TWO-PRIMARY ALGEBRAIC K-THEORY OF POINTED SPACES

John Rognes

Abstract. We compute the mod 2 cohomology of Waldhausen's algebraic K-theory spectrum $A(*)$ of the category of finite pointed spaces, as a module over the Steenrod algebra. This also computes the mod 2 cohomology of the smooth Whitehead spectrum of a point, denoted $\mathrm{Wh}^{\mathrm{Diff}}(*)$. Using an Adams spectral sequence we compute the 2-primary homotopy groups of these spectra in dimensions $* \le 18$, and up to extensions in dimensions $19 \le * \le 21$. As applications we show that the linearization map $L\colon A(*) \to K(\mathbb{Z})$ induces the zero homomorphism in mod 2 spectrum cohomology in positive dimensions, the space level Hatcher–Waldhausen map $hw\colon G/O \to \Omega\mathrm{Wh}^{\mathrm{Diff}}(*)$ does not admit a four-fold delooping, and there is a 2-complete spectrum map $M\colon \mathrm{Wh}^{\mathrm{Diff}}(*) \to \Sigma g/o_{\oplus}$ which is precisely 9-connected. Here $g/o_{\oplus}$ is a spectrum whose underlying space has the 2-complete homotopy type of G/O.

Introduction

Let $A(X)$ be Waldhausen's algebraic K-theory of spaces functor evaluated on the space X, see [Wa1]. When X is a manifold, $A(X)$ provides the fundamental link between algebraic K-theory and the geometric topology of X — in particular with the concordance space, the h-cobordism space and the automorphism space of X, see [Wa3]. We are therefore interested in evaluating its homotopy type. It is the aim of this paper to compute the 2-primary homotopy type of $A(X)$ in the case when $X = *$ is the one-point space. We achieve this by computing the mod 2 spectrum cohomology of $A(*)$ as a module over the mod 2 Steenrod algebra. The result is a complete calculation valid in all dimensions; we also compute the homotopy groups of $A(*)$ modulo odd torsion in dimensions $* \le 18$, and up to extensions in dimensions $19 \le * \le 21$.

We begin by discussing some definitions and interpretations of $A(X)$, in order to explain why this is an important homotopy type.

One way to define $A(X)$ is as the algebraic K-theory of a category with cofibrations and weak equivalences $R_f(X)$, whose objects are retractive spaces over X subject to a relative finiteness condition, see [Wa5]. When $X = *$ this category $R_f(*)$ is the category of finite pointed CW-complexes and pointed cellular maps, and is the category of pointed spaces alluded to in the title. The cofibrations are the cellular embeddings, and the weak equivalences are the homotopy equivalences.

Let $hR_f(X)$ be the subcategory of $R_f(X)$ obtained by restricting the morphisms to be homotopy equivalences, and let $|hR_f(X)|$ denote its geometric realization. As a space, $A(X)$ is defined as the loop space $\Omega|hS_\bullet R_f(X)|$, where $S_\bullet$ is Waldhausen's simplicial construction of the same name. This construction can be iterated, and in fact $A(X)$ is an infinite loop space with nth delooping $|hS^{(n)}_\bullet R_f(X)|$ for each $n \geq 1$. There is a canonical map

$$e \colon |hR_f(X)| \to A(X)$$

from the geometric realization of the category of finite retractive spaces over X and homotopy equivalences to the infinite loop space $A(X)$.

There is a natural isomorphism $\pi_0 A(X) \cong \mathbb{Z}$, and for every object $Y \in hR_f(X)$ the image under $\pi_0(e)$ of the corresponding point in $|hR_f(X)|$ is the Euler characteristic $\chi(Y, X) = \tilde{\chi}(Y/X)$ of Y relative to X. From this point of view the map e is a lift of the usual Euler characteristic that takes values in the integers, to a map which takes values in the infinite loop space $A(X)$. Furthermore, a diagram of spaces and homotopy equivalences given as a functor $F \colon \mathcal{C} \to hR_f(X)$ gives rise to a map $e \circ |F| \colon |\mathcal{C}| \to A(*)$, which will detect more information than just the Euler characteristics of the individual spaces in the diagram. For example a finite pointed G-CW complex Y gives rise to a map $BG \to A(*)$ whose homotopy class is a refined invariant of Y. We think of e as a homotopy theoretic improvement on the Euler characteristic, able also to detect information about diagrams of spaces and homotopy equivalences, rather than just individual spaces, and $A(X)$ is the receptacle for this improved Euler characteristic.

In fact $A(*)$ is a kind of universal receptacle for homotopy invariants of finite pointed spaces that take values in infinite loop spaces and are subject to the following additivity condition: for each cofiber sequence $Y' \to Y \to Y''$ we have $[Y'] + [Y''] = [Y]$ where $[Y] \in \pi_0 A(*)$ denotes the path component in $A(*)$ of the invariant applied to Y. Of course, the corresponding universal invariant taking values in an abelian group is just the reduced Euler characteristic. We shall not make the universality claim more precise in this introduction, but note that a similar discussion applies for $A(X)$ and suitably additive homotopy invariants of finite retractive spaces over X.

Hereafter it will be more convenient to work with spectra than with infinite loop spaces. The given deloopings of the infinite loop space $A(X)$ determine a unique connective spectrum, and from now on $A(X)$ will refer to this spectrum. The body of this paper is also written in terms of spectra rather than infinite loop spaces, partly because a few non-connective spectra will appear.

Suspension of retractive spaces over X induces an equivalence on the level of algebraic K-theory, and so $A(X)$ can also be considered as the algebraic K-theory of a category of spectra over X. It is simplest to make this precise for $X = *$, when $A(*)$ is equivalent to the algebraic K-theory of the category of finite CW-spectra, with respect to suitable notions of cofibrations and stable equivalences, see [Wa4].

Let $\mathbb{S}$ be the sphere spectrum in some good closed symmetric monoidal category of spectra and spectrum maps, for example the S-modules of [EKMM] or the Γ-spaces of [Se] and [Ly]. In either case the ring spectrum $\mathbb{S}$ is a monoid object with respect to the internal smash product, and a spectrum is a module over $\mathbb{S}$, so we can sensibly refer to spectra as $\mathbb{S}$-modules. Then $A(*)$ can be described as the algebraic K-theory of a category of $\mathbb{S}$-modules subject to suitable finiteness conditions, and briefly $A(*)$ is the algebraic K-theory of the ring spectrum $\mathbb{S}$. See [BHM] for a discussion in terms of FSPs.

More generally, for a unital and associative ring spectrum A we may consider a category of finitely generated free A-modules, and form its algebraic K-theory, see [D2]. These ring spectra are the monoids in one of the symmetric monoidal categories of spectra considered above, and may conveniently be called $\mathbb{S}$-algebras. For each ring R in the algebraic sense, the Eilenberg–MacLane spectrum HR is an $\mathbb{S}$-algebra whose algebraic K-theory agrees with Quillen's $K(R)$, see [Q2]. For a simplicial monoid G the unreduced suspension spectrum $\Sigma^\infty(G_+)$ is an $\mathbb{S}$-algebra whose algebraic K-theory agrees with Waldhausen's $A(X)$ for $X = BG$. Thus $\mathbb{S}$-algebras encompass the previous examples of inputs for algebraic K-theory. Now $\mathbb{S}$ is a commutative $\mathbb{S}$-algebra, so its algebraic K-theory $K(\mathbb{S}) = A(*)$ is itself a ring spectrum, and furthermore the algebraic K-theory $K(A)$ of any $\mathbb{S}$-algebra is a module spectrum over $A(*)$. Hence every algebraic K-theory spectrum considered so far is a module spectrum over $A(*)$, which further emphasizes the special role played by $A(*)$.

The relationship of $A(X)$ to geometric topology is based on the splitting of spectra $A(X) \simeq \Sigma^\infty(X_+) \vee \mathrm{Wh}^{\mathrm{Diff}}(X)$ for the smooth category, and the cofiber sequence of spectra

$$A(*) \wedge X_+ \xrightarrow{\alpha} A(X) \to \mathrm{Wh}^{\mathrm{PL}}(X)$$

for the piecewise linear category, see [Wa3] and [Wa6]. Here α is the assembly map, one construction of which uses that $A(X)$ is a homotopy functor in X, see [WW2].

The spectra $\mathrm{Wh}^{\mathrm{Diff}}(X)$ and $\mathrm{Wh}^{\mathrm{PL}}(X)$ are the smooth and PL Whitehead spectra, respectively. The topological Whitehead spectrum $\mathrm{Wh}^{\mathrm{Top}}(X)$ is equivalent to the PL one by [KS] and [BuLa]. Thus knowledge of $A(*)$ determines $\mathrm{Wh}^{\mathrm{Diff}}(*)$ and is the ingredient needed to pass from $A(X)$ to $\mathrm{Wh}^{\mathrm{PL}}(X) \simeq \mathrm{Wh}^{\mathrm{Top}}(X)$. The underlying infinite loop spaces of these Whitehead spectra are called Whitehead spaces, and it is perhaps more common to work in terms of these.

When X is a smooth manifold, $\Omega^\infty \mathrm{Wh}^{\mathrm{Diff}}(X)$ gives the homotopy functor that best approximates the space $C^{\mathrm{Diff}}(X)$ of smooth concordances (= pseudoisotopies) of X. By Igusa's stability theorem [Ig] there is a stabilization map

$$\Sigma_X^{\mathrm{Diff}} \colon C^{\mathrm{Diff}}(X) \to \Omega^2\Omega^\infty \mathrm{Wh}^{\mathrm{Diff}}(X)$$

which is at least roughly $n/3$-connected where n is the dimension of X. Similar results relate $\mathrm{Wh}^{\mathrm{PL}}(X)$ and $\mathrm{Wh}^{\mathrm{Top}}(X)$ to the PL- and topological concordance spaces $C^{\mathrm{PL}}(X)$ and $C^{\mathrm{Top}}(X)$ when X is a PL- or topological manifold, respectively.

Furthermore there is a geometrically significant involution on $A(X)$, related through the Whitehead spectra to the involution on concordance spaces arising from 'turning a concordance upside-down', see [H] and [Vog]. By [WW1] there is a map

$$\Phi_X^{\mathrm{Diff}} \colon \widetilde{\mathrm{Diff}}(X)/\mathrm{Diff}(X) \to \Omega^\infty(EC_{2+} \wedge_{C_2} \Omega \mathrm{Wh}^{\mathrm{Diff}}(X))$$

which is at least as connected as the stabilization map considered by Igusa. The C_2-action on $\Omega\mathrm{Wh}^{\mathrm{Diff}}(X)$ on the right is given by the involution, and the homotopy orbit construction is formed on the spectrum level. This is a space level interpretation of the output of the Hatcher spectral sequence [H], which works on the level of homotopy groups.

The space $\widetilde{\mathrm{Diff}}(X)/\mathrm{Diff}(X)$ measures the difference between the topological group $\mathrm{Diff}(X)$ of diffeomorphisms of the smooth manifold X and the simplicial group $\widetilde{\mathrm{Diff}}(X)$ of 'block diffeomorphisms', which is computable in terms of surgery theory, see [H]. Thus knowledge of the homotopy orbits for the involution acting on the spectrum $\mathrm{Wh}^{\mathrm{Diff}}(X)$, or equivalently on the spectrum $A(X)$, can be viewed as giving knowledge of the homotopy type of the space of diffeomorphisms $\mathrm{Diff}(X)$ in dimensions up to roughly $n/3$, where n is the dimension of X. Similar results apply for the spaces of PL homeomorphisms of PL manifolds and homeomorphisms of topological manifolds. See [WW3] for a more detailed survey.

In this paper we shall determine the homotopy type of the 2-primary completion of the spectrum $\mathrm{Wh}^{\mathrm{Diff}}(*)$. Since the Whitehead spectrum is a homotopy functor and preserves connectivity of maps, for any smooth n-manifold X which is roughly $n/3$-connected the map Φ_X^{Diff} composed with the natural map

$$\Omega^\infty(EC_{2+} \wedge_{C_2} \Omega\mathrm{Wh}^{\mathrm{Diff}}(X)) \to \Omega^\infty(EC_{2+} \wedge_{C_2} \Omega\mathrm{Wh}^{\mathrm{Diff}}(*))$$

is roughly $n/3$-connected. Thus when our 2-primary calculation is extended to a calculation of the C_2-homotopy orbits of $\mathrm{Wh}^{\mathrm{Diff}}(*)$, we will have complete information about the 2-primary homotopy type of the space of diffeomorphisms $\mathrm{Diff}(X)$ of roughly $n/3$-connected manifolds up to dimension roughly $n/3$. We leave these calculations for a future paper.

We now turn to a description of the contents of the present paper.

We are able to access the homotopy type of $A(*)$ by means of a comparison of algebraic K-theory with the topological cyclic homology theory of Bökstedt, Hsiang and Madsen [BHM], relying on a theorem of Dundas [D1]. In Chapter 1 we review these notions, and are led in Theorem 1.11 to the homotopy cartesian square

$$\begin{tikzcd} A(*) \arrow[r, "L"] \arrow[d, "\mathrm{trc}_*"] & K(\mathbb{Z}) \arrow[d, "\mathrm{trc}_{\mathbb{Z}}"] \\ TC(*) \arrow[r, "L"] & TC(\mathbb{Z})\,. \end{tikzcd} \tag{0.1}$$

Here TC denotes the topological cyclic homology functor, and the natural transformation trc is the cyclotomic trace map of [BHM]. After 2-adic completion we can identify the primary homotopy type of $A(*)$ because the 2-primary homotopy types of the three other spectra in this diagram are known, together with sufficient information about the maps in the diagram. More specifically, the homotopy type of $TC(*)$ was determined in [BHM], for odd primes p the p-adic completion of $TC(\mathbb{Z})$ was computed in [BM], and the 2-adic completion was determined in [R5]. The 2-adic completion of $K(\mathbb{Z})$ was found in [RW], by arguments based on Voevodsky's proof of the Milnor conjecture [Voe] and the Bloch–Lichtenbaum spectral sequence [BlLi]. The 2-adic map $\mathrm{trc}_{\mathbb{Z}}\colon K(\mathbb{Z}) \to TC(\mathbb{Z})$ was also studied in [R5], in sufficient detail that we can describe $A(*)$ as an extension of $TC(*)$ by the common homotopy fiber of the maps labelled trc_* and $\mathrm{trc}_{\mathbb{Z}}$ in the diagram above.

