THE WEIGHT AND RANK FILTRATIONS

JOHN ROGNES

Abstract. We compare the weight and stable rank filtrations of algebraic K-theory, and relate the Beilinson–Soulé vanishing conjecture to the author's connectivity conjecture.

1. Introduction

Let $\mathbb{F}$ be a field. For $j \geq 0$ let $K_j(\mathbb{F})$ be the (higher) algebraic K-groups of $\mathbb{F}$ [Quillen (1970)]. There is a decreasing *weight filtration*

$$K_j(\mathbb{F}) \supset F^1 K_j(\mathbb{F}) \supset \cdots \supset F^w K_j(\mathbb{F}) \supset \cdots \supset F^j K_j(\mathbb{F}) \supset 0$$

associated to the λ- and Adams-operations on $K_j(\mathbb{F})$ [Grothendieck, Quillen/Hiller (1981)].

There is also an increasing *stable rank filtration*

$$0 \subset F_1 K_j(\mathbb{F}) \subset \cdots \subset F_r K_j(\mathbb{F}) \subset \cdots \subset F_j K_j(\mathbb{F}) \subset K_j(\mathbb{F})\,.$$

given by the image filtration

$$F_r K_j(\mathbb{F}) = \operatorname{im}(\pi_j F_r \mathbf{K}(\mathbb{F}) \to \pi_j \mathbf{K}(\mathbb{F}))$$

associated to a sequence of spectra

$$* \to F_1\mathbf{K}(\mathbb{F}) \to \cdots \to F_r\mathbf{K}(\mathbb{F}) \to \cdots \to \mathbf{K}(\mathbb{F})$$

called the *spectrum level rank filtration* [Rognes (1992)].

Conjecture 1.1 (Beilinson–Soulé)**.**

$$F^w K_j(\mathbb{F}) = K_j(\mathbb{F})$$

for $2w \leq j+1$.

A stronger form asserts this equality also for $2w = j+2$ when $j > 0$. The conjecture is known with finite coefficients, so it suffices to verify it rationally, i.e., after tensoring over $\mathbb{Z}$ with $\mathbb{Q}$. We write $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Conjecture 1.2 (Connectivity)**.**

$$F_r K_j(\mathbb{F}) = K_j(\mathbb{F})$$

for $2r \geq j+1$.

A stronger form asserts rational equality also for $2r = j$ when $j > 0$.

Conjecture 1.3 (Stable Rank)**.**

$$F^w K_j(\mathbb{F})_{\mathbb{Q}} = F_r K_j(\mathbb{F})_{\mathbb{Q}}$$

for $w + r = j+1$.

I will provide evidence for the connectivity and stable rank conjectures, which, if true, will imply the Beilinson–Soulé vanishing conjecture.

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2. Algebraic K-theory

Let $\mathscr{P}(\mathbb{F})$ be the category of finitely generated projective $\mathbb{F}$-modules, i.e., the category of finite-dimensional $\mathbb{F}$-vector spaces.

The dimension $\dim V$ of an object V defines an additive invariant in $K_0(\mathbb{F}) \cong \mathbb{Z}$.

The determinant $\det A$ of an automorphism $A\colon V \to V$ defines an additive invariant in $K_1(\mathbb{F}) \cong \mathbb{F}^\times$.

The classifying space $|i\mathscr{P}(\mathbb{F})|$ of the subcategory $i\mathscr{P}(\mathbb{F})$ of isomorphisms in $\mathscr{P}(\mathbb{F})$ is built with one q-simplex Δ^q for each chain

$$V_0 \overset{\cong}{\to} V_1 \overset{\cong}{\to} \dots \overset{\cong}{\to} V_q$$

of q composable morphisms in $i\mathscr{P}(\mathbb{F})$. The inclusion of the full subcategory generated by the objects $\mathbb{F}^r$, with automorphism groups $GL_r(\mathbb{F})$, induces an equivalence

$$\coprod_{r \ge 0} BGL_r(\mathbb{F}) \simeq |i\mathscr{P}(\mathbb{F})| \,.$$

Direct sum of vector spaces, $(V, W) \mapsto V \oplus W$, makes $|i\mathscr{P}(\mathbb{F})|$ a (coherently homotopy commutative) topological monoid. To group complete $\pi_0 \cong \{r \ge 0\}$ and $\pi_1 \cong GL_r(\mathbb{F})$ (for a suitable choice of base point), we map $K(\mathbb{F})_0 = |i\mathscr{P}(\mathbb{F})|$ to a grouplike (coherently homotopy commutative) topological monoid, i.e., a (infinite) loop space $\Omega K(\mathbb{F})_1 = \Omega|iS_\bullet\mathscr{P}(\mathbb{F})|$, to be defined below. The map $K(\mathbb{F})_0 \to \Omega K(\mathbb{F})_1$ is a group completion.

Definition 2.1. $K_j(\mathbb{F}) = \pi_j \Omega K(\mathbb{F})_1 = \pi_{j+1} K(\mathbb{F})_1$, where $K(\mathbb{F})_1 = |iS_\bullet\mathscr{P}(\mathbb{F})|$ is given by Waldhausen's $S_\bullet$-construction.

3. The algebraic K-theory spectrum

Waldhausen's $S_\bullet$-construction applied to $\mathscr{P}(F)$ is a simplicial category

$$[q] \mapsto iS_q\mathscr{P}(F) \,.$$

The category in degree q has objects the sequences of injective homomorphisms

$$0 = V_0 \rightarrowtail V_1 \rightarrowtail \dots \rightarrowtail V_q$$

in $\mathscr{P}(F)$, together with compatible choices of quotients V_j/V_i for $0 \le i \le j \le q$. Morphisms are vertical isomorphisms

$$\begin{array}{ccccccc} 0 = V_0 & \rightarrowtail & V_1 & \rightarrowtail & \dots & \rightarrowtail & V_q \\ \downarrow \cong & & \downarrow \cong & & & & \downarrow \cong \\ 0 = W_0 & \rightarrowtail & W_1 & \rightarrowtail & \dots & \rightarrowtail & W_q \end{array}$$

of horizontal diagrams.

The construction can be iterated $n \ge 1$ times. We define

$$K(\mathbb{F})_n = |iS^{(n)}_\bullet \mathscr{P}(\mathbb{F})|$$

as the classifying space of the simplicial category

$$[q] \mapsto iS^{(n)}_q \mathscr{P}(\mathbb{F})$$

with objects in degree q given by n-dimensional cubical diagrams $[q]^n \to \mathscr{P}(\mathbb{F})$. In the case $n = 2$

$$\begin{array}{ccccccc} 0 & = & 0 & = & \dots & = & 0 \\ \| & & \downarrow & & & & \downarrow \\ 0 & \rightarrowtail & V_{1,1} & \rightarrowtail & \dots & \rightarrowtail & V_{1,q} \\ \| & & \downarrow & & & & \downarrow \\ \vdots & & \vdots & & \ddots & & \vdots \\ \| & & \downarrow & & & & \downarrow \\ 0 & \rightarrowtail & V_{q,1} & \rightarrowtail & \dots & \rightarrowtail & V_{q,q} \end{array}$$

we require to have injective homomorphisms $V_{i-1,j} \to V_{i,j}$ and $V_{i,j-1} \to V_{i,j}$ and injective pushout homomorphisms

$$V_{i-1,j} \oplus_{V_{i-1,j-1}} V_{i,j-1} \to V_{i,j} ,$$

for all $1 \le i, j \le q$. For higher n there are similar conditions for d-dimensional subcubes for all $1 \le d \le n$.

Definition 3.1. The algebraic K-theory spectrum of $\mathbb{F}$ is the spectrum

$$\mathbf{K}(\mathbb{F}) = \{n \mapsto K(\mathbb{F})_n = |iS^{(n)}_\bullet \mathscr{P}(\mathbb{F})|\} .$$

It is positive fibrant, in the sense that $K(\mathbb{F})_n \to \Omega K(\mathbb{F})_{n+1}$ is an equivalence for each $n \ge 1$. Hence

$$K_j(\mathbb{F}) = \pi_j \mathbf{K}(\mathbb{F}) = \pi_{j+n} K(\mathbb{F})_n$$

for each $n \ge 1$.

This constructions produces a symmetric spectrum: the group Σ_n permutes the order of the n instances of the $S_\bullet$-construction.

4. Weight filtration

Let $k \ge 0$. The k-th exterior power $V \mapsto \Lambda^k V$ induces λ-operations

$$\lambda^k \colon K_j(\mathbb{F}) \to K_j(\mathbb{F}) .$$

For $V = L_1 \oplus \cdots \oplus L_r$ a direct sum of lines,

$$\Lambda^k V = \bigoplus_{1 \le i_1 < \cdots < i_k \le r} L_{i_1} \otimes \cdots \otimes L_{i_k}$$

corresponds to the k-th elementary symmetric polynomial

$$\sigma_k(x_1, \dots, x_r) = \sum_{1 \le i_1 < \cdots < i_k \le r} x_{i_1} \dots x_{i_r} .$$

The k-th Adams operation

$$\psi^k \colon K_j(\mathbb{F}) \to K_j(\mathbb{F})$$

is induced by $L_1 \oplus \cdots \oplus L_r \mapsto L_1^{\otimes k} \oplus \cdots \oplus L_r^{\otimes k}$ and corresponds to the k-th power sum polynomial

$$s_k(x_1, \dots, x_r) = \sum_{i=1}^r x_i^k .$$

It can thus be expressed in terms of the λ-operations.

The weight filtration $\{F^w K_j(\mathbb{F})\}_{w \ge 0}$ on $K_j(\mathbb{F})$ is constructed by means of the λ^k. The Adams operations satisfy

$$\psi^k(x) \equiv k^w x \mod F^{w+1} K_j(\mathbb{F})$$

for $x \in F^w K_j(\mathbb{F})$. Hence ψ^k acts as multiplication by k^w on $F^w K_j(\mathbb{F})/F^{w+1} K_j(\mathbb{F})$, for all $k \ge 0$.

Rationally the weight filtration splits as a direct sum of common eigenspaces for the Adams operations. Let

$$K_j(\mathbb{F})^{(w)}_{\mathbb{Q}} = \{x \in K_j(\mathbb{F})_{\mathbb{Q}} \mid \psi^k(x) = k^w x \text{ for all } k\}$$

be the *weight w rational eigenspace*. Then

$$F^w K_j(\mathbb{F})_{\mathbb{Q}} = \bigoplus_{v \ge w} K_j(\mathbb{F})^{(v)}_{\mathbb{Q}}$$

is the subspace of weights $\ge w$, and

$$\frac{F^w K_j(\mathbb{F})_{\mathbb{Q}}}{F^{w+1} K_j(\mathbb{F})_{\mathbb{Q}}} \cong K_j(\mathbb{F})^{(w)}_{\mathbb{Q}} .$$

Soulé proved that $K_j(\mathbb{F})^{(w)}_{\mathbb{Q}} = 0$ for $w < 0$ and for $w > j$, i.e., $K_j(\mathbb{F})_{\mathbb{Q}}$ only contains classes of weight $0 \le w \le j$.

Example 4.1. ψ^k acts (additively) on $K_1(\mathbb{F})$ by $\psi^k(x) = kx$, since it acts (multiplicatively) on $\mathbb{F}^\times$ by $u \mapsto u^k$. Hence all of $K_1(\mathbb{F})_\mathbb{Q} = K_1(\mathbb{F})^{(1)}_\mathbb{Q}$ has weight 1. The product $x_1 \cdots x_j \in K_j(\mathbb{F})$ of j classes $x_1, \dots, x_j \in K_1(\mathbb{F})$ has weight j, since $\psi^k(x_1 \cdots x_j) = \psi^k(x_1) \cdots \psi^k(x_j) = (kx_1) \cdots (kx_j) = k^j x_1 \cdots x_j$.

Milnor K-theory $K^M_*(\mathbb{F})$ is defined to be the quotient of the tensor algebra on $F^\times$, over $\mathbb{Z}$, by the ideal generated by $u \otimes (1-u)$ for $u \in \mathbb{F} \setminus \{0, 1\}$. It is graded commutative, so in degree j there are surjections

$$(\mathbb{F}^\times)^{\otimes j} \to \Lambda^j \mathbb{F}^\times \to K^M_j(\mathbb{F}) \,.$$

By the Steinberg relation $\{u, 1-u\} = 0$ in $K_2(\mathbb{F})$, these all map to $K_j(\mathbb{F})$, and land in the weight j eigenspace.

5. Motivic cohomology

By analogy with the Atiyah–Hirzebruch spectral sequence from singular cohomology to topological K-theory for a topological space, there is a *motivic spectral sequence*

$$E^2_{s,t}(mot) = H^{t-s}_{mot}(\mathbb{F}, \mathbb{Z}(t)) \Longrightarrow K_{s+t}(\mathbb{F}) \,.$$

It is of homological type, concentrated in the first quadrant ($s \geq 0$ and $t \geq 0$), and collapses rationally at the E^2-term ($d^r = 0$ after rationalization for $r \geq 2$).

t					
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
4	$H^4_{mot}(\mathbb{F};\mathbb{Z}(4))$	$H^3_{mot}(\mathbb{F};\mathbb{Z}(4))$	$H^2_{mot}(\mathbb{F};\mathbb{Z}(4))$	$H^1_{mot}(\mathbb{F};\mathbb{Z}(4))$	$\dots$
		d^2			
3	$H^3_{mot}(\mathbb{F};\mathbb{Z}(3))$	$H^2_{mot}(\mathbb{F};\mathbb{Z}(3))$	$H^1_{mot}(\mathbb{F};\mathbb{Z}(3))$	$H^0_{mot}(\mathbb{F};\mathbb{Z}(3))$	$\dots$
2	$H^2_{mot}(\mathbb{F};\mathbb{Z}(2))$	$H^1_{mot}(\mathbb{F};\mathbb{Z}(2))$	$H^0_{mot}(\mathbb{F};\mathbb{Z}(2))$	$H^{-1}_{mot}(\mathbb{F};\mathbb{Z}(2))$	$\dots$
1	$H^1_{mot}(\mathbb{F};\mathbb{Z}(1))$	$H^0_{mot}(\mathbb{F};\mathbb{Z}(1))$	$H^{-1}_{mot}(\mathbb{F};\mathbb{Z}(1))$	$H^{-2}_{mot}(\mathbb{F};\mathbb{Z}(1))$	$\dots$
0	$H^0_{mot}(\mathbb{F};\mathbb{Z}(0))$	$H^{-1}_{mot}(\mathbb{F};\mathbb{Z}(0))$	$H^{-2}_{mot}(\mathbb{F};\mathbb{Z}(0))$	$H^{-3}_{mot}(\mathbb{F};\mathbb{Z}(0))$	$\dots$
$E^2_{s,t}$	0	1	2	3	s

Rationally, the motivic cohomology groups can be defined in terms of the weight filtration.

Definition 5.1.

$$H^{t-s}_{mot}(\mathbb{F}, \mathbb{Z}(t))_\mathbb{Q} = H^{t-s}_{mot}(\mathbb{F}, \mathbb{Q}(t)) = K_{s+t}(\mathbb{F})^{(t)}_\mathbb{Q}$$

so that

$$H^i_{mot}(\mathbb{F}; \mathbb{Q}(w)) = K_{2w-i}(\mathbb{F})^{(w)}_\mathbb{Q} \,.$$

Remark 5.2. These groups give the E^2-term of a rational motivic spectral sequence collapsing to $K_{s+t}(\mathbb{F})_\mathbb{Q}$.

The definition is not effective. $K_*(\mathbb{F})$ with its Adams operations is hard to calculate and to work with.

By Soulé's result, $H^{t-s}_{mot}(\mathbb{F}; \mathbb{Q}(t))$ can only be nonzero for $0 \leq t \leq s+t$, i.e., for $s \geq 0$ and $t \geq 0$.

Equivalently, $H^i_{mot}(\mathbb{F}, \mathbb{Q}(w))$ can only be nonzero for $w \geq 0$ and $i \leq w$. This does not ensure that $H^i_{mot} = 0$ for $i < 0$!

6. Mixed motives

It is expected that there exists a category MM of *mixed motives*, such that the motivic cohomology groups are given by the Ext-groups

$$H^i_{mot}(\mathbb{F};\mathbb{Q}(w)) \cong \mathrm{Ext}^i_{MM}(\mathbb{Q},\mathbb{Q}(w))$$

classifying i-fold extensions from (the pure motive associated to) $\mathbb{Q}(w)$ to $\mathbb{Q}$ in this category.

If this is true, $H^i_{mot}(\mathbb{F};\mathbb{Q}(w)) = 0$ for $i < 0$, which is equivalent to $K_{2w-i}(\mathbb{F})^{(w)}_{\mathbb{Q}} = 0$ for $i < 0$, hence also to $K_j(\mathbb{F})^{(w)}_{\mathbb{Q}} = 0$ for $2w < j$.

Furthermore, $H^0_{mot}(\mathbb{F};\mathbb{Q}(w)) \cong \mathrm{Hom}_{MM}(\mathbb{Q},\mathbb{Q}(w)) = 0$ for $w > 0$, so $K_{2w}(\mathbb{F})^{(w)}_{\mathbb{Q}} = 0$ for $w > 0$, which means that $K_j(\mathbb{F})^{(w)}_{\mathbb{Q}} = 0$ for $2w \le j$ when $w > 0$.

These assertions are the content of the Beilinson–Soulé vanishing conjecture.

Conjecture 6.1 (Beilinson–Soulé). *$K_j(\mathbb{F})^{(w)}_{\mathbb{Q}} = 0$ for $w < j/2$ (and for $w \le j/2$ when $j > 0$).*

This is equivalent to the rational version of the conjecture as first stated.

7. Higher Chow groups

A construction of integral motivic cohomology groups is given by Bloch's higher Chow groups.

Definition 7.1. For each $q \ge 0$ let

$$\Delta^q_{\mathbb{F}} = \mathrm{Spec}\,\mathbb{F}[x_0,\dots,x_q]/(x_0+\dots+x_q = 1)$$

be the affine q-simplex over $\mathbb{F}$. It is isomorphic to $\mathbb{A}^q_{\mathbb{F}} = \mathrm{Spec}\,\mathbb{F}[x_1,\dots,x_q]$, but the $\Delta^q_{\mathbb{F}}$ combine more naturally to a precosimplicial variety:

$$\Delta^0_{\mathbb{F}} \underset{d_1}{\overset{d_0}{\rightrightarrows}} \Delta^1_{\mathbb{F}} \underset{d_2}{\overset{d_0}{\underset{d_1}{\to\!\!\!\!\!\rightrightarrows}}} \Delta^2_{\mathbb{F}} \qquad \dots$$

Let

$$z^p(\mathbb{F},q) = \{\text{codimension } p \text{ cycles } V \subset \Delta^q_{\mathbb{F}} \text{ meeting each face } \Delta^a_{\mathbb{F}} \to \Delta^q_{\mathbb{F}} \text{ transversely}\}\,.$$

A cycle is an integral sum of irreducible subvarieties. Pullback of V along the cofaces $d_i\colon \Delta^{q-1}_{\mathbb{F}} \to \Delta^q_{\mathbb{F}}$ defines face operators $d_i\colon z^p(\mathbb{F},q) \to z^{p-1}(\mathbb{F},q)$ that assemble to a presimplicial abelian group

$$z^p(\mathbb{F},0) \underset{d_1}{\overset{d_0}{\leftleftarrows}} z^p(\mathbb{F},1) \underset{d_2}{\overset{d_0}{\underset{d_1}{\leftleftarrows\!\!\!\!\!\leftarrow}}} z^p(\mathbb{F},2) \qquad \dots$$

There is an associated chain complex $(z^p(\mathbb{F},*),\partial)$

$$0 \longleftarrow z^p(\mathbb{F},0) \overset{\partial_1}{\longleftarrow} z^p(\mathbb{F},1) \overset{\partial_2}{\longleftarrow} z^p(\mathbb{F},2) \longleftarrow \dots$$

with $\partial_1 = d_0 - d_1$, $\partial_2 = d_0 - d_1 + d_2$, etc.

Bloch's higher Chow groups are the homology groups

$$CH^p(\mathbb{F},q) = \frac{\ker \partial_q}{\mathrm{im}\,\partial_{q+1}} = H_q(z^p(\mathbb{F},*),\partial)$$

of this chain complex.

(In what generality is $CH^p(X,0) = CH^p(X)$?)

Definition 7.2. Integral motivic cohomology groups can be defined as

$$H^i_{mot}(\mathbb{F};\mathbb{Z}(w)) = CH^w(\mathbb{F},2w-i)\,.$$

Remark 7.3. These give an integral motivic spectral sequence converging to $K_*(\mathbb{F})$.

Rationally they agree with the weight eigenspace definition of rational motivic cohomology.

There are no codimension p subvarieties in $\Delta^q_{\mathbb{F}}$ for $q < p$, so $z^p(\mathbb{F},q) = 0$ and $CH^p(\mathbb{F},q) = 0$ for $q < p$. Hence $H^i_{mot}(\mathbb{F},\mathbb{Z}(w)) = 0$ for $2w - i < w$, i.e., for $i > w$.

This definition does not tell us whether $H^i_{mot} = 0$ for $i < 0$, or equivalently, if $CH^p(\mathbb{F},q) = 0$ for $2p < q$.

Theorem 7.4 (Suslin)**.** *With finite coefficients,*

$$CH^p(\mathbb{F}, q; \mathbb{Z}/m) \cong H^{2p-q}_{et}(\mathbb{F}; \mathbb{Z}/m(p))$$

when $q \geq p$, i.e., when $2p-q \leq p$. In particular, $CH^p(\mathbb{F}, q; \mathbb{Z}/m) = 0$ for $2p < q$, so $H^i_{mot}(\mathbb{F}, \mathbb{Z}/m(w)) = 0$ for $i < 0$.

Conjecture 7.5 (Beilinson/Lichtenbaum)**.** *There are complexes (of Zariski/étale sheaves)*

$$\dots \leftarrow \Gamma(w, \mathbb{F})^w \overset{\delta}{\leftarrow} \dots \overset{\delta}{\leftarrow} \Gamma(w, \mathbb{F})^0 \leftarrow \dots$$

with cohomology calculating motivic cohomology

$$H^i(\Gamma(w, \mathbb{F})^*, \delta) \cong H^i_{mot}(\mathbb{F}, \mathbb{Z}(w)) .$$

Remark 7.6. We can let $\Gamma(0, \mathbb{F}) = \mathbb{Z}$ and $\Gamma(1, \mathbb{F}) = \mathbb{F}^\times$ (in cohomological degree 1). Lichtenbaum has a proposed complex $\Gamma(2, F)$. Goncharov has proposed complexes $\Gamma_{pol}(w, \mathbb{F})$ associated to polylogarithms, i.e., functions like

$$Li_w(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^w} .$$

Here $Li_1(z) = -\ln(1 - z)$.

8. Quillen's rank filtration

Recall that $K(\mathbb{F})_1 = |iS_\bullet \mathscr{P}(\mathbb{F})|$ is the classifying space of the simplicial category with q-simplices having objects

$$\sigma \colon 0 = V_0 \rightarrowtail V_1 \rightarrowtail \dots \rightarrowtail V_q .$$

Definition 8.1. Let $F_r K(\mathbb{F})_1 \subset K(\mathbb{F})_1$ be the subspace consisting of simplices where $\dim V_q \leq r$ (so that $\dim V_i \leq r$ for all i).

$$* = F_0 K(\mathbb{F})_1 \subset \dots \subset F_{r-1} K(\mathbb{F})_1 \subset F_r K(\mathbb{F})_1 \subset \dots \subset K(\mathbb{F})_1 .$$

Proposition 8.2.

$$F_r K(\mathbb{F})_1 / F_{r-1} K(\mathbb{F})_1 \simeq \Sigma^2 B(\mathbb{F}^r)_{hGL_r(\mathbb{F})} = EGL_r(\mathbb{F})_+ \wedge_{GL_r(\mathbb{F})} \Sigma^2 B(\mathbb{F}^r)$$

is the (based) homotopy orbit space for $GL_r(\mathbb{F})$ acting on the double suspension of the Tits building $B(\mathbb{F}^r)$.

Sketch proof. The (non-basepoint) q-simplices of $F_r K(\mathbb{F})_1 / F_{r-1} K(\mathbb{F})_1$ are generated from the objects

$$\sigma \colon 0 = V_0 \rightarrowtail V_1 \rightarrowtail \dots \rightarrowtail V_q$$

with $\dim V_q = r$, together with the $GL_r(\mathbb{F})$-action on the latter.

This is equivalent to the $GL_r(\mathbb{F})$-homotopy orbits of the subspace with q-simplices

$$\sigma \colon 0 = V_0 \subset V_1 \subset \dots \subset V_{q-1} \subset V_q = \mathbb{F}^r .$$

The i-th face operator deletes V_i. The 0-th and q-th face operators map to the base point if $0 \neq V_1$ or $V_{q-1} \neq \mathbb{F}^r$, respectively.

This is equivalent to Σ^2 of the simplicial set with $(q-2)$-simplices the chains

$$0 \subsetneq V_1 \subset \dots \subset V_{q-1} \subsetneq \mathbb{F}^r ,$$

which is the nerve of the set of proper, nontrivial subspaces V of $\mathbb{F}^r$, partially ordered by inclusion, i.e., the Tits building $B(\mathbb{F}^r)$. An element $A \in GL_r(\mathbb{F})$ acts on the partially ordered set by mapping V to $A(V)$, and has the induced action on $B(\mathbb{F}^r)$. □

Example 8.3. $\Sigma^2 B(\mathbb{F}^1) \cong \Delta^1 / \partial \Delta^1 \cong S^1$.

Theorem 8.4 (Solomon–Tits)**.**

$$B(\mathbb{F}^r) \simeq \bigvee_\alpha S^{r-2} .$$

Definition 8.5.

$$\mathrm{St}_r(\mathbb{F}) = \tilde{H}_{r-2} B(\mathbb{F}^r) \cong \tilde{H}_r \Sigma^2 B(\mathbb{F}^r) \cong \bigoplus_\alpha \mathbb{Z}$$

is the *Steinberg representation* of $GL_r(\mathbb{F})$.

Corollary 8.6. *The homology*

$$\tilde{H}_*(F_rK(\mathbb{F})_1/F_{r-1}K(\mathbb{F})_1) \cong \tilde{H}_*(\Sigma^2 B(\mathbb{F}^r)_{hGL_r(\mathbb{F})}) \cong H^{gp}_{*-r}(GL_r(\mathbb{F}); \mathrm{St}_r(\mathbb{F}))$$

is concentrated in degrees $* \geq r$*. Hence*

$$H_{j+1}(F_rK(\mathbb{F})_1) \to H_{j+1}K(\mathbb{F})_1$$

is surjective for $j+1 = r$*, and an isomorphism for* $j+1 < r$*. Thus*

$$F_rH_{j+1}K(\mathbb{F})_1 = \mathrm{im}(H_{j+1}(F_rK(\mathbb{F})_1) \to H_{j+1}K(\mathbb{F})_1)$$

is equal to $H_{j+1}K(\mathbb{F})_1$ *for* $r \geq j+1$*.*

This enters in the proof of the following theorem.

Theorem 8.7 (Quillen)**.** *Let* $\mathcal{O}_F$ *be the ring of integers in a number field* F*. For each* $j \geq 0$ *the group* $K_j(\mathcal{O}_F)$ *is finitely generated.*

The connectivity conjecture asserts a stronger convergence result, namely $F_rK_j(\mathbb{F}) = K_j(\mathbb{F})$ for $2r \geq j+1$, but for the more powerful stable rank filtration.

9. The spectrum level rank filtration

Also recall that $\mathbf{K}(\mathbb{F}) = \{n \mapsto K(\mathbb{F})_n = |iS^{(n)}_\bullet \mathscr{P}(\mathbb{F})|\}$ where $iS^{(n)}_\bullet \mathscr{P}(\mathbb{F})$ has q-simplices the category with objects

$$\sigma\colon [q]^n \to \mathscr{P}(\mathbb{F})$$
$$(i_1, \dots, i_n) \mapsto V_{i_1,\dots,i_n}$$

plus choices of subquotients, subject to lists of conditions.

Definition 9.1 (Rognes (1992))**.** Let $F_rK(\mathbb{F})_n \subset K(\mathbb{F})_n$ be the subspace where $\dim V_{q,\dots,q} \leq r$ (so that $\dim V_{i_1,\dots,i_n} \leq r$ for all $(i_1, \dots, i_n)$). Let

$$F_r\mathbf{K}(\mathbb{F}) = \{n \mapsto F_rK(\mathbb{F})_n\}$$

be the associated (pre-)spectrum. The sequence

$$* \rightarrowtail F_1\mathbf{K}(\mathbb{F}) \rightarrowtail \dots \rightarrowtail F_{r-1}\mathbf{K}(\mathbb{F}) \rightarrowtail F_r\mathbf{K}(\mathbb{F}) \rightarrowtail \dots \rightarrowtail \mathbf{K}(\mathbb{F})$$

is the *spectrum level rank filtration.*

Recall that $\pi_j\mathbf{X} = \mathrm{colim}_n\, \pi_{j+n}X_n$ for a prespectrum $\mathbf{X} = \{n \mapsto X_n\}$.

Definition 9.2. Let

$$F_rK_j(\mathbb{F}) = \mathrm{im}(\pi_jF_r\mathbf{K}(\mathbb{F}) \to \pi_j\mathbf{K}(\mathbb{F}))$$

so that

$$0 \subset F_1K_j(\mathbb{F}) \subset \dots \subset F_rK_j(\mathbb{F}) \subset \dots \subset K_j(\mathbb{F})\,.$$

This is the *stable rank filtration.*

Proposition 9.3.

$$F_r\mathbf{K}(\mathbb{F})/F_{r-1}\mathbf{K}(\mathbb{F}) \simeq \mathbf{D}(\mathbb{F}^r)_{hGL_r(\mathbb{F})} = EGL_r(\mathbb{F})_+ \wedge_{GL_r(\mathbb{F})} \mathbf{D}(\mathbb{F}^r)$$

is the homotopy orbit spectrum for $GL_r(\mathbb{F})$ *acting on the* stable building $\mathbf{D}(\mathbb{F}^r)$.

Sketch proof. At level n, $F_rK(\mathbb{F})_n/F_{r-1}K(\mathbb{F})_n$ realizes a simplicial category with q-simplices diagrams

$$\sigma\colon (i_1, \dots, i_n) \mapsto V_{i_1,\dots,i_n}$$

with $\dim V_{q,\dots,q} = r$. It is equivalent to the subcategory where $V_{q,\dots,q} = \mathbb{F}^r$ and each $V_{i_1,\dots,i_n}$ is a subspace of $\mathbb{F}^r$, with morphisms given by the $GL_r(\mathbb{F})$-action on $\mathbb{F}^r$ and its subspaces. □

Definition 9.4. We define $\mathbf{D}(\mathbb{F}^r) = \{n \mapsto D(\mathbb{F}^r)_n\}$ by letting $D(\mathbb{F}^r)_n$ be a simplicial set with q-simplices diagrams $\sigma\colon [q]^n \to \mathrm{Sub}(\mathbb{F}^r) \subset \mathscr{P}(\mathbb{F})$ consisting of subspaces $V_{i_1,\dots,i_n}$ of $\mathbb{F}^r$ and inclusions between these. The case $n = 2$ appears as follows:

$$\begin{array}{ccccccc}
0 & = & 0 & = & \dots & = & 0 \\
\| & & \cap & & & & \cap \\
0 & \subset & V_{1,1} & \subset & \dots & \subset & V_{1,q} \\
\| & & \cap & & & & \cap \\
\vdots & & \vdots & & \ddots & & \vdots \\
\| & & \cap & & & & \cap \\
0 & \subset & V_{q,1} & \subset & \dots & \subset & V_{q,q}
\end{array}$$

with $\sigma\colon (i,j) \mapsto V_{i,j}$. In general we require (0) that $V_{i_1,\dots,i_n} = 0$ if some $i_s = 0$, and $V_{q,\dots,q} = \mathbb{F}^r$, (1) that $V_{i_1,\dots,i_s-1,\dots,i_n} \subset V_{i_1,\dots,i_s,\dots,i_n}$ is an inclusion, (2) that the pushout morphism

$$V_{\dots,i_s-1,\dots,i_t,\dots} \oplus_{V_{\dots,i_s-1,\dots,i_t-1,\dots}} V_{\dots,i_s,\dots,i_t-1,\dots} \to V_{\dots,i_s,\dots,i_t,\dots}$$

is injective, etc. (to (n)). We call these the *lattice conditions*.

Example 9.5. $\mathbf{D}(\mathbb{F}^1) \cong \mathbf{S}$ (the sphere spectrum), so $F_1\mathbf{K}(\mathbb{F}) \simeq \mathbf{S}_{hGL_1(\mathbb{F})} = \Sigma^\infty(B\mathbb{F}^\times)_+$. Rationally, $\pi_j F_1\mathbf{K}(\mathbb{F}) \cong \pi_j^S(B\mathbb{F}^\times_+)$ is isomorphic to $H_j(B\mathbb{F}^\times) = H_j^{gp}(\mathbb{F}^\times)$, which is also rationally isomorphic to $\Lambda^j\mathbb{F}^\times$. Hence $F_1K_j(\mathbb{F}) \subset K_j(\mathbb{F})$ agrees rationally with the image of Milnor K-theory:

$$F_1K_j(\mathbb{F})_\mathbb{Q} = K_j^M(\mathbb{F})_\mathbb{Q}$$

as subgroups of $K_j(\mathbb{F})_\mathbb{Q}$.

