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The Segal conjecture for smash powers

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Dedicated to our PhD advisor and grand-advisor Gunnar Carlsson, on the occasion of his 70th birthday

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Abstract

We prove that the comparison map from G-fixed points to G-homotopy fixed points, for the G-fold smash power of a bounded below spectrum B, becomes an equivalence after p-completion if G is a finite p-group and $H_*(B; \mathbb{F}_p)$ is of finite type. We also prove that the map becomes an equivalence after $I(G)$-completion if G is any finite group and $\pi_*(B)$ is of finite type, where $I(G)$ is the augmentation ideal in the Burnside ring.

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1 | INTRODUCTION

Let G be a finite group, let B be a flat orthogonal spectrum, and let

$$B^{\wedge G} = \bigwedge_{g \in G} B$$

be the G-fold smash power of B, that is, the smash product of one copy of B for each element of G. The group G acts from the left on $B^{\wedge G}$ by permuting the smash factors, and the resulting orthogonal spectrum with G-action prolongs essentially uniquely to an orthogonal G-spectrum indexed on any given choice of a complete G-universe. This construction is originally due to Marcel Bökstedt (ca. 1987, cf. [10, §2.4]), who worked in the context of functors with smash product. In the context of orthogonal spectra, it is the special case $B^{\wedge G} = N_{\{e\}}^G B$ of the Hill–Hopkins–Ravenel [11] norm. When $B = S$ is the sphere spectrum, this construction produces the G-equivariant sphere spectrum $S^{\wedge G} = S_G$.

For any G-spectrum X, there is a comparison map

$$\gamma : X^G = F(S^0, X)^G \longrightarrow F(EG_+, X)^G = X^{hG}$$

from G-fixed points to G-homotopy fixed points, induced by the collapse map $c : EG_+ \to S^0$. Let p be a prime and suppose for a little while that G is a p-group. We then say that the (generalized) Segal conjecture holds for the G-spectrum X if the comparison map γ becomes an equivalence after p-completion. When $X = S_G$, this is equivalent to Graeme Segal's Burnside ring conjecture for the p-group G, in the strong form proved by Gunnar Carlsson. We adapt the overall strategy [1, 5, 6, 19] from Carlsson's proof to establish the following result, which specializes to his theorem in the case $B = S$.

Theorem 1.1. *Let p be a prime, G a finite p-group, and B a flat orthogonal spectrum. Suppose that $\pi_*(B)$ is bounded below and that $H_*(B; \mathbb{F}_p)$ is of finite type. Then the Segal conjecture holds for the smash power G-spectrum $B^{\wedge G}$. In other words, the comparison map*

$$\gamma : \left(B^{\wedge G}\right)^G \longrightarrow (B^{\wedge G})^{hG}$$

becomes an equivalence after p-completion.

The proof is given near the end of Section 3 for G not elementary abelian, and at the end of Section 5 for $G \cong (C_p)^k$ with $k \geqslant 1$. The main novelty of our work concerns how we deal with the fact that in general the G-spectrum $B^{\wedge G}$ is not split in the sense of May–McClure [16, Definition 10], so that [6, Theorem B] does not apply, even though this is so for $B = S$.

When $G \cong C_p$, the theorem was proved earlier by Sverre Lunøe-Nielsen and the second author [14, Theorem 5.13], and the case $G \cong C_{p^n}$ was proved by these authors together with Marcel Bökstedt and Robert Bruner in [3, Theorem 2.7]. For $G \cong C_p$, the finite-type hypothesis on $H_*(B; \mathbb{F}_p)$ was subsequently lifted by Nikolaus–Scholze in [18, Theorem III.1.7]. We do not know whether the finite-type hypothesis can be removed for G containing elementary abelian p-groups $(C_p)^k$ of rank $k \geqslant 2$.

Now return to the case of a general finite group G. Let $I(G) \subset A(G)$ denote the augmentation ideal in the Burnside ring. For any G-spectrum X, the comparison map

$$\gamma : X^G \xrightarrow{\iota} (X^\wedge_{I(G)})^G \xrightarrow{\xi^*} X^{hG}$$

extends naturally over the spectrum-level $I(G)$-completion map here denoted as ι, cf. Greenlees–May [8, §4]. We now say that the (generalized) Segal conjecture holds for the G-spectrum X when the natural extension ξ^* is an equivalence. When $I(G)$-completion induces $I(G)$-adic completion at the level of G-equivariant homotopy groups, May–McClure [16, p. 217] refer to this assertion about ξ^* as the completion conjecture, and such results are referred to as completion theorems in [8]. In particular, the Segal (or completion) conjecture for the G-spectrum S_G is equivalent to the strong form of Segal's Burnside ring conjecture for the general finite group G.

When G is a p-group, it follows from work of Kári Ragnarsson [21, Theorem C], adapting [16, Proposition 14] to the nonsplit case, that the two formulations just given of the Segal conjecture agree for bounded below G-spectra X with $\pi_*(X)$ of finite type, since the comparison map γ becomes an equivalence after p-completion if and only if it becomes one after $I(G)$-adic completion. See Proposition 6.2. Hence, we can apply [16, Theorem 13] to deduce the following form of the Segal conjecture for general finite groups and their smash power spectra.

Theorem 1.2. *Let G be a finite group and B a flat orthogonal spectrum. Suppose that $\pi_*(B)$ is bounded below and of finite type. Then the Segal conjecture holds for the smash power G-spectrum $B^{\wedge G}$. In other words, the natural map*

$$\xi^* : \left((B^{\wedge G})^{\wedge}_{I(G)}\right)^G \longrightarrow (B^{\wedge G})^{hG}$$

is an equivalence, inducing an isomorphism

$$\pi_*(\gamma)^{\wedge}_{I(G)} : \pi_*\left((B^{\wedge G})^G\right)^{\wedge}_{I(G)} \xrightarrow{\cong} \pi_*\left((B^{\wedge G})^{hG}\right)$$

of $I(G)$-adically completed homotopy groups.

We give the proof at the end of Section 6. We make the assumption that $\pi_*(B)$ is of finite type in order to ensure that the spectrum-level $I(G)$-completion induces algebraic $I(G)$-adic completion at the level of G-equivariant homotopy groups, so as to be able to refer directly to the algebraic induction theory of [16]. Presumably, this can be sidestepped by carrying out the induction theory closer to the spectrum level.

The first author obtained a proof of Theorem 1.1 around April 2013, and lectured on the result at a conference in December 2015 [2]. The second author returned to the argument in May and June 2022, finding some simplifications that have been incorporated into the present account. We apologize for the long delay in publication. In his July 2022 ICM address [17, Remark 7.11], Thomas Nikolaus conjectured that Theorem 1.1 also holds without the finite-type assumption on mod p homology. As mentioned above, we do not know how to remove this hypothesis.

The hallmark signs of Gunnar Carlsson's breakthrough approach to the classical Segal conjecture are evident throughout our paper. We heartily congratulate him on the occasion of his anniversary.

2 | ISOTROPY SEPARATION AND *S*-FUNCTORS

To prove the Segal conjecture for the G-spectra $B^{\wedge G}$, we follow Carlsson and inductively assume that it holds for the J-spectra $C^{\wedge J}$ for all proper subquotient groups $J = K/H$ of G. This is useful, because of the following proposition.

Proposition 2.1. *Let $H \lhd K \subset G$, let $J = K/H$, and let B be a flat orthogonal spectrum.*

(a) *The restriction $\mathrm{res}^G_K(B^{\wedge G})$ along $K \subset G$ of the G-spectrum $B^{\wedge G}$ is equivalent to the K-spectrum $C^{\wedge K}$, where*

$$C = B^{\wedge G/K} = \bigwedge_{Kg \in G/K} B$$

is the smash product of one copy of B for each right coset of K in G.