At odd primes p, the missing information needed to determine the p-primary homotopy type of $A(*)$ is the identification of the p-adic completion of $K(\mathbb{Z})$, i.e., a proof of the p-primary Lichtenbaum–Quillen conjecture for the integers, and the determination of how $A(*)$ is an extension of $TC(*)$ by the homotopy fiber of $\mathrm{trc}_{\mathbb{Z}}$, after p-adic completion. Since $A(*)$ has finite type [Dw], and is rationally equivalent to $K(\mathbb{Z})$, this would suffice to determine the integral homotopy type of $A(*)$.

Also in Chapter 1 we make precise a part of the calculation of $TC(*)$ from [BHM], relating its p-adic completion to the Thom spectrum $\mathbb{C}P^\infty_{-1} = Th(-\gamma^1)$ of minus the canonical complex line bundle over $\mathbb{C}P^\infty$. See Theorem 1.16 and Corollary 1.21, which when combined yield a homotopy equivalence $TC(*) \simeq \Sigma^\infty S^0 \vee \Sigma\mathbb{C}P^\infty_{-1}$ after p-adic completion.

In Chapter 2 we analyze the 2-primary homotopy type of $\mathbb{C}P^\infty_{-1}$ by classical methods. We obtain its homotopy groups in dimensions $* \leq 20$ in Theorem 2.13, by use of the Atiyah–Hirzebruch spectral sequence for stable homotopy associated to the skeleton filtration of $\mathbb{C}P^\infty_{-1}$ by the subspectra $\mathbb{C}P^s_{-1}$ for $s \geq -1$. The E^1-term in this spectral sequence is given in terms of the stable homotopy groups of spheres, π_*^S, and the differentials depend on the attaching maps for the cells in $\mathbb{C}P^\infty_{-1}$. This involves primary and secondary operations in homotopy, somewhat along the lines of Toda's book [To], and we build on previous work for $\mathbb{C}P^\infty$ by Mosher [Mo] and Mukai [Mu1], [Mu2] and [Mu3].

It is much easier to describe $\mathbb{C}P^\infty_{-1}$ cohomologically, and in Proposition 2.15 we find that the mod 2 spectrum cohomology of $\mathbb{C}P^\infty_{-1}$ is cyclic as an A-module, where A is the mod 2 Steenrod algebra, and we describe the annihilator ideal C of the generator in Definition 2.14. The squaring operations Sq^i with i odd together with the admissible monomials Sq^I of length ≥ 2 form a basis for C as an $\mathbb{F}_2$-vector space. Thus $H^*_{spec}(\mathbb{C}P^\infty_{-1}; \mathbb{F}_2) \cong \Sigma^{-2} A/C$ as left graded A-modules. This allows us to describe the E_2-term of the Adams spectral sequence for the 2-adically completed homotopy of $\mathbb{C}P^\infty_{-1}$ in a range in Tables 2.18(a) and (b). Combined with the results from the Atiyah–Hirzebruch spectral sequence, we are also able to determine the differentials that land in homotopical degree $t - s \leq 20$ in this spectral sequence. The details of this computation will be applied in Chapter 5, where Adams filtration and sparseness in the Adams spectral sequence will make it easier for us to study the homotopy type of $A(*)$ (and $\mathrm{Wh}^{\mathrm{Diff}}(*)$) in terms of its spectrum cohomology and the differentials in its Adams spectral sequence, rather than by means of the long exact sequences in homotopy arising from Dundas' homotopy cartesian square (0.1).

In Chapter 3 we familiarize ourselves with the spectrum hofib(trc) defined as the homotopy fiber of the (implicitly 2-completed) map

$$\mathrm{trc}_{\mathbb{Z}} \colon K(\mathbb{Z}) \to TC(\mathbb{Z}) .$$

By Dundas' theorem this is also the homotopy fiber of the map $\mathrm{trc}_* \colon A(*) \to TC(*)$. The principal result is Theorem 3.13, which expresses this common homotopy fiber as the homotopy fiber of the spectrum map $\delta \colon \Sigma^{-2} ku \to \Sigma^4 ko$ given as a suitably connected cover of the explicit composite map

$$\Sigma^4 r \circ \beta^{-2} \circ (\psi^3 - 1) \circ \beta^{-1} \colon \Sigma^{-2} KU \to \Sigma^4 KO .$$

From this description it is easy to extract other homotopical information about hofib(trc), such as its homotopy groups (Corollary 3.16), its spectrum cohomology (Theorem 4.4), or its Adams spectral sequence (Tables 3.18(a) and (b)).

The calculations in Chapter 3 are based on the spectrum level description of $K(\mathbb{Z}[\frac{1}{2}])$ given in Theorem 3.4, and of $K(\mathbb{Q}_2)$ given in Theorem 3.6, which were obtained in [RW] and [R5, 8.1] respectively. The calculation of $K(\mathbb{Z}[\frac{1}{2}])$ relied on the proven Lichtenbaum–Quillen conjecture in this case [RW], using essential inputs from algebraic geometry [Voe] and [BlLi], while the identification of $K(\mathbb{Q}_2)$ in

[R5] amounted to the calculation of $TC(\mathbb{Z})$ completed at 2, which used topological cyclic homology and calculational spectral sequence techniques from stable homotopy theory. The results in Chapter 3 also rely on knowing how the natural map $j' \colon K(\mathbb{Z}[\frac{1}{2}]) \to K(\mathbb{Q}_2)$ acts on the level of homotopy groups, which was determined in [R5, 7.7 and 9.1]. Those results depended on knowing the structure of the K-theory spectra involved, not just their homotopy groups, and were feasible because the prime 2 is so small, or perhaps because it is regular.

These inputs allow us to obtain a spectrum level description of the homotopy fiber of j' in Propositions 3.10 and 3.11, with a more convenient reformulation given in Proposition 3.12. The arguments rely on knowing the endomorphism algebras of the 2-completed connective topological K-theory spectra ko and ku, as well as all the maps between them, which stems from [MST]. Using Quillen's localization sequence in algebraic K-theory, and Hesselholt and Madsen's link between $K(\mathbb{Z}_2)$ and $TC(\mathbb{Z})$ from [HM, Thm. D], we rework the description of $\mathrm{hofib}(j')$ into a spectrum level description of hofib(trc) in Theorem 3.13, as desired.

In Chapter 4 we use the cofiber sequence (3.14)

$$\mathbb{C}P^\infty_{-1} \xrightarrow{i} \mathrm{hofib}(\mathrm{trc}) \xrightarrow{j} \mathrm{Wh}^{\mathrm{Diff}}(*)\,,$$

and the splitting $A(*) \simeq \Sigma^\infty S^0 \vee \mathrm{Wh}^{\mathrm{Diff}}(*)$, to reduce the identification of $A(*)$ to that of $\mathbb{C}P^\infty_{-1}$, which was studied in Chapter 2, to that of hofib(trc), which was settled in Chapter 3, and the map i between the two. At the prime 2 we are in the fortunate situation that the mod 2 spectrum cohomology of $\mathbb{C}P^\infty_{-1}$ is cyclic as an A-module on a generator in degree -2, so because $\mathrm{Wh}^{\mathrm{Diff}}(*)$ is 2-connected it follows that i induces a surjection on cohomology in all degrees. Thus we can omit any discussion of the linearization map $L \colon TC(*) \to TC(\mathbb{Z})$ in Dundas' homotopy cartesian square, and still obtain a complete cohomological description of $\mathrm{Wh}^{\mathrm{Diff}}(*)$.

This is achieved in the main Theorem 4.5. We have an isomorphism of left graded A-modules

$$H^*_{spec}(A(*); \mathbb{F}_2) \cong H^*_{spec}(\Sigma^\infty S^0; \mathbb{F}_2) \oplus H^*_{spec}(\mathrm{Wh}^{\mathrm{Diff}}(*); \mathbb{F}_2)$$

where $H^*_{spec}(\Sigma^\infty S^0; \mathbb{F}_2) = \mathbb{F}_2$ is the trivial A-module in dimension zero, and there is a unique nontrivial extension of left graded A-modules

$$\Sigma^{-2} C/A(Sq^1, Sq^3) \to H^*_{spec}(\mathrm{Wh}^{\mathrm{Diff}}(*); \mathbb{F}_2) \to \Sigma^3 A/A(Sq^1, Sq^2)$$

characterizing $H^*_{spec}(\mathrm{Wh}^{\mathrm{Diff}}(*); \mathbb{F}_2)$. Here $C \subset A$ is the annihilator ideal of the generator for $H^*_{spec}(\mathbb{C}P^\infty_{-1}; \mathbb{F}_2)$, introduced in Definition 2.14. The assertion of the theorem is that algebraically there are precisely two such extensions of left graded A-modules, and $H^*_{spec}(\mathrm{Wh}^{\mathrm{Diff}}(*); \mathbb{F}_2)$ is the one that does not split.

In Chapter 5 we turn to a homotopical analysis of the smooth Whitehead spectrum $\mathrm{Wh}^{\mathrm{Diff}}(*)$, and thus also of $A(*)$. Our approach is to study the Adams spectral sequence (5.5)

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*_{spec}(\mathrm{Wh}^{\mathrm{Diff}}(*); \mathbb{F}_2), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathrm{Wh}^{\mathrm{Diff}}(*))^\wedge_2\,.$$

Here we can in principle compute the E_2-term in a large range of bidegrees, but there will be many families of differentials and a complete determination of the homotopy groups of $\mathrm{Wh}^{\mathrm{Diff}}(*)$ is out of reach.

The cofiber sequence (3.14) displayed above has the special property that its connecting map induces the zero map in mod 2 spectrum cohomology, so its associated long exact sequence in cohomology breaks up into short exact sequences, which in turn induce long exact sequences of Ext_A-groups. Thus the E_2-terms of the Adams spectral sequences for $\mathbb{C}P^\infty_{-1}$, hofib(trc) and $\mathrm{Wh}^{\mathrm{Diff}}(*)$ are linked in a long exact sequence (5.6). The spectral sequence for hofib(trc) was completely described in Chapter 3, and in Chapter 5 we use the long exact sequence of E_2-terms to translate the information from Chapter 2 about differentials in the Adams spectral sequence for $\mathbb{C}P^\infty_{-1}$ to information about differentials in the Adams spectral sequence (5.5) for $\mathrm{Wh}^{\mathrm{Diff}}(*)$. This is a convenient approach, because the Adams spectral sequence of hofib(trc) is concentrated above the line $t-s = 2s+3$, while the differentials in the spectral sequence for $\mathbb{C}P^\infty_{-1}$ mostly originate below this line. The only subtle point concerns whether certain h_1-divisible classes in bidegrees $(s,t) = (4k, 12k+3)$ of (5.5) are hit by differentials, but a comparison with [R5, 9.1] reveals that they indeed do survive to the E_∞-term. Thus the complexity of determining the homotopy groups of $\mathrm{Wh}^{\mathrm{Diff}}(*)$ is in practice equivalent to that of determining the homotopy groups of $\mathbb{C}P^\infty_{-1}$, which is a well-explored but not exhaustively analyzed problem.

The Adams E_2-term for $\mathrm{Wh}^{\mathrm{Diff}}(*)$ is displayed in part in Tables 5.7(a) and (b), and the nonzero differentials landing in homotopical dimension $t-s \le 21$ are listed in Proposition 5.9. This leads to a calculational conclusion in Theorem 5.10, where the 2-completed homotopy groups of $\mathrm{Wh}^{\mathrm{Diff}}(*)$ are listed in dimensions $* \le 18$, and up to group extensions in dimensions $19 \le * \le 21$. Previously only the homotopy groups in dimensions ≤ 3 were known, see [BW]. We do not give names to the classes identified in $\pi_*(\mathrm{Wh}^{\mathrm{Diff}}(*))$, but in Theorem 7.5 we show that the (space level) Hatcher–Waldhausen map $hw\colon G/O \to \Omega\mathrm{Wh}^{\mathrm{Diff}}(*)$ constructed in [Wa3, §3] induces an isomorphism on 2-primary homotopy groups in dimensions $* \le 8$, and an injection on 2-primary homotopy groups in dimensions $* \le 13$. Thus the better known homotopy groups of $G/O \simeq BSO \times \mathrm{Cok}J$ account for much of the low-dimensional homotopy of $\mathrm{Wh}^{\mathrm{Diff}}(*)$.

In Chapter 6 we use the known spectrum level description of $K(\mathbb{Z})$ completed at 2 to compute its mod 2 spectrum cohomology in Theorem 6.4, and to show in Corollary 6.8 that the linearization map $L\colon A(*) \to K(\mathbb{Z})$ induces the zero map in mod 2 spectrum cohomology in positive dimensions. Thus the linearization map does not itself provide a good cohomological approximation to $A(*)$. In Remark 6.9 we explain why the Hatcher–Waldhausen map hw does not admit a four-fold delooping, using that multiplication by the Hopf map $\sigma \in \pi_7^S$ is nonzero on $\pi_4(\Omega\mathrm{Wh}^{\mathrm{Diff}}(*))$, but is zero on $\pi_4(G/O)$. We also explain how this relates to the results of [R1], where an infinite loop map from G/O to a different infinite loop space structure on $\Omega\mathrm{Wh}^{\mathrm{Diff}}(*)$ was obtained.

