \to H\mathbb{F}_p$. We have graded $\mathbb{F}_p$-modules

$$H_*(X;\mathbb{F}_p) = \pi_*(H\mathbb{F}_p \wedge X)$$

and

$$H^*(X;\mathbb{F}_p) = [X, H\mathbb{F}_p]_{-*}$$

for all spectra X. There are universal coefficient and Künneth isomorphisms

$$H^*(X;\mathbb{F}_p) \cong \mathrm{Hom}(H_*(X;\mathbb{F}_p), \mathbb{F}_p)$$

and

$$H_*(X;\mathbb{F}_p) \otimes H_*(Y;\mathbb{F}_p) \cong H_*(X \wedge Y;\mathbb{F}_p)$$

for all spectra X and Y, with Hom and $\otimes$ formed over $\mathbb{F}_p$. These extend the usual results for the reduced mod p (co-)homology of based spaces.

The mod p Steenrod algebra is the non-commutative algebra

$$A = H^*(H\mathbb{F}_p;\mathbb{F}_p) = [H\mathbb{F}_p, H\mathbb{F}_p]_{-*}$$

of stable cohomology operations, with multiplication ϕ defined by composition. It admits a cocommutative coproduct, and the dual Steenrod algebra

$$A_* = H_*(H\mathbb{F}_p;\mathbb{F}_p) = \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$$

is the Hom-dual commutative algebra, with a non-cocommutative coproduct ψ. In other words, A and A_* are dual Hopf algebras. The (canonical) coproduct χ on A_* is induced by the twist map $\tau\colon H\mathbb{F}_p \wedge H\mathbb{F}_p \to H\mathbb{F}_p \wedge H\mathbb{F}_p$.

For each spectrum X, composition of stable maps defines a homomorphism

$$A \otimes H^*(X;\mathbb{F}_p) = [H\mathbb{F}_p, H\mathbb{F}_p]_{-*} \otimes [X, H\mathbb{F}_p]_{-*} \xrightarrow{\circ} [X, H\mathbb{F}_p]_{-*} = H^*(X;\mathbb{F}_p)$$

that makes $H^*(X;\mathbb{F}_p)$ a left A-module. In the special case $X = H\mathbb{F}_p$, this recovers the product ϕ. Dually, for each spectrum X the natural map

$$X \cong S \wedge X \xrightarrow{\eta\wedge 1} H\mathbb{F}_p \wedge X$$

induces a homomorphism

$$\nu\colon H_*(X;\mathbb{F}_p) \to H_*(H\mathbb{F}_p \wedge X;\mathbb{F}_p) \cong A_* \otimes H_*(X;\mathbb{F}_p)\,.$$

In the special case $X = H\mathbb{F}_p$, this is the coproduct $\psi\colon A_* \to A_* \otimes A_*$. In general, the coaction map ν makes $H_*(X;\mathbb{F}_p)$ into an A_*-comodule.

For spectra X of finite type, for which $H_*(X;\mathbb{F}_p)$ is finite in each degree, we can go back and forth between the A-module $H^*(X;\mathbb{F}_p)$ and the A_*-comodule $H_*(X;\mathbb{F}_p)$, via Hom-duality. However, when $H_*(X;\mathbb{F}_p)$ not finite in each degree, it is better to work in homology. We therefore will view mod p homology as a functor from spectra to A_*-comodules. This covariant point of view is also more convenient when discussing algebra structures, etc. ((Explain A_*-comodule structure on tensor products and Hom-duals?))

To go back from homology to homotopy we can then use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_p, H_*(X;\mathbb{F}_p)) \Longrightarrow \pi_{t-s}(X_p)\,.$$

When X is of finite type, $\pi_*(X_p) \cong \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, otherwise the p-completion is more subtle.

III.3. The dual Steenrod algebra.

The structure of the dual Steenrod algebra was described by Milnor. For p odd, we have

$$A_* = P(\xi_k \mid k \geq 1) \otimes E(\tau_k \mid k \geq 0)$$

where $|\xi_k| = 2p^k - 2$ and $|\tau_k| = 2p^k - 1$. Here $P(-)$ denotes the polynomial algebra over $\mathbb{F}_p$ on the listed generators, and $E(-)$ denotes the exterior algebra over $\mathbb{F}_p$. The coproduct is given by the formulas

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j$$

and

$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_{i+j=k} \xi_i^{p^j} \otimes \tau_j \,.$$

In all such formulas, ξ_0 is to be read as 1. The class τ_0 is dual to the Bockstein operation β, while ξ_1^k is dual to the Steenrod reduced power operation P^k.

((Discuss relation to $H_*(K(\mathbb{F}_p, 1); \mathbb{F}_p)$.))

It will be more convenient for us to work with the conjugate generators $\bar{\xi}_k = \chi\xi_k$ and $\bar{\tau}_k = \chi\tau_k$. We still have

$$A_* = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq 0)$$

with $|\bar{\xi}_k| = 2p^k - 2$ and $|\bar{\tau}_k| = 2p^k - 1$, but

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i}$$

and

$$\psi(\bar{\tau}_k) = 1 \otimes \bar{\tau}_k + \sum_{i+j=k} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i} \,.$$

As before, $\bar{\xi}_0$ is to be read as 1.

((Include formulas for $p = 2$, too.))

III.4. Homology of THH.

Let B be a symmetric ring spectrum. The 0-simplex inclusion $\eta\colon B \to THH(B)$ and the circle action $\alpha\colon S^1_+ \wedge THH(B) \to THH(B)$ induce A_*-comodule homomorphisms $\eta : H_*(B; \mathbb{F}_p) \to H_*(THH(B); \mathbb{F}_p)$ and

$$\alpha\colon E(s_1) \otimes H_*(THH(B); \mathbb{F}_p) \to H_*(THH(B); \mathbb{F}_p) \,,$$

respectively. Here $H_*(S^1_+; \mathbb{F}_p) = E(s_1)$, with s_1 in homological degree 1. The stable map $d\colon \Sigma THH(B) \to THH(B)$ induces the A_*-comodule homomorphism

$$d\colon H_*(THH(B); \mathbb{F}_p) \to H_{*+1}(THH(B); \mathbb{F}_p)$$

given by $d(x) = \alpha(s_1 \otimes x)$.

The Hopf map $\eta\colon S^1 \to S^0$ induces the zero map on homology, so $dd = 0$ at the level of homology, i.e., d is a differential on $H_*(THH(B); \mathbb{F}_p)$.

The adjoint map $\tilde{\alpha}\colon THH(B) \to F(S^1_+, THH(B))$ to the circle action, and the canonical stable equivalence $\nu\colon DS^1_+ \wedge THH(B) \to F(S^1_+, THH(B))$, induce an A_*-comodule homomorphism

$$\nu^{-1}\tilde{\alpha}\colon H_*(THH(B); \mathbb{F}_p) \to E(\iota_1) \otimes H_*(THH(B); \mathbb{F}_p)$$

given by

$$(\nu^{-1}\tilde{\alpha})(x) = 1 \otimes x + \iota_1 \otimes dx\,.$$

Here $DS^1_+ = F(S^1_+, S)$ denotes the functional dual, and

$$H_*(DS^1_+; \mathbb{F}_p) \cong H^{-*}(S^1; \mathbb{F}_p) = E(\iota_1)\,,$$

with ι_1 in homological degree (-1), dual to s_1.