(b) *The geometric H-fixed point spectrum $\Phi^H(C^{\wedge K})$ of the K-spectrum $C^{\wedge K}$ is equivalent to the J-spectrum $C^{\wedge J}$.*

(c) *If $\pi_*(B)$ is bounded below and $H_*(B; \mathbb{F}_p)$ is of finite type, then $\pi_*(C)$ is bounded below and $H_*(C; \mathbb{F}_p)$ is of finite type.*

(d) *If $\pi_*(B)$ is bounded below and of finite type, then $\pi_*(C)$ is bounded below and of finite type.*

Proof. See [10, Proposition 2.5] or [11, Proposition B.209] for part (b). The remaining claims are clear. □

As usual, let EG denote any G-CW space with $EG^{\{e\}}$ contractible and EG^K empty for each nontrivial subgroup $K \subset G$. Also, let $E\mathcal{P}$ denote any G-CW space with $E\mathcal{P}^K$ contractible for each proper subgroup $K \subset G$ and $E\mathcal{P}^G$ empty. Define $\widetilde{EG}$ and $\widetilde{E\mathcal{P}}$ by the homotopy cofiber sequences

$$EG_+ \xrightarrow{c} S^0 \longrightarrow \widetilde{EG}$$
$$E\mathcal{P}_+ \xrightarrow{c} S^0 \longrightarrow \widetilde{E\mathcal{P}}\,,$$

where the collapse maps c send EG and $E\mathcal{P}$ to the nonbase point of S^0. Following [5, §III], let $\rho = \ker(\epsilon\colon \mathbb{C}\{G\} \to \mathbb{C})$ be the reduced regular complex representation of G. Then $S^{\infty\rho}$ is a model for $\widetilde{E\mathcal{P}}$, conveniently filtered by the G-CW subspaces

$$S^0 \subset \cdots \subset S^{m\rho} \subset \cdots \subset S^{\infty\rho}\,. \tag{2.1}$$

Proposition 2.2 ([5, Theorem A(b)], [6, Lemma 1.9]). *Let G be a nontrivial p-group and suppose that Theorem 1.1 holds for each proper subgroup of G. Let B be a flat orthogonal spectrum with $\pi_*(B)$ bounded below and $H_*(B;\mathbb{F}_p)$ of finite type, and suppose also that*

$$F(S^{\infty\rho}, B^{\wedge G})^G$$

becomes trivial after p-completion. Then

$$\gamma\colon (B^{\wedge G})^G \longrightarrow (B^{\wedge G})^{hG}$$

becomes an equivalence after p-completion.

Proof. Consider the commutative square

$$\begin{array}{ccc} F(S^0, B^{\wedge G})^G & \xrightarrow{\gamma} & F(EG_+, B^{\wedge G})^G \\ \downarrow & & \downarrow \\ F(E\mathcal{P}_+, B^{\wedge G})^G & \longrightarrow & F(EG_+ \wedge E\mathcal{P}_+, B^{\wedge G})^G\,. \end{array}$$

The right-hand arrow is an equivalence because $EG \times E\mathcal{P} \simeq_G EG$. The lower arrow is a homotopy limit of maps

$$(B^{\wedge G})^K \cong F(G/K_+, B^{\wedge G})^G \longrightarrow F(EG_+ \wedge G/K_+, B^{\wedge G})^G \simeq (B^{\wedge G})^{hK}\,,$$

with K ranging over the proper subgroups of G, and therefore, becomes an equivalence after p-completion by the inductive hypothesis and Proposition 2.1(a,c). The left-hand arrow becomes an equivalence after p-completion if and only if its homotopy fiber, namely, $F(S^{\infty\rho}, B^{\wedge G})^G$, becomes trivial after p-completion. □

We continue to follow Carlsson's strategy of isotropy separation, considering the homotopy cofiber sequence

$$F(S^{\infty\rho}, \Sigma^{-1}\widetilde{EG} \wedge B^{\wedge G})^G \xrightarrow{\delta} F(S^{\infty\rho}, EG_+ \wedge B^{\wedge G})^G \xrightarrow{c} F(S^{\infty\rho}, B^{\wedge G})^G . \tag{2.2}$$

Clearly, $F(S^{\infty\rho}, B^{\wedge G})^G$ becomes trivial after p-completion if and only if the connecting map δ becomes an equivalence after p-completion, and this is what we will verify. We note that

$$F(S^{\infty\rho}, \Sigma^{-1}\widetilde{EG} \wedge B^{\wedge G})^G \simeq \operatorname*{holim}_m F(S^{m\rho}, \Sigma^{-1}\widetilde{EG} \wedge B^{\wedge G})^G$$

and

$$\begin{aligned} F(S^{\infty\rho}, EG_+ \wedge B^{\wedge G})^G &\simeq \operatorname*{holim}_m F(S^{m\rho}, EG_+ \wedge B^{\wedge G})^G \\ &\simeq \operatorname*{holim}_m (\Sigma^{2m} EG_+ \wedge (\Sigma^{-2m}B)^{\wedge G})^G \\ &\simeq \operatorname*{holim}_m \Sigma^{2m} EG_+ \wedge_G (\Sigma^{-2m}B)^{\wedge G} \\ &= \operatorname*{holim}_m \Sigma^{2m} D_G(\Sigma^{-2m}B) . \end{aligned}$$

Here the first two equivalences are induced by the filtration (2.1), the third equivalence follows from an identification $S^{m\rho} \wedge S^{2m} \cong (S^{2m})^{\wedge G}$, and the fourth equivalence is a case of the Adams transfer equivalence [12, Theorem II.7.1]. The final identity uses the notation

$$D_G B = EG_+ \wedge_G B^{\wedge G}$$

for the G-fold extended power of any spectrum B, where G is viewed as a subgroup of the symmetric group on $|G|$ elements. In particular, $D_G S = BG_+$. The map in the limit system that corresponds to restriction along $S^{m\rho} \subset S^{(m+1)\rho}$ is then the twisted diagonal map

$$\Sigma^{2(m+1)} D_G(\Sigma^{-2(m+1)}B) = \Sigma^{2m}\Sigma^2 D_G(\Sigma^{-2(m+1)}B) \xrightarrow{\Delta} \Sigma^{2m} D_G(\Sigma^2\Sigma^{-2(m+1)}B) \simeq \Sigma^{2m} D_G(\Sigma^{-2m}B)$$

of [4, Definition II.3.1], associated to the based CW space S^2. For brevity, we introduce the following notations.

Definition 2.3. Let

$$V(G,B) = \operatorname*{holim}_m F(S^{m\rho}, \Sigma^{-1}\widetilde{EG} \wedge B^{\wedge G})^G$$

and

$$W(G,B) = \operatorname*{holim}_m \Sigma^{2m} D_G(\Sigma^{-2m}B)$$

define functors of B, so that there is a natural homotopy cofiber sequence

$$V(G,B) \xrightarrow{\delta} W(G,B) \longrightarrow F(S^{\infty\rho}, B^{\wedge G})^G .$$

For G-spectra X, the spectra $F(S^{\infty\rho}, \widetilde{EG} \wedge X)^G$, hence also the spectra $V(G, B)$, have been fully analyzed by means of Carlsson's theory of S-functors [5, §§IV–VI]. Recall that an elementary abelian p-group is a group of the form $G \cong (C_p)^k$. The rank $k \geqslant 1$ Tits building $\mathcal{T}_k$ is the classifying space of the partially ordered set of proper, nontrivial subgroups of $(C_p)^k$, and by the Solomon–Tits theorem [23], its double suspension

$$\Sigma^2 \mathcal{T}_k \simeq \bigvee^{p^{\binom{k}{2}}} S^k$$

has the homotopy type of a finite wedge sum of k-spheres. Here $p^{\binom{k}{2}}$ denotes p raised to the power $\binom{k}{2} = k(k-1)/2$. The wedge sum in the following result suggested the use of the letter "V" in $V(G, B)$.

Theorem 2.4 ([5, §§IV–VI], [6, §§3–4]). *Let G be a nontrivial p-group and suppose that Theorem 1.1 holds for each proper subquotient of G. Let B be a flat orthogonal spectrum with $\pi_*(B)$ bounded below and $H_*(B; \mathbb{F}_p)$ of finite type.*

(a) *If $G = (C_p)^k$, then there are natural equivalences*

$$V(G, B)^\wedge_p \simeq F(\Sigma^2 \mathcal{T}_k, B)^\wedge_p \simeq \bigvee^{p^{\binom{k}{2}}} \Sigma^{-k} B^\wedge_p .$$

(b) *If G is not elementary abelian, then $V(G, B)^\wedge_p \simeq *$.*

Proof. This is the special case $k_G = B^{\wedge G}$, $j = k_{G/G} = \Phi^G(B^{\wedge G})$ of Caruso–May–Priddy's [6, Theorem A], in view of the equivalence $\Phi^G(B^{\wedge G}) \simeq B$ recalled in Proposition 2.1(b). □

As pointed out in [6, Remark 8.4], for $G = (C_p)^k$, there is a natural action of $GL_k(\mathbb{Z}/p)$ on the terms in the sequence (2.2), and the maps are $GL_k(\mathbb{Z}/p)$-equivariant. This uses that the G-actions on $B^{\wedge G}, EG_+ \to S^0 \to \widetilde{EG}$ and $S^{\infty\rho}$ all extend to permutation actions by the symmetric group $\Sigma_{|G|}$. Hence, the normalizer N of G in $\Sigma_{|G|}$ acts naturally on the G-fixed point spectra in (2.2), and these actions factor through the Weyl group N/G. This normalizer is classically known as the holomorph of G, and is isomorphic to the semidirect product $\mathrm{Aut}(G) \ltimes G$ for the tautological action of the automorphism group $\mathrm{Aut}(G)$ on G. In the case $G = (C_p)^k$, the normalizer is the semidirect product $N \cong GL_k(\mathbb{Z}/p) \ltimes (C_p)^k$. Here, the Weyl group $N/G = GL_k(\mathbb{Z}/p)$ acts linearly on $(C_p)^k$, via $\mathbb{Z}/p = \mathrm{End}(C_p)$.

Similarly, $GL_k(\mathbb{Z}/p)$ acts on the partially ordered set of proper, nontrivial subgroups of $(C_p)^k$, hence also on $\mathcal{T}_k$ and $F(\Sigma^2\mathcal{T}_k, B)^\wedge_p$, and the first equivalence in Theorem 2.4(a) respects these $GL_k(\mathbb{Z}/p)$-actions. The induced action on

$$\mathrm{St}_k := H_k(\Sigma^2 \mathcal{T}_k; \mathbb{Z}) \cong \bigoplus^{p^{\binom{k}{2}}} \mathbb{Z}$$

is the Steinberg representation. Let $U_k(\mathbb{Z}/p) \subset GL_k(\mathbb{Z}/p)$ be the subgroup of upper triangular matrices with "ones" on the diagonal. This is a p-Sylow subgroup, of order $p^{\binom{k}{2}}$, and the Solomon–Tits theorem also says that the restriction along $U_k(\mathbb{Z}/p) \subset GL_k(\mathbb{Z}/p)$ of the Steinberg representation is the regular integral representation.

Addendum 2.5. *The homotopy cofiber sequence (2.2) is $GL_k(\mathbb{Z}/p)$-equivariant, and the equivalence in Theorem 2.4(a) can be written $U_k(\mathbb{Z}/p)$-equivariantly as*

$$V(G,B)^\wedge_p \simeq U_k(\mathbb{Z}/p)_+ \wedge \Sigma^{-k} B^\wedge_p .$$

Proof. As reviewed above, the Solomon–Tits equivalence can be written $U_k(\mathbb{Z}/p)$-equivariantly as $\Sigma^2 \mathcal{T}_k \simeq U_k(\mathbb{Z}/p)_+ \wedge S^k$, and the rest of the analysis respects this action. □

As stated at the beginning of this section, we assume throughout the remainder of the paper that Theorem 1.1 holds for each proper subquotient of G. In particular, the inductive hypotheses in Proposition 2.2 and Theorem 2.4 are satisfied when G is a p-group.

3 | TOWERS OF EXTENDED POWERS

In the papers [5, §III], [6, §8], the spectrum $F(S^{\infty\rho}, EG_+ \wedge X)^G$ is analyzed under the hypothesis that the G-spectrum X is split. This enables a translation into nonequivariant terms, involving the X-homology of a tower

$$\cdots \longrightarrow BG^{-(m+1)\rho} \longrightarrow BG^{-m\rho} \longrightarrow \cdots \longrightarrow BG_+$$

of Thom spectra. The smash power G-spectra $X = B^{\wedge G}$ are not generally split. (For example, with $B = H = H\mathbb{F}_p$ and $G = C_p$, we have $\pi_0((B^{\wedge G})^G) \cong \mathbb{Z}/p^2$ by a variant of [10, Theorem 3.3], and this group does not contain $\pi_0(B^{\wedge G}) \cong \mathbb{F}_p$ as a direct summand.) We shall therefore instead follow Steenrod [24] and calculate with the mod p cohomology of the tower

$$\cdots \longrightarrow \Sigma^{2(m+1)} D_G(\Sigma^{-2(m+1)} B) \xrightarrow{\Delta} \Sigma^{2m} D_G(\Sigma^{-2m} B) \longrightarrow \cdots \longrightarrow D_G B \tag{3.1}$$

of G-fold extended power spectra. Recall that we write $W(G,B)$ for the homotopy limit of this tower.

Let p be a prime, briefly write $H_*(-) = H_*(-;\mathbb{F}_p)$ and $H^*(-) = H^*(-;\mathbb{F}_p)$, and let $L = e(\rho) \in H^{2(|G|-1)}_{gp}(G)$ be the mod p Euler class of the G-representation ρ. If $G = (C_p)^k$ is elementary abelian and p is odd, then its group cohomology

$$H^*_{gp}(G) = E(x_1, \dots, x_k) \otimes P(y_1, \dots, y_k)$$

is a tensor product of exterior and polynomial algebras, with $|x_i| = 1$, $|y_i| = 2$, and $\beta(x_i) = y_i$ for each $1 \leqslant i \leqslant k$. If instead $p = 2$, then

$$H^*_{gp}(G) = P(x_1, \dots, x_k)$$

with $|x_i| = 1$ and $\beta(x_i) = x_i^2$ for each $1 \leqslant i \leqslant k$. In either case

$$L = \prod_{x \neq 0} \beta(x) ,$$

where x ranges over the $(p^k - 1)$ nonzero elements in $H^1_{gp}(G) = \mathbb{F}_p\{x_1, \dots, x_k\}$. Let A denote the mod p Steenrod algebra.

Proposition 3.1. *Let G be a p-group and B a bounded below spectrum with $H_*(B; \mathbb{F}_p)$ of finite type.*

(a) *There is a natural A- and $H^*_{gp}(G)$-linear isomorphism*

$$H^*(D_G B)[L^{-1}] \cong \operatorname*{colim}_m H^*(\Sigma^{2m} D_G(\Sigma^{-2m} B)) =: H^*_c(W(G, B)).$$

(b) *There is an $H^*_{gp}(G)$-linear isomorphism*

$$H^*(D_G B)[L^{-1}] \cong H^*_{gp}(G)[L^{-1}]\{b^{\otimes G} \mid b \in \mathcal{B}\},$$

where $\mathcal{B}$ is a homogeneous basis for $H^*(B)$ and $b^{\otimes G}$ denotes the tensor product of one copy of b for each element $g \in G$.

(c) *If G is not elementary abelian, then $H^*(D_G B)[L^{-1}] = 0$.*

(d) *If $G = (C_p)^k$, then the isomorphisms in (a) and (b) are $GL_k(\mathbb{Z}/p)$-linear.*

Proof.

(a) For each $m \geqslant 0$, there is a Thom isomorphism $H^*(\Sigma^{2m} D_G(\Sigma^{-2m} B)) \cong H^{*+2m(|G|-1)}(D_B G)$ under which the homomorphism induced by Δ is given by multiplication by L, cf. [4, Lemma II.5.6]. Hence, the continuous cohomology $H^*_c(W(G, B))$ is isomorphic to the colimit of the sequence

$$\cdots \longleftarrow H^{*+2(m+1)(|G|-1)}(D_G B) \xleftarrow{L} H^{*+2m(|G|-1)}(D_G B) \longleftarrow \cdots \longleftarrow H^*(D_G B),$$

that is, the localization of $H^*(D_G B)$ away from the Euler class L.