Following Miller and Priddy [MP], we describe in (6.3) a spectrum $g/o_\oplus$ as the homotopy fiber of the 2-completed unit map $\Sigma^\infty S^0 \to K(\mathbb{Z})$. Its underlying space $G/O_\oplus$ has the same 2-adic homotopy type as the usual G/O. Although there is no spectrum map $\Sigma g/o_\oplus \to \mathrm{Wh}^{\mathrm{Diff}}(*)$ inducing a π_3-isomorphism, we construct in Chapter 7 a 2-complete spectrum map $M\colon \mathrm{Wh}^{\mathrm{Diff}}(*) \to \Sigma g/o_\oplus$ which induces an isomorphism on mod 2 spectrum cohomology in all dimensions $* \le 9$. This is a best possible approximation, since the cohomology groups differ in dimension 10. The comparison of $\mathrm{Wh}^{\mathrm{Diff}}(*)$ with $\Sigma g/o_\oplus$ finally allows us to evaluate the Hatcher–

Waldhausen map on 2-completed homotopy groups in dimensions $* \leq 13$, leading to the previously cited Theorem 7.5.

Acknowledgement. The main part of this work was done in December 1997 during visits to Aarhus and Bielefeld. The author thanks M. Bökstedt, I. Madsen, J. Tornehave and F. Waldhausen for helpful discussions and hospitality.

1. Algebraic K-theory and topological cyclic homology

We commence by discussing the cyclotomic trace map from algebraic K-theory to topological cyclic homology, and a special case of Dundas' theorem comparing relative algebraic K-theory to relative topological cyclic homology.

1.1. Γ-spaces and $\mathbb{S}$-algebras. Let $\mathcal{S}_*$ be the category of pointed simplicial sets, and let Γ^{op} be the category of finite pointed sets $k_+ = \{0, 1, \dots, k\}$ based at 0, and base-point preserving functions. This is the opposite of Segal's category Γ from [Se]. Let $\Gamma\mathcal{S}_*$ be the category of Γ-spaces, i.e., functors $F \colon \Gamma^{op} \to \mathcal{S}_*$ with $F(0_+) = *$. Each Γ-space F naturally extends to a functor $F \colon \mathcal{S}_* \to \mathcal{S}_*$, which when evaluated on spheres determines a connective (pre-)spectrum $\{n \mapsto F(S^n)\}$. We write $\pi_*(F)$ for the homotopy groups of this spectrum. The natural inclusion $\Gamma^{op} \to \mathcal{S}_*$ is a Γ-space denoted $\mathbb{S}$, whose associated spectrum is the sphere spectrum. The groups $\pi_*(\mathbb{S})$ are the stable homotopy groups of spheres.

There is a smash product $\wedge$ of Γ-spaces defined by Lydakis in [Ly], making $(\Gamma\mathcal{S}_*, \wedge, \mathbb{S})$ a symmetric monoidal category. A monoid A in this symmetric monoidal category will be called an $\mathbb{S}$-algebra. Its associated spectrum is an associative ring spectrum, conveniently thought of as an algebra over the sphere spectrum.

1.2. Examples of $\mathbb{S}$-algebras. When G is a simplicial group the functor $\Sigma^\infty(G_+)$ given by $\Sigma^\infty(G_+)(k_+) = G_+ \wedge k_+$ is a Γ-space. The group multiplication and unit define the structure maps

$$\mu \colon \Sigma^\infty(G_+) \wedge \Sigma^\infty(G_+) \to \Sigma^\infty(G_+)$$

and $\eta \colon \mathbb{S} \to \Sigma^\infty(G_+)$ making $\Sigma^\infty(G_+)$ an $\mathbb{S}$-algebra. Its associated ring spectrum is the unreduced suspension spectrum on G, with product map induced by the multiplication on G.

When R is a (discrete) ring the functor HR given by $HR(k_+) = R\{1, \dots, k\}$ (the free R-module on the non-basepoint elements in k_+) is a Γ-space. The ring multiplication and unit define the structure maps

$$\mu \colon HR \wedge HR \to HR$$

and $\eta \colon \mathbb{S} \to HR$ making HR an $\mathbb{S}$-algebra. Its associated ring spectrum is the Eilenberg–Mac Lane spectrum representing ordinary cohomology with coefficients in R.

Let G be a simplicial group, with group of path components $\pi_0(G)$, and let $R = \mathbb{Z}[\pi_0(G)]$ be its integral group ring. The *linearization map* is the map of $\mathbb{S}$-algebras $L \colon \Sigma^\infty(G_+) \to HR$ taking $g \wedge i \in G_+ \wedge k_+$ to $[g] \cdot i \in R\{1, \dots k\}$, where $g \in G$, $i \in \{1, \dots k\}$ and $[g]$ denotes the path component of g viewed as an element of $\pi_0(G) \subset R$.

1.3. Algebraic K-theory, topological Hochschild homology and topological cyclic homology. Let A be an $\mathbb{S}$-algebra. The extended functor $A\colon \mathcal{S}_* \to \mathcal{S}_*$ comes equipped with a product and unit map making it an FSP (functor with smash product) in the sense of [B2]. In [BHM] Bökstedt, Hsiang and Madsen functorially define the algebraic K-theory spectrum $K(A)$, topological Hochschild homology spectrum $THH(A)$ and topological cyclic homology spectrum $TC(A,p)$ of an FSP A. Here p is any prime. An integral functor $A \mapsto TC(A)$ has been defined by Goodwillie (unpublished), together with a natural p-adic equivalence $TC(A) \to TC(A,p)$ for each prime p.

When G is a simplicial group with classifying space $X = BG$, we write $A(X) = K(\Sigma^\infty(G_+))$, $THH(X) = THH(\Sigma^\infty(G_+))$ and $TC(X,p) = TC(\Sigma^\infty(G_+),p)$. Here $A(X)$ is naturally homotopy equivalent to Waldhausen's algebraic K-theory spectrum $A(X)$ of the space X [Wa1], i.e., the algebraic K-theory of the category of finite retractive spaces over X.

When R is a ring we write $K(R) = K(HR)$, $THH(R) = THH(HR)$ and $TC(R,p) = TC(HR,p)$. Here $K(R)$ is naturally homotopy equivalent to Quillen's algebraic K-theory spectrum $K(R)$ of the ring R [Q2], i.e., the algebraic K-theory of the category of finitely generated projective R-modules.

We recall from [BHM, 3.7] that there are C-equivariant homotopy equivalences

$$THH(X) \simeq_C \Sigma^\infty_C(\Lambda X_+) \tag{1.4}$$

for each finite subgroup $C \subset S^1$. Here Σ^∞_C denotes the C-equivariant suspension spectrum, and $C \subset S^1$ acts on the free loop space ΛX by rotating the loops.

1.5. Trace maps. A trace map $\mathrm{tr}_X\colon A(X) \to THH(X)$ was defined by Waldhausen in [Wa2], and Bökstedt defined a trace map $\mathrm{tr}_A\colon K(A) \to THH(A)$ in [B2], as a natural transformation of functors from FSPs to spectra. The cyclotomic trace map trc_A of [BHM] gives a factorization

$$K(A) \xrightarrow{\mathrm{trc}_A} TC(A,p) \xrightarrow{\beta_A} THH(A)$$

of tr_A, although the map to $TC(A,p)$ was initially only defined up to homotopy. The map β_A is a projection map from the homotopy limit defining $TC(A,p)$. When $A = \Sigma^\infty(G_+)$ with $X = BG$ or $A = HR$ we substitute X or R, respectively, for A in the notations trc_A, β_A and tr_A. Thus $\mathrm{trc}_X\colon A(X) \to TC(X,p)$, etc.

In the case $A = \Sigma^\infty(G_+)$ with $X = BG$ the six authors of [6A] gave a model for the cyclotomic trace map trc_X as a natural transformation in X. When $A = HR$, Dundas and McCarthy [DuMc] gave models for $K(R)$ and $TC(R)$ such that trc_R is a natural transformation. Finally Dundas [D2] has provided a construction of functors K, THH and TC from $\mathbb{S}$-algebras to spectra, and natural transformations $\mathrm{trc}\colon K \to TC$, $\beta\colon TC \to THH$ and $\mathrm{tr}\colon K \to THH$ with $\mathrm{tr} = \beta \circ \mathrm{trc}$, which agree up to natural homotopy equivalence with the preceding definitions.

1.6. Dundas' theorem. The following theorem of Dundas [D1] generalizes to maps of $\mathbb{S}$-algebras a theorem of McCarthy [Mc] valid for maps of simplicial rings. Both results are analogous to an older theorem about rational algebraic K-theory due to Goodwillie [Go].

Theorem 1.7 (Dundas). *Let $\phi\colon A \to B$ be a map of $\mathbb{S}$-algebras, such that the ring homomorphism $\pi_0(\phi)\colon \pi_0(A) \to \pi_0(B)$ is a surjection with nilpotent kernel. Then the commutative square of spectra*

$$\begin{tikzcd} K(A) \arrow[r, "\phi"] \arrow[d, "\mathrm{trc}_A"] & K(B) \arrow[d, "\mathrm{trc}_B"] \\ TC(A) \arrow[r, "\phi"] & TC(B) \end{tikzcd}$$

is homotopy cartesian.

Corollary 1.8 (Dundas). *Let G be a simplicial group, and write $X = BG$ and $R = \mathbb{Z}[\pi_1(X)] = \mathbb{Z}[\pi_0(G)]$. The linearization map $L\colon \Sigma^\infty(G_+) \to HR$ induces a homotopy cartesian square*

$$\begin{tikzcd} A(X) \arrow[r, "L"] \arrow[d, "\mathrm{trc}_X"] & K(R) \arrow[d, "\mathrm{trc}_R"] \\ TC(X) \arrow[r, "L"] & TC(R)\,. \end{tikzcd}$$

In particular, the vertical homotopy fiber $\mathrm{hofib}(\mathrm{trc}_X)$ *only depends on the fundamental group $\pi_1(X)$, for a pointed connected space X.*

For the last claim we used that any pointed connected space X is homotopy equivalent to BG for a simplicial group G, e.g. the Kan loop group of X. See [Wa7].

1.9. Whitehead spectra. There are natural cofiber sequences of spectra

$$\Sigma^\infty(X_+) \xrightarrow{i} A(X) \to \mathrm{Wh}^{\mathrm{Diff}}(X)$$

and

$$A(*) \wedge X_+ \xrightarrow{\alpha} A(X) \to \mathrm{Wh}^{\mathrm{PL}}(X)$$

where $\mathrm{Wh}^{\mathrm{Diff}}(X)$ is the *smooth Whitehead spectrum* of X, and $\mathrm{Wh}^{\mathrm{PL}}(X)$ is the *piecewise linear Whitehead spectrum* of X. The sequences are constructed geometrically in [Wa3], where $\mathrm{Wh}^{\mathrm{Diff}}(X)$ is interpreted in terms of stabilized smooth concordance spaces and stabilized spaces of smooth h-cobordisms, and similarly in the piecewise linear case. The identification of the upper left hand homology theory in X with $\Sigma^\infty(X_+)$ uses the 'vanishing of the mystery homology theory' established in [Wa6].

The composite

$$\Sigma^\infty(X_+) \xrightarrow{i} A(X) \xrightarrow{\mathrm{tr}_X} THH(X) \simeq \Sigma^\infty(\Lambda X_+) \xrightarrow{ev} \Sigma^\infty(X_+)$$

is homotopic to the identity. Here $ev\colon \Lambda X \to X$ is the map evaluating a free loop $S^1 \to X$ at the identity $1 \in S^1$. Hence $ev \circ \mathrm{tr}_X$ provides a natural splitting for the first cofiber sequence above, as in

$$A(X) \simeq \Sigma^\infty(X_+) \vee \mathrm{Wh}^{\mathrm{Diff}}(X)\,.$$

We can therefore identify $\mathrm{Wh}^{\mathrm{Diff}}(X)$ with the homotopy fiber of the splitting map $ev \circ \mathrm{tr}_X$.

1.10. The smooth Whitehead spectrum of a point. Suppose $G = 1$, so that $X = *$. Then $ev\colon \Lambda X \to X$ is the identity map, $THH(*) \simeq \Sigma^\infty S^0$, and the splitting above identifies $\mathrm{Wh}^{\mathrm{Diff}}(*)$ with the homotopy fiber of tr_*. We obtain a map of horizontal (co-)fiber sequences of spectra:

$$\begin{tikzcd} \mathrm{Wh}^{\mathrm{Diff}}(*) \arrow[r] \arrow[d, "\widetilde{\mathrm{trc}}"] & A(*) \arrow[r, "\mathrm{tr}_*"] \arrow[d, "\mathrm{trc}_*"] & THH(*) \arrow[d, equal] \\ \widetilde{TC}(*) \arrow[r] & TC(*) \arrow[r, "\beta_*"] & THH(*) . \end{tikzcd}$$

Here $\widetilde{TC}(*)$ is defined as the homotopy fiber of β_*, and $\widetilde{\mathrm{trc}}$ is the induced map of homotopy fibers over trc_* and the identity map on $THH(*)$. The unit map $\Sigma^\infty S^0 \to A(*) \to TC(*)$ and β_* yield a splitting

$$TC(*) \simeq \Sigma^\infty S^0 \vee \widetilde{TC}(*) .$$

Theorem 1.11. *The two squares*

$$\begin{tikzcd} \mathrm{Wh}^{\mathrm{Diff}}(*) \arrow[r] \arrow[d, "\widetilde{\mathrm{trc}}"] & A(*) \arrow[r, "L"] \arrow[d, "\mathrm{trc}_*"] & K(\mathbb{Z}) \arrow[d, "\mathrm{trc}_{\mathbb{Z}}"] \\ \widetilde{TC}(*) \arrow[r] & TC(*) \arrow[r, "L"] & TC(\mathbb{Z}) \end{tikzcd}$$

are homotopy cartesian, and induce homotopy equivalences of vertical homotopy fibers

$$\mathrm{hofib}(\widetilde{\mathrm{trc}}) \xrightarrow{\simeq} \mathrm{hofib}(\mathrm{trc}_*) \xrightarrow{\simeq} \mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}}) .$$

We denote either of these by hofib(trc). $\square$

1.12. The topological cyclic homology of a point. The topological cyclic homology $TC(X,p)$ of a pointed connected space X was computed by Bökstedt, Hsiang and Madsen in [BHM]. We recall their result, making precise a point that was omitted in the published argument. See [Ma, §4.4] for more details about the following review.

Fix a prime p. From (1.4) there is an equivalence $THH(X)^{C_{p^n}} \simeq \Sigma^\infty_{C_{p^n}}(\Lambda X_+)^{C_{p^n}}$ for each $n \geq 0$. The Segal–tom Dieck splitting

$$\Sigma^\infty_{C_{p^n}}(\Lambda X_+)^{C_{p^n}} \simeq \prod_{k=0}^{n} \Sigma^\infty(EC_{p^k} \times_{C_{p^k}} \Lambda X^{C_{p^{n-k}}})_+$$

and the power map homeomorphisms $\Delta_p^{n-k}\colon \Lambda X \cong \Lambda X^{C_{p^{n-k}}}$ combine to give an equivalence

$$THH(X)^{C_{p^n}} \simeq \prod_{k=0}^{n} \Sigma^\infty(EC_{p^k} \times_{C_{p^k}} \Lambda X)_+ . \tag{1.13}$$

The pth power map $\Delta_p\colon \Sigma^\infty \Lambda X_+ \to \Sigma^\infty \Lambda X_+$ is induced by taking a free loop $S^1 \to X$ to its precomposition by the standard degree p map $S^1 \to S^1$. Let

$$t_p\colon \Sigma^\infty(EC_{p^n} \times_{C_{p^n}} \Lambda X)_+ \to \Sigma^\infty(EC_{p^{n-1}} \times_{C_{p^{n-1}}} \Lambda X)_+$$

be the Becker–Gottlieb transfer for the principal C_p-bundle $EC_{p^{n-1}} \times_{C_{p^{n-1}}} \Lambda X \to EC_{p^n} \times_{C_{p^n}} \Lambda X$. There are *restriction* and *Frobenius* maps $R, F\colon THH(X)^{C_{p^n}} \to THH(X)^{C_{p^{n-1}}}$. Up to homotopy these are given by the formulas:

$$\begin{aligned} R(x_0, x_1, \dots, x_n) &= (x_0, x_1, \dots, x_{n-1}) \\ F(x_0, x_1, \dots, x_n) &= (\Delta_p(x_0) + t_p(x_1), t_p(x_2), \dots, t_p(x_n)) . \end{aligned}$$

Here x_k refers to the factor in $\Sigma^\infty(EC_{p^k} \times_{C_{p^k}} \Lambda X)_+$ in the equivalence (1.13), and the formulas must be interpreted as giving maps defined in terms of this splitting.

Writing

$$TR(X,p) = \operatorname*{holim}_{n,R} THH(X)^{C_{p^n}} \simeq \prod_{n=0}^{\infty} \Sigma^\infty(EC_{p^n} \times_{C_{p^n}} \Lambda X)_+ \tag{1.14}$$

we have $R(x_0, x_1, x_2, \dots) = (x_0, x_1, x_2, \dots)$ and $F(x_0, x_1, x_2, \dots) = (\Delta_p(x_0) + t_p(x_1), t_p(x_2), t_p(x_3), \dots)$ up to homotopy. The topological cyclic homology spectrum $TC(X,p)$ is defined as the homotopy equalizer

$$TC(x,p) \xrightarrow{\pi} TR(X,p) \underset{F}{\overset{R}{\rightrightarrows}} TR(X,p) ,$$

and is homotopy equivalent to the homotopy fiber of $1-F\colon TR(X,p) \to TR(X,p)$. Let $T, D\colon TR(X,p) \to TR(X,p)$ be given up to homotopy by the formulas:

$$\begin{aligned} T(x_0, x_1, x_2, \dots) &= (t_p(x_1), t_p(x_2), t_p(x_3), \dots) \\ D(x_0, x_1, x_2, \dots) &= (\Delta_p(x_0), 0, 0, \dots) . \end{aligned}$$

The following observation lets us calculate $TC(X,p)$.

Lemma 1.15. *The composite* $(1-T)\circ(1-D)\colon TR(X,p) \to TR(X,p)$ *is homotopic to* $(1-F)$.

Proof. In terms of the splitting (1.14) it is clear that $(1-D)(x_0, x_1, x_2, \dots) = (x_0 - \Delta_p(x_0), x_1, x_2, \dots)$ is mapped by $(1-T)$ to $(x_0 - \Delta_p(x_0) - t_p(x_1), x_1 - t_p(x_2), x_2 - t_p(x_3), \dots)$, which is homotopic to $(1-F)(x_0, x_1, x_2, \dots)$. $\square$

Given such a choice of commuting homotopy for the right hand square below, there is an induced map of horizontal (co-)fiber sequences

$$\begin{tikzcd} TC(X,p) \arrow[r, "\pi"] \arrow[d, "\alpha_X"] & TR(X,p) \arrow[r, "1-F"] \arrow[d, "1-D"] & TR(X,p) \arrow[d, equal] \\ C(X,p) \arrow[r] & TR(X,p) \arrow[r, "1-T"] & TR(X,p) . \end{tikzcd}$$

Here we have written $C(X,p)$ for the homotopy limit $\operatorname{holim}_{n,t_p} \Sigma^\infty(EC_{p^n} \times_{C_{p^n}} \Lambda X)_+$, which is homotopy equivalent to the homotopy fiber of $1-T$ in view of (1.14). When α_X is determined by the right hand commuting homotopy, the left hand square is homotopy cartesian (with respect a suitable commuting homotopy). Let $pr\colon TR(X,p) \to THH(X) \simeq \Sigma^\infty \Lambda X_+$ denote projection to the zeroth term in the homotopy limit defining $TR(X,p)$. Then there is clearly a homotopy cartesian square

$$\begin{tikzcd} TR(X,p) \arrow[r, "pr"] \arrow[d, "1-D"] & \Sigma^\infty \Lambda X_+ \arrow[d, "1-\Delta_p"] \\ TR(X,p) \arrow[r, "pr"] & \Sigma^\infty \Lambda X_+ \, , \end{tikzcd}$$

again with respect to a suitable commuting homotopy. We can combine these two homotopy cartesian squares horizontally. Then the upper composite $\beta_X = pr \circ \pi\colon TC(X,p) \to TR(X,p) \to THH(X) \simeq \Sigma^\infty \Lambda X_+$ agrees with the natural transformation β of 1.5. The lower composite is the projection $pr_0\colon C(X,p) \to \Sigma^\infty \Lambda X_+$ from the homotopy limit system over the Becker–Gottlieb transfer maps to its zeroth term.

Theorem 1.16. [BHM, §5] *Let X be a pointed connected space and write $C(X,p) = \operatorname{holim}_{n,t_p} \Sigma^\infty(EC_{p^n} \times_{C_{p^n}} \Lambda X)_+$. The diagram*

$$\begin{tikzcd} TC(X,p) \arrow[r, "\alpha_X"] \arrow[d, "\beta_X"] & C(X,p) \arrow[d, "pr_0"] \\ \Sigma^\infty \Lambda X_+ \arrow[r, "1-\Delta_p"] & \Sigma^\infty \Lambda X_+ \end{tikzcd}$$

homotopy commutes, and there exists a commuting homotopy making the diagram homotopy cartesian.

This is now clear. (The proofs in [BHM] and [Ma] only show that the horizontal homotopy fibers in this diagram are homotopy equivalent, not necessarily by the map induced by β_X and pr_0.) Specializing to $X = *$ we have the following corollary, which is what we will use in the rest of the present paper.

Corollary 1.17. *There is a (co-)fiber sequence of spectra*

$$\widetilde{TC}(*,p) \to \operatorname*{holim}_{n,t_p} \Sigma^\infty(BC_{p^n}{}_+) \xrightarrow{pr_0} \Sigma^\infty S^0 \, .$$

For each $n \geq 0$ there is a dimension-shifting S^1-transfer map

$$\mathrm{trf}^n_{S^1} : \Sigma^\infty(\Sigma(\mathbb{C}P^\infty_+)) \to \Sigma^\infty(BC_{p^n}{}_+)$$

associated to the S^1-bundle $BC_{p^n} \to BS^1 \simeq \mathbb{C}P^\infty$. See [K], [LMS] or [Mu1]. These induce a map

$$\Sigma^\infty(\Sigma(\mathbb{C}P^\infty_+)) \to \operatorname*{holim}_{n,t_p} \Sigma^\infty(BC_{p^n}{}_+)$$

which is a homotopy equivalence after p-adic completion. Hence we can identify the map pr_0 above with the S^1-transfer map $\mathrm{trf}^0_{S^1}$, briefly denoted trf_{S^1}, after p-adic completion. Combined with the p-adic equivalence $TC(*) \to TC(*,p)$ we obtain:

Corollary 1.18. [BHM, 5.15] *There is a homotopy equivalence*

$$\widetilde{TC}(*) \simeq \mathrm{hofib}(\mathrm{trf}_{S^1} : \Sigma^\infty(\Sigma(\mathbb{C}P^\infty_+)) \to \Sigma^\infty S^0)$$

after p-adic completion, for each prime p.

1.19. A Thom spectrum. Let $\mathbb{C}P^\infty_k$ denote the truncated complex projective space with one cell in each even dimension greater than or equal to $2k$, interpreted as a spectrum when $k < 0$. There is a homotopy equivalence

$$\mathbb{C}P^\infty_k \simeq Th(k\gamma^1)$$

where the right hand side is the Thom spectrum of k times the canonical complex line bundle over $\mathbb{C}P^\infty$, see [At]. We shall be concerned with the case $k = -1$, i.e., with the spectrum $\mathbb{C}P^\infty_{-1}$, which can be thought of as the Thom spectrum of minus the canonical line bundle on $\mathbb{C}P^\infty$.

Theorem 1.20 (Knapp). *There is a homotopy equivalence*

$$\Sigma\mathbb{C}P^\infty_{-1} \simeq \mathrm{hofib}(\mathrm{trf}_{S^1} : \Sigma^\infty(\Sigma(\mathbb{C}P^\infty_+)) \to \Sigma^\infty S^0)\,.$$

See [K, 2.9] for a proof. Bringing these results together we have shown:

Corollary 1.21. *There is a homotopy equivalence*

$$(\Sigma\mathbb{C}P^\infty_{-1})^\wedge_p \simeq \widetilde{TC}(*)^\wedge_p$$

of p-adically completed spectra. □

2. Two-primary homotopy of $\mathbb{C}P^\infty_{-1}$

In this chapter we study the 2-primary homotopy type of the Thom spectrum $\mathbb{C}P^\infty_{-1}$ of minus the canonical complex line bundle over $\mathbb{C}P^\infty$. We first use a reindexed Atiyah–Hirzebruch spectral sequence for stable homotopy to compute the 2-completed homotopy groups $\pi_*(\mathbb{C}P^\infty_{-1})^\wedge_2$ in dimensions $* \leq 20$, and next compare with the Adams spectral sequence with the same abutment to determine the differentials in the latter spectral sequence in the same range of dimensions.

Only the simple cohomology calculations in Proposition 2.15 and Lemma 2.16 are needed for Chapters 3 and 4, including Theorem 4.5. The harder homotopy calculations in this chapter will first be applied in Chapter 5 and onwards. The reader who is primarily interested in the spectrum cohomology of $A(*)$, rather than its homotopy groups, can therefore read 2.14 – 2.16 and then skip ahead to Chapter 3.

The reindexed Atiyah–Hirzebruch spectral sequence in question is derived from the stable homotopy exact couple associated to the filtration of $\mathbb{C}P^\infty_{-1}$ by the subspectra $\mathbb{C}P^s_{-1}$, for $s \geq -1$. Its E^1-term is

$$E^1_{s,t} = \pi_{s+t}(\mathbb{C}P^s_{-1}/\mathbb{C}P^{s-1}_{-1}) \cong \pi^S_{t-s} \tag{2.1}$$

for $s \geq -1$, and zero elsewhere. Here $\pi^S_k = \pi_k(\Sigma^\infty S^0)$ is the kth stable stem.

To determine the differentials in the reindexed Atiyah–Hirzebruch spectral sequence, we compare with the computation by Mosher [Mo] of the differentials in the corresponding spectral sequence for the stable homotopy of $\mathbb{C}P^\infty$. The E^1-term of the latter spectral sequence is obtained from (2.1) by restricting to filtrations $s \geq 1$, i.e., by omitting the columns $s = -1$ and $s = 0$, and the collapse map $j\colon \mathbb{C}P^\infty_{-1} \to \mathbb{C}P^\infty$ induces a map of spectral sequences. From here on we often use the same notation for a based space and its suspension spectrum, such as writing S^0 for $\Sigma^\infty S^0$.

The differentials in (2.1) landing in filtration $s = 0$ are always zero, due to the splitting $\mathbb{C}P^\infty_0 = \mathbb{C}P^\infty_+ \simeq \mathbb{C}P^\infty \vee S^0$. The differentials in (2.1) landing in filtration $s = -1$ arise from the connecting map in the cofiber sequence $S^{-2} \to \mathbb{C}P^\infty_{-1} \to \mathbb{C}P^\infty_+$. This is the wedge sum of the (desuspended) S^1-transfer map $\mathbb{C}P^\infty \to S^{-1}$, and the (desuspended) multiplication by η map $S^0 \to S^{-1}$. The image of the S^1-transfer map was computed in dimensions $* \leq 20$ by Mukai in [Mu1], [Mu2] and [Mu3], and we use these results to determine the differentials in (2.1) landing in filtration $s = -1$ in the same range of dimensions.

For ease of reference we use similar notation for classes in our spectral sequence (2.1) as in [Mo]. Thus we write $\beta_s \in E^1_{s,s+t}$ for the class corresponding to $\beta \in \pi^S_t$, and write $\mathbb{Z}/n(\beta)$ for a cyclic group of order n with generator β. In Tables 2.5 and 2.12 we briefly write $n(\beta)$ for $\mathbb{Z}/n(\beta)$ and (β) for $\mathbb{Z}(\beta)$, to save some space. Hereafter we concentrate on the 2-primary components, and all spectra and groups are implicitly 2-completed. Differentials are mostly given only up to multiplication by a 2-adic unit.

In dimensions $* \leq 22$, we will use the following presentation for the stable stems π^S_*, following the tables in [To, XIV] and [Ra, A3.3].

$\pi^S_0 = \mathbb{Z}(\iota)$, $\pi^S_1 = \mathbb{Z}/2(\eta)$, $\pi^S_2 = \mathbb{Z}/2(\eta^2)$, $\pi^S_3 = \mathbb{Z}/8(\nu)$, $\pi^S_4 = 0$, $\pi^S_5 = 0$, $\pi^S_6 = \mathbb{Z}/2(\nu^2)$, $\pi^S_7 = \mathbb{Z}/16(\sigma)$, $\pi^S_8 = \mathbb{Z}/2(\bar\nu) \oplus \mathbb{Z}/2(\epsilon)$, $\pi^S_9 = \mathbb{Z}/2(\nu^3) \oplus \mathbb{Z}/2(\eta\epsilon) \oplus \mathbb{Z}/2(\mu)$, $\pi^S_{10} = \mathbb{Z}/2(\eta\mu)$, $\pi^S_{11} = \mathbb{Z}/8(\zeta)$, $\pi^S_{12} = 0$, $\pi^S_{13} = 0$, $\pi^S_{14} = \mathbb{Z}/2(\sigma^2) \oplus \mathbb{Z}/2(\kappa)$, $\pi^S_{15} = \mathbb{Z}/32(\rho) \oplus \mathbb{Z}/2(\eta\kappa)$, $\pi^S_{16} = \mathbb{Z}/2(\eta^*) \oplus \mathbb{Z}/2(\eta\rho)$, $\pi^S_{17} = \mathbb{Z}/2(\eta\eta^*) \oplus \mathbb{Z}/2(\nu\kappa) \oplus \mathbb{Z}/2(\eta^2\rho) \oplus \mathbb{Z}/2(\bar\mu)$, $\pi^S_{18} = \mathbb{Z}/8(\nu^*) \oplus \mathbb{Z}/2(\eta\bar\mu)$, $\pi^S_{19} = \mathbb{Z}/2(\bar\sigma) \oplus \mathbb{Z}/8(\bar\zeta)$, $\pi^S_{20} = \mathbb{Z}/8(\bar\kappa)$, $\pi^S_{21} = \mathbb{Z}/2(\nu\nu^*) \oplus \mathbb{Z}/2(\eta\bar\kappa)$ and $\pi^S_{22} = \mathbb{Z}/2(\nu\bar\sigma) \oplus \mathbb{Z}/2(\eta^2\bar\kappa)$.

For a fixed r, the d^r-differentials in the spectral sequence for $\pi^S_*(\mathbb{C}P^\infty)$ are periodic in the filtration degree s, see [Mo, 4.4], and this periodicity propagates to the spectral sequence (2.1). Hence Mosher's description of the d^1-, d^2- and d^3-differentials for $\mathbb{C}P^\infty$ in [Mo, 5.1, 5.2, and 5.4] extends to give the formulas 2.2, 2.3 and 2.4 for the corresponding differentials in (2.1). Let $\beta \in \pi^S_*$.

Proposition 2.2. *$d^1(\beta_s) = 0$ for s odd and $d^1(\beta_s) = \eta\beta_{s-1}$ for s even.*

Proposition 2.3. *$d^2(\beta_s) = \nu\beta_{s-2}$ for $s \equiv 0, 1, 4, 5 \bmod 8$, $d^2(\beta_s) = 2\nu\beta_{s-2}$ for $s \equiv 3, 6 \bmod 8$ and $d^2(\beta_s) = 0$ for $s \equiv 2, 7 \bmod 8$.*

Proposition 2.4. *$d^3(\beta_s) = 0$ for s odd. If s is even then $d^3(\beta_s) = \gamma_{s-3}$, where $\gamma \in \langle \eta, \nu, \beta\rangle$ for $s \equiv 0 \bmod 8$, $\gamma \in \langle \nu, \eta, \beta\rangle$ for $s \equiv 2 \bmod 8$, $\gamma \in \langle 2\nu, \eta, \beta\rangle + \langle \eta, \nu, \beta\rangle$ for $s \equiv 4 \bmod 8$ and $\gamma \in \langle \nu, \eta, \beta\rangle + \langle \eta, 2\nu, \beta\rangle$ for $s \equiv 6 \bmod 8$.*

The d^1-differentials in (2.1) are given by the following multiplicative relations in π^S_*, see [Ra] and [To].

$\eta\cdot\iota = \eta$, $\eta\cdot\eta = \eta^2$, $\eta\cdot\eta^2 = 4\nu$, $\eta\cdot\nu = 0$, $\eta\cdot\nu^2 = 0$, $\eta\cdot\sigma = \bar\nu + \epsilon$, $\eta\cdot\bar\nu = \nu^3$, $\eta\cdot\epsilon = \eta\epsilon$, $\eta\cdot\nu^3 = 0$, $\eta\cdot\eta\epsilon = 0$, $\eta\cdot\mu = \eta\mu$, $\eta\cdot\eta\mu = 4\zeta$, $\eta\cdot\zeta = 0$, $\eta\cdot\sigma^2 = 0$, $\eta\cdot\kappa = \eta\kappa$, $\eta\cdot\rho = \eta\rho$, $\eta\cdot\eta\kappa = 0$, $\eta\cdot\eta^* = \eta\eta^*$, $\eta\cdot\eta\rho = \eta^2\rho$, $\eta\cdot\eta\eta^* = 4\nu^*$, $\eta\cdot\nu\kappa = 0$, $\eta\cdot\eta^2\rho = 0$, $\eta\cdot\bar\mu = \eta\bar\mu$, $\eta\cdot\nu^* = 0$, $\eta\cdot\eta\bar\mu = 4\bar\zeta$, $\eta\cdot\bar\sigma = 0$, $\eta\cdot\bar\zeta = 0$, $\eta\cdot\bar\kappa = \eta\bar\kappa$, $\eta\cdot\nu\nu^* = 0$ and $\eta\cdot\eta\bar\kappa = \eta^2\bar\kappa$.

For example, $\bar{\sigma} = \langle \nu, \eta\sigma, \sigma \rangle$, so $\eta \cdot \bar{\sigma} = -\langle \eta, \nu, \eta\sigma \rangle \sigma = 0$ with zero indeterminacy.

The d^2-differentials in (2.1) are given by the following multiplicative relations in π_*^S, see [Ra] and [To].

$\nu \cdot \iota = \nu$, $\nu \cdot \nu = \nu^2$, $\nu \cdot \nu^2 = \nu^3$, $\nu \cdot \sigma = 0$, $\nu \cdot \bar{\nu} = 0$, $\nu \cdot \nu^3 = 0$, $\nu \cdot \eta\epsilon = 0$, $\nu \cdot \mu = 0$, $\nu \cdot \zeta = 0$, $\nu \cdot \sigma^2 = 0$, $\nu \cdot \kappa = \nu\kappa$, $\nu \cdot \rho = 0$, $\nu \cdot \eta\kappa = 0$, $\nu \cdot \eta^* = 0$, $\nu \cdot \nu\kappa = 4\bar{\kappa}$, $\nu \cdot \eta^2\rho = 0$, $\nu \cdot \bar{\mu} = 0$, $\nu \cdot \nu^* = \nu\nu^*$, $\nu \cdot \eta\bar{\mu} = 0$, $\nu \cdot \bar{\sigma} = \nu\bar{\sigma}$ and $\nu \cdot \bar{\zeta} = 0$.

The d^3-differentials are given by the following secondary compositions, from [MT], [Mo, 10.1] and [To].

$\langle \nu, \eta, \nu \rangle = \bar{\nu}$, $\langle \eta, \nu, 2\nu \rangle = \langle \eta, 2\nu, \nu \rangle = \{\epsilon, \bar{\nu}\}$, $\langle \nu, \eta, \zeta \rangle \subseteq \{0, \eta\rho\}$, $\langle \eta, \nu, \zeta \rangle = \{0, \eta\rho\}$, $\langle \nu, \eta, \sigma^2 \rangle = \bar{\sigma}$, $\langle \nu, \eta, 2\rho \rangle = \{0, 4\bar{\kappa}\}$ and $\langle \nu, \eta, \eta\kappa \rangle = \pm 2\bar{\kappa}$ by [MT].

The resulting E^4-term is shown in Table 2.5, accounting for all differentials landing in total degree $s + t \le 20$.

In lemmas 2.6 to 2.11, we only consider differentials landing in total degree $s + t \le 20$.

Lemma 2.6. *The nonzero d^4-differentials in* (2.1) *are* $d^4(2\iota_3) = 2\sigma_{-1}$, $d^4(4\iota_5) = 8\sigma_1$, $d^4(4\iota_6) = 8\sigma_2$, $d^4(\iota_7) = 2\sigma_3$, $d^4(8\iota_8) = 8\sigma_4$, $d^4(4\iota_9) = 4\sigma_5$, $d^4(2\iota_{10}) = 2\sigma_6$ *and* $d^4(\sigma_3) = \sigma_{-1}^2$.

Proof. The d^4-differentials landing in filtration $s \ge 1$ and total degree $s + t \le 19$ are determined by those in the spectral sequence for $\pi_*^S(\mathbb{C}P^\infty)$, and are given in [Mo, 5.6 and 6.4].

In total degree 20, $d^4(\zeta_5) = 0$ by the computation of $\pi_{20}^S(\mathbb{C}P^5)$ following [Mu3, 4.2], and $d^4(\sigma_7) = 0$ by the proof of [Mu3, 4.3] (the formula $\gamma_6\sigma = 2i\tilde{\sigma}'\sigma$).

The differentials landing in filtration $s = 0$ are always zero, as noted above. The differentials landing in filtration $s = -1$ are determined by the computation of the S^1-transfer in [Mu1] and [Mu2]. Thus $d^4(2\iota_3) = 2\sigma_{-1}$ by [Mu1, 13.1(iii)], $d^4(\sigma_3) = \sigma_{-1}^2$ by the proof of [Mu2, 5.3] (the formula $g_4\tilde{\sigma}' = \sigma^2$), $d^4(\bar{\nu}_3) = 0$ by the proof of [Mu2, 5.3] (the formula $g_8 i\tilde{\bar{\nu}} = \eta\kappa$), $d^4(\mu_3) = 0$ by the proof of [Mu2, 5.4] (the formula $g_4\tilde{\mu} = 0$), and $d^4(\zeta_3) = 0$ by the proof of [Mu2, 5.5] (the formula $g_4\pi_{17}^S(\mathbb{C}P^3) = 0$). □

Lemma 2.7. *The nonzero d^5-differentials in* (2.1) *are* $d^5(8\iota_6) = \mu_1$, $d^5(16\iota_8) = \mu_3$ *and* $d^5(16\iota_{10}) = \mu_5$.

Proof. The d^5-differentials landing in filtration $s \ge 1$ and total degree $s + t \le 19$ are determined by those in the spectral sequence for $\pi_*^S(\mathbb{C}P^\infty)$, and are given in [Mo, 6.5].

In total degree 20, $d^5(\eta\epsilon_6) = 0$ by the calculation of $\pi_{20}^S(\mathbb{C}P^6)$ following [Mu3, 4.2].

The differentials landing in filtration $s = -1$ are $d^5(8\iota_4) = 0$ by [Mu1, 13.1(iv)], $d^5(2\sigma_4) = 0$ by the proof of [Mu2, 5.4] (the formula $g_5\widetilde{2\sigma}' \equiv 0 \bmod \mu\sigma$), $d^5(\nu_4^3) = 0$ and $d^5(\eta\epsilon_4) = 0$ by the proof of [Mu2, 5.5] (the formulas $g_5\tilde{\nu}^3 \equiv 0 \bmod \{4\nu^*, \eta\bar{\mu}\}$ and $g_5\lambda = 0$, where λ was chosen as a coextension of $\eta^2\sigma$ before [Mu2, 4.7]), and $d^5(\zeta_4) = 0$ by [Mu3, 5.1]. □

Lemma 2.8. *The nonzero d^6-differentials in* (2.1) *are* $d^6(8\iota_5) = \zeta_{-1}$, $d^6(8\iota_7) = 2\zeta_1$, $d^6(32\iota_8) = 2\zeta_2$, $d^6(16\iota_9) = \zeta_3$, $d^6(32\iota_{10}) = 4\zeta_4$, $d^6(2\nu_5) = \kappa_{-1}$, $d^6(\sigma_5) = \nu_{-1}^*$ *and* $d^6(\sigma_7) = 2\nu_1^*$.

Proof. The differentials landing in filtration $s \ge 1$ and total degree $s + t \le 19$ come from [Mo, 6.6], and $d^6(\sigma_7) = 2\nu_1^*$ by [Mu3, 4.3] and its proof (namely, $\gamma_6\sigma =$

0											
0	$4(2\bar\kappa_0)$										
$2(\bar\kappa_{-1})$	$2(\bar\sigma_0) \oplus 8(\bar\zeta_0)$	$2(2\nu_1^*)$									
$4(\bar\zeta_{-1})$	$8(\nu_0^*)$	$2(\bar\mu_1)$	0	$32(\rho_3)$							
$4(\nu^*_{-1})$	$2(\nu\kappa_0) \oplus 2(\eta^2\rho_0)$	$2(\eta_1^*)$	$16(2\rho_2)$	$2(\sigma_3^2) \oplus 2(\kappa_3)$	0						
$2(\bar\mu_{-1})$	0	$32(\rho_1)$	0	0	0	$4(\zeta_5)$					
$2(\eta^*_{-1})$	$16(2\rho_0) \oplus 2(\eta\kappa_0)$	$2(\sigma_1^2)$	0	0	$8(\zeta_4)$	0	$2(\eta\epsilon_6)$				
$32(\rho_{-1})$	$2(\sigma_0^2)$	0	0	$4(\zeta_3)$	0	$2(\mu_5)$	0	$16(\sigma_7)$			
$2(\sigma^2_{-1}) \oplus 2(\kappa_{-1})$	0	0	$8(\zeta_2)$	0	$2(\nu_4^3) \oplus 2(\eta\epsilon_4)$	0	$8(2\sigma_6)$	0	0		
0	0	$4(\zeta_1)$	0	$2(\mu_3)$	0	$16(\sigma_5)$	0	0	0	0	
0	$8(\zeta_0)$	0	$2(\eta\epsilon_2)$	$2(\bar\nu_3)$	$8(2\sigma_4)$	$2(\nu_5^2)$	0	0	$2(4\nu_8)$	0	0
$4(\zeta_{-1})$	0	$2(\mu_1)$	0	$16(\sigma_3)$	0	0	0	0	0	0	$(2\iota_{10})$
0	$2(\nu_0^3) \oplus 2(\eta\epsilon_0)$	0	$8(2\sigma_2)$	0	0	0	$2(\nu_6)$	0	0	$(4\iota_9)$	
$2(\mu_{-1})$	0	$16(\sigma_1)$	0	0	0	$2(2\nu_5)$	0	0	$(8\iota_8)$		
0	$8(2\sigma_0)$	$2(\nu_1^2)$	0	0	0	0	0	(ι_7)			
$16(\sigma_{-1})$	$2(\nu_0^2)$	0	0	0	0	0	$(4\iota_6)$				
0	0	0	0	0	0	$(4\iota_5)$					
0	0	0	0	0	$(8\iota_4)$						
0	$8(\nu_0)$	0	0	$(2\iota_3)$							
0	0	0	$(2\iota_2)$								
0	0	$(4\iota_1)$									
0	$(2\iota_0)$										
(ι_{-1})											

TABLE 2.5. E^4 in total degrees $s+t \leq 20$.

$2i\tilde\sigma'\sigma = 2i\nu^*$).

Also $d^6(8\iota_5) = \zeta_{-1}$ by [Mu1, 13.1(v)], $d^6(2\nu_5) = \kappa_{-1}$ by the proof of [Mu2, 5.3] (the formula $g_8(\widetilde{i2\nu}''') \equiv \kappa \bmod \sigma^2$), $d^6(\nu_5^2) = 0$ by the proof of [Mu2, 5.4] (the formula $g_9 i\tilde{\nu}^2 \equiv \omega^*\eta \bmod i\nu\kappa$), and $d^6(\sigma_5) = \pm\nu^*_{-1}$ by the proof of [Mu2, 5.5] (the formula $g_6\tilde{\sigma}'' \equiv x\nu^* \bmod \eta\bar{\mu}$ where x is odd). □

Lemma 2.9. *The only nonzero d^7-differential in (2.1) is $d^7(\nu_6) = \eta^*_{-1}$.*

Proof. We have $d^7(\nu_6) = \eta^*_{-1}$ by [Mu2, 5.4] and its proof (the formula $g_7\tilde{\nu}'' = \omega^*$). All other d^7-differentials are zero by [Mo, 6.7] or bidegree reasons. □

Lemma 2.10. *The nonzero d^8-differentials in (2.1) are $d^8(16\iota_7) = 2\rho_{-1}$, $d^8(64\iota_9) = 16\rho_1$ and $d^8(64\iota_{10}) = 16\rho_2$.*

Proof. These follow from [Mu1, 4.3] since 2ρ generates the complex image of J in dimension 15, and from [Mo, 6.8]. □

Lemma 2.11. *The remaining nonzero differentials in (2.1) are $d^9(2^7\iota_{10}) = \bar{\mu}_1$ and $d^{10}(2^7\iota_9) = \bar{\zeta}_{-1}$.*

Proof. These follow from [Mo, 6.9] and [Mu1, 4.3], since $\bar{\zeta}$ generates the complex image of J in dimension 19. □

This leaves us with the E^∞-term shown in Table 2.12, in total degrees $s+t \le 20$. Recall the convention that $n(\beta)$ denotes a cyclic group of order n, generated by β.

Theorem 2.13. *The 2-primary homotopy groups of $\mathbb{C}P^\infty_{-1}$ in dimensions $* \le 20$ are as follows:*

$$
\begin{aligned}
\pi_{-2}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}(\iota_{-1})\\
\pi_{-1}(\mathbb{C}P^\infty_{-1}) &= 0\\
\pi_{0}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}(2\iota_{0})\\
\pi_{1}(\mathbb{C}P^\infty_{-1}) &= 0\\
\pi_{2}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}(4\iota_{1})\\
\pi_{3}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/8(\nu_0)\\
\pi_{4}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}(2\iota_{2})\\
\pi_{5}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/2(\sigma_{-1})\\
\pi_{6}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/2(\nu_0^2) \oplus \mathbb{Z}(16\iota_3)\\
\pi_{7}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/2(\mu_{-1}) \rtimes \mathbb{Z}/8(2\sigma_0)\\
&\cong \mathbb{Z}/16(2\sigma_0)\\
\pi_{8}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/2(\nu_1^2) \oplus \mathbb{Z}(8\iota_4)\\
\pi_{9}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/2(\nu_0^3) \oplus \mathbb{Z}/2(\eta\epsilon_0) \oplus \mathbb{Z}/8(\sigma_1)\\
\pi_{10}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}(32\iota_5)\\
\pi_{11}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/8(\zeta_0) \oplus \mathbb{Z}/4(2\sigma_2)\\
\pi_{12}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}(16\iota_6)\\
\pi_{13}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/2(\rho_{-1}) \rtimes \mathbb{Z}/2(\zeta_1) \rtimes \mathbb{Z}/2(\eta\epsilon_2)\\
&\cong \mathbb{Z}/2(\rho_{-1}) \rtimes \mathbb{Z}/4(\eta\epsilon_2)\\
\pi_{14}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/2(\sigma_0^2) \oplus \mathbb{Z}/2(\bar{\nu}_3) \oplus \mathbb{Z}(2^8\iota_7)\\
\pi_{15}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/2(\bar{\mu}_{-1}) \rtimes \mathbb{Z}/16(2\rho_0) \oplus \mathbb{Z}/2(\eta\kappa_0) \rtimes \mathbb{Z}/2(\zeta_2) \rtimes \mathbb{Z}/4(2\sigma_4)\\
&\cong \mathbb{Z}/32(2\rho_0) \oplus \mathbb{Z}/2(\eta\kappa_0) \rtimes \mathbb{Z}/2(\zeta_2) \rtimes \mathbb{Z}/4(2\sigma_4)
\end{aligned}
$$

0											
0	$4(2\bar{\kappa}_0)$										
$2(\bar{\kappa}_{-1})$	$2(\bar{\sigma}_0)$ $\oplus$ $8(\bar{\zeta}_0)$	0									
0	$8(\nu_0^*)$	0	0								
0	$2(\nu\kappa_0)$ $\oplus$ $2(\eta^2\rho_0)$	$2(\eta_1^*)$	$8(2\rho_2)$	$2(\sigma_3^2)$ $\oplus$ $2(\kappa_3)$							
$2(\bar{\mu}_{-1})$	0	$16(\rho_1)$	0	0	0						
0	$16(2\rho_0)$ $\oplus$ $2(\eta\kappa_0)$	$2(\sigma_1^2)$	0	0	$4(\zeta_4)$	0					
$2(\rho_{-1})$	$2(\sigma_0^2)$	0	0	0	0	0	0				
0	0	0	$2(\zeta_2)$	0	$2(\nu_4^3)$ $\oplus$ $2(\eta\epsilon_4)$	0	0	0			
0	0	$2(\zeta_1)$	0	0	0	0	0	0	0		
0	$8(\zeta_0)$	0	$2(\eta\epsilon_2)$	$2(\bar{\nu}_3)$	$4(2\sigma_4)$	$2(\nu_5^2)$	0	0	$2(4\nu_8)$	0	
0	0	0	0	0	0	0	0	0	0	0	$(2^8\iota_{10})$
0	$2(\nu_0^3)$ $\oplus$ $2(\eta\epsilon_0)$	0	$4(2\sigma_2)$	0	0	0	0	0	0	$(2^9\iota_9)$	
$2(\mu_{-1})$	0	$8(\sigma_1)$	0	0	0	0	0	0	$(2^7\iota_8)$		
0	$8(2\sigma_0)$	$2(\nu_1^2)$	0	0	0	0	0	$(2^8\iota_7)$			
$2(\sigma_{-1})$	$2(\nu_0^2)$	0	0	0	0	0	$(16\iota_6)$				
0	0	0	0	0	0	$(32\iota_5)$					
0	0	0	0	0	$(8\iota_4)$						
0	$8(\nu_0)$	0	0	$(16\iota_3)$							
0	0	0	$(2\iota_2)$								
0	0	$(4\iota_1)$									
0	$(2\iota_0)$										
(ι_{-1})											

TABLE 2.12. E^∞ in total degrees $s+t \le 20$.

$$\pi_{16}(\mathbb{C}P^\infty_{-1}) = \mathbb{Z}/2(\sigma_1^2) \oplus \mathbb{Z}/2(\nu_5^2) \oplus \mathbb{Z}(2^7\iota_8)$$

$$\pi_{17}(\mathbb{C}P^\infty_{-1}) = \mathbb{Z}/2(\nu\kappa_0) \oplus \mathbb{Z}/2(\eta^2\rho_0) \oplus \mathbb{Z}/16(\rho_1) \rtimes \mathbb{Z}/2(\nu_4^3) \oplus \mathbb{Z}/2(\eta\epsilon_4)$$

$$\begin{aligned}
&\cong \mathbb{Z}/2(\nu\kappa_0) \oplus \mathbb{Z}/2(\eta^2\rho_0) \oplus \mathbb{Z}/32(\eta\epsilon_4) \oplus \mathbb{Z}/2(\nu_4^3) \\
\pi_{18}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/2(\bar\kappa_{-1}) \rtimes \mathbb{Z}/8(\nu_0^*) \rtimes \mathbb{Z}/2(\eta_1^*) \oplus \mathbb{Z}(2^9\iota_9) \\
\pi_{19}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/2(\bar\sigma_0) \oplus \mathbb{Z}/8(\bar\zeta_0) \oplus \mathbb{Z}/8(2\rho_2) \rtimes \mathbb{Z}/4(\zeta_4) \rtimes \mathbb{Z}/2(4\nu_8) \\
&\cong \mathbb{Z}/2(\bar\sigma_0) \oplus \mathbb{Z}/8(\bar\zeta_0) \oplus \mathbb{Z}/64(4\nu_8) \\
\pi_{20}(\mathbb{C}P^\infty_{-1}) &= \mathbb{Z}/4(2\bar\kappa_0) \rtimes \mathbb{Z}/2(\sigma_3^2) \oplus \mathbb{Z}/2(\kappa_3) \oplus \mathbb{Z}(2^8\iota_{10}) .
\end{aligned}$$

Proof. Up to extensions, this can be read off from the E^∞-term above.

In dimensions $* = 9, 11, 14, 17, 19$ the subgroup in filtration $s = 0$ is split off by the composite map $\mathbb{C}P^\infty_{-1} \to \mathbb{C}P^\infty_+ \to S^0$, followed by a retraction of π_*^S onto the kernel of $\eta\colon \pi_*^S \to \pi_{*+1}^S$.

The extension in dimension 7 will follow from the proof of 2.21 below, in view of h_0-multiplications in the Adams spectral sequence for $\pi_*(\mathbb{C}P^\infty_{-1})$.

The right hand extension in dimension 13 can be read off from

$$\pi_{13}^S(\mathbb{C}P^2) \cong \mathbb{Z}/8(\eta\epsilon_2) \oplus \mathbb{Z}/2(\nu_2^3),$$

see [Mu2, p.197].

The left hand extension in dimension 15 can be read off from $\pi_{19}^S(\mathbb{C}P^2) \cong \pi_{15}^S(\mathbb{C}P^0_{-1})$, see [Mu3, p.133].

The splitting in dimension 16 can be deduced from the injection $\pi_{16}(\mathbb{C}P^\infty_{-1}) \to \pi_{16}(\mathbb{C}P^\infty) \cong (\mathbb{Z}/2)^3 \oplus \mathbb{Z}$, see [Mu2, 1(ii)].

The right hand extension in dimension 17 can be read off from $\pi_{17}^S(\mathbb{C}P^4)$, see [Mu2, 4.7 and 4.8]. Note that $\eta^2\sigma = \nu^3 + \eta\epsilon$, so twice the coextension λ of $\eta^2\sigma$ is twice a coextension of $\eta\epsilon$.

The middle and right hand extensions in dimension 19 follow from [Mu3, 3.2]. $\square$

We proceed to compare these results with the Adams spectral sequence for $\pi_*^S(\mathbb{C}P^\infty_{-1})^\wedge_2$. Let $A = \mathcal{A}(2)$ be the mod 2 Steenrod algebra, generated by the Steenrod squaring operations Sq^i. For each sequence of natural numbers $I = (i_1, \dots, i_n)$ let $Sq^I = Sq^{i_1} \circ \cdots \circ Sq^{i_n}$ be the composite operation. The sequence I, or the operation Sq^I, is said to be *admissible* if $i_s \geq 2i_{s+1}$ for all $0 \leq s < n$. The set of admissible Sq^I form a vector space basis for A.

Definition 2.14. Let C be the left ideal in A with vector space basis the set of admissible Sq^I such that $I = (i_1, \dots, i_n)$ has length $n \geq 2$, or $I = (i)$ with i odd. Then A/C is a cyclic left A-module, with vector space basis the set of Sq^i with $i \geq 0$ even.

Let us briefly write $H^*(X)$ for the mod 2 spectrum cohomology $H^*_{spec}(X; \mathbb{F}_2)$ of a spectrum X. It is naturally a graded left A-module.

Proposition 2.15.

$$H^*(\mathbb{C}P^\infty_{-1}) \cong \Sigma^{-2} A/C$$

as graded left A-modules.

Proof. It is clear that $H^n(\mathbb{C}P^\infty_{-1}) \cong \mathbb{F}_2$ for $n \geq -2$ even, and 0 otherwise. In $H^*(\Sigma^\infty \mathbb{C}P^\infty_+) \cong \mathbb{F}_2\{y^j \mid j \geq 0\}$ with $\deg(y) = 2$ the squaring operations are given by $Sq^{2i-1}(y^j) = 0$ and $Sq^{2i}(y^j) = \binom{j}{i} y^{i+j}$. By James periodicity and stability of the squaring operations the same formulas apply in

$$H^*(\mathbb{C}P^\infty_{-1}) \cong \mathbb{F}_2\{y^j \mid j \geq -1\},$$

including the case $j = -1$. Then $Sq^{2i}(y^{-1}) = y^{i-1}$ since $\binom{-1}{i} \equiv 1 \bmod 2$. To prove the proposition it remains to show that $Sq^I(y^{-1}) = 0$ when $I = (i_1, \dots, i_n)$ is admissible of length ≥ 2. Let $z = Sq^{i_n}(y^{-1})$. Then z has dimension $(i_n - 2)$ and lifts to the ordinary cohomology $H^*(\mathbb{C}P^\infty_+; \mathbb{F}_2)$ of the space $\mathbb{C}P^\infty_+$, which is an unstable A-module. Thus $Sq^{i_{n-1}}(z) = 0$ since $i_{n-1} > i_n - 2$, and so $Sq^I(y^{-1}) = 0$. $\square$

Lemma 2.16. *In $\mathbb{C}P^\infty_{-1}$, the lowest k-invariant*

$$k^1 \colon \Sigma^{-2}H\mathbb{Z} \to \Sigma H\mathbb{Z}$$

is nontrivial, and has mod 2 reduction the class of Sq^3 mod ASq^1.

Proof. The lowest homotopy group of $\mathbb{C}P^\infty_{-1}$ is detected by a map $\mathbb{C}P^\infty_{-1} \to \Sigma^{-2}H\mathbb{Z}$. On cohomology it induces a surjection $\Sigma^{-2}A/ASq^1 \to \Sigma^{-2}A/C$, whose kernel $\Sigma^{-2}C/ASq^1$ begins with $\Sigma^{-2}Sq^3$ mod ASq^1 in degree 1. This is the cohomology operation represented by the lowest k-invariant k^1. $\square$

Consider the Adams spectral sequence

$$E_2^{s,t} = \operatorname{Ext}_A^{s,t}(H^*(\mathbb{C}P^\infty_{-1}), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathbb{C}P^\infty_{-1})^\wedge_2 \,. \tag{2.17}$$

Its E_2-term can be computed in a range from a (minimal) resolution of $\Sigma^{-2}A/C$, either by hand or by Bruner's Ext-calculator program [Br]. The E_2-term in homotopical degrees $t - s \leq 20$ is shown in Tables 2.18(a) and (b). The notation ${}_sx$ represents a class arising in the Adams E_2-term for $\mathbb{C}P^s_{-1}$, mapping to the class usually named x in the Adams E_2-term for $\mathbb{C}P^s_{-1}/\mathbb{C}P^{s-1}_{-1} \cong \Sigma^{2s}S^0$. The distinction between classes marked as '•' or as '∘' will be explained in Chapter 5, following the long exact sequence (5.6). Classes supporting or hit by differentials are shaded gray. In bidegree $(3, 14)$ a d_2-differential hits the sum of the two given generators, which are therefore both half-shaded.

The cofiber sequence of spectra

$$\mathbb{C}P^0_{-1} \xrightarrow{i} \mathbb{C}P^\infty_{-1} \xrightarrow{j} \mathbb{C}P^\infty$$

induces a short exact sequence in mod 2 spectrum cohomology, and thus gives a long exact sequence of Ext-groups relating the Adams E_2-term (2.17) to the Adams E_2-terms

$$'E_2^{s,t} = \operatorname{Ext}_A^{s,t}(H^*(\mathbb{C}P^0_{-1}), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathbb{C}P^0_{-1})^\wedge_2 \tag{2.19}$$

and

$$''E_2^{s,t} = \operatorname{Ext}_A^{s,t}(H^*(\mathbb{C}P^\infty), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathbb{C}P^\infty)^\wedge_2 \,. \tag{2.20}$$

Knowledge of the stable homotopy of $\mathbb{C}P^0_{-1} \simeq \Sigma^{-4}\mathbb{C}P^2$ and $\mathbb{C}P^\infty$ in a range allows us to determine the differentials in the spectral sequences $'E_*$ and $''E_*$ in a similar range. This is comparatively easy for $\mathbb{C}P^0_{-1}$, and was done for $\mathbb{C}P^\infty$ by Mosher in [Mo]. Using the long exact sequence of E_2-terms

$$\dots \to {}'E_2^{s,t} \xrightarrow{i_*} E_2^{s,t} \xrightarrow{j_*} {}''E_2^{s,t} \xrightarrow{\partial} {}'E_2^{s+1,t} \to \dots$$

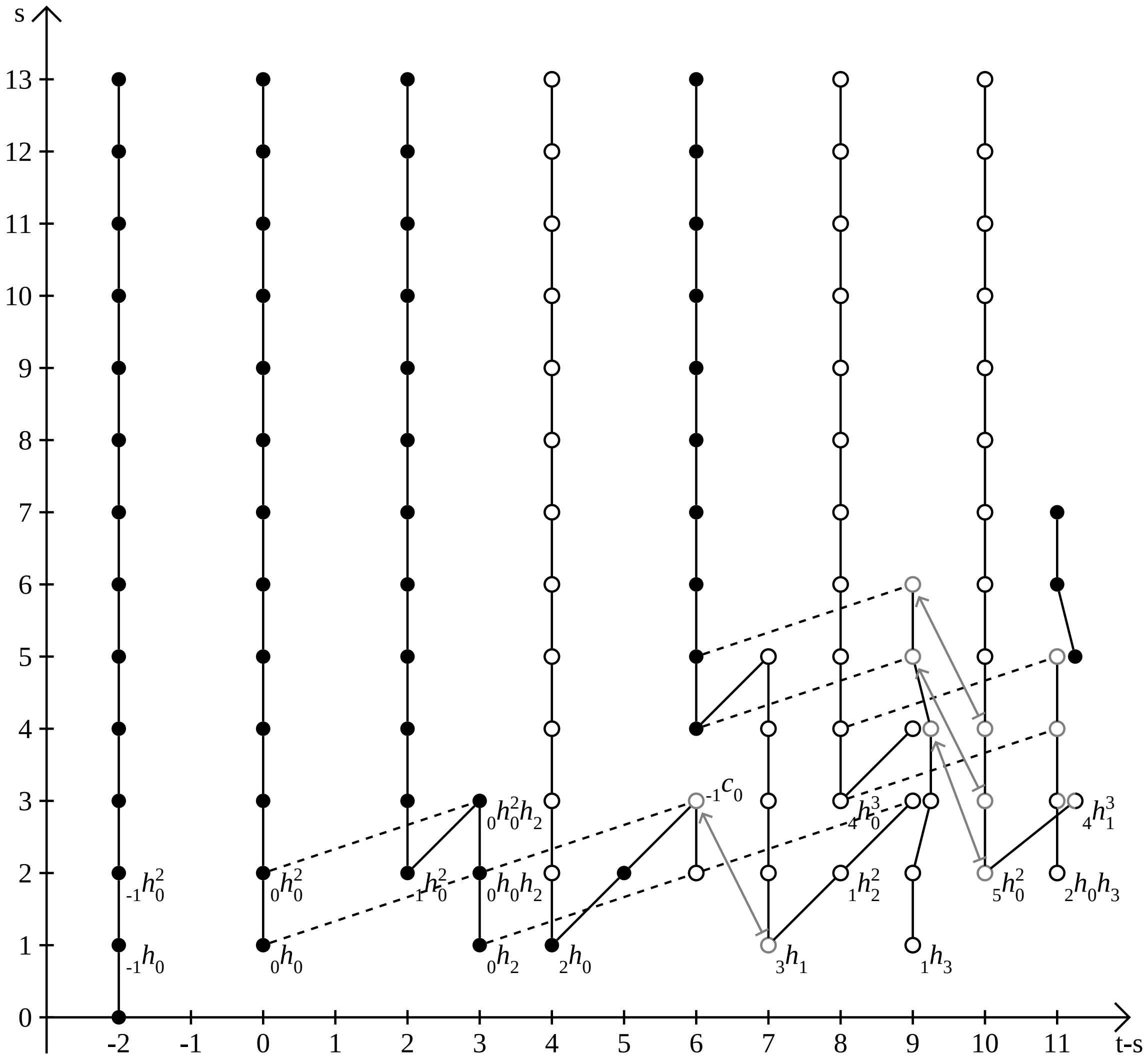

TABLE 2.18(A). The Adams E_2-term for $\mathbb{C}P^{\infty}_{-1}$

and the geometric boundary theorem [Ra, 2.3.4] we can transfer some of these differentials to (2.17). (The careful reader should come equipped with the Ext charts for $\mathbb{C}P^{0}_{-1}$ and $\mathbb{C}P^{\infty}$ to check the details in the following proof.)

Proposition 2.21. *In the Adams spectral sequence* (2.17) *the nonzero differentials landing in homotopical degree* ≤ 20 *are:*

(i) $d_2^{1,8}({}_3h_1) = {}_{-1}c_0$.

(ii) $d_2^{2,12}({}_5h_0^2) = h_0^3 \cdot {}_1h_3$, $d_2^{3,13}({}_5h_0^3) = h_0^4 \cdot {}_1h_3 = {}_{-1}Ph_2$ *and* $d_2^{4,14}({}_5h_0^4) = h_0^5 \cdot {}_1h_3 = {}_{-1}h_0Ph_2$.

(iii) $d_2^{1,13}({}_6h_0) = h_0 \cdot {}_2h_0h_3 + h_1 \cdot {}_5h_0^2$, $d_2^{2,14}({}_6h_0^2) = h_0^2 \cdot {}_2h_0h_3$, $d_2^{3,15}({}_6h_0^3) = h_0^3 \cdot {}_2h_0h_3$ *and* $d_2^{2,15}({}_5h_0h_2) = {}_{-1}d_0$.

(iv) $d_2^{1,16}({}_6h_2) = h_0 \cdot {}_0h_3^2$, $d_3^{2,17}(h_0 \cdot {}_6h_2) = {}_0h_0d_0$, $d_3^{3,18}(h_0^2 \cdot {}_6h_2) = h_0 \cdot {}_0h_0d_0$ *and* $d_3^{4,19}(h_0^3 \cdot {}_6h_2) = h_0^2 \cdot {}_0h_0d_0 = {}_{-1}Pc_0$.

(v) $d_2^{1,18}({}_5h_3) = h_0 \cdot {}_1h_3^2$, $d_2^{2,19}(h_0 \cdot {}_5h_3) = h_0^2 \cdot {}_1h_3^2 = {}_{-1}f_0$ *and* $d_3^{3,20}(h_0^2 \cdot {}_5h_3) = {}_1h_0^2d_0$.

(vi) $d_2^{4,22}(x) = h_0^2 \cdot {}_4h_1c_0$ *with* $h_0 \cdot x \neq 0$, $d_2^{5,23}(h_0 \cdot x) = h_0^3 \cdot {}_4h_1c_0$, $d_2^{6,24}({}_9h_0^6) = h_0^7 \cdot {}_5h_3$, $d_2^{7,25}(h_0 \cdot {}_9h_0^6) = h_0^8 \cdot {}_5h_3 = {}_{-1}P^2h_2$ *and* $d_2^{8,26}(h_0^2 \cdot {}_9h_0^6) = h_0^9 \cdot {}_5h_3 = {}_{-1}h_0P^2h_2$.

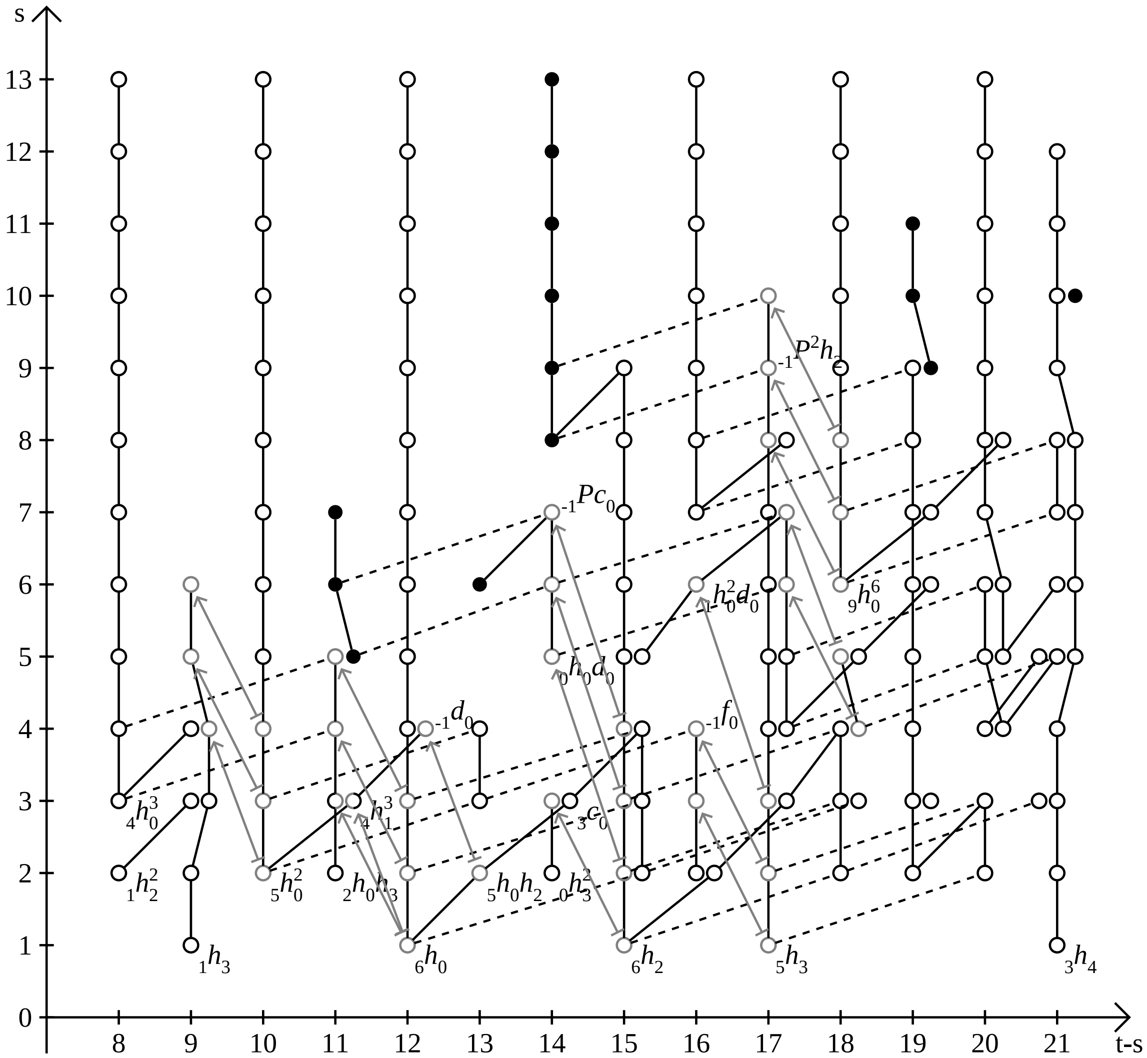

TABLE 2.18(B). The Adams E_2-term for $\mathbb{C}P^\infty_{-1}$

(vii) $d_2^{4,24} \neq 0$, $d_2^{5,25} \neq 0$, $d_2^{6,26} \neq 0$, $d_2^{7,27} \neq 0$, $d_2^{1,22}({}_3h_4) \neq 0$ *and* $d_2^{6,27} \neq 0$ *all have rank* 1.

Proof. We compare the Adams E_2-term in Table 2.18 with its abutment 2.13. Each h_0-torsion class in the E_∞-term of (2.17) comes from an h_0-torsion class in the E_2-term, and so is represented by a 2-torsion class in $\pi_*(\mathbb{C}P^\infty_{-1})$. (The proof of this assertion goes by induction over the subspectra $\mathbb{C}P^s_{-1}$ of $\mathbb{C}P^\infty_{-1}$.)

In each degree $t-s \leq 5$ the order of the 2-torsion in $\pi_*(\mathbb{C}P^\infty_{-1})$ equals the order of the h_0-torsion in Table 2.18, hence there are no nonzero differentials in this range.

(i): In degree $t-s=6$ the 2-torsion in the abutment is $\mathbb{Z}/2$, while the E_2-term has two h_0-torsion generators, so one of these must be hit by a differential. For bidegree reasons the only possibility is $d_2^{1,8}({}_3h_1) = {}_{-1}c_0$, and then there is no room for further differentials landing in degrees $t-s \leq 8$.

In degree 7 of the E_∞-term there is then a nonzero multiplication by h_0^3, showing that the extension in $\pi_7(\mathbb{C}P^\infty_{-1})$ is cyclic.

(ii) and (iii): We turn to degrees $9 \leq t-s \leq 13$. The Adams spectral sequence for $\mathbb{C}P^\infty$, denoted $''E_*$ in (2.20), has differentials $''d_2({}_5h_0^2) = {}_1h_0^3h_3$ and $''d_2({}_6h_0) =$

$h_0 \cdot {}_2h_0h_3 + h_1 \cdot {}_5h_0^2$. This uses $\pi_9^S(\mathbb{C}P^\infty) = \mathbb{Z}/8$ and $\pi_{11}^S(\mathbb{C}P^\infty) = \mathbb{Z}/4$, see [Mo, 7.2].

The map of spectral sequences $j_*\colon E_2 \to {}''E_2$ is an isomorphism in bidegrees $(2,12)$ and $(1,13)$, so these differentials lift to E_2.

Regarding the first ${}''d_2$-differential, both basis elements in $E_2^{4,13} \cong \mathbb{F}_2\{h_0^3 \cdot {}_1h_3, h_1 \cdot {}_4h_0^3\}$ map to ${}_1h_0^3h_3$ in ${}''E_2^{4,13}$. Hence $d_2({}_5h_0^2)$ equals one or the other of these basis elements. It cannot be $h_1 \cdot {}_4h_0^3$, because then $d_2({}_5h_0^3) = 0$ by h_0-multiplication, and more classes would survive to the E_∞-term in degree 9 than the abutment $\pi_9(\mathbb{C}P_{-1}^\infty) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8$ allows. Thus $d_2^{2,12}({}_5h_0^2) = h_0^3 \cdot {}_1h_3$. Multiplication by h_0 implies $d_2^{3,13}({}_5h_0^3) = h_0^4 \cdot {}_1h_3$ and $d_2^{4,14}({}_5h_0^4) = h_0^5 \cdot {}_1h_3$ in (2.17).

In bidegrees $(2,12)$, $(2,13)$ and $(3,14)$ the map j_* is an isomorphism, so the second ${}''d_2$-differential lifts to $d_2^{1,13}({}_6h_0) = h_0 \cdot {}_2h_0h_3 + h_1 \cdot {}_5h_0^2$. Multiplication by h_0, h_0^2 and h_1 implies $d_2^{2,14}({}_6h_0^2) = h_0^2 \cdot {}_2h_0h_3$, $d_2^{3,15}({}_6h_0^3) = h_0^3 \cdot {}_2h_0h_3$ and $d_2^{2,15}({}_5h_0h_2) = {}_{-1}d_0$, respectively. There is no room for further differentials landing in degree $t - s \leq 13$.

(iv): We turn to degrees $14 \leq t - s \leq 15$. For bidegree reasons the class ${}_0h_3^2 \in E_2^{2,16}$ survives to E_∞, and the classes $h_0 \cdot {}_0h_3^2$ and ${}_3c_0$ in $E_2^{3,17}$ can only be affected by a d_2-differential from ${}_6h_2 \in E_2^{1,16}$. The 2-torsion in $\pi_{14}(\mathbb{C}P_{-1}^\infty)$ is $(\mathbb{Z}/2)^2$, so the class $h_0 \cdot {}_0h_3^2$ cannot survive to E_∞, i.e., there is a nonzero differential $d_2({}_6h_2) = h_0 \cdot {}_0h_3^2$ in E_*.

The Adams spectral sequence for $\mathbb{C}P_{-1}^0$, denoted ${}'E_*$ in (2.19), has a differential ${}'d_3({}_0h_0h_4) = {}_0h_0d_0$. (This lifts the usual differential $d_3(h_0h_4) = h_0d_0$ in the Adams spectral sequence for π_*^S. Multiplying this by h_0^2 gives the differential ${}'d_3({}_0h_0^3h_4) = {}_{-1}Pc_0$, arising from the hidden multiplicative relation $\eta \cdot \{h_0^3h_4\} = \{Pc_0\}$ in the stable 16-stem.)

The map of spectral sequences $i_*\colon {}'E_2 \to E_2$ is injective in bidegree $(2,17)$, taking ${}_0h_0h_4$ to $h_0 \cdot {}_6h_2$. For in ${}'E_2$ we know that $h_2 \cdot {}_0h_0h_4 = h_0 \cdot {}_0h_2h_4$. Thus the image of ${}_0h_0h_4$ in E_2 is such that h_2 times it is divisible by h_0, and by inspection this property characterizes $h_0 \cdot {}_6h_2 \in E_2^{2,17}$. Thus we have another nonzero differential $d_3(h_0 \cdot {}_6h_2) = {}_0h_0d_0$ in E_*. Multiplication by h_0 and h_0^2 leads to the differentials $d_3(h_0^2 \cdot {}_6h_2) = h_0 \cdot {}_0h_0d_0$ and $d_3(h_0^3 \cdot {}_6h_2) = h_0^2 \cdot {}_0h_0d_0 = {}_{-1}Pc_0$, respectively. There is no room for further differentials landing in degrees $14 \leq t - s \leq 15$.

(v): Next we consider differentials landing in degree $t - s = 16$. For bidegree reasons the two classes $h_1 \cdot {}_6h_2$ and ${}_1h_3^2$ in $E_2^{2,18}$ survive to E_∞, and since $\pi_{16}(\mathbb{C}P_{-1}^\infty) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2$, the remaining h_0-torsion classes are hit by differentials. Thus $d_2^{1,18}({}_5h_3) = h_0 \cdot {}_1h_3^2$ and $d_2^{2,19}(h_0 \cdot {}_5h_3) = h_0^2 \cdot {}_1h_3^2 = {}_{-1}f_0$.

To determine the last differential landing in degree 16, we compare once again with the Adams spectral sequence ${}''E_*$ for $\pi_*(\mathbb{C}P^\infty)$. Comparing the ${}''E_2$-term and the ${}''E_\infty$-term given in Table 7.2 of [Mo] we deduce that there are differentials ${}''d_2({}_1h_4) = {}_1h_0h_3^2$, ${}''d_3(h_0 \cdot {}_1h_4) = {}_1h_0d_0$ and ${}''d_3(h_0^2 \cdot {}_1h_4) = {}_1h_0^2d_0$. In particular, the cited table asserts that ${}''d_2({}_4h_1c_0) = 0$ does not interfere with the second ${}''d_3$-differential. Also ${}''d_2(h_2 \cdot {}_71) = h_2 \cdot {}''d_2({}_71) = 0$, and ${}''d_3(h_2 \cdot {}_71) = 0$ follows from $\pi_{16}^S(\mathbb{C}P^\infty) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^3$.

The map $j_*\colon E_2 \to {}''E_2$ is an isomorphism in degree $t - s = 17$ and Adams filtration $s \leq 4$, while in degree $t - s = 16$ the kernel consists of the class $h_0^2 \cdot {}_1h_3^2 = {}_{-1}f_0$ only. Comparing $d_2^{1,18}$ to ${}''d_2^{1,18}$ it follows that ${}_5h_3 \in E_2^{1,18}$ maps to ${}_1h_4 \bmod h_2 \cdot {}_71 \in {}''E_2^{1,18}$. Thus $d_3(h_0^2 \cdot {}_5h_3)$ maps under j_* to ${}_1h_0^2d_0$, and we have proven that $d_3^{3,20}(h_0^2 \cdot {}_5h_3) = {}_1h_0^2 d_0$ in E_*.

(vi): In degree $t-s=17$, the abutment has order 2^8 and the E_2-term has 16 classes. Hence there are five differentials landing in degree 17, in addition to the three differentials we have just found leaving that degree. The 2-torsion in the abutment in degree 18 has order 2^5, and the E_2-term has seven h_0-torsion classes. Hence at most two differentials leave the h_0-torsion in degree 18, and at least three differentials leave the h_0-periodic part of the E_2-term. For bidegree reasons this extreme case is precisely what occurs, so $d_2^{6,24}({}_9h_0^6) \neq 0$, $d_2^{7,25}({}_9h_0^7) = {}_{-1}P^2h_2$ and $d_2^{8,26}({}_9h_0^8) = {}_{-1}h_0P^2h_2$, and there are no nonzero differentials landing in degree $t-s=18$.

To precisely pin down the differential $d_2^{6,24}$ we use the same argument as for $d_2^{2,12}$. The map $j_*\colon E_2 \to {}''E_2$ is an isomorphism in bidegree $(6,24)$ and surjective in bidegree $(8,25)$, so the relation $h_1 \cdot {}_8h_0^7 = h_0^7 \cdot {}_1h_4$ in ${}''E_2^{8,25}$ and the differential ${}''d_2({}_9h_0^6) = h_0^7 \cdot {}_1h_4$ implies that $d_2({}_9h_0^6)$ is either $h_1 \cdot {}_8h_0^7$ or $h_0^7 \cdot {}_5h_3$ in $E_2^{8,25}$. Multiplying with h_0 and comparing with $d_2^{7,25}$ eliminates the first possibility, so in fact $d_2^{6,24}({}_9h_0^6) = h_0^7 \cdot {}_5h_3$.

Considering h_0- and h_2-multiplications in the E_2-term, either $d_2=0$ on all h_0-torsion classes in degree $t-s=18$, or $d_2^{4,22}(x) = h_0^2 \cdot {}_4h_1c_0$ on the classes $x \in E_2^{4,22}$ not divisible by h_0, and $d_2^{5,23}(h_0 \cdot x) = h_0^3 \cdot {}_4h_1c_0$. In the former case, the d_3-differential $d_3^{2,17}$ would propagate by h_2- and h_0-multiplications to three nonzero d_3-differentials from the h_0-torsion in degