TOPOLOGICAL MODULAR FORMS - I

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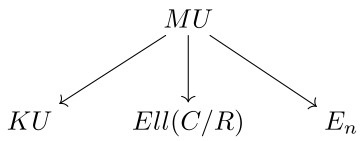
1. Complex cobordism and Elliptic cohomology



1.1. **Formal group laws.** Let G be 1-dimensional Lie group, and let $x\colon U \to \mathbb{R}$ be a coordinate chart in a neighborhood U of the identity element e, with $x(e) = 0$. The group multiplication $G \times G \to G$ defines the germ of a map $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, with $F(x(g_1), x(g_2)) = x(g_1 \cdot g_2)$ near $(0, 0)$. If G is real analytic, the map germ is given by a power series

$$F(x_1, x_2) = x_1 + x_2 + \sum_{i,j\geq 1} a_{i,j} x_1^i x_2^j \,.$$

The identity component of G is commutative, so $a_{i,j} = a_{j,i}$. When viewed as a formal power series in $\mathbb{R}[[x_1, x_2]]$, this F is an example of a 1-dimensional formal group law defined over $\mathbb{R}$. A 1-dimensional commutative formal group law F over a commutative ring R is a formal power series

$$F(x_1, x_2) \in R[[x_1, x_2]]$$

such that $F(x_1, 0) = x_1$, $F(0, x_2) = x_2$, $F(x_1, x_2) = F(x_2, x_1)$ and $F(F(x_1, x_2), x_3) = F(x_1, F(x_2, x_3))$. (This idea generalizes to higher dimensions, but only 1-dimensional commutative formal groups will be relevant here. Formal groups are intermediate between Lie groups (or algebraic groups) and Lie algebras.)

1.2. **The Lazard ring.** Let $FGL(R)$ be the set of (1-dimensional commutative) formal group laws over R. A ring homomorphism $\phi\colon R \to R'$ induces a function $\phi^*\colon FGL(R) \to FGL(R')$, with $(\phi^* F)(x_1, x_2) = x_1 + x_2 + \sum_{i,j\geq 1} \phi(a_{i,j})$. The covariant functor

$$R \longmapsto FGL(R)$$

is corepresentable: there is a universal formal group law

$$F_L(x_1, x_2) = x_1 + x_2 + \sum_{i,j\geq 1} a_{i,j} x_1^i x_2^j$$

defined over a ring $L = \mathbb{Z}[a_{i,j} \mid i, j \geq 1]/{\sim}$, called the Lazard ring, where the relations $\sim$ are those required for F_L to satisfy commutativity and associativity.

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Lazard proved that L is a polynomial ring in countably many variables. The universal propery asserts that the natural map

$$\begin{aligned} \mathrm{Hom}(L, R) &\xrightarrow{\cong} FGL(R) \\ \phi &\longmapsto F = \phi^* F_L \end{aligned}$$

is a bijection for each R. In other words, $\mathrm{Spec}(R) \mapsto FGL(R)$ is a (pre-)sheaf on schemes over $\mathrm{Spec}(\mathbb{Z})$, represented by the affine scheme $\mathrm{Spec}(L)$. We can identify L with (the global sections of) the structure sheaf $\mathscr{O}_{FGL}$.

1.3. **Complex (co-)bordism.** The complex bordism spectrum MU is an E_∞ ring spectrum (= coherently homotopy commutative, brave new ring) given by the sequence of Thom spaces

$$MU_{2n} = Th(\gamma^n \downarrow BU(n)) = EU(n)_+ \wedge_{U(n)} S^{2n}$$

for $n \geq 0$, where γ^n is the tautological $\mathbb{C}^n$-bundle over the infinite complex Grassmannian $BU(n) = Gr_n(\mathbb{C}^\infty)$. The associated homology and cohomology theories, called complex bordism and complex cobordism, associate to each finite CW complex X the graded abelian groups

$$\begin{aligned} MU_*(X) &= \pi_*(MU \wedge X_+) = \operatorname*{colim}_n \pi_{*+2n}(MU_{2n} \wedge X_+) \\ MU^{-*}(X) &= \pi_* F(X_+, MU) = \operatorname*{colim}_n \pi_{*+2n} \mathrm{Map}(X_+, MU_{2n}) \,. \end{aligned}$$

Its coefficient ring

$$MU_* = \pi_*(MU) = \operatorname*{colim}_n \pi_{*+2n}(MU_{2n}) \cong \Omega^U_*$$

is the graded ring of bordism classes of stably almost complex manifolds. Milnor and Novikov, independently, proved that it is a polynomial ring in countably many variables.