If B is a commutative symmetric ring spectrum, then $\tilde{\alpha}$ and ν are commutative ring spectrum maps. Here we give $F(S^1_+, THH(B))$ and $DS^1_+ = F(S^1_+, S)$ the pointwise multiplications, induced from the products on $THH(B)$ and S, respectively, and the diagonal coproduct on S^1_+. It follows that $\nu^{-1}\tilde{\alpha}$ is a $H_*(B; \mathbb{F}_p)$-algebra homomorphism, so

$$(\nu^{-1}\tilde{\alpha})(xy) = 1 \otimes xy + \iota_1 \otimes d(xy)$$

equals

$$\begin{aligned}(\nu^{-1}\tilde{\alpha})(x) \cdot (\nu^{-1}\tilde{\alpha})(y) &= (1 \otimes x + \iota_1 \otimes dx) \cdot (1 \otimes y + \iota_1 \otimes dy) \\ &= 1 \otimes xy + \iota_1 \otimes (dx \cdot y + (-1)^{|x|} x \cdot dy)\,.\end{aligned}$$

Hence

$$d(xy) = dx \cdot y + (-1)^{|x|} x \cdot dy\,,$$

and d acts as a derivation on $H_*(THH(B); \mathbb{F}_p)$.

III.5. The Bökstedt spectral sequence.

Proposition. *Let B be a symmetric ring spectrum. (a) The skeleton filtration of $THH(B)$ induces a spectral sequence of A_*-comodules, with*

$$E^1_{q,*}(B) = H_*(B; \mathbb{F}_p) \otimes \bar{H}_*(B; \mathbb{F}_p) \otimes \dots \otimes \bar{H}_*(B; \mathbb{F}_p)$$

(with q copies of $\bar{H}_*(B; \mathbb{F}_p) = \operatorname{coker}(\eta_*\colon \mathbb{F}_p \to H_*(B; \mathbb{F}_p))$) and

$$E^2_{q,*}(B) = HH_q(H_*(B; \mathbb{F}_p))\,,$$

converging strongly to $H_*(THH(B); \mathbb{F}_p)$.

(b) The edge homomorphism

$$E^1_{0,*}(B) = H_*(B; \mathbb{F}_p) \to HH_0(H_*(B; \mathbb{F}_p)) = E^2_{0,*}(B) \to H_*(THH(B); \mathbb{F}_p)$$

equals the homomorphism induced by the inclusion $\eta\colon B \to THH(B)$ of 0-simplices.

(c) Connes' operator

$$E^2_{q,*}(B) = HH_q(H_*(B; \mathbb{F}_p)) \to HH_{q+1}(H_*(B; \mathbb{F}_p)) = E^2_{q+1,*}(B)$$

abuts to the homomorphism $d\colon H_*(THH(B);\mathbb{F}_p) \to H_{*+1}(THH(B);\mathbb{F}_p)$ *induced by the map* $d\colon \Sigma THH(B) \to THH(B)$.

Proof. This is the spectral sequence associated to the unraveled exact couple of A_*-comodules obtained by applying $H_*(-;\mathbb{F}_p)$ to the cofiber sequences

$$|sk_{q-1}THH(B)| \to |sk_q THH(B)| \to S^q \wedge THH(B)^{nd}_q ,$$

where $THH(B)^{nd}_q = B \wedge \bar{B} \wedge \cdots \wedge \bar{B}$ models the non-degenerate q-simplices. Here $\bar{B}$ denotes the cofiber of $\eta\colon S \to B$, which we implicitly take to be a cofibration. Hence

$$\begin{aligned} E^1_{q,*}(B) &= H_{q+*}(S^q \wedge THH(B)^{nd}_q;\mathbb{F}_p) \\ &\cong H_*(B \wedge \bar{B} \wedge \cdots \wedge \bar{B};\mathbb{F}_p) \\ &\cong H_*(B;\mathbb{F}_p) \otimes \bar{H}_*(B;\mathbb{F}_p) \otimes \cdots \otimes \bar{H}_*(B;\mathbb{F}_p) . \end{aligned}$$

By a standard inspection, the d^1-differential is induced by the alternating sum of the simplicial face maps, hence recovers the boundary operator in the normalized Hochschild complex for $H_*(B;\mathbb{F}_p)$. ((Reference to Segal?)) Thus the E^2-term is given by the homology of that complex, i.e., by the indicated Hochschild homology groups.

((Circle action, edge homomorphism.)) □

When B is commutative, the Bökstedt spectral sequence reflects the multiplicative structure on $THH(B)$. The algebra structure on Hochschild homology is given by the shuffle product.

Proposition. *Let B be a commutative symmetric ring spectrum. (a) The Bökstedt spectral sequence*

$$E^2_{q,*}(B) = HH_q(H_*(B;\mathbb{F}_p)) \Longrightarrow H_*(THH(B);\mathbb{F}_p)$$

is an augmented commutative A_*-comodule $H_*(B;\mathbb{F}_p)$-algebra spectral sequence.

(b) If each term $E^r_{**}(B)$ is flat over $H_*(B;\mathbb{F}_p)$, then $E^*_{**}(B)$ is a commutative A_*-comodule $H_*(B;\mathbb{F}_p)$-Hopf algebra spectral sequence.

((See Angeltveit–Rognes (2005).))

((There is a similar E-based spectral sequence converging to $E_*(THH(B))$, for each S-algebra E, see [EKMM].))

III.6. First calculations.

We write $P(x) = \mathbb{F}_p[x]$ and $E(x) = P(x)/(x^2 = 0) = \mathbb{F}_p\{1, x\}$ for the polynomial algebra and the exterior algebra on a generator x, respectively. We also write $P_h(x) = P(x)/(x^h = 0)$ for the truncated polynomial algebra of height h. Furthermore we write

$$\Gamma(y) = \mathbb{F}_p\{\gamma_k(y)) \mid k \ge 0\}$$

for the divided power algebra on y, with $\gamma_i(y) \cdot \gamma_j(y) = (i,j)\gamma_{i+j}(y)$, where $(i,j) = (i+j)!/(i!j!)$. We identify $\gamma_0(y) = 1$ and $\gamma_1(y) = y$. There is an isomorphism

$$\Gamma(y) = \bigotimes_{e \ge 0} P_p(\gamma_{p^e}(y)) .$$

(($\Gamma(y)$ is Hom dual to $P(\eta)$, where y is dual to η.))

To compute the E^2-term of a Bökstedt spectral sequence, the following standard calculations are useful.

Lemma. *(a)*
$$HH_*(P(x)) \cong P(x) \otimes E(dx)$$
for $|x|$ even, with $dx \in HH_1(P(x))$ represented by $1 \otimes x$ in the Hochschild complex. The coproduct is given by $\psi(dx) = dx \otimes 1 + 1 \otimes dx$, i.e., dx is $P(x)$-coalgebra primitive.

(b)
$$HH_*(E(x)) \cong E(x) \otimes \Gamma(dx)$$
for $|x|$ odd, with $\gamma_k(dx) \in HH_k(E(x))$ represented by $1 \otimes x \otimes \cdots \otimes x$ (with k copies of x) in the Hochschild complex. The coproduct is given by
$$\psi(\gamma_k(dx)) = \sum_{i+j=k} \gamma_i(dx) \otimes \gamma_j(dx)\,,$$
i.e., dx is $E(x)$-coalgebra primitive.

((What about $p = 2$?))

Proof. We can compute $HH_*(P(x))$ as $P(x)$-bimodule Tor. There is a short free resolution
$$0 \to P(x) \otimes \mathbb{F}_p\{dx\} \otimes P(x) \xrightarrow{\partial} P(x) \otimes P(x) \xrightarrow{\mu} P(x) \to 0$$
of $P(x)$, viewed as a $P(x)$-bimodule by the algebra multiplication μ, where $\partial(dx) = 1 \otimes x - x \otimes 1$. It maps to the two-sided bar resolution by taking dx to $1 \otimes x \otimes 1$. Tensoring with $P(x)$ over $P(x) \otimes P(x)^{op}$ we get the complex
$$0 \to P(x) \otimes \mathbb{F}_p\{dx\} \xrightarrow{0} P(x) \to 0$$
with homology $P(x) \otimes E(dx)$, mapping to the Hochschild complex by taking dx to $1 \otimes x$. The induced map in homology is an isomorphism, and identifies $P(x) \otimes E(dx)$ with $HH_*(P(x))$. ((Discuss product and coproduct.))

((Do $E(x)$ too.)) □

((Reference to Cartan–Eilenberg. See Proposition 3.3 of Ausoni (2005) for a more complicated case.))

We now study the example $B = H\mathbb{F}_p$, which is a commutative symmetric ring spectrum. Its homology algebra is
$$H_*(H\mathbb{F}_p; \mathbb{F}_p) = A_* = P(\bar\xi_k \mid k \geq 1) \otimes E(\bar\tau_k \mid k \geq 0)\,.$$
Thus the Bökstedt spectral sequence has E^2-term
$$E^2_{**}(\mathbb{F}_p) = HH_*(A_*) \cong A_* \otimes E(d\bar\xi_k \mid k \geq 1) \otimes \Gamma(d\bar\tau_k \mid k \geq 0)\,.$$
It is free, thus flat, over A_*.

The A_*-algebra generators of this E^2-term are the exterior classes $d\bar\xi_k$ in filtration $q = 1$, and the divided powers $\gamma_{p^e}(d\bar\tau_k)$ in filtration $q = p^e$, for $e \geq 0$.

The A_*-coalgebra primitives constitute the free A_*-module generated by the classes $d\bar\xi_k$ and $d\bar\tau_k$, all in filtration $q = 1$. Among these, the A_*-comodule primitives constitute the $\mathbb{F}_p$-module generated by the same classes.

Suppose that the shortest nonzero differentials in the Bökstedt spectral sequence are of length $r \geq 2$, i.e., are d^r-differentials, so that $E^2(\mathbb{F}_p) = E^r(\mathbb{F}_p)$. The A_*-comodule A_*-Hopf algebra structure on the E^2-term then remains (unchanged) at the E^r-term, and the d^r-differentials are generated, by this structure, by differentials from A_*-algebra generators to A_*-coalgebra primitives that are also A_*-comodule primitives.

Any such generating differentials must map from a class $\gamma_{p^e}(d\bar{\tau}_k)$ with $e \geq 1$, to a class $d\bar{\xi}_k$ or $d\bar{\tau}_k$, not necessarily with the same index k. The total degrees of these classes are $2p^{k+e}$, $2p^k - 1$ and $2p^k$, respectively, so the only possibilities are the differentials

$$d^r(\gamma_{p^e}(d\bar{\tau}_k)) = d\bar{\xi}_{k+e}$$

where $r = p^e - 1$, for some $e \geq 1$. More precisely, this formula might hold up to a unit in $\mathbb{F}_p$, which we suppress. In fact, the differential structure is as rich as possible, i.e., for each $k \geq 0$ the differential above for $e = 1$ occurs.

Proposition. *(a) In the Bökstedt spectral sequence for $B = H\mathbb{F}_p$, with*

$$E^2_{**}(\mathbb{F}_p) = A_* \otimes E(d\bar{\xi}_k \mid k \geq 1) \otimes \Gamma(d\bar{\tau}_k \mid k \geq 0)$$

there are nonzero differentials

$$d^{p-1}(\gamma_p(d\bar{\tau}_k)) = d\bar{\xi}_{k+1}$$

for each $k \geq 0$.

(b) By the A_*-coalgebra structure, these imply differentials

$$d^{p-1}(\gamma_j(d\bar{\tau}_k)) = d\bar{\xi}_{k+1} \cdot \gamma_{j-p}(d\bar{\tau}_k)$$

for each $j \geq p$, $k \geq 0$, which leave the E^p-term

$$E^p_{**}(\mathbb{F}_p) = A_* \otimes P_p(d\bar{\tau}_k \mid k \geq 0)\,.$$

For filtration reasons, this equals the E^∞-term.

To establish the differentials in (a), we appeal to the homology operations in $H_*(THH(B); \mathbb{F}_p)$ that are derived from its E_∞-ring spectrum structure. These also allow us to make the following deductions.

Proposition. *(a) There are multiplicative extensions*

$$(d\bar{\tau}_k)^p = d\bar{\tau}_{k+1}$$

in $H_*(THH(\mathbb{F}_p); \mathbb{F}_p)$, for all $k \geq 0$, so

$$H_*(THH(\mathbb{F}_p); \mathbb{F}_p) \cong A_* \otimes P(d\bar{\tau}_0)$$

as a commutative A_*-comodule A_*-Hopf algebra, with $|d\bar{\tau}_0| = 2$.

(b) There is an $\mathbb{F}_p$-algebra isomorphism

$$\pi_* THH(\mathbb{F}_p) \cong P(\mu_0)$$

where μ_0, in degree 2, is represented by $d\bar{\tau}_0$.

III.7. Power operations.

A commutative symmetric ring spectrum B has a canonical structure as an E_∞ ring spectrum. In terms of the positive model structure on commutative symmetric ring spectra, this can be seen as follows: A (positively) cofibrant B has the property that the Σ_j-action on the j-th smash power

$$B^{\wedge j} = B \wedge \cdots \wedge B \,,$$

permuting the smash factors, is free off the base point at each level. Hence the map

$$E\Sigma_{j+} \wedge_{\Sigma_j} B^{\wedge j} \to B^{\wedge j}/\Sigma_j$$

that collapses $E\Sigma_j$ to a point, is a weak equivalence. The (strictly) commutative product $\mu\colon B \wedge B \to B$ induces maps

$$B^{\wedge j}/\Sigma_j \to B$$

and the composite maps

$$\xi_j \colon E\Sigma_{j+} \wedge_{\Sigma_j} B^{\wedge j} \to B$$

for $j \geq 0$ provide the structure maps for an E_∞ ring spectrum structure on B, for the Barratt–Eccles operad $\mathcal{E}$ with j-th space $\mathcal{E}(j) = E\Sigma_j$. The point of replacing $B^{\wedge j}/\Sigma_j$ with the extended power

$$D_j(B) = E\Sigma_{j+} \wedge_{\Sigma_j} B^{\wedge j}$$

is that the homology of the latter can be readily computed. We focus on the case $j = p$, where $D_p(B)$ extends the p-fold smash power $B^{\wedge p}$. By a transfer argument, there is a split surjection

$$H_*(EC_{p+} \wedge_{C_p} B^{\wedge p}; \mathbb{F}_p)) \to H_*(E\Sigma_{p+} \wedge_{\Sigma_p} B^{\wedge p}; \mathbb{F}_p) \,,$$

where $C_p \subset \Sigma_p$ is the cyclic subgroup generated by $T = (12 \dots p)$. As a model for EC_p we can take $S^\infty = S(\mathbb{C}^\infty)$, with its usual C_p-CW structure. The associated mod p cellular complex is $W_* = C_*(EC_p; \mathbb{F}_p)$, with $W_i = \mathbb{F}_p[C_p]\{e_i\}$ for each $i \geq 0$, $d(e_i) = (1 - T)e_{i-1}$ for i odd and $d(e_i) = (1 + T + \cdots + T^{p-1})e_{i-1}$ for $i \geq 2$ even.

There is then an isomorphism

$$H_*(EC_{p+} \wedge_{C_p} B^{\wedge p}; \mathbb{F}_p) \cong H_*(W_* \otimes_{C_p} H_*(B; \mathbb{F}_p)^{\otimes p})$$

and these homology groups are generated by the cycles $e_i \otimes x^{\otimes p}$ for $i \geq 0$ (and $e_0 \otimes x_1 \otimes \cdots \otimes x_p$), where x (and the $x_1, \dots, x_p$) ranges through a basis for $H_*(B; \mathbb{F}_p)$. It follows that

$$H_*(D_p(B); \mathbb{F}_p) = H_*(E\Sigma_{p+} \wedge_{\Sigma_p} B^{\wedge p}; \mathbb{F}_p)$$

is generated by the cycles $e_i \otimes x^{\otimes p}$ for $i \equiv -1, 0 \mod 2p-2$ and $|x|$ even, and for $i \equiv p-2, p-1 \mod 2p-2$ and $|x|$ odd (and $e_0 \otimes x_1 \otimes \cdots \otimes x_p$). We define $Q_i(x)$ in $H_*(B; \mathbb{F}_p)$ to be the image under

$$(\xi_p)_* \colon H_*(D_p(B); \mathbb{F}_p) \to H_*(B; \mathbb{F}_p)$$

of the class of $e_i \otimes x^{\otimes p}$, for these i and x. We can write the degree of $Q_i(x)$ as

$$i + p|x| = |x| + r(2p-2) - \epsilon$$

for unique $r \geq 0$ and $\epsilon \in \{0,1\}$, and it is traditional to rewrite Q_i in terms of upper indices as follows:

$$\beta^\epsilon Q^r(x) = Q_i(x) = (\xi_p)_*(e_i \otimes x^{\otimes p}) .$$

Here $\beta = \beta^1 Q^0$ is the mod p homology Bockstein operation, of degree (-1), and $Q^r = \beta^0 Q^r$ has degree $r(2p-2)$.

These extended power homology operations

$$\beta^\epsilon Q^r \colon H_*(B; \mathbb{F}_p) \to H_*(B; \mathbb{F}_p)$$

(for commutative symmetric ring spectra B) are known as Dyer–Lashof operations. Dyer and Lashof first described these operations for $p = 2$, while the odd-primary case is due to Araki and Kudo. See Steinberger (1986) for a list of the formal properties of these operations.

For $|x| = 2r$ even (with $\epsilon = 0$ and $i = 0$), the operation $Q^r(x) = Q_0(x)$ is the image under $(\xi_p)_*$ of the class of $e_0 \otimes x^{\otimes p}$. Since ξ_p extends the usual p-fold product $B^{\wedge p} \to B$, it follows that in this case

$$Q^r(x) = x^p$$

in the algebra structure on $H_*(B; \mathbb{F}_p)$. The other Dyer–Lashof operations are of higher degree.

In the special case of $B = H\mathbb{F}_p$, the operations

$$\beta^\epsilon Q^r \colon A_* \to A_*$$

were first described by Leif Kristensen, see Steinberger's paper for a published reference. We need the following facts:

$$Q^{p^k}(\bar{\xi}_k) = \bar{\xi}_{k+1}$$

for $k \geq 1$,

$$Q^{p^k}(\bar{\tau}_k) = \bar{\tau}_{k+1}$$

for $k \geq 0$, and

$$\beta(\bar{\tau}_k) = \bar{\xi}_k$$

for $k \geq 0$, where $\bar{\xi}_0 = 1$, as usual. These formulas are all forced by the formal properties of the Dyer–Lashof operations, such as the Nishida relations.

In the special case $B = DX_+ = F(X_+, S)$, for a finite CW complex X, the Dyer–Lashof operations on $H_*(B; \mathbb{F}_p)$ are compatible with the Steenrod operations on $H^*(X; \mathbb{F}_p)$ under the isomorphisms

$$H_*(B; \mathbb{F}_p) = \pi_*(H\mathbb{F}_p \wedge F(X_+, S)) \cong \pi_* F(X, H\mathbb{F}_p) = H^{-*}(X; \mathbb{F}_p) .$$

More precisely, $\beta^\epsilon Q^r$ corresponds to $\beta^\epsilon P^{-r}$.

The maps

$$THH(B) \xrightarrow{\tilde{\alpha}} F(S^1_+, THH(B)) \underset{\simeq}{\xleftarrow{\nu}} DS^1_+ \wedge THH(B)$$

are maps of commutative symmetric ring spectra, hence the induced homomorphism

$$\nu^{-1}\tilde{\alpha} \colon H_*(THH(B); \mathbb{F}_p) \to E(\iota_1) \otimes H_*(THH(B); \mathbb{F}_p)$$

in homology commutes with the Dyer–Lashof operations. In view of the formula $(\nu^{-1}\tilde{\alpha})(x) = 1 \otimes x + \iota_1 \otimes dx$, this implies the relation

$$\beta^\epsilon Q^r(dx) = d(\beta^\epsilon Q^r x)$$

for all $r \geq 0$, $\epsilon \in \{0, 1\}$, $x \in H_*(THH(B); \mathbb{F}_p)$. The argument uses the Cartan formula for the Dyer–Lashof operations in the homology of $DS^1_+ \wedge THH(B)$, and the relation with Steenrod operations mentioned above for $X = S^1_+$.

As a corollary, we can deduce the multiplicative relation

$$(d\bar{\tau}_k)^p = Q^{p^k}(d\bar{\tau}_k) = dQ^{p^k}(\bar{\tau}_k) = d\bar{\tau}_{k+1}$$

for each $k \geq 0$.

III.8 Differentials.

Following Ausoni (2005), we show that there are differentials

$$d^{p-1}(\gamma_p(d\bar{\tau}_k)) = d\bar{\xi}_{k+1}$$

for $k \geq 0$, in the Bökstedt spectral sequence for $H_*(THH(\mathbb{F}_p); \mathbb{F}_p)$. In the abutment, we can compute that

$$d\bar{\xi}_{k+1} = d(\beta\bar{\tau}_{k+1}) = \beta(d\bar{\tau}_{k+1}) = \beta((d\bar{\tau}_k)^p) = 0$$

since β acts as a derivation, and we are working in characteristic p. Hence $d\bar{\xi}_{k+1}$ must be hit by a differential.

In the lowest case, $k = 0$, the only possible source of this differential is $\gamma_p(d\bar{\tau}_0)$. It follows from the coalgebra structure that there are also differentials

$$d^{p-1}(\gamma_j(d\bar{\tau}_0)) = d\bar{\xi}_1 \cdot \gamma_{j-p}(d\bar{\tau}_0)$$

for all $j \geq p$. The homology of this differential acting on $E(d\bar{\xi}_1) \otimes \Gamma(d\bar{\tau}_0)$ is thus $P_p(d\bar{\tau}_0)$.

Turning to the next case, $k = 1$, only possible sources of differentials hitting $d\bar{\xi}_2$ were $\gamma_p(d\bar{\tau}_1)$ and $\gamma_{p^2}(d\bar{\tau}_0)$, but we have just seen that the latter class already supported a d^{p-1}-differential. Hence there is a differential $d^{p-1}(\gamma_p(d\bar{\tau}_1)) = d\bar{\xi}_2$, and the coalgebra structure implies that there are also differentials $d^{p-1}(\gamma_j(d\bar{\tau}_1)) = d\bar{\xi}_2 \cdot \gamma_{j-p}(d\bar{\tau}_1)$ for all $j \geq p$. The homology of this differential acting on $E(d\bar{\xi}_2) \otimes \Gamma(d\bar{\tau}_1)$ is $P_p(d\bar{\tau}_1)$.

By induction on k it follows that we have precisely the d^{p-1}-differentials listed above, killing the classes $d\bar{\xi}_{k+1}$, together with their coalgebraic consequences

$$d^{p-1}(\gamma_j(d\bar{\tau}_k)) = d\bar{\xi}_{k+1} \cdot \gamma_{j-p}(d\bar{\tau}_k)$$

for all $k \geq 0$ and $j \geq p$. By the Künneth formula, we are left with the E^p-term

$$E^p_{**}(\mathbb{F}_p) = A_* \otimes P_p(d\bar\tau_k \mid k \geq 1)\,.$$

Here all the algebra generators are in Hochschild filtration 1, so there is no room for further differentials, and this also equals the E^∞-term.

In view of the multiplicative relations $(d\bar\tau_k)^p = d\bar\tau_{k+1}$ in $H_*(THH(\mathbb{F}_p);\mathbb{F}_p)$, we deduce that

$$H_*(THH(\mathbb{F}_p);\mathbb{F}_p) = A_* \otimes P(d\bar\tau_0)$$

as a (bi-)commutative A_*-comodule A_*-Hopf algebra.

The class $d\bar\tau_0$ is A_*-comodule primitive, since the coaction can be calculated as

$$\nu(d\bar\tau_0) = (1 \otimes d)(\psi(\bar\tau_0)) = 1 \otimes d\bar\tau_0 + \bar\tau_0 \otimes d1 = 1 \otimes d\bar\tau_0\,.$$

Hence the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_p, H_*(THH(\mathbb{F}_p);\mathbb{F}_p)) \Longrightarrow \pi_{t-s}THH(\mathbb{F}_p)$$

collapses to the $s = 0$-line, where $\pi_*THH(\mathbb{F}_p)$ is identified with the subalgebra $P(d\bar\tau_0)$ of A_*-comodule primitives in $H_*(THH(\mathbb{F}_p);\mathbb{F}_p)$. We let

$$\mu_0 = d\bar\tau_0 \in \pi_2THH(\mathbb{F}_p)$$

be the generating homotopy class, and conclude that

$$\pi_*THH(\mathbb{F}_p) = P(\mu_0)\,.$$

Heuristically, we might think of $H\mathbb{F}_p$ as an S-algebra built from S by attaching a 1-cell labeled $\bar\tau_0$ by a degree p map, together with higher cells. The Kähler differentials of $H\mathbb{F}_p$ over S then receive a contribution called $d\bar\tau_0$, in bidegree $(1,1)$. Going from the symmetric algebra on the Kähler differentials to THH, the calculation above shows that this class freely generates all of $THH(\mathbb{F}_p)$.

IV. The ku-algebra structures on ku/p (October 11th)

((Discuss the Lazarev obstruction theory classifying A_∞ ku-ring spectrum structures on ku/p, working up its Postnikov tower. The extensions of the $(2m-2)$-th Postnikov section $P = P^{2m-2}ku/p$ to the $(2m)$-th section are classified by the A_∞ ku-algebra derivations

$$\mathcal{A}\mathrm{Der}_{ku}(P, \Sigma^{2m+1}H\mathbb{Z}/p) \cong THH^{2m+2}_{ku}(P, H\mathbb{Z}/p) \cong \mathbb{Z}/p\{y_0^{m+1}, y_{1,m}\}$$

that map to the underlying ku-module extension in

$$H^{2m+1}_{ku}(P, H\mathbb{Z}/p) \cong \mathbb{Z}/p\{Q_{1,m}\}\,.$$

Here $y_{1,m} \mapsto Q_{1,m}$, so there are p choices of ku-algebra extensions for each Postnikov stage.))

V. Thom spectra (October 16th)

We focus on the real case. Complex Thom spectra can be handled in much the same way, taking $MU_n = F(S^n, MU(n))$ as the n-th space.

V.1. The Thom spectrum MO.

Let $EO(n)$ be the simplical space with q-simplices $\mathrm{Map}([q], O(n)) \cong O(n)^{q+1}$. The group $O(n)$ acts freely on the contractible space $EO(n)$, both from the left and from the right, and these actions commute. Let $BO(n) = EO(n)/O(n)$ be the orbit space for the right action. The remaining left $O(n)$-action on $BO(n)$ is given by the adjoint (= conjugation) action of $O(n)$ on $O(n)$, together with functoriality of the bar construction. We will write $BO(n)^{ad}$ for this left $O(n)$-space.

The Thom spectrum MO is the orthogonal spectrum

$$n \mapsto MO_n = MO(n) = EO(n)_+ \wedge_{O(n)} S^n$$

with left $O(n)$-action induced from the left $O(n)$-action on $EO(n)$. The n-th stucture map $\sigma\colon S^1 \wedge MO_n \to MO_{1+n}$ equals the composite

$$S^1 \wedge EO(n)_+ \wedge_{O(n)} S^n \cong EO(n)_+ \wedge_{O(n)} (S^1 \wedge S^n) \to EO(1+n)_+ \wedge_{O(1+n)} S^{1+n} .$$

The underlying symmetric spectrum is the same sequence of based spaces, with left Σ_n-action given by restricting the $O(n)$-action along the inclusion $\Sigma_n \subset O(n)$. There are product maps

$$\begin{aligned}(EO(m)_+ \wedge_{O(m)} S^m) &\wedge (EO(n)_+ \wedge_{O(n)} S^n) \\ &\cong E(O(m) \times O(n))_+ \wedge_{O(m)\times O(n)} (S^m \wedge S^n) \\ &\to EO(m+n)_+ \wedge_{O(m+n)} S^{m+n}\end{aligned}$$

that make MO a commutative orthogonal ring spectrum.