10. The component filtration

To analyze the stable building $\mathbf{D}(\mathbb{F}^r)$ we associate some invariants to the simplices $\sigma\colon [q]^n \to \mathrm{Sub}(\mathbb{F}^r)$.

Definition 10.1. The *rank jump* at $\vec{p} = (i_1,\dots,i_n) \in [q]^n$ is the dimension of the cokernel of the n-cube pushout morphism to $V_{i_1,\dots,i_n}$, i.e., the alternating sum

$$\sum_{\epsilon_1,\dots,\epsilon_n\in\{0,1\}} (-1)^{\epsilon_1+\dots+\epsilon_n} \dim V_{i_1-\epsilon_1,\dots,i_n-\epsilon_n} \,.$$

It is non-negative by the lattice conditions, and the sum over all $\vec{p}$ of the rank jumps is $r = \dim V_{q,\dots,q}$. Hence there are r distinguished points $\vec{p}_1,\dots,\vec{p}_r \in [q]^n$, counted with multiplicities, where the rank jumps are positive.

(The ordering of $\vec{p}_1,\dots,\vec{p}_r$ is not well-defined.)

A preordering is a reflexive and transitive relation. It amounts to a small category with at most one morphism from i to j for each pair of objects (i,j).

Definition 10.2. The r distinguished points $\vec{p}_1,\dots,\vec{p}_r$ inherit a preordering from the product partial ordering on $[q]^n$. Let the *path component count* of σ, denoted $c(\sigma)$, be the number of path components of (the classifying space of the category associated to) this preordering. Clearly $1 \le c(\sigma) \le r$.

Face operators in $D(\mathbb{F}^r)_n$ may merge distinguished points, which in turn may reduce the path component count.

Definition 10.3. Let $F_cD(\mathbb{F}^r)_n \subset D(\mathbb{F}^r)_n$ be the simplicial subset consisting of simplices σ with path component count $c(\sigma) \le c$. Let

$$F_c\mathbf{D}(\mathbb{F}^r) = \{n \mapsto F_cD(\mathbb{F}^r)_n\}$$

be the associated (pre-)spectrum. The sequence

$$* \rightarrowtail F_1\mathbf{D}(\mathbb{F}^r) \rightarrowtail \dots \rightarrowtail F_{c-1}\mathbf{D}(\mathbb{F}^r) \rightarrowtail F_c\mathbf{D}(\mathbb{F}^r) \rightarrowtail \dots \rightarrowtail F_r\mathbf{D}(\mathbb{F}^r) = \mathbf{D}(\mathbb{F}^r)$$

is the *component filtration* of the stable building $\mathbf{D}(\mathbb{F}^r)$.

Example 10.4. $F_1\mathbf{D}(\mathbb{F}^r) \simeq \Sigma^\infty\Sigma B(\mathbb{F}^r) \simeq \bigvee_\alpha \mathbf{S}^{r-1}$.

Theorem 10.5.

$$F_c\mathbf{D}(\mathbb{F}^r)/F_{c-1}\mathbf{D}(\mathbb{F}^r) \simeq \bigvee_\beta \mathbf{S}^{r+c-2}$$

for $1 \le c \le r$.

Sketch proof. There is a finer filtration of $\mathbf{D}(\mathbb{F}^r)$ (than the component filtration) given by restricting the (isomorphism classes of) preorders on $\{1, \dots, r\}$ given by setting $s \preceq t$ if $\vec{p}_s \le \vec{p}_t$. The filtration subquotients of this *preorder filtration* can be completely analyzed, in terms of configuration spaces and smash products of Tits buildings. The preorders that are not componentwise (pre-)linear contribute stably trivial filtration subquotients. The stable homology of configuration spaces contributes Lie representations, and the smash products of Tits building contribute tensor products of Steinberg representations. See [Rognes (1992)] for details. □

Hence $H_*\mathbf{D}(\mathbb{F}^r)$ is the homology of a free chain complex

$$0 \to Z_{2r-2} \xrightarrow{\partial} \dots \xrightarrow{\partial} Z_{r-1} \to 0\,,$$

with

$$Z_{r+c-2} = H_{r+c-2}(F_c\mathbf{D}(\mathbb{F}^r)/F_{c-1}\mathbf{D}(\mathbb{F}^r)) \quad (\cong \bigoplus_\beta \mathbb{Z})$$

for $1 \le c \le r$. In particular,

$$Z_{2r-2} = \mathbb{Z}[GL_r(\mathbb{F})/T_r] \otimes_{\Sigma_r} \mathrm{Lie}_r^*$$

and $Z_{r-1} = \mathrm{St}_r(\mathbb{F})$. Here $T_r \subset GL_r(\mathbb{F})$ is the diagonal torus, and Lie_r^* is the dual of the Lie representation of the symmetric group Σ_r. The group $GL_r(\mathbb{F})$ acts naturally on this complex.

Corollary 10.6. $H_*\mathbf{D}(\mathbb{F}^r)$ *is concentrated in the range* $r - 1 \le * \le 2r - 2$.

11. The connectivity conjecture

In [Rognes (1992)] we made the following conjecture.

Conjecture 11.1 (Connectivity)**.** $H_*\mathbf{D}(\mathbb{F}^r)$ *is concentrated in degree* $(2r - 2)$.

Equivalently, the complex

$$0 \to H_{2r-2}\mathbf{D}(\mathbb{F}^r) \to Z_{2r-2} \xrightarrow{\partial} \dots \xrightarrow{\partial} Z_{r-1} \to 0$$

is exact, $\mathbf{D}(\mathbb{F}^r)$ is $(2r - 3)$-connected, and $\mathbf{D}(\mathbb{F}^r) \simeq \bigvee_\gamma \mathbf{S}^{2r-2}$.

Theorem 11.2 (Rognes)**.** *The connectivity conjecture is true for* $r = 1$, 2 *and* 3.

Definition 11.3. Let

$$\Delta_r(\mathbb{F}) = H_{2r-2}\mathbf{D}(\mathbb{F}^r) \quad (\cong \bigoplus_\gamma \mathbb{Z})$$

be the *stable Steinberg representation* of $GL_r(\mathbb{F})$.

Example 11.4. $\Delta_1(\mathbb{F}) = \mathbb{Z}$ and $\Delta_2(\mathbb{F})$ is H_1 of the complete graph on the set $\mathbb{P}^1(\mathbb{F})$ of lines $L \subset \mathbb{F}^2$.

Corollary 11.5. *If the connectivity conjecture holds, then*

$$H_*(F_r\mathbf{K}(\mathbb{F})/F_{r-1}\mathbf{K}(\mathbb{F})) \cong H_*(\mathbf{D}(\mathbb{F}^r)_{hGL_r(\mathbb{F})}) \cong H^{gp}_{*-2r+2}(GL_r(\mathbb{F}); \Delta_r(\mathbb{F}))$$

is concentrated in degrees $* \ge 2r - 2$. *Then* $F_r\mathbf{K}(\mathbb{F}) \to \mathbf{K}(\mathbb{F})$ *is* $(2r - 1)$*-connected, so*

$$F_rK_j(\mathbb{F}) = \mathrm{im}(\pi_j F_r\mathbf{K}(\mathbb{F}) \to \pi_j\mathbf{K}(\mathbb{F}))$$

is equal to $K_j(\mathbb{F})$ *for* $j \le 2r - 1$*, or equivalently, for* $2r \ge j + 1$.

Remark 11.6. For $r \ge 2$, if $H^{gp}_0(GL_r(\mathbb{F}); \Delta_r(\mathbb{F})) = \Delta_r(\mathbb{F})_{GL_r(\mathbb{F})}$ is torsion, hence rationally trivial, then $F_rK_j(\mathbb{F})_\mathbb{Q} = K_j(\mathbb{F})_\mathbb{Q}$ also for $j = 2r$, i.e., for $2r \ge j$.

12. The stable rank conjecture

Applying homology to the sequence of homotopy cofiber sequences

$$\begin{array}{ccccccccccc} * & \longrightarrow & F_1\mathbf{K}(\mathbb{F}) & \longrightarrow & F_2\mathbf{K}(\mathbb{F}) & \longrightarrow \dots \longrightarrow & F_r\mathbf{K}(\mathbb{F}) & \longrightarrow & \dots & \longrightarrow & \mathbf{K}(\mathbb{F}) \\ & & \downarrow \simeq & & \downarrow & & \downarrow & & & & \\ & & \Sigma^\infty B\mathbb{F}^\times_+ & & \mathbf{D}(\mathbb{F}^2)_{hGL_2(\mathbb{F})} & & \mathbf{D}(\mathbb{F}^r)_{hGL_r(\mathbb{F})} & & & & \end{array}$$

with $F_r\mathbf{K}(\mathbb{F})$ in filtration $s = r-1$ we obtain the *homological rank spectral sequence*

$$E^1_{s,t}(rk) = H_{s+t}(\mathbf{D}(\mathbb{F}^{s+1})_{hGL_{s+1}(\mathbb{F})}) \Longrightarrow_s H_{s+t}\mathbf{K}(\mathbb{F})\,.$$

It is of homological type, concentrated in the first quadrant ($s \ge 0$ and $t \ge 0$). Assuming the connectivity conjecture, the E^1-term can be rewritten as

$$E^1_{s,t}(rk) = H^{gp}_{t-s}(GL_{s+1}(\mathbb{F}); \Delta_{s+1}(\mathbb{F}))\,,$$

hence is in fact concentrated in the wedge $s \ge 0$ and $t \ge s$. The Hurewicz homomorphism $K_{s+t}(\mathbb{F}) = \pi_{s+t}\mathbf{K}(\mathbb{F}) \to H_{s+t}\mathbf{K}(\mathbb{F})$ is a rational equivalence, so after rationalization the rank spectral sequence converges to $K_{s+t}(\mathbb{F})_{\mathbb{Q}}$.

$$\begin{array}{cccccc}
t & \dots & \dots & \dots & \dots & \dots \\
4 & H^{gp}_4(\mathbb{F}^\times) \longleftarrow & H^{gp}_3(GL_2\mathbb{F};\Delta_2\mathbb{F}) \longleftarrow & H^{gp}_2(GL_3\mathbb{F};\Delta_3\mathbb{F}) \longleftarrow & H^{gp}_1(GL_4\mathbb{F};\Delta_4\mathbb{F}) \longleftarrow & \dots \\
3 & H^{gp}_3(\mathbb{F}^\times) \longleftarrow & H^{gp}_2(GL_2\mathbb{F};\Delta_2\mathbb{F}) \longleftarrow & H^{gp}_1(GL_3\mathbb{F};\Delta_3\mathbb{F}) \longleftarrow & \Delta_4(\mathbb{F})_{GL_4\mathbb{F}} & \dots \\
2 & H^{gp}_2(\mathbb{F}^\times) \longleftarrow & H^{gp}_1(GL_2\mathbb{F};\Delta_2\mathbb{F}) \longleftarrow & \Delta_3(\mathbb{F})_{GL_3\mathbb{F}} & 0 & \dots \\
1 & \mathbb{F}^\times \xleftarrow{\;d^1\;} & \Delta_2(\mathbb{F})_{GL_2\mathbb{F}} & 0 & 0 & \dots \\
0 & \mathbb{Z} & 0 & 0 & 0 & \dots \\
E^1_{s,t}(rk) & 0 & 1 & 2 & 3 & s
\end{array}$$

Example 12.1. $E^1_{0,t}(rk) = H_t(B\mathbb{F}^\times) = H^{gp}_t(\mathbb{F}^\times)$ is rationally isomorphic to $\Lambda^t\mathbb{F}^\times$.

The E^1-term suggests the following definition of the motivic complexes sought by Beilinson and Lichtenbaum.

Definition 12.2. For each $w \ge 0$ define the *rank complex* $(\Gamma_{rk}(w,\mathbb{F})^*, \delta)$ by

$$\Gamma_{rk}(w,\mathbb{F})^i = E^1_{w-i,w}(rk)$$

and $\delta^i = d^1_{w-i,w} \colon \Gamma_{rk}(w,\mathbb{F})^i \to \Gamma_{rk}(w,\mathbb{F})^{i+1}$.

By definition, $\Gamma_{rk}(w,\mathbb{F})^i = 0$ for $i > w$. If the connectivity conjecture holds, then

$$\Gamma_{rk}(w,\mathbb{F})^i \cong H^{gp}_i(GL_{w-i+1}(\mathbb{F}); \Delta_{w-i+1}(\mathbb{F}))$$

is nonzero only for $0 \le i \le w$.

Definition 12.3. Let the *rank cohomology* $H^*_{rk}(\mathbb{F};\mathbb{Z}(w))$ be the cohomology of this cochain complex:

$$H^i_{rk}(\mathbb{F};\mathbb{Z}(w)) = \frac{\ker \delta^i}{\operatorname{im} \delta^{i-1}} = H^i(\Gamma_{rk}(w,\mathbb{F})^*, \delta)\,.$$

These groups give the E^2-term of the homological rank spectral sequence

$$E^2_{s,t}(rk) = H^{t-s}_{rk}(\mathbb{F};\mathbb{Z}(t)) \Longrightarrow_s H_{s+t}\mathbf{K}(\mathbb{F})\,.$$

If the connectivity conjecture holds, then this spectral sequence is concentrated in the region $0 \le s \le t$ (with $s < t$ for $t > 0$ if $\Delta_r(\mathbb{F})_{GL_r(\mathbb{F})}$ is torsion).