1.4. **Quillen's theorem.** There is a class $x \in MU^2(\mathbb{C}P^\infty)$ (a complex orientation, in the sense of Dold) such that

$$\begin{aligned} MU^{-*}(\mathbb{C}P^\infty) &\cong MU_*[[x]] \\ MU^{-*}(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) &\cong MU_*[[x_1, x_2]] \end{aligned}$$

with $x_1 = x \times 1$ and $x_2 = 1 \times x$, and the classifying map $m \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ for the tensor product of complex line bundles induces a ring homomorphism

$$m^* \colon MU^{-*}(\mathbb{C}P^\infty) \longrightarrow MU^{-*}(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

that takes x to a formal group law

$$F_{MU}(x_1, x_2) \in MU_*[[x_1, x_2]]$$

defined over MU_*. It expresses the tensor product law for the first Chern class in complex cobordism. Quillen proved that the representing homomorphism

$$\phi \colon L \xrightarrow{\cong} MU_*$$

with $\phi^* F_L = F_{MU}$ is an isomorphism. In other words, the complex bordism spectrum, with its standard complex orientation, realizes the Lazard ring and the universal formal group law. The algebraic structure sheaf $\mathscr{O}_{FGL} = L$ can be lifted to a topological structure sheaf $\mathscr{O}^{top}_{FGL} = MU$ in E_∞ ring spectra.

1.5. **The Conner–Floyd theorem.** The multiplicative group $\mathbb{G}_m(\mathbb{R}) = GL_1(\mathbb{R})$ admits a coordinate $x\colon U \to \mathbb{R}$ near $e = 1$ given by $x(g) = g - 1$, with $F_m(x_1, x_2) = (x_1+1)(x_2+1) - 1 = x_1 + x_2 + x_1 x_2$, called the multiplicative formal group law. It is defined over $\mathbb{Z}$, and the classifying ring homomorphism $\phi_m\colon L \to \mathbb{Z}$ corresponds to a ring homomorphism $Td\colon MU_* = \Omega^U_* \to \mathbb{Z}$ called the Todd genus. To get a degree-preserving homomorphism we let $R = \mathbb{Z}[u^{\pm 1}]$ with $|u| = 2$ and consider

$$F_m(x_1, x_2) = x_1 + x_2 + u x_1 x_2$$

defined over $\mathbb{Z}[u^{\pm 1}]$, classified by a graded ring homomorphism $Td\colon MU_* \to \mathbb{Z}[u^{\pm 1}]$. Conner and Floyd proved that there are natural isomorphisms

$$MU_*(X) \otimes_{MU_*} \mathbb{Z}[u^{\pm 1}] \cong KU_*(X)$$
$$MU^{-*}(X) \otimes_{MU_*} \mathbb{Z}[u^{\pm 1}] \cong KU^{-*}(X)\,,$$

where KU denotes complex topological K-theory. In particular, the algebraic tensor products on the left hand sides define generalized homology and cohomology theories.

The coherent direct sum and tensor product of vector bundles makes KU an E_∞ ring spectrum. The (graded) algebraic structure sheaf $\mathscr{O}_{\mathbb{G}_m} = \mathbb{Z}[u^{\pm 1}]$ can be lifted to a topological structure sheaf $\mathscr{O}^{top}_{\mathbb{G}_m} = KU$ in E_∞ ring spectra.

1.6. **Landweber's exact functor theorem.** It is perhaps surprising that this works, because $\mathbb{Z}[u^{\pm 1}]$ is not flat as an MU_*-module, so one might expect exactness to fail. Landweber discovered necessary and sufficient conditions on an MU_*-module R_* for the functor

$$X \mapsto MU_*(X) \otimes_{MU_*} R_*$$

to define a generalized homology theory, which is then represented by a spectrum R, so that $R_*(X) \cong MU_*(X) \otimes_{MU_*} R_*$. In particular, $\pi_*(R) \cong R_*$. (We omit to explain Landweber's regularity conditions, but will return to them later.)

1.7. **The Honda formal group law.** Let F be a formal group law defined over R. Write $x_1 +_F x_2 = F(x_1, x_2)$ for the formal sum of x_1 and x_2. For each $m \ge 1$ the m-series

$$[m]_F(x) = x +_F + \cdots +_F x \in R[[x]]$$

expresses multiplication by m in the formal group law. For each prime p and each integer $n \ge 1$ there is a height n Honda formal group law

$$F_n(x_1, x_2) \in \mathbb{F}_{p^n}[[x_1, x_2]]$$

defined over the finite field $\mathbb{F}_{p^n}$, with p-series $[p]_{F_n}(x) = x^{p^n}$. There is a graded version defined over $\mathbb{F}_{p^n}[u^{\pm 1}]$, with $|u| = 2$, with p-series

$$[p]_{F_n}(x) = u^{p^n - 1} x^{p^n}\,.$$

The classifying homomorphism

$$\phi_n\colon MU_* = L \longrightarrow \mathbb{F}_{p^n}[u^{\pm 1}]$$

does not satisfy Landweber's conditions.

1.8. **Lubin–Tate theory.** However, the universal deformation

$$\tilde{F}_n(x_1, x_2) \in \mathbb{WF}_{p^n}[[u_1, \dots, u_{n-1}]][u^{\pm 1}][[x_1, x_2]]$$

of the formal group law F_n, constructed by Lubin and Tate, is classified by a homomorphism

$$\tilde{\phi}_n \colon MU_* = L \longrightarrow \mathbb{WF}_{p^n}[[u_1, \dots, u_{n-1}]][u^{\pm 1}] = E_{n*}$$

that does satisfy Landweber's conditions. Here $\mathbb{WF}_{p^n}$ denotes the ring of Witt vectors on $\mathbb{F}_{p^n}$, which is an unramified extension of degree n of the ring of p-adic integers $\mathbb{Z}_p = \mathbb{WF}_p$, and $u_1, \dots, u_{n-1}$ are deformation parameters with $|u_i| = 0$. Hence the functor

$$X \mapsto MU_*(X) \otimes_{MU_*} E_{n*} = E_{n*}(X)$$

defines a generalized homology theory, represented by a spectrum E_n, called the n-th Lubin–Tate theory. Hopkins, Miller and Goerss proved that E_n is an E_∞ ring spectrum. The (graded) algebraic structure sheaf $\mathscr{O}_{\tilde{F}_n} = \mathbb{WF}_{p^n}[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$ can be lifted to a topological structure sheaf $\mathscr{O}^{top}_{\tilde{F}_n} = E_n$ in E_∞ ring spectra.

The information about finite CW-complexes seen by E_n is said to be of chromatic complexity $\le n$, and is the same information as that seen by closely related spectra called the n-th Morava E-theory, and the n-th Johnson–Wilson spectrum $E(n)$.

Example: $E_1 = KU^\wedge_p$, with $\pi_*(E_1) = E_{1*} = \mathbb{Z}_p[u^{\pm 1}]$, and $E(1)$ is the Adams summand of $KU_{(p)}$, with $\pi_*(E(1)) = \mathbb{Z}_{(p)}[v_1^{\pm 1}]$.

1.9. **Elliptic (co-)homology.** An elliptic curve C defined over a commutative ring R is a flat and proper scheme over $\operatorname{Spec}(R)$, with a chosen unit section, such that each fiber is a smooth genus 1 curve. By choosing a local coordinate x near the unit section, the standard group multiplication $C \times_{\operatorname{Spec}(R)} C \to C$ is given by a rational function, which can be expanded in a power series to define a 1-dimensional commutative formal group law

$$F_{C/R}(x_1, x_2) \in R[[x_1, x_2]] \,.$$

There is also a graded version, defined over $R[u^{\pm 1}]$, with $|u| = 2$. The associated homomorphism

$$\phi_C \colon MU_* = L \longrightarrow R[u^{\pm 1}]$$

may or may not satisfy Landweber's conditions (cf. Franke). When they are satisfied, the resulting theories

$$Ell(C/R)_*(X) = MU_*(X) \otimes_{MU_*} R[u^{\pm 1}]$$
$$Ell(C/R)^{-*}(X) = MU^{-*}(X) \otimes_{MU_*} R[u^{\pm 1}]$$

are called elliptic (co-)homology theories, and are represented by "even periodic" spectra $Ell(C/R)$, with $\pi_*(Ell(C/R)) \cong R[u^{\pm 1}]$.

Examples of elliptic cohomology theories were introduced by Landweber, Ravenel and Stong, and by Ochanine, as receptacles for Witten's genus for string manifolds, which he informally defined in terms of index theory on free loop spaces (ca. 1986). By Deuring–Eichler, the height of an elliptic formal group law at a closed point $\operatorname{Spec}(k) \to \operatorname{Spec}(R)$ is 1 or 2, so elliptic cohomology theories are closely related to the Lubin–Tate theories $E_1 = KU^\wedge_p$ and E_2.

1.10. **Weierstrass curves.** Let $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ and consider the Weierstrass curve $\bar{W}$ of points $(X : Y : Z)$ satisfying

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 ,$$

given by the projective closure of

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 ,$$

defined over A. We set $|x| = 4$, $|y| = 6$ and $|a_i| = 2i$, so that the equation is homogeneous of degree 12. The discriminant $\Delta \in A$ is a polynomial with $|\Delta| = 24$. The Weierstrass curve $\bar{W}$ has singularities, but becomes smooth when we invert Δ. Let W be the resulting elliptic curve defined over $A[\Delta^{-1}]$. The associated formal group law $F_{W/A[\Delta^{-1}]}$ and homomorphism

$$\phi_W \colon MU_* = L \longrightarrow A[\Delta^{-1}]$$

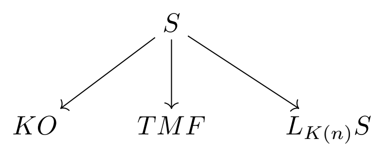
satisfies Landweber's conditions, hence defines the Weierstrass elliptic (co-)homology theory

$$Ell_*(X) = MU_*(X) \otimes_{MU_*} A[\Delta^{-1}]$$
$$Ell^{-*}(X) = MU^{-*}(X) \otimes_{MU_*} A[\Delta^{-1}]$$

with 24-periodic coefficient ring $Ell_* = \mathbb{Z}[a_1, \dots, a_4, a_6][\Delta^{-1}]$.