V.2. The suspension spectrum of BO.

Let $S[BO]^{ad}$ be the orthogonal spectrum

$$n \mapsto S^n \wedge BO(n)^{ad}$$

with $O(n)$ acting diagonally on $S(n)$ and $BO(n)^{ad}$. There are obvious structure maps

$$\sigma\colon S^1 \wedge S^n \wedge BO(n)^{ad} \to S^{1+n} \wedge BO(1+n)^{ad} .$$

There are product maps

$$\begin{aligned}(S^m \wedge BO(m)^{ad}) &\wedge (S^n \wedge BO(n)^{ad}) \\ &\cong S^m \wedge S^n \wedge B(O(m) \times O(n))^{ad} \\ &\to S^{m+n} \wedge BO(m+n)^{ad}\end{aligned}$$

that make $S[BO]^{ad}$ a commutative orthogonal ring spectrum.

Let $BO = \mathrm{colim}_n\, BO(n)$, and let $\Sigma^\infty BO_+$ be the unreduced orthogonal suspension spectrum of BO, with n-th space $S^n \wedge BO_+$. Here $O(n)$ acts only on the S^n-factor, fixing BO.

The underlying sequential spectrum of $S[BO]^{ad}$ maps to the underlying sequential spectrum of $\Sigma^\infty BO_+$, via the inclusion

$$S^n \wedge BO(n)_+ \to S^n \wedge BO_+$$

at level n. The induced map of homotopy groups

$$\operatorname*{colim}_n \pi_{k+n}(S^n \wedge BO(n)_+) \to \operatorname*{colim}_n \pi_{k+n}(S^n \wedge BO_+)$$

is an isomorphism for each integer k, so these sequential spectra are stably equivalent. But the homotopy category of orthogonal spectra is equivalent to the homotopy category of sequential spectra, so it follows that also the orthogonal spectra $S[BO]^{ad}$ and $\Sigma^\infty BO_+$ are stably equivalent.

Note that the orthogonal ring spectrum structure on $S[BO]^{ad}$ is commutative and easy to describe, while that on $\Sigma^\infty BO_+$ is only an E_∞ ring spectrum structure.

V.3. The Thom diagonal.

The Thom diagonal map

$$\theta_n \colon MO(n) \to MO(n) \wedge BO(n)^{ad}_+$$

extends the proper map

$$(id, \pi) \colon E(\gamma^n) \to E(\gamma^n) \times BO(n)^{ad}$$

over $E(\gamma^n) = EO(n) \times_{O(n)} \mathbb{R}^n \subset MO(n)$. It is $O(n)$-equivariant for the diagonal action in the target, when $O(n)$ acts on $BO(n)$ by the left adjoint action, as indicated.

We continue the Thom map with the usual inclusions

$$MO(n) \wedge BO(n)^{ad}_+ \to F(S^n, MO(n) \wedge S^n \wedge BO(n)^{ad}_+) \to F(S^n, (MO \wedge S[BO]^{ad})_{2n})$$

to get a map of orthogonal ring spectra

$$\theta \colon MO \to \{n \mapsto F(S^n, (MO \wedge S[BO]^{ad})_{2n})\} .$$

For any orthogonal spectrum X there is a map of orthogonal spectra

$$\tilde\sigma \colon X = \{n \mapsto X_n\} \to \{n \mapsto F(S^n, X_{2n})\}$$

that at the n-th level is adjoint to the iterated structure map

$$\sigma^n \colon S^n \wedge X_n \to X_{n+n} = X_{2n} .$$

The latter is $O(n)$-equivariant via the diagonal embedding $O(n) \to O(n) \times O(n) \subset O(2n)$, so the adjoint map $\tilde\sigma^n \colon X_n \to F(S^n, X_{2n})$ is also $O(n)$-equivariant. When X is an orthogonal ring spectrum, the product maps

$$F(S^m, X_{2m}) \wedge F(S^n, X_{2n}) \to F(S^m \wedge S^n, X_{2m} \wedge X_{2n}) \to F(S^{m+n}, X_{2(m+n)})$$

make $\{n \mapsto F(S^n, X_{2n})\}$ an orthogonal ring spectrum, too. ((Check!)) The map $\tilde\sigma$ induces an isomorphism

$$\operatorname*{colim}_n \pi_{k+n}(X_n) \to \operatorname*{colim}_n \pi_{k+n}(F(S^n, X_{2n})) \cong \operatorname*{colim}_n \pi_{k+2n}(X_{2n})$$

of homotopy groups for each integer k, hence is a stable equivalence.

There results a chain of maps of commutative orthogonal ring spectra

$$MO \xrightarrow{\beta} \{n \mapsto F(S^n, (MO \wedge S[BO]^{ad})_{2n})\} \xleftarrow[\simeq]{\tilde\sigma} MO \wedge S[BO]^{ad} ,$$

and a stable equivalence of orthogonal (E_∞ ring-)spectra

$$MO \wedge S[BO]^{ad} \simeq MO \wedge \Sigma^\infty BO_+ \cong MO \wedge BO_+ .$$

V.4. A Hopf–Galois structure.

We wish to view β, or $\tilde{\sigma}^{-1}\beta$, as the coaction in a Hopf–Galois extension $S \to MO$, with Hopf algebra $H = S[BO]^{ad}$.

((Is there a good coproduct $\psi \colon H \to H \wedge H$, lifting the diagonal map $\Sigma^\infty BO_+ \to \Sigma^\infty (BO \times BO)_+ \cong \Sigma^\infty BO_+ \wedge \Sigma^\infty BO_+$?))

((ETC))

VI. THH of the integers

((Consider the case $B = H\mathbb{Z}$, with $H_*(H\mathbb{Z}) = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq 1) \subset A_*$. The Bökstedt E^2-term equals $HH_*(H_*(H\mathbb{Z})) = H_*(H\mathbb{Z}) \otimes E(d\bar{\xi}_k \mid k \geq 1) \otimes \Gamma(d\bar{\tau}_k \mid k \geq 1)$. The differentials $d^{p-1}(\gamma_p(d\bar{\tau}_k)) = d\bar{\xi}_{k+1}$ for $k \geq 1$, leave $E^\infty_{**}(\mathbb{Z}) = H_*(H\mathbb{Z}) \otimes E(d\bar{\xi}_1) \otimes P_p(d\bar{\tau}_k \mid k \geq 1)$ and $H_*(THH(\mathbb{Z})) = H_*(H\mathbb{Z}) \otimes E(d\bar{\xi}_1) \otimes P(d\bar{\tau}_1)$. We find generating A_*-comodule primitives

$$\begin{aligned} \lambda_1 &= 1 \wedge d\bar{\xi}_1 \\ \mu_1 &= 1 \wedge d\bar{\tau}_1 + \tau_0 \wedge d\bar{\xi}_1 \end{aligned}$$

in $H_*(V(0) \wedge THH(\mathbb{Z}))$, where $H_*(V(0)) = E(\tau_0)$, so

$$V(0)_* THH(\mathbb{Z}) = E(\lambda_1) \otimes P(\mu_1)\,.$$

A Bockstein spectral sequence arguement implies

$$\pi_* THH(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } * = 0, \\ \mathbb{Z}/n & \text{for } * = 2n-1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

))

VII. THH for local fields

VII.1. Kähler forms.