(b) Following Steenrod [24, §VIII.3], we have a natural isomorphism

$$H^*(D_G B) \cong H^*_{gp}(G; H^*(B)^{\otimes G}).$$

A basis for $H^*(B)^{\otimes G}$ is given by the tensor products $b' = \otimes_{g \in G} b_g$, where each $b_g \in \mathcal{B}$ lies in the chosen basis, and the action by G permutes these generators (up to signs). Hence, $H^*_{gp}(G; H^*(B)^{\otimes G})$ splits as a direct sum of summands

$$H^*_{gp}(G; \mathbb{F}_p[G/K]\{b'\}) \cong H^*_{gp}(K; \mathbb{F}_p\{b'\}),$$

where K is the stabilizer of b'. If K is a proper subgroup of G, then L restricts trivially to $H^*_{gp}(K)$, and the summand $H^*_{gp}(G; \mathbb{F}_p[G/K]\{b'\})$ is annihilated by localization away from L. Only the summands with $b' = b^{\otimes G}$ survive, each of which contributes $H^*_{gp}(G)[L^{-1}]\{b'\}$ to $H^*(D_G B)[L^{-1}]$.

(c) If G is not elementary abelian, then $L \in H^*_{gp}(G)$ is nilpotent by the Quillen–Venkov theorem [5, Lemma III.1], [20] hence $H^*_{gp}(G)[L^{-1}] = 0$.

(d) The action of each element in $GL_k(\mathbb{Z}/p)$ permutes the elements in $G = (C_p)^k$, hence also permutes the tensor factors in $b^{\otimes G}$, all of which are equal. ☐

To pass from continuous cohomology to homotopy groups, we make use of an inverse limit of Adams spectral sequences associated to the tower (3.1), as in [5, §III], [6, §7], and [14, §2].

Proposition 3.2. *Let G be a p-group and B a bounded below spectrum with $H_*(B;\mathbb{F}_p)$ of finite type. There is a natural, strongly convergent, inverse limit Adams spectral sequence*

$$E_2^{s,t} = \operatorname{Ext}_A^{s,t}(H^*(D_G B)[L^{-1}], \mathbb{F}_p) \Longrightarrow \pi_{t-s} W(G,B)^\wedge_p .$$

Proof. Each spectrum $\Sigma^{2m} D_G(\Sigma^{-2m} B)$ is bounded below with mod p homology of finite type. Its mod p Adams spectral sequence

$$E_2^{*,*}(m) = \operatorname{Ext}_A^{*,*}(H^*(\Sigma^{2m} D_G(\Sigma^{-2m} B)), \mathbb{F}_p) \Longrightarrow \pi_* \Sigma^{2m} D_G(\Sigma^{-2m} B)^\wedge_p$$

is therefore strongly convergent, with $E_2^{*,*}(m)$ finite in each bidegree. By [6, Proposition 7.1], in the slightly generalized form from [14, Proposition 2.2], the algebraic limit groups $E_r^{*,*} = \lim_m E_r^{*,*}(m)$ (and the induced differentials d_r) also form a spectral sequence, with E_2-term

$$\begin{aligned} E_2^{*,*} &= \lim_m \operatorname{Ext}_A^{*,*}(H^*(\Sigma^{2m} D_G(\Sigma^{-2m} B)), \mathbb{F}_p) \\ &\cong \operatorname{Ext}_A^{*,*}(\operatorname{colim}_m H^*(\Sigma^{2m} D_G(\Sigma^{-2m} B)), \mathbb{F}_p) \\ &= \operatorname{Ext}_A^{*,*}(H_c^*(W(G,B)), \mathbb{F}_p) \cong \operatorname{Ext}_A^{*,*}(H^*(D_G B)[L^{-1}], \mathbb{F}_p) . \end{aligned}$$

Moreover, this spectral sequence converges strongly to

$$\pi_* \operatorname{holim}_m \Sigma^{2m} D_G(\Sigma^{-2m} B)^\wedge_p = \pi_* W(G,B)^\wedge_p ,$$

as asserted. □

Proposition 3.3. *Let G be a p-group that is not elementary abelian, and let B be a bounded below spectrum with $H_*(B;\mathbb{F}_p)$ of finite type. Then $W(G,B)^\wedge_p \simeq *$.*

Proof. By Proposition 3.1(c), we have $H^*(D_G B)[L^{-1}] = 0$, so $E_2^{*,*} = 0$ in the inverse limit spectral sequence of Proposition 3.2, which by strong convergence implies $\pi_* W(G,B)^\wedge_p = 0$. □

We can now collect some of the threads, as in the proof of [5, Theorem C].

Proof of Theorem 1.1 for G not elementary abelian. Let G be a p-group that is not elementary abelian, and suppose that Theorem 1.1 holds for each proper subquotient of G. Let B be a bounded below flat orthogonal spectrum with $H_*(B;\mathbb{F}_p)$ of finite type. By Theorem 2.4(b), $V(G,B)^\wedge_p \simeq *$. By Proposition 3.3, $W(G,B)^\wedge_p \simeq *$. Hence, $F(S^{\infty\rho}, B^{\wedge G})^G$ becomes trivial after p-completion, by the homotopy cofiber sequence in Definition 2.3. Therefore, $\gamma \colon (B^{\wedge G})^G \to (B^{\wedge G})^{hG}$ becomes an equivalence after p-completion, by Proposition 2.2. □

In the elementary abelian case, with $G \cong (C_p)^k$, we need better control of the connecting map δ. Suppose that B is bounded below with $H_*(B;\mathbb{F}_p)$ of finite type. Then $V(G,B)^\wedge_p$ and $\Sigma^{2m} D_G(\Sigma^{-2m} B)$ are also bounded below with mod p homology of finite type, in view of our inductive hypothesis on G and Theorem 2.4(a). For each $m \geqslant 0$, the composite map

$$V(G,B) \xrightarrow{\delta} W(G,B) \longrightarrow \Sigma^{2m} D_G(\Sigma^{-2m}B)$$

induces a morphism

$$\mathrm{Ext}_A^{*,*}(H^*(V(G,B)),\mathbb{F}_p) \longrightarrow \mathrm{Ext}_A^{*,*}(H^*(\Sigma^{2m}D_G(\Sigma^{-2m}B)),\mathbb{F}_p) = E_2^{*,*}(m)$$

of strongly convergent Adams spectral sequences. Passing to the limit over m, these define a natural morphism of spectral sequences

$$E_2(\delta):\ \mathrm{Ext}_A^{*,*}(H^*(V(G,B)),\mathbb{F}_p) \longrightarrow \mathrm{Ext}_A^{*,*}(H_c^*(W(G,B)),\mathbb{F}_p) = E_2^{*,*}$$

converging to $\pi_*(\delta)^\wedge_p : \pi_* V(G,B)^\wedge_p \to W(G,B)^\wedge_p$. By construction, $E_2(\delta) = f_B^*$ is induced by the homomorphism f_B specified in the following definition.

Definition 3.4. Let $f_B = \delta^*\kappa$ be the natural A- and $\mathrm{Aut}(G)$-linear homomorphism defined by the composition

$$f_B:\ H_c^*(W(G,B)) \xrightarrow{\kappa} H^*(W(G,B)) \xrightarrow{\delta^*} H^*(V(G,B))\,,$$

where

$$\kappa:\ H_c^*(W(G,B)) = \operatorname*{colim}_m H^*(\Sigma^{2m}D_G(\Sigma^{-2m}B)) \longrightarrow H^*(W(G,B))$$

is the canonical map from the continuous to the ordinary mod p cohomology associated to the tower (3.1).