Each elliptic curve is isomorphic to the pullback of the smooth part of a Weierstrass curve, so $Ell = Ell(W/A[\Delta^{-1}])$ has a weak universal property. However, elliptic curves admit nontrivial automorphisms, so the isomorphism is not unique, and therefore Ell is not a fully universal elliptic cohomology spectrum. To account for this we must pass to the moduli stack of elliptic curves.

2. The sphere spectrum and topological modular forms



2.1. **Strict isomorphisms.** A second coordinate $x' \colon U' \to \mathbb{R}$ near $e \in G$ can be compared with the first coordinate $x \colon U \to \mathbb{R}$, by the germ of a map $f \colon \mathbb{R} \to \mathbb{R}$ with $f(x(g)) = x'(g)$ near 0. We write

$$f(x) = b_0x + \sum_{i \geq 1} b_i x^{i+1} ,$$

with b_0 invertible. If $F' \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the formal group law associated to G and x', then $f \colon F \to F'$ is a formal isomorphism, in the sense that $f(F(x_1, x_2)) = F'(f(x_1), f(x_2))$. We concentrate on the strict case, when $b_0 = 1$.

For formal group laws F and F' defined over R, a strict isomorphism $f \colon F \to F'$ is a formal power series

$$f(x) \in R[[x]]$$

such that $f(0) = 0$, $f(F(x_1, x_2)) = F'(f(x_1), f(x_2))$ and $(df/dx)(0) = 1$. Here f and F determine F', by the formula

$$F'(x_1, x_2) = f(F(f^{-1}(x_1), f^{-1}(x_2))) \in R[[x_1, x_2]] .$$

2.2. **The moduli stack of formal group laws.** For each R, let $\mathscr{M}_{FGL}(R)$ be the (small) groupoid of formal group laws and the strict isomorphisms between them, all defined over R. The set of objects is $\operatorname{obj} \mathscr{M}_{FGL}(R) = FGL(R)$, and the set of morphisms from F to F' is the set of strict isomorphisms

$$f(x) = x + \sum_{i \geq 1} b_i x^{i+1}$$

defined over R, from F to F'. Let $\operatorname{mor} \mathscr{M}_{FGL}(R)$ denote the set of all morphisms in $\mathscr{M}_{FGL}(R)$. A ring homomorphism $\theta \colon R \to R'$ induces a functor $\theta^* \colon \mathscr{M}_{FGL}(R) \to \mathscr{M}_{FGL}(R')$. On objects it is given by $F \mapsto \theta^* F$, as before. On morphisms it is given by $f \mapsto \theta^* f$, where $(\theta^* f)(x) = x + \sum_{i \geq 1} \theta(b_i) x^{i+1} \in R'[[x]]$. We obtain a covariant functor

$$R \longmapsto \mathscr{M}_{FGL}(R)$$

from commutative rings to groupoids. In other words, $\operatorname{Spec}(R) \mapsto \mathscr{M}_{FGL}(R)$ is a (pre-)stack.

There is a universal strict isomorphism $f_{LB} \colon F_{LB} \to F'_{LB}$ defined over the ring $LB = L \otimes B$ where $B = \mathbb{Z}[b_i \mid i \geq 1]$, with $f(x) = x + \sum_{i \geq 1} b_i x^{i+1}$. Here $F_{LB} = \eta_L^* F_L$, where the "left unit" $\eta_L \colon L \to LB$ is the usual inclusion. It follows that F'_{LB} is given by

$$F'_{LB}(x_1, x_2) = f_{LB}(F_{LB}(f_{LB}^{-1}(x_1), f_{LB}^{-1}(x_2))) \in LB[[x_1, x_2]] \,.$$

This is a formal group law over LB, so there is a unique ring homomorphism

$$\eta_R \colon L \longrightarrow LB$$

such that $F'_{LB} = \eta_R^* F_L$. (Here $\eta_R(a_{1,1}) = a_{1,1} + 2b_1$, but explicit formulas for the "right unit" quickly get complicated.) The universal property asserts that the natural map

$$\begin{aligned} \operatorname{Hom}(LB, R) &\longrightarrow \operatorname{mor} \mathscr{M}_{FGL}(R) \\ \theta &\longmapsto \big(\theta^* f_{LB} \colon (\theta\eta_L)^* F_L \longrightarrow (\theta\eta_R)^* F_L\big) \end{aligned}$$

is a bijection.