Conjecture 12.4 (Stable Rank)**.** *The motivic spectral sequence and the stable rank spectral sequence are rationally isomorphic, starting from the E^2-terms:*

$$\begin{array}{ccc} E^2_{s,t}(mot)_{\mathbb{Q}} = H^{t-s}_{mot}(\mathbb{F},\mathbb{Z}(t))_{\mathbb{Q}} & \Longrightarrow & K_{s+t}(\mathbb{F})_{\mathbb{Q}} \\ \cong \big\downarrow (?) & & \big\downarrow \cong \\ E^2_{s,t}(rk)_{\mathbb{Q}} = H^{t-s}_{rk}(\mathbb{F},\mathbb{Z}(t))_{\mathbb{Q}} & \Longrightarrow & H_{s+t}\mathbf{K}(\mathbb{F})_{\mathbb{Q}} \end{array}$$

Theorem 12.5. *The stable rank conjecture holds for $s = 0$. More precisely,*

$$\begin{array}{ccc} E^2_{0,j}(mot) & \twoheadrightarrow & E^{\rho}_{0,j}(mot) \\ \big\downarrow & & \big\downarrow \\ E^2_{0,j}(rk) & \twoheadrightarrow & E^{\rho}_{0,j}(rk) \end{array}$$

consists of rational isomorphisms for all $\rho \ge 2$.

Sketch proof. Consider the diagram

$$\begin{array}{ccccc} \Lambda^j\mathbb{F}^\times & \twoheadrightarrow & K^M_j(\mathbb{F}) & \longrightarrow & K_j(\mathbb{F}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ H^{gp}_j\mathbb{F}^\times & \longrightarrow & E^2_{0,j}(rk) & \longrightarrow & H_j\mathbf{K}(\mathbb{F})\,. \end{array}$$

□

The connectivity and stable rank conjectures together imply (the rational form of) the Beilinson–Soulé vanishing conjecture.

An advantage of the stable rank point of view is that $\mathbf{D}(\mathbb{F}^r)$ is described only in terms of linear subspaces $V \subset \mathbb{F}^r$, as opposed to general subvarieties of $\Delta^q_{\mathbb{F}}$.

13. The common basis complex

By covering the stable building $\mathbf{D}(\mathbb{F}^r)$ by the $GL_r(\mathbb{F})$-translates of a *stable apartment* $\mathbf{A}(r)$, we obtain the following elementary description of the stable building.

Definition 13.1. Let the *common basis complex* $D'(\mathbb{F}^r)$ be the simplicial complex with vertices the proper, nontrivial subspaces $0 \subsetneq V \subsetneq \mathbb{F}^r$, such that a set $\{V_0,\dots,V_p\}$ of vertices spans a p-simplex if and only if these vector spaces admit a *common basis*, i.e., there exists a basis $\mathcal{B} = \{b_1,\dots,b_r\}$ for $\mathbb{F}^r$ such that for each $i = 0,\dots,p$ there is a subset of $\mathcal{B}$ that is a basis for V_i.

Theorem 13.2. *$\Sigma^\infty\Sigma D'(\mathbb{F}^r) \simeq \mathbf{D}(\mathbb{F}^r)$.*

Sketch proof. The stable apartment $\mathbf{A}(r)$ the a (pre-)spectrum with n-th space $A(r)_n$ a simplicial set with q-simplices diagrams $\sigma\colon [q]^n \to \mathrm{Sub}(\{1,\dots,r\})$ consisting of subsets of $\{1,\dots,r\}$ and inclusions between these. We know that $A(r)_n \cong S^{rn}$, so $\mathbf{A}(1) \cong \mathbf{S}$ and $\mathbf{A}(r) \simeq *$ for $r \ge 2$. (This, incidentally, gives a proof of the Barratt–Priddy–Quillen theorem.)

The free $\mathbb{F}$-vector space functor $\mathrm{Sub}(\{1,\dots,r\}) \to \mathrm{Sub}(\mathbb{F}^r)$ induces an embedding $\mathbf{A}(r) \rightarrowtail \mathbf{D}(\mathbb{F}^r)$, and the translates $\{g\mathbf{A}(r) \mid g \in GL_r(\mathbb{F})\}$ cover $\mathbf{D}(\mathbb{F}^r)$. A $(p+1)$-fold intersection

$$g_0\mathbf{A}(r) \cap \cdots \cap g_p\mathbf{A}(r)$$

is isomorphic to $\mathbf{S}$ if there is a proper, nontrivial subspace $V \subset \mathbb{F}^r$ such that for each $0 \le s \le p$ there is a basis for V given by a subset of the columns of $g_s \in GL_r(\mathbb{F})$. Otherwise, the intersection is (stably) contractible. Hence $\mathbf{D}(\mathbb{F}^r) \simeq \Sigma^\infty\Sigma D''(\mathbb{F}^r)$, where $D''(\mathbb{F}^r)$ is the simplicial complex with vertices the elements g of $GL_r(\mathbb{F})$, such that $\{g_0,\dots,g_p\}$ span a p-simplex if and only if there is a $0 \subsetneq V \subsetneq \mathbb{F}^r$ such that for each $0 \le s \le p$ a subset of the columns of g_s is a basis for V.

For each $0 \subsetneq V \subsetneq \mathbb{F}^r$ the set of $g \in GL_r(\mathbb{F})$ such that a subset of the columns of g is a basis for V span a contractible subspace $C(V) \subset D''(\mathbb{F}^r)$. A p-fold intersection

$$C(V_0) \cap \cdots \cap C(V_p)$$

is contractible if there exists a single $g \in GL_r(\mathbb{F})$ such that for each $0 \le t \le p$ a subset of the columns of g is a basis for V_t. In other words, the intersection is contractible if $\{V_0, \dots, V_p\}$ admit a common basis. Otherwise the intersection is empty. This proves that $D''(\mathbb{F}^r) \simeq D'(\mathbb{F}^r)$. □

Conjecture 13.3 (Connectivity). *$\tilde{H}_* D'(\mathbb{F}^r)$ is concentrated in degree $(2r-3)$.*

Example 13.4. For $r = 2$, $D'(\mathbb{F}^2)$ is the complete graph on the set $\mathbb{P}^1(\mathbb{F})$ of lines $L \subset \mathbb{F}^2$. It is connected, hence its homology $\tilde{H}_* D'(\mathbb{F}^2)$ is concentrated in degree 1. Thus $\Delta_2(\mathbb{F})$ is the homology of the complete graph on $\mathbb{P}^1(\mathbb{F})$, as previously claimed.

References

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Department of Mathematics, University of Oslo, Norway
E-mail address: rognes@math.uio.no