((Define derivations, differentials, relation to HH_1, higher forms, map to HH_*, étale, polynomial and smooth examples, HKR-theorem.))

VII.2. Rings of integers in local fields.

We follow Hesselholt–Madsen:

Let A be a complete DVR (discrete valuation ring) with perfect residue field k of characteristic p, and fraction field K of characteristic 0. For example, K can be a finite extension of $\mathbb{Q}_p$ and A its valuation ring. In that case k is a finite field.

Let $\pi = \pi_K$ be a uniformizer, so $v(\pi) = 1$ where $v = v_K$ is the normalized valuation, $\pi A = (\pi) \subset A$ is the maximal ideal, $k = A/(\pi)$ and $K = A[\pi^{-1}]$:

$$k \xleftarrow{i} A \xrightarrow{j} K\,.$$

Let $W = W(k)$ be the Witt ring of k, with uniformizer p. We can write

$$A = W[x]/(\phi(x))\,,$$

where the irreducible polynomial of π over W is an Eisenstein polynomial

$$\phi(x) = x^e - p\theta(x)$$

in $W[x]$, where $\theta(x)$ is of degree $< e$ and $\theta(0)$ is a unit in W. It follows that $\theta(\pi)$ is also a unit. Hence $\pi^e = p\theta(\pi)$ and $p = \pi^e\theta(\pi)^{-1}$ in A. The formal derivative is $\phi'(x) = ex^{e-1} - p\theta'(x)$, so

$$\phi'(\pi) = e\pi^{e-1} - p\theta'(\pi)$$

and

$$\Omega^1_{A/W} \cong A/(\phi'(\pi))\{d\pi\}\,.$$

If $p \nmid e$, we say that A is tamely ramified over W. Then $v(\phi'(\pi)) = v(\pi^{e-1}) = e-1$, $(p) \subsetneq (\phi'(\pi))$, and $\Omega^1_{A/W} \cong k\{1, \pi, \dots, \pi^{e-2}\}$.

Example 1: $K = \mathbb{Q}_p$, $A = W = \mathbb{Z}_p$, $\pi = p$, $e = 1$, $\phi(x) = x - p$, $\theta(x) = 1$, $\phi'(\pi) = 1$.

Example 2: $K = \mathbb{Q}_p(\zeta_p)$, $A = \mathbb{Z}_p[\zeta_p]$, $\pi = 1 - \zeta_p$, $e = p-1$, $\phi(x) = x^{p-1} - px^{p-2} + \dots + (-1)^{p-1}p$, $\theta(x) = x^{p-2} + \dots + (-1)^{-p}$, $\phi'(\pi) = (p-1)\pi^{p-2} - p\theta'(\pi)$, $\Omega^1_{A/W} \cong \mathbb{F}_p^{p-1}$.

If $p \mid e$ we say that A is wildly ramified over W. Then $v(\phi'(\pi)) \geq v(p) = e$ and $(\phi'(\pi)) \subseteq (p)$.

Example 3: $K = \mathbb{Q}_p(\sqrt[p]{p})$, $A = \mathbb{Z}_p[\sqrt[p]{p}]$, $\pi = \sqrt[p]{p}$, $e = p$, $\phi(x) = x^p - p$, $\theta(x) = 1$, $\phi'(\pi) = p\pi^{p-1} = \pi^{2p-1}$.

Following Lindenstrauss–Madsen:

$HH_*(\mathbb{Z}_p) \to \mathbb{Z}_p$ is a p-adic equivalence, so $HH_*(W) \to HH_*^{\mathbb{Z}_p}(W) \cong W$ is a p-adic equivalence, the Witt ring W is unramified over $\mathbb{Z}_p$. Similarly, $HH_*(A) \to HH_*^W(A)$ is a p-adic equivalence.

From $A = W[x]/(\phi(x))$ we get a 2-periodic resolution of A over $A^e = A \otimes_W A$,

$$A \xleftarrow{\mu} A^e \xleftarrow{1\otimes x - x\otimes 1} A^e \xleftarrow{\frac{1\otimes\phi(x)-\phi(x)\otimes 1}{1\otimes x - x\otimes 1}} A^e \leftarrow \dots$$

and calculate

$$HH_*^W(A) = \begin{cases} A & \text{for } * = 0, \\ A/(\phi'(\pi)) & \text{for } * = 2n-1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus this is also $HH_*(A)$, up to p-adic equivalence.

If $p \mid e$, then

$$V(0)_*HH(A) \cong A/p \otimes E(d\pi) \otimes \Gamma(\mu_A)\,.$$

Here $\mu_A \in V(0)_2HH(A)$ has Bockstein image a generator of the p-torsion in Ω^1_A, say $(\phi'(\pi)/p)d\pi$. Note that $(\Omega^*_A)/p \cong A/p \otimes E(d\pi)$.

In view of the pushout square of commutative S-algebras

$$\begin{array}{ccc} THH(\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ THH(A) & \longrightarrow & HH(A) \end{array}$$

with $V(0)_*THH(\mathbb{Z}) = E(\lambda_1)\otimes P(\mu_1)$, this fits well with the Lindenstrauss–Madsen calculation

$$V(0)_*THH(A) \cong A/p \otimes E(d\pi) \otimes P(\mu_A)\,.$$

Note that $(\Omega^*_A)/p \cong A/p \otimes E(d\pi)$. ((Does μ_1 map to μ_A^p? Does λ_1 suspend to $\gamma_p(\mu_A)$? Can one extract $THH(A)$ from the other three terms in the square above?))

For $p \nmid e$ the answers are more complicated.

Get a weak equivalence

$$B \wedge_A THH(A) \to THH(B)$$

for $B \subset L$ like $A \subset K$, when $K \to L$ is an unramified (thus étale) extension, but not more generally.

((What about $HH_*(k)$ (in the derived sense, over the ground ring $\mathbb{Z}$) and $HH_*(A|K)$?))

VII.3. Log poles.

Quillen's K-theory localization theorem gives a cofiber sequence

$$K(k) \xrightarrow{i_*} K(A) \xrightarrow{j^*} K(K)\,.$$

In homotopy, we get the exact sequence

$$0 \to A^\times \xrightarrow{j^*} K^\times \xrightarrow{v} \mathbb{Z} \to 0$$

where $v\colon \pi \mapsto 1$.

By analogy, Hesselholt and Madsen define the target $THH(A|K)$ for a trace map from $K(K)$ to sit in a cofiber sequence:

$$\begin{tikzcd} THH(k) \arrow[r, "i_*"] & THH(A) \arrow[r, "j"] \arrow[dr, "j^*"'] & THH(A|K) \arrow[r, "\partial"] \arrow[d] & \Sigma THH(k) \\ & & THH(K) & \end{tikzcd}$$

In homotopy we get an extension

$$0 \to \Omega^1_A \xrightarrow{j} \Omega^1_{(A|K)} \xrightarrow{res} k \to 0$$

where

$$\Omega^1_{(A|K)/W} \cong A/(\pi\phi'(\pi))\{\mathrm{dlog}\,\pi\}\,,$$

$j\colon d\pi \mapsto \pi\,\mathrm{dlog}\,\pi$ and $res\colon\ \mathrm{dlog}\,\pi \mapsto 1$.

Now

$$\pi\phi'(\pi) = e\pi^e - p\pi\theta'(\pi) = \pi^e(e - \pi\theta'(\pi)\theta(\pi)^{-1})$$

so $(\pi\phi'(\pi)) \subseteq (\pi^e) = (p)$ for all values of e. Furthermore,

$$\mathrm{dlog}\,p = \mathrm{dlog}(\pi^e\theta(\pi)^{-1}) = (e - \pi\theta'(\pi)\theta(\pi)^{-1})\,\mathrm{dlog}\,\pi\,,$$

so $\operatorname{dlog} p$ generates the p-torsion in $\Omega^1_{(A|K)}$. (The term $\pi\theta'(\pi)\theta(\pi)^{-1}$ is known as the elasticity of $\theta(\pi)$.)

Thus $V(0)_1 THH(A|K) \cong A/p\{\operatorname{dlog} \pi\}$, while $V(0)_2 THH(A|K) \cong A/p\{\kappa_0\}$, where the Bockstein image of κ_0 equals $\operatorname{dlog} p$.

Hesselholt–Madsen prove

$$V(0)_* THH(A|K) \cong A/p \otimes E(\operatorname{dlog} \pi) \otimes P(\kappa_0)\,.$$

Note that $(\Omega^*_{(A|K)})/p \cong A/p \otimes E(\operatorname{dlog} \pi)$, while $V(0)_* THH(\mathbb{Z}_p|\mathbb{Q}_p) \cong P(\kappa_0)$. This formula turns out to hold for all e, divisible by p or not.

One gets a weak equivalence

$$B \wedge_A THH(A|K) \to THH(B|L)$$

for $B \subset L$ like $A \subset K$, when $K \to L$ is a tamely ramified extension, but not more generally.

((Check whether $THH(A|K) \to THH(B|L)^{hG}$ is a weak equivalence for a faithful G-Galois extension $K \to L$, wildly ramified or not.))

((See how κ_0 comes from $V(0)_2 THH(\mathbb{Z}_p|\mathbb{Q}_p)$. Make the homotopy exact sequence for $THH(k)$, $THH(A)$, $THH(A|K)$ clear.))

((Also consider the case of

$$k \leftarrow A = k[x] \to K = k[x, x^{-1}]$$

with $\Omega^1_A = A\{dx\}$, $\Omega^1_{(A|K)} = A\{\operatorname{dlog} x\}$ and $\Omega^1_K = K\{dx\}$.))

VII.4. The trace map.

The trace map $tr\colon K(A) \to THH(A) \to HH(A)$ extends the map $BGL_1(A) \to THH(A) \to HH(A)$ that takes $[a]$ in the bar complex, representing the symbol $\{a\}$ in $K_1(A)$, to $a^{-1} \otimes a$ in the Hochschild complex. This equals the Kähler form $a^{-1}da$, so

$$tr\colon K_1(A) \to \Omega^1_A = THH_1(A) = HH_1(A)$$

satisfies $tr(\{a\}) = a^{-1}da$, for $a \in A^\times$.

Let $U^1_A \subset A^\times$ be the kernel of $A^\times \to k^\times$. The exponential map $\exp(x) = \sum_{n\ge 0} x^n/n!$ defines a group isomorphism $\exp\colon \pi A \to U^1_A$ (for p odd), and the composite

$$\pi A \underset{\cong}{\xrightarrow{\exp}} U^1_A \subset A^\times \to K_1(A) \xrightarrow{tr} \Omega^1_A$$

equals the composite

$$\pi A \subset A \xrightarrow{d} \Omega^1_A\,,$$

because $\exp(\pi x)^{-1} d \exp(\pi x) = d(\pi x)$.

The extended trace map $tr\colon K(K) \to THH(A|K)$ takes $\{\pi\}$ to $\operatorname{dlog} \pi$. ((Justify, almost by definition.))

We have $U^1_A/(U^1_A)^p \cong A^\times/A^{\times p}$, since k is perfect, and $A^\times/A^{\times p} \cong V(0)_1 K(A)$, since A is local(?), so there is an isomorphism $A/p \cong V(0)_1 K(A)$ given by $x \mapsto \{\exp(\pi x)\}$.

This provides a complete calculation of

$$tr\colon A/p \cong V(0)_1 K(A) \to (\Omega^1_A)/p$$

with $tr(\{\exp(\pi x)\}) = d(\pi x)$, and the extension

$$tr\colon V(0)_1 K(K) \to (\Omega^1_{(A|K)})/p \cong A/p\{\operatorname{dlog} \pi\}$$

with $tr(\{\pi\}) = \operatorname{dlog} \pi$. Note that the map $A/p \cong V(0)_1 K(A) \to (\Omega^1_A)/p$, given by $x \mapsto d(\pi x)$, is not A-linear, but is a degree ≤ 1 differential operator.

VIII. Topological André–Quillen (co-)homology (November 22nd)

Let $A \to B$ be a map of commutative S-algebras, X a based space. Each square in

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & X \otimes_A B \arrow[d] \\
B \arrow[r] & B \wedge_A B \arrow[r] \arrow[d, "\mu"'] & X \otimes_B (B \wedge_A B) \arrow[d] \\
 & B \arrow[r] & X \tilde{\otimes}_B (B \wedge_A B)
\end{tikzcd}$$

is a pushout in commutative A-algebras. The composite in the middle column is the identity, so the composite in the right hand column is an equivalence

$$X \otimes_A B \simeq X \tilde{\otimes}_B (B \wedge_A B)\,.$$

Thus the suspension spectrum of $B \wedge_A B$ in the category $\mathcal{C}_B/B$ of augmented commutative B-algebras

$$E_B^\infty(B \wedge_A B) = \{n \mapsto S^n \tilde{\otimes}_B (B \wedge_A B)\}\,,$$

is equivalent to the sequential spectrum

$$\{n \mapsto S^n \otimes_A B\}\,.$$

Here $S^1 \otimes_A B = THH^A(B)$, and we think of $S^n \otimes_A B$ as the n-th higher A-based THH of B.

To compute, we use

$$\mathrm{Tor}^{S^n \otimes_A B}(B, B) = S^{n+1} \otimes_A B$$

to get a spectral sequence

$$E^2_{**} = \mathrm{Tor}^{\pi_*(S^n \otimes_A B)}_{**}(\pi_*(B), \pi_*(B)) \Longrightarrow \pi_*(S^{n+1} \otimes_A B)$$

for $n \geq 0$. Passing to augmentation ideals and their underlying B-modules,

$$TAQ^A(B) \simeq \{n \mapsto I_B(S^n \otimes_A B)\}$$

as B-module (bi-)spectra.