2.3. **A Hopf algebroid.** The commutative rings L and LB corepresent the object and morphisms sets, respectively, in the covariant functor $R \mapsto \mathscr{M}_{FGL}(R)$ to groupoids. Equivalently, $\operatorname{Spec}(L)$ and $\operatorname{Spec}(LB)$ represent the object and morphism sets in the (pre-)stack

$$\operatorname{Spec}(R) \longmapsto \mathscr{M}_{FGL}(R) \,.$$

The homomorpisms $\eta_L \colon L \to LB$ and $\eta_R \colon L \to LB$ corepresent the source and target maps, respectively, in these groupoids:

$$\begin{aligned} s, t \colon \mathscr{M}_{FGL}(R) &\longrightarrow FGL(R) \\ (f \colon F \to F') &\longmapsto F, F' \,. \end{aligned}$$

The projection $\epsilon \colon LB \to L$ with $\epsilon(b_i) = 0$ for each $i \geq 1$ corepresents the identity morphism map:

$$\begin{aligned} id \colon FGL(R) &\longrightarrow \mathscr{M}_{FGL}(R) \\ F &\longmapsto (id_F \colon F \to F) \,. \end{aligned}$$

The tensor product $LB \otimes_L LB$, where the first copy of LB is viewed as a (right) L-module using η_R, and the second copy of LB is viewed as a (left) L-module using

η_L, corepresents composable pairs (f, f') of strict isomorphisms, where $f\colon F \to F'$ and $f'\colon F' \to F''$. The composite $f' \circ f\colon F \to F''$ is a strict isomorphism, so by the universal property of f_{LB} there is a unique ring homomorphism (the "coproduct", or diagonal)

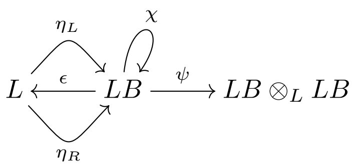
$$\psi\colon LB \longrightarrow LB \otimes_L LB$$

such that $\psi^* f_{LB} = f' \circ f$ ((suitably interpreted)). Finally, each strict isomorphism $f\colon F \to F'$ has an inverse, $f^{-1}\colon F' \to F$, so there is a unique ring homomorphism (the "conjugation", or antipode)

$$\chi\colon LB \longrightarrow LB$$

such that $\chi^* f_{LB} = f_{LB}^{-1}$, which corepresents the passage from f to f^{-1}.

The pair (L, LB), together with the structural morphisms



and the relations they satisfy, corresponding to two-sided unitality and associativity for composition of morphisms, together with the existence of a two-sided inverse, constitute a Hopf algebroid. In total, this is the corepresenting object for the functor $\mathscr{M}_{FGL}$ from commutative rings to groupoids. Equivalently, $(\mathrm{Spec}(L), \mathrm{Spec}(LB))$ and the morphisms η_L, η_R, ϵ, ψ and χ give a presentation of $\mathscr{M}_{FGL}$ as an affine stack.

2.4. **Complex bordism, revisited.** Let S be the sphere spectrum, given by the sequence of spheres $S_n = S^n$ for $n \geq 0$, and let $\eta\colon S \to MU$ and $\mu\colon MU \wedge MU \to MU$ be the unit map and the multiplication map for the E_∞ ring spectrum MU. Consider the maps

$$\begin{aligned}
1 \wedge \eta\colon MU &\cong MU \wedge S \longrightarrow MU \wedge MU \\
\eta \wedge 1\colon MU &\cong S \wedge MU \longrightarrow MU \wedge MU \\
\mu\colon MU \wedge MU &\longrightarrow MU \\
1 \wedge \eta \wedge 1\colon MU \wedge MU &\cong MU \wedge S \wedge MU \\
&\longrightarrow MU \wedge MU \wedge MU \cong (MU \wedge MU) \wedge_{MU} (MU \wedge MU) \\
\gamma\colon MU \wedge MU &\longrightarrow MU \wedge MU
\end{aligned}$$

where γ denotes the twist isomorphism. Quillen also showed that there is a strict isomorphism

$$f\colon (1 \wedge \eta)^* F_{MU} \longrightarrow (\eta \wedge 1)^* F_{MU}$$

defined over $MU_* MU = \pi_*(MU \wedge MU)$, and the representing homomorphism

$$\theta\colon LB \xrightarrow{\cong} MU_* MU$$

is an isomorphism. Furthermore, the Hopf algebroid structure maps of (L, LB) are precisely those obtained by applying π_* to the diagram

$$MU \underset{\eta\wedge 1}{\overset{1\wedge\eta}{\rightrightarrows}} MU \wedge MU \circlearrowleft^{\gamma} \qquad MU \xleftarrow{\mu} MU \wedge MU \xrightarrow{1\wedge\eta\wedge 1} (MU \wedge MU) \wedge_{MU} (MU \wedge MU)$$

One relevant fact is that $MU_*MU \cong LB$ is flat as a left L-module using η_L (hence also as a right L-module, using η_R, due to the existence of the conjugation χ), so that the product

$$MU_*MU \otimes_{MU_*} MU_*MU \xrightarrow{\cong} \pi_*((MU \wedge MU) \wedge_{MU} (MU \wedge MU))$$

is an isomorphism. We say that $(MU_*, MU_*MU) \cong (L, LB)$ is a flat Hopf algebroid.

The (graded) algebraic pair of structure sheaves $\mathscr{O}_{\mathscr{M}_{FGL}} = (L, LB)$, together with the prestack/Hopf algebroid structure maps, can be lifted to a topological pair of structure sheaves $\mathscr{O}^{top}_{\mathscr{M}_{FGL}} = (MU, MU \wedge MU)$ in E_∞ ring spectra, together with the prestack/Hopf algebroid structure maps.

2.5. **Global sections.** The global sections in the algebraic moduli stack $\mathscr{M}_{FGL}$ is the equalizer

$$\Gamma(\mathscr{M}_{FGL}, \mathscr{O}) = \mathrm{eq}(L \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} LB) \cong \mathbb{Z}\,.$$

The natural generalization of global sections for the topological moduli stack is the totalization

$$\begin{aligned}\Gamma(\mathscr{M}_{FGL}, \mathscr{O}^{top}) &= \mathrm{Tot}(MU \rightrightarrows MU \wedge MU \substack{\rightarrow\\\rightarrow\\\rightarrow} MU \wedge MU \wedge MU \quad \dots) \\ &\simeq S\end{aligned}$$

of the cosimplicial E_∞ ring spectrum given in codegree s by the smash product

$$(MU \wedge MU) \wedge_{MU} \cdots \wedge_{MU} (MU \wedge MU) \cong MU^{\wedge(s+1)}\,.$$

As indicated, this totalization is equivalent to the sphere spectrum. The associated spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{LB}(L, L) \Longrightarrow_s \pi_{t-s}(S)$$

is the Adams–Novikov spectral sequence. Here the E_2-term can in principle be calculated as the cohomology of the cobar complex

$$0 \to L \xrightarrow{d^0} LB \xrightarrow{d^1} LB \otimes_L LB \xrightarrow{d^2} \dots\,.$$

The edge homomorphism from topological to algebraic global sections is the degree map

$$\pi_*(S) \longrightarrow E_2^{0,*} \cong \mathbb{Z}\,.$$

It is a rational isomorphism, but the kernel contains extremely subtle p-torsion for all primes p.

2.6. **Real topological K-theory.** ((Recover $KO = KU^{hC_2}$ as global sections of the moduli stack $\mathscr{M}_{\mathbb{G}_m} \sim BC_2$ of multiplicative groups.))

2.7. **Morava's change-of-rings theorem.** ((Devinatz–Hopkins: Recover $L_{K(n)}S = E_n^{hG_n}$ as the homotopy fixed points for the action of the extended Morava stabilized group.))

2.8. **Landweber's exact functor theorem, revisited.** Miller (with Hopkins?) has reformulated the conditions for Landweber's exact functor theorem as saying that for any given ring homomorphism $MU_* = L \to R$, the functor

$$X \mapsto MU_*(X) \otimes_{MU_*} R$$

defines a generalized homology theory $R_*(X)$ if and only if the morphism

$$\mathrm{Spec}(R) \longrightarrow \mathscr{M}_{FGL}$$

is flat, in the sense of morphisms of algebraic stacks.

2.9. **The moduli stack of elliptic curves.** For each R, let $\mathscr{M}_{Ell}(R)$ be the groupoid of elliptic curves and isomorphisms between them, all defined over R. For each ring homomorphism $\phi\colon R \to R'$ there is a functor

$$\phi^*\colon \mathscr{M}_{Ell}(R) \longrightarrow \mathscr{M}_{Ell}(R')$$

given by pullback of flat and proper schemes over $\phi^{op}\colon \mathrm{Spec}(R') \to \mathrm{Spec}(R)$.

For each R, we obtain a functor

$$F(R)\colon \mathscr{M}_{Ell}(R) \longrightarrow \mathscr{M}_{FGL}(R)$$

that takes an elliptic curve C to its associated formal group law F_C, and takes an isomorphism $f\colon C \to C'$ of elliptic curves to the induced strict isomorphism $f\colon F_C \to F_{C'}$, all defined over R.

WARNING: This step has been oversimplified. To get a formal group *law* F_C we must equip the elliptic curves with local coordinates. To get a *strict* isomorphism we must keep track of tangential data. These issues are related.

2.10. **The Goerss–Hopkins–Miller theorem.** The stack morphism

$$F\colon \mathscr{M}_{Ell} \longrightarrow \mathscr{M}_{FGL}$$

is flat. Hence for any flat morphism $C\colon \mathrm{Spec}(R) \to \mathscr{M}_{Ell}$, which corresponds to a suitable elliptic curve C defined over R, the composite

$$FC\colon \mathrm{Spec}(R) \to \mathscr{M}_{Ell} \to \mathscr{M}_{FGL}$$

is flat. Thus Landweber's exact functor theorem applies, and the functor

$$X \mapsto MU_*(X) \otimes_{MU_*} R[u^{\pm 1}] = Ell(C/R)_*(X)$$

is a (multiplicative) homology theory, represented by a (ring) spectrum $Ell(C/R)$. We get a presheaf of (multiplicative) (co-)homology theories

$$(C\colon \mathrm{Spec}(R) \xrightarrow{\text{flat}} \mathscr{M}_{Ell}) \overset{\mathscr{O}^*}{\longmapsto} Ell(C/R)_*(-)$$

on the flat site of $\mathscr{M}_{Ell}$, which lifts the graded version

$$(C\colon \mathrm{Spec}(R) \xrightarrow{\text{flat}} \mathscr{M}_{Ell}) \overset{\mathscr{O}}{\longmapsto} R[u^{\pm 1}]$$

of the algebraic structure sheaf of $\mathscr{M}_{Ell}$.