4 | COMPARISON OF Tor^A-EQUIVALENCES

We now assume that $G = (C_p)^k$ with $k \geqslant 1$, so that $\mathrm{Aut}(G) = GL_k(\mathbb{Z}/p)$, and that Theorem 1.1 holds for each proper subquotient of G. We will show that

$$\delta^\wedge_p :\ V(G,B)^\wedge_p \longrightarrow W(G,B)^\wedge_p$$

is an equivalence for suitable B by using the classical Segal conjecture to show that the A- and $GL_k(\mathbb{Z}/p)$-linear homomorphism

$$f_B:\ H^*(D_G B)[L^{-1}] \cong H_c^*(W(G,B)) \longrightarrow H^*(V(G,B)) \cong \bigoplus^{p^{\binom{k}{2}}} \Sigma^{-k} H^*(B)\,,$$

cf. Definition 3.4, Proposition 3.1(a,d), and Theorem 2.4(a), is a Tor^A-equivalence in the key cases $B = S$ and $B = H = H\mathbb{F}_p$. More precisely, as a $U_k(\mathbb{Z}/p)$-module, the target can be rewritten as

$$H^*(V(G,B)) \cong \Sigma^{-k} H^*(B)[U_k(\mathbb{Z}/p)]\,,$$

cf. Addendum 2.5. The proof will be an application of the following comparison theorem of Priddy–Wilkerson. (As an aside, we recall that for p-groups U, an $\mathbb{F}_p[U]$-module is projective if and only if it is free.)

Theorem 4.1 ([19, Theorem III(i)]). *Let A be the mod p Steenrod algebra, let U be a p-group, and let $A[U] = A \otimes \mathbb{F}_p[U]$ denote the group algebra. Let $f : M \to N$ be a surjective $A[U]$-module homomorphism, where M and N are projective as $\mathbb{F}_p[U]$-modules. If*

$$f_*^U : \operatorname{Tor}^A_{*,*}(\mathbb{F}_p, M^U) \longrightarrow \operatorname{Tor}^A_{*,*}(\mathbb{F}_p, N^U)$$

is an isomorphism, then

$$f_* : \operatorname{Tor}^A_{*,*}(\mathbb{F}_p, M) \longrightarrow \operatorname{Tor}^A_{*,*}(\mathbb{F}_p, N)$$

is an isomorphism, too.

Recall from [1, Proposition 1.2] that the conclusion about f_*, i.e., that it is a Tor^A-equivalence, also implies that

$$f^* : \operatorname{Ext}_A^{*,*}(N, Q) \longrightarrow \operatorname{Ext}_A^{*,*}(M, Q) \tag{4.1}$$

is an isomorphism for each A-module Q that is bounded below and of finite type. This will be applied with $Q = \mathbb{F}_p$ to show that a morphism of Adams spectral sequences is an isomorphism.

For brevity we hereafter set

$$U := U_k(\mathbb{Z}/p) \subset GL_k(\mathbb{Z}/p)\,. \tag{4.2}$$

Remark 4.2. Our approach to specifying f_B differs from that of [19, (1.7)], where f for $B = S$ is instead defined via the evidently surjective projection

$$f : H^*_{gp}(G)[L^{-1}] \cong H^*_c(W(G,S)) \longrightarrow \mathbb{F}_p \otimes_A H^*_{gp}(G)[L^{-1}]$$

onto the A-module coinvariants, followed by an (a posteriori) identification of the target with $\Sigma^{-k} \operatorname{St}_k \otimes \mathbb{F}_p \cong H^*(V(G,S))$. This will not work for many other spectra B, including $B = H$, since A generally acts nontrivially on $H^*(V(G,B))$.

To verify that our homomorphism f_B is surjective for $B = H$, we now rely on the classical Segal conjecture in the case $B = S$, including the delicate comparison in [6, §§5–6] of S_G with $F(EG_+, H)$, representing stable equivariant cohomotopy and mod p Borel cohomology, respectively.

Proposition 4.3. *$f_S : H^*_{gp}(G)[L^{-1}] \to \Sigma^{-k}\mathbb{F}_p[U]$ is surjective.*

Proof. The edge homomorphism of the inverse limit Adams spectral sequence

$$E_2^{*,*} = \operatorname{Ext}_A^{*,*}(H^*_c(W(G,S)), \mathbb{F}_p) \Longrightarrow \pi_* W(G,S)^\wedge_p$$

from Proposition 3.2 factors as

$$\pi_* W(G,S)^\wedge_p \xrightarrow{h} \mathrm{Hom}_A(H^*(W(G,S)), \mathbb{F}_p) \xrightarrow{\kappa^*} \mathrm{Hom}_A(H^*_c(W(G,S)), \mathbb{F}_p),$$

where h is induced by the Hurewicz homomorphism. By Adams–Gunawardena–Miller [1, Theorem 1.1(a,b)], there is a Tor^A-equivalence

$$H^*_c(W(G,S)) \cong H^*_{gp}(G)[L^{-1}] \longrightarrow \bigoplus^{p^{\binom{k}{2}}} \Sigma^{-k}\mathbb{F}_p .$$

Hence, the inverse limit Adams spectral sequence is isomorphic to the direct sum of $p^{\binom{k}{2}}$ copies of the Adams spectral sequence for S^{-k}, and this implies that $W(G,S)^\wedge_p \simeq \bigvee^{p^{\binom{k}{2}}} \Sigma^{-k} S^\wedge_p$. In particular, the homomorphisms $\bar{h}$ and $\kappa^*\bar{h}$ in

$$\pi_{-k} W(G,S)/p \xrightarrow{\bar{h}} \mathrm{Hom}^{-k}_A(H^*(W(G,S)), \mathbb{F}_p) \xrightarrow{\kappa^*} \mathrm{Hom}^{-k}_A(H^*_c(W(G,S)), \mathbb{F}_p)$$

are isomorphisms, hence so is κ^*. It follows that κ is surjective. Finally, $\delta^\wedge_p$ is an equivalence for $B = S$, by the classical Segal conjecture, so $\delta^* : H^*(W(G,S)) \to H^*(V(G,S))$ is an isomorphism. Hence, $f_S = \delta^*\kappa$ is surjective. □

Proposition 4.4. *$f_H : H^*(D_G H)[L^{-1}] \to \Sigma^{-k} A[U]$ is surjective.*

Proof. By naturality of f_B with respect to the mod p Hurewicz map $h : S \to H$, the A-module diagram

$$\begin{array}{ccc} H^*(D_G H)[L^{-1}] \cong H^*_c(W(G,H)) & \xrightarrow{f_H} & H^*(V(G,H)) \cong \Sigma^{-k} A[U] \\ \downarrow h^* & & \downarrow h^* \\ H^*_{gp}(G)[L^{-1}] \cong H^*_c(W(G,S)) & \xrightarrow{f_S} & H^*(V(G,S)) \cong \Sigma^{-k}\mathbb{F}_p[U] \end{array}$$

commutes, where $A = H^*(H)$ and $\mathbb{F}_p = H^*(S)$. The left-hand homomorphism h^* and f_S are both surjective, by Propositions 3.1(b) and 4.3. Hence, the image of f_H contains all of the A-module generators of $\Sigma^{-k} A[U]$, in degree $-k$, which by A-linearity implies that f_H is surjective. □

The projectivity hypothesis in Theorem 4.1 follows easily from the special case considered by Priddy–Wilkerson.

Proposition 4.5. *Let B be bounded below with $H_*(B; \mathbb{F}_p)$ of finite type. Then $H^*(D_G B)[L^{-1}]$ and $\Sigma^{-k} H^*(B)[U]$ are both projective as $\mathbb{F}_p[U]$-modules.*

Proof. Suppose p is odd. By [19, Proposition 2.4], the homomorphism

$$P(y_1, \dots, y_k)[L^{-1}]^{GL_k(\mathbb{Z}/p)} \longrightarrow P(y_1, \dots, y_k)[L^{-1}]$$

is a $GL_k(\mathbb{Z}/p)$-Galois extension of commutative rings, in the sense of [7, Theorem 1.3, Definition 1.4]. It follows from [7, Theorem 2.2, Theorem 4.2(a)] that

$$P(y_1, \dots, y_k)[L^{-1}]^U \longrightarrow P(y_1, \dots, y_k)[L^{-1}]$$

is a U-Galois extension, so that $P(y_1, \dots, y_k)[L^{-1}]$ is a projective $\mathbb{F}_p[U]$-module. Hence [19, Proposition 2.5] implies that

$$H^*_{gp}(G)[L^{-1}] = E(x_1, \dots, x_k) \otimes P(y_1, \dots, y_k)[L^{-1}]$$

and

$$H^*(D_G B)[L^{-1}] \cong H^*_{gp}(G)[L^{-1}]\{b^{\otimes G} \mid b \in \mathcal{B}\}$$

are also projective as $\mathbb{F}_p[U]$-modules. The case $p = 2$ is a little simpler, replacing $P(y_1, \dots, y_k)$ by $P(x_1, \dots, x_k)$, omitting the factor $E(x_1, \dots, x_k)$, and noting that our Euler class L is the square of the class considered by Priddy–Wilkerson. The claim for $\Sigma^{-k} H^*(B)[U] \cong \mathbb{F}_p[U]\{\Sigma^{-k} b \mid b \in \mathcal{B}\}$ is immediate. □

Our next aim is to generalize results of Li–Singer and Adams–Gunawardena–Miller to identify $H^*(D_G B)[L^{-1}]^U$ as an A-module with the k-fold iterated desuspended C_p-Singer construction $T^k(H^*(B))$, which is Tor^A-equivalent to $\Sigma^{-k} H^*(B)$. The notation $T(M)$, for any A-module M, is that of [1, §2], and is equal to the A-module denoted as $\Sigma^{-1} R_+(M)$ in [14, Definition 3.1]. We shall use the expressions

$$\begin{aligned} T(H^*(B)) &\cong H^*_c(W(C_p, B)) = \operatorname*{colim}_m H^*(\Sigma^{2m} D_{C_p}(\Sigma^{-2m} B)) \\ &\cong H^*(D_{C_p} B)[L_1^{-1}] \cong H^*_{gp}(C_p; H^*(B)^{\otimes p})[L_1^{-1}] \\ &\cong H^*_{gp}(C_p)[L_1^{-1}]\{b^{\otimes p} \mid b \in \mathcal{B}\} \end{aligned}$$

from [14, Theorem 5.9], extending [4, Theorem II.5.1], as presentations of this version of the Singer construction. Here, $L_1 = -\beta(x_1)^{p-1} \in H^{2(p-1)}_{gp}(C_p)$ is the case $k = 1$ of the Euler class L.

Definition 4.6. For any group H, let $C_p \wr H = C_p \ltimes H^p$ denote the wreath product, that is, the semidirect product where C_p acts on the pth power H^p by cyclically permuting the factors. Let $C_p \times H \to C_p \wr H$ be the diagonal inclusion mapping (g, h) to $(g; h, \dots, h)$, and let

$$d \colon (C_p)^k = C_p \times \cdots \times C_p \longrightarrow C_p \wr \cdots \wr C_p = \wr^k C_p$$

denote its $(k-1)$-fold iterate, with $H = \wr^i C_p$ at the ith instance.

Lemma 4.7. *View d as an inclusion of subgroups of Σ_{p^k}. The normalizer of $G = (C_p)^k$ in the p-Sylow subgroup $\wr^k C_p$ of Σ_{p^k} is $U_k(\mathbb{Z}/p) \ltimes (C_p)^k$, with Weyl group the p-Sylow subgroup $U = U_k(\mathbb{Z}/p)$ of $GL_k(\mathbb{Z}/p)$.*

Proof. The normalizer of $(C_p)^k$ in Σ_{p^k} is $GL_k(\mathbb{Z}/p) \ltimes (C_p)^k$, and the lemma follows by restricting to elements in the p-Sylow subgroup $\wr^k C_p$ of Σ_{p^k}. □

The diagonal inclusion d induces a natural map of extended powers

$$D_{(C_p)^k}B \simeq E\Sigma_{p^k+} \wedge_{(C_p)^k} B^{\wedge p^k} \longrightarrow E\Sigma_{p^k+} \wedge_{\wr^k C_p} B^{\wedge p^k} \simeq D_{C_p}(\cdots D_{C_p}(B)\cdots)\,,$$

and a morphism of collapsing [15, Lemma 1.1(iii)] homotopy orbit spectral sequences, from

$$'E_2^{*,*} = H^*_{gp}(\wr^k C_p; H^*(B)^{\otimes p^k}) \Longrightarrow H^*(E\Sigma_{p^k+} \wedge_{\wr^k C_p} B^{\wedge p^k})$$

to

$$''E_2^{*,*} = H^*_{gp}((C_p)^k; H^*(B)^{\otimes p^k}) \Longrightarrow H^*(E\Sigma_{p^k+} \wedge_{(C_p)^k} B^{\wedge p^k})\,,$$

given at the E_2-terms by the homomorphism

$$d^* : H^*_{gp}(\wr^k C_p; H^*(B)^{\otimes p^k}) \longrightarrow H^*_{gp}((C_p)^k; H^*(B)^{\otimes p^k})\,. \tag{4.3}$$

More generally, for each $m \geqslant 0$, the diagonal inclusion induces a map

$$d_{B,m} : \Sigma^{2m}D_{(C_p)^k}\Sigma^{-2m}B \longrightarrow \Sigma^{2m}D_{C_p}(\cdots D_{C_p}(\Sigma^{-2m}B)\cdots) \simeq \Sigma^{2m}D_{C_p}\Sigma^{-2m}(\cdots \Sigma^{2m}D_{C_p}\Sigma^{-2m}(B)\cdots)$$

to the k-fold iterate of $\Sigma^{2m}D_{C_p}\Sigma^{-2m}(-)$ applied to B, and these are compatible under the twisted diagonal maps Δ. Passing to cohomology, we obtain homomorphisms

$$d^*_{B,m} : H^*(\Sigma^{2m}D_{C_p}\Sigma^{-2m}(\cdots \Sigma^{2m}D_{C_p}\Sigma^{-2m}(B)\cdots)) \longrightarrow H^*(\Sigma^{2m}D_{(C_p)^k}\Sigma^{-2m}B)$$

factoring through the U-invariants of the target, and passing to colimits over m, we obtain an A-module homomorphism

$$\begin{aligned} d^*_B : T^k(H^*(B)) &\cong \operatorname*{colim}_{m_1,\dots,m_k} H^*(\Sigma^{2m_1}D_{C_p}\Sigma^{-2m_1}(\cdots \Sigma^{2m_k}D_{C_p}\Sigma^{-2m_k}(B)\cdots)) \\ &\cong \operatorname*{colim}_m H^*(\Sigma^{2m}D_{C_p}(\cdots D_{C_p}(\Sigma^{-2m}B)\cdots)) \\ &\longrightarrow \operatorname*{colim}_m H^*(\Sigma^{2m}D_{(C_p)^k}\Sigma^{-2m}B) = H^*(D_GB)[L^{-1}] \end{aligned}$$

from the k-fold iterate of T applied to $H^*(B)$, with image contained in the U-invariants of the target. Here we use that the Singer construction T commutes with sequential colimits, and that the k-tuples $(m,\dots,m)$ are cofinal among the $(m_1,\dots,m_k)$.

Proposition 4.8. *Let B be bounded below with $H_*(B;\mathbb{F}_p)$ of finite type. The homomorphism d^*_B factors through a natural isomorphism of A-modules*

$$T^k(H^*(B)) \xrightarrow{\cong} H^*(D_GB)[L^{-1}]^U\,.$$

Proof. In the case $B = S$, William Singer's result [22, Proposition 9.1] (for $p = 2$) and the extension [1, Theorem 1.4] of [13, (1.6)] (for p odd) show that

$$d_S^* : T^k(\mathbb{F}_p) \xrightarrow{\cong} H^*_{gp}(G)[L^{-1}]^U \subset H^*_{gp}(G)[L^{-1}]$$

maps $T^k(\mathbb{F}_p)$ isomorphically to the U-invariants of $H^*_{gp}(G)[L^{-1}]$. In general, we can rewrite the lift of d_B^* as

$$H^*_{gp}(C_p; (\cdots H^*_{gp}(C_p; H^*(B)^{\otimes p})[L_1^{-1}] \cdots)^{\otimes p})[L_1^{-1}] \longrightarrow H^*_{gp}((C_p)^k; H^*(B)^{\otimes p^k})[L^{-1}]^U ,$$

or as

$$T^k(\mathbb{F}_p)\{b^{\otimes p^k} \mid b \in \mathcal{B}\} \longrightarrow H^*_{gp}(G)[L^{-1}]^U\{b^{\otimes p^k} \mid b \in \mathcal{B}\} .$$

Here, both sides are filtered by the cohomological degree of $b^{\otimes p^k}$, and the homomorphism respects these filtrations. Then, just as in (4.3),

$$d_B^*(x \cdot b^{\otimes p^k}) \equiv d_S^*(x) \cdot b^{\otimes p^k} \quad \text{modulo lower filtrations}$$

for $x \in T^k(\mathbb{F}_p)$ and $b \in H^q(B)$, and it follows by induction on the degree q that the lift of d_B^* is an isomorphism. □

Proposition 4.9.

$$f_H^U : H^*(D_G H)[L^{-1}]^U \longrightarrow \Sigma^{-k} A[U]^U \cong \Sigma^{-k} A$$

is a Tor^A*-equivalence.*

Proof. For any A-module M, the Tor^A-equivalence $\epsilon : T(M) \to \Sigma^{-1} M$ of [1, Theorem 1.3] can be iterated k-fold to give a Tor^A-equivalence $\epsilon^k : T^k(M) \to \Sigma^{-k} M$. When combined with Proposition 4.8, this gives a Tor^A-equivalence

$$H^*(D_G B)[L^{-1}]^U \cong T^k(H^*(B)) \xrightarrow{\epsilon^k} \Sigma^{-k} H^*(B) .$$

It follows that the source and target of f_B^U are abstractly Tor^A-equivalent, but it remains to verify, in the special case $B = H$, that f_H^U induces this equivalence. Using (4.1) with $Q = \Sigma^{-k} A$, we see that

$$\mathrm{Hom}_A(H^*(D_G H)[L^{-1}]^U, \Sigma^{-k} A) \cong \mathrm{Hom}_A(\Sigma^{-k} A, \Sigma^{-k} A) \cong \mathbb{F}_p ,$$

so that f_H^U is a multiple in $\mathbb{F}_p$ times a Tor^A-equivalence. The conclusion now follows, since Propositions 4.4 and 4.5 imply that f_H^U is nonzero. □

5 | THE ELEMENTARY ABELIAN CASE

We continue to assume that $G = (C_p)^k$ with $k \geqslant 1$, and that Theorem 1.1 holds for each proper subquotient of G.

Proposition 5.1. *$f_H : H^*(D_G H)[L^{-1}] \to \Sigma^{-k} A[U]$ is a* Tor^A*-equivalence.*

Proof. This follows from the Priddy–Wilkerson comparison theorem, that is, Theorem 4.1, for the $A[U]$-module homomorphism $f = f_H$, since f_H is surjective by Proposition 4.4, its source and target are $\mathbb{F}_p[U]$-projective by Proposition 4.5, and its U-invariant part f_H^U is a Tor^A-equivalence by Proposition 4.9. ☐

Theorem 5.2. *$\delta_p^\wedge : V(G,H)_p^\wedge \to W(G,H)_p^\wedge$ is an equivalence.*

Proof. By Proposition 3.2 and Definition 3.4, we have a morphism

$$E_2(\delta) = f_H^* : \ \mathrm{Ext}_A^{*,*}(\Sigma^{-k} A[U], \mathbb{F}_p) \longrightarrow \mathrm{Ext}_A^{*,*}(H^*(D_G H)[L^{-1}], \mathbb{F}_p)$$

of Adams and inverse limit Adams spectral sequences, converging to the homomorphism

$$\pi_*(\delta_p^\wedge) : \Sigma^{-k}\mathbb{F}_p[U] \cong \pi_* V(G,H)_p^\wedge \longrightarrow \pi_* W(G,H)_p^\wedge .$$

Here f_H^* is an isomorphism by Proposition 5.1 and (4.1), which implies that $\pi_*(\delta_p^\wedge)$ is an isomorphism, as claimed. ☐

This proves Theorem 1.1 for $G = (C_p)^k$ and $B = H$. Given this toehold result, we can deduce the theorem for bounded below B with $H_*(B; \mathbb{F}_p)$ of finite type by the inductive strategy of Nikolaus–Scholze [18, §III.1]. Recall that $\mathcal{T}_k$ denotes the rank k Tits building, which is a finite complex.

Proposition 5.3. *Let $B' \to B \to B''$ be a homotopy cofiber sequence of bounded below spectra with mod p homology of finite type. Then*

$$V(G,B')_p^\wedge \longrightarrow V(G,B)_p^\wedge \longrightarrow V(G,B'')_p^\wedge$$

and

$$W(G,B') \longrightarrow W(G,B) \longrightarrow W(G,B'')$$

are homotopy cofiber sequences.

Proof. It is clear that

$$F(\Sigma^2\mathcal{T}_k, B') \longrightarrow F(\Sigma^2\mathcal{T}_k, B) \longrightarrow F(\Sigma^2\mathcal{T}_k, B'')$$

is a homotopy cofiber sequence. This implies the corresponding result for $V(G,-)_p^\wedge$ by Theorem 2.4.

For the second claim, we follow the proof of [4, Proposition II.3.11]. We may assume $B' \to B$ is a cofibration, with $B/B' = B''$. There is a filtration

$$D_G(B') = \Gamma^{p^k}(B) \to \cdots \to \Gamma^{i+1}(B) \to \Gamma^i(B) \to \cdots \to \Gamma^0(B) = D_G(B)$$

with quotients

$$\Gamma^i(B)/\Gamma^{i+1}(B) \simeq EG_+ \wedge_G \bigvee^{\binom{p^k}{i}} (B')^{\wedge i} \wedge (B'')^{\wedge p^k - i}.$$

For $i = 0$, this is $D_G(B'')$. For $0 < i < p^k$, it is a finite wedge sum of spectra, each of the form

$$EG_+ \wedge_K (B')^{\wedge i} \wedge (B'')^{\wedge p^k - i}$$

with $K \subset G$ a proper subgroup. These filtrations and splittings are compatible with the twisted diagonal maps Δ. Since the inclusion $e\colon S^0 \to S^\rho$ is K-equivariantly null-homotopic, it follows that

$$\operatorname*{holim}_m \frac{\Sigma^{2m}\Gamma^i(\Sigma^{-2m}B)}{\Sigma^{2m}\Gamma^{i+1}(\Sigma^{-2m}B)} \simeq *$$

for each $0 < i < p^k$. Hence $W(G,B)/W(G,B') \to W(G,B'')$ is an equivalence. □

Proposition 5.4. *Let B be a bounded below spectrum, with Postnikov tower*

$$B \to \cdots \to \tau_{\leqslant n+1}B \to \tau_{\leqslant n}B \to \cdots .$$

The natural maps

$$V(G,B)^\wedge_p \xrightarrow{\simeq} \operatorname*{holim}_n V(G,\tau_{\leqslant n}B)^\wedge_p$$

and

$$W(G,B) \xrightarrow{\simeq} \operatorname*{holim}_n W(G,\tau_{\leqslant n}B)$$

are equivalences.

Proof. It is clear that

$$F(\Sigma^2\mathcal{T}_k, B) \xrightarrow{\simeq} \operatorname*{holim}_n F(\Sigma^2\mathcal{T}_k, \tau_{\leqslant n}B)$$

is an equivalence, and, by Theorem 2.4, this implies the corresponding result for $V(G,-)^\wedge_p$.

For the second claim, note that since B is bounded below, the connectivity of

$$\Sigma^{2m}D_G(\Sigma^{-2m}B) \longrightarrow \Sigma^{2m}D_G(\Sigma^{-2m}\tau_{\leqslant n}B)$$

increases to infinity with n, for each fixed $m \geqslant 0$. Hence,

$$\Sigma^{2m}D_G(\Sigma^{-2m}B) \xrightarrow{\simeq} \operatorname*{holim}_n \Sigma^{2m}D_G(\Sigma^{-2m}\tau_{\leqslant n}B)$$

is an equivalence. The result follows by passing to the homotopy limit over m and interchanging the order of the two homotopy limits. □

Theorem 5.5. *Let $G = (C_p)^k$ with $k \geqslant 1$, suppose that Theorem 1.1 holds for each proper subquotient of G, and let B be a flat orthogonal spectrum with $\pi_*(B)$ bounded below and $H_*(B; \mathbb{F}_p)$ of finite type. Then*

$$\delta_p^\wedge : V(G,B)_p^\wedge \to W(G,B)_p^\wedge$$

is an equivalence.

Proof. It suffices to prove that $\delta/p : V(G,B)/p \to W(G,B)/p$ is an equivalence. In view of Proposition 5.3 and the homotopy cofiber sequence

$$B \xrightarrow{p} B \longrightarrow B/p\,,$$

this is equivalent to checking that $\delta_p^\wedge$ for B/p is an equivalence. Each Postnikov section $\tau_{\leqslant n}(B/p)$ has only finitely many nonzero homotopy groups, each of order a finite power of p. Hence, the result for $\tau_{\leqslant n}(B/p)$ follows by induction from Theorem 5.2 and Proposition 5.3. The result for B/p then follows from Proposition 5.4. □

Proof of Theorem 1.1 for G elementary abelian. Let $G = (C_p)^k$ with $k \geqslant 1$, suppose that Theorem 1.1 holds for each proper subquotient of G, and let B be a bounded below flat orthogonal spectrum with $H_*(B; \mathbb{F}_p)$ of finite type. By Theorem 5.5, the map $\delta_p^\wedge : V(G,B)_p^\wedge \to W(G,B)_p^\wedge$ is an equivalence. Hence, $F(S^{\infty\rho}, B^{\wedge G})^G$ becomes trivial after p-completion, by the homotopy cofiber sequence in Definition 2.3. Therefore, $\gamma : (B^{\wedge G})^G \to (B^{\wedge G})^{hG}$ becomes an equivalence after p-completion, by Proposition 2.2. □

6 | THE FINITE GROUP CASE

We now assume that G is any finite group and that B is a flat orthogonal spectrum that is bounded below and of finite type. We aim to prove Theorem 1.2 concerning the G-spectrum $X = B^{\wedge G}$.

Let p be a prime, and let $K \subset G$ be a p-Sylow subgroup. By Proposition 2.1(a,d), the restriction $\mathrm{res}^G_K(B^{\wedge G}) = C^{\wedge K}$ is the K-fold smash power of $C \cong B^{\wedge G/K}$, which is also bounded below and of finite type. In particular, $H_*(C; \mathbb{F}_p)$ is of finite type, so by Theorem 1.1 the map

$$\gamma : (C^{\wedge K})^K \longrightarrow (C^{\wedge K})^{hK}$$

becomes an equivalence after p-completion.

The K-spectrum $C^{\wedge K}$ will not generally be split, so to translate between p-adic and $I(K)$-adic completion, we need a replacement for [16, Proposition 14]. Our Proposition 6.2 will be deduced from the following result of Ragnarsson, relating the spectrum-level $I(K)$-completion of Greenlees–May [8, §1] to p-completion.

Theorem 6.1 ([21, Theorem C]). *Let K be a p-group and Y a bounded below K-spectrum. Then there is a natural homotopy cofiber sequence*

$$(EK_+ \wedge Y)^K \longrightarrow (Y^\wedge_{I(K)})^K \longrightarrow ((\widetilde{EK} \wedge Y)^K)^\wedge_p .$$

Proposition 6.2. *Let K be a p-group and Y a bounded below K-spectrum with $\pi_*(Y)$ of finite type. Suppose that $\gamma : Y^K \to Y^{hK}$ becomes an equivalence after p-completion. Then*

$$\xi^* : (Y^\wedge_{I(K)})^K \xrightarrow{\simeq} Y^{hK}$$

is an equivalence.

Proof. For brevity, let $Z = F(EK_+, Y)$. Note that the K-Tate construction

$$Y^{tK} = (\widetilde{EK} \wedge Z)^K = (\widetilde{EK} \wedge F(EK_+, Y))^K ,$$

is already p-complete, since K is a p-group and $\pi_*(Y)$ is bounded below and of finite type, cf. [9]. Using Theorem 6.1, we have vertical maps

$$\begin{array}{ccccc}
(EK_+ \wedge Y)^K & \longrightarrow & Y^K & \longrightarrow & (\widetilde{EK} \wedge Y)^K \\
\| & & \downarrow \iota & & \downarrow \tilde{\iota} \\
(EK_+ \wedge Y)^K & \longrightarrow & (Y^\wedge_{I(K)})^K & \longrightarrow & ((\widetilde{EK} \wedge Y)^K)^\wedge_p \\
\simeq \downarrow & & \downarrow \xi^* & & \downarrow \tilde{\xi}^* \\
(EK_+ \wedge Z)^K & \longrightarrow & Z^K & \longrightarrow & (\widetilde{EK} \wedge Z)^K
\end{array}$$

of horizontal homotopy cofiber sequences. The left-hand vertical map is always an equivalence, and the middle vertical composite $\gamma = \xi^* \iota$ becomes an equivalence after p-completion by assumption. Hence, also the right-hand composite $\tilde{\gamma} = \tilde{\xi}^* \tilde{\iota}$ becomes an equivalence after p-completion. But $\tilde{\gamma}^\wedge_p = \tilde{\xi}^*$, since $(\widetilde{EK} \wedge Z)^K$ is p-complete, so $\tilde{\xi}^*$ is an equivalence. It follows that ξ^* is an equivalence, as claimed. □

To verify the equivariant bounded below hypothesis for $Y = C^{\wedge K}$, and a finite-type hypothesis needed for [8, Theorem 1.6(ii)], we can use the following variant of the folklore result proved in [21, Proposition 3.1].

Lemma 6.3. *Let K be a finite group and Y a K-spectrum. Let H range over all subgroups of K. Then every fixed-point spectrum Y^H is bounded below (and of finite type) if and only if every geometric fixed-point spectrum $\Phi^H(Y)$ is bounded below (and of finite type).*

Proof. It suffices to prove this with "bounded below" replaced by "connective." Let $\mathcal{P}$ be the family of proper subgroups of K. By induction on K, we may assume that Y^H and $\Phi^H(Y)$ are connective (and of finite type) for all $H \in \mathcal{P}$. In the homotopy cofiber sequence

$$(E\mathcal{P}_+ \wedge Y)^K \longrightarrow Y^K \longrightarrow \Phi^K(Y),$$

the left-hand term is built from nonnegative suspensions of $(K/H_+ \wedge Y)^K \simeq Y^H$, where $H \in \mathcal{P}$ (and the suspension degrees increase to infinity), hence is connective (and of finite type). Thus, Y^K is connective (and of finite type) if and only if $\Phi^K(Y)$ is connective (and of finite type). □

Proof of Theorem 1.2. We keep the notation from the beginning of this section. By Proposition 2.1, each geometric fixed-point spectrum $\Phi^H(C^{\wedge K})$ is bounded below and of finite type, so by Lemma 6.3, the K-spectrum $Y = C^{\wedge K}$ is bounded below and of finite type. Theorem 1.1 for K and C and Proposition 6.2 then prove that

$$\xi^* : (Y^\wedge_{I(K)})^K \xrightarrow{\simeq} Y^{hK}$$

is an equivalence. Moreover, by [8, Theorem 1.6(ii)],

$$\pi_*((Y^\wedge_{I(K)})^K) \cong \pi_*(Y^K)^\wedge_{I(K)}$$

is given algebraically by $I(K)$-adic completion. Hence, Theorem 1.2 holds for the K-spectrum Y given by the restriction of the G-spectrum $X = B^{\wedge G}$. The algebraic part of this statement is the completion conjecture for Y, in the terminology of May–McClure [16, p. 217]. Since this applies for all Sylow subgroups K of G, the completion conjecture also holds for the G-spectrum X by [16, Theorem 13], so that

$$\pi_*(\gamma)^\wedge_{I(G)} : \pi_*(X^G)^\wedge_{I(G)} \xrightarrow{\cong} \pi_*(X^{hG})$$

is an isomorphism. Since the G-spectrum X is also bounded below and of finite type, we have

$$\pi_*((X^\wedge_{I(G)})^G) \cong \pi_*(X^G)^\wedge_{I(G)},$$

by a second appeal to [8, Theorem 1.6(ii)]. Hence,

$$\xi^* : (X^\wedge_{I(G)})^G \xrightarrow{\simeq} X^{hG}$$

is an equivalence, which proves Theorem 1.2 for G and B. □

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