One might try to lift the presheaf of (co-)homology theories to a presheaf of spectra. We know this can be done one object at a time, but to do this functorially in the morphisms of $\mathscr{M}_{Ell}$ requires hard work.

Hopkins and Miller proved that if one passes to the coarser site consisting only of étale morphisms $C\colon \operatorname{Spec}(R) \to \mathscr{M}_{Ell}$, and asks the apparently harder question of lifting the presheaf of multiplicative cohomology theories to a presheaf of A_∞ ring spectra, then such a lift does exist. This involves an obstruction theory related to Hochschild cohomology, and in this framework the obstruction groups (to existence) vanish. Thus there exists a presheaf of A_∞ ring spectra

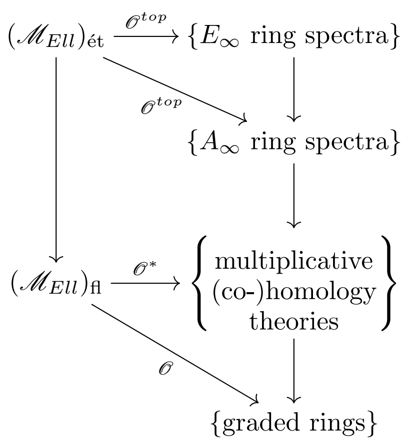
$$(C\colon \operatorname{Spec}(R) \overset{\text{étale}}{\longrightarrow} \mathscr{M}_{Ell}) \overset{\mathscr{O}^{top}}{\longmapsto} Ell(C/R)$$

on the étale site of $\mathscr{M}_{Ell}$.

Goerss, Hopkins and Miller then developed a refined obstruction theory for lifting the presheaf of multiplicative cohomology theories to a presheaf of E_∞ ring spectra. This involves an obstruction theory related to André–Quillen cohomology, and again the obstruction groups (to existence) vanish. Thus there exists a presheaf of E_∞ ring spectra

$$(C\colon \operatorname{Spec}(R) \overset{\text{étale}}{\longrightarrow} \mathscr{M}_{Ell}) \overset{\mathscr{O}^{top}}{\longmapsto} Ell(C/R)$$

on the étale site of $\mathscr{M}_{Ell}$:



The algebraic structure sheaf of $(\mathscr{M}_{Ell})_{\text{ét}}$ can thus be lifted to a topological strucure sheaf $\mathscr{O}^{top}$ in E_∞ ring spectra.

In both cases the obstruction groups to uniqueness do not vanish. Lurie uses a modified construction, using ∞-categories, to get an obstruction problem where both the obstructions to existence and the obstructions to uniqueness vanish, hence giving a uniquely defined construction of the sheaf $\mathscr{O}^{top}$ of E_∞ ring spectra.

2.11. **Topological Modular Forms.**

$$\begin{array}{ccc} S & \longrightarrow & MU \\ \downarrow & & \downarrow \\ TMF & \longrightarrow & Ell(C/R) \end{array}$$

The global sections in the algebraic moduli stack $\mathscr{M}_{Ell}$ is the ring

$$\Gamma(\mathscr{M}_{Ell}, \mathscr{O}) = MF_* \cong \frac{\mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]}{(c_4^3 - c_6^2 = 1728\Delta)}$$

of integral modular forms, as calculated by Deligne/Tate. Here $|c_4| = 8$ and $|c_6| = 12$.

The global sections in the corresponding topological moduli stack is by definition the Topological Modular Forms spectrum

$$TMF = \Gamma(\mathscr{M}_{Ell}, \mathscr{O}^{top}) .$$

There is a descent spectral sequence

$$E_2^{s,t} = H^s_{\text{ét}}(\mathscr{M}_{Ell}, \omega^{\otimes t/2}) \Longrightarrow_s \pi_{t-s}(TMF) ,$$

called the elliptic spectral sequence. (The terms with t odd are zero.) The edge homomorphism from topological to algebraic global sections is a map

$$TMF_* = \pi_*(TMF) \longrightarrow E_2^{0,*} = MF_*$$

from topological modular forms to integral modular forms. It is a rational isomorphism. Both the kernel and cokernel are nontrivial, but consist only of 2- and 3-torsion. For example, $\Delta \in MF_{24}$ is not in the image from TMF_*, but 24Δ and Δ^{24} are.

Borcherds has shown that the theta-functions of even unimodular lattices, which are modular forms, satisfy certain congruences. Hopkins has shown that these congruences are those satisfied by the image of TMF_* in MF_*. Hence theta-functions of even unimodular lattices "are" topological modular forms.

2.12. **Weierstrass curves, revisited.** Under the linear coordinate change

$$\begin{aligned} x &\mapsto x + r \\ y &\mapsto y + sx + t \end{aligned}$$

with $r, s, t \in R$ a Weierstrass curve

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_1, \dots, a_4, a_6 \in R$ is mapped isomorphically to another Weierstrass curve

$$y^2 + a'_1xy + a'_3y = x^3 + a'_2x^2 + a'_4x + a'_6$$

also defined over R. Recall that $|a_i| = 2i$, and set $|r| = 4$, $|s| = 2$ and $|t| = 6$. Let $\mathscr{M}_{Wei}(R)$ be the groupoid of Weierstrass curves and linear isomorphisms, all defined over R. Letting R vary we obtain a Weierstrass (pre-)stack $\mathscr{M}_{Wei}$. It is corepresented by a Hopf algebra (A, Λ), where

$$\begin{aligned} A &= \mathbb{Z}[a_1, \dots, a_4, a_6] \\ \Lambda &= A[r, s, t] . \end{aligned}$$

Here $\eta_L \colon A \to \Lambda$ is the usual inclusion, while $\eta_R : A \to \Lambda$ maps a_i to a'_i. (For example, $\eta_R(a_1) = a'_1 = a_1 + 2s$.)

Inverting $\Delta \in A$, we obtain a (pre-)stack $\mathscr{M}_{Wei}[\Delta^{-1}]$ of elliptic curves and isomorphisms, corepresented by $(A[\Delta^{-1}], \Lambda[\Delta^{-1}])$. The functor

$$\mathscr{M}_{Wei}[\Delta^{-1}] \xrightarrow{\sim} \mathscr{M}_{Ell}$$

is an equivalence of (pre-)stacks, and induces cohomology isomorphisms

$$H^s_{\text{ét}}(\mathscr{M}_{Ell}, \omega^{\otimes t/2}) \xrightarrow{\cong} \operatorname{Ext}^{s,t}_{\Lambda}(A, A)[\Delta^{-1}] .$$

The ring of integral modular forms is thus the equalizer

$$MF_* = \mathrm{eq}(\,A[\Delta^{-1}] \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} \Lambda[\Delta^{-1}]\,)$$

$$= \mathrm{eq}(\,A \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} \Lambda\,)[\Delta^{-1}]$$

Furthermore, the E_2-term of the elliptic spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_\Lambda^{s,t}(A, A)[\Delta^{-1}] \Longrightarrow_s \pi_{t-s}(TMF)$$

is obtained from the cohomology of the cobar complex

$$0 \to A \xrightarrow{d^0} \Lambda \xrightarrow{d^1} \Lambda \otimes_A \Lambda \xrightarrow{d^2} \dots$$

by inverting Δ.

2.13. **Connective topological modular forms.** It is desirable to realize the unlocalized form of the elliptic spectral sequence as a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_\Lambda^{s,t}(A, A) \Longrightarrow_s \pi_{t-s}(tmf)$$

for a connective E_∞ ring spectrum tmf, with $TMF_* \cong \pi_*(tmf)[\Delta^{-24}]$.

This can be done, but requires an intermediate step. There is a Deligne–Mumford compactification $\bar{\mathscr{M}}_{Ell}$ of the moduli stack of elliptic curves, where additional curves with nodal singularities are permitted. The sheaf $\mathscr{O}^{top}$ extends over $\bar{\mathscr{M}}_{Ell}$, and one can consider its E_∞ ring spectrum

$$Tmf = \Gamma(\bar{\mathscr{M}}_{Ell}, \mathscr{O}^{top})$$

of global sections. Here $\mathscr{M}_{Ell} \subset \bar{\mathscr{M}}_{Ell}$ is the substack where Δ is invertible, so $TMF \simeq Tmf[\Delta^{-24}]$. The compactness of $\bar{\mathscr{M}}_{Ell}$ leads to a form of Serre duality for $\pi_*(Tmf)$, called Anderson duality (see papers by Stojanoska), and the connective cover

$$tmf = Tmf[0, \infty)$$

of Tmf is the sought-after connective topological modular forms spectrum.

Hopkins–Mahowald (see a paper by Mathew) showed that

$$H^*(tmf; \mathbb{F}_2) \cong \mathscr{A} \otimes_{A(2)} \mathbb{F}_2 = \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2, Sq^4\}$$

as a module over the mod 2 Steenrod algebra $\mathscr{A}$, where $A(2) = \langle Sq^1, Sq^2, Sq^4\rangle$ is a finite subalgebra of $\mathscr{A}$. This can be taken as the basis for a calculation of the 2-completed homotopy groups of tmf, by means of the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(tmf; \mathbb{F}_2), \mathbb{F}_2) \Longrightarrow_s \pi_{t-s}(tmf)^\wedge_2\,.$$

Here the E_2-term can be rewritten as

$$E_2^{s,t} \cong \mathrm{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)\,.$$

We return to this in later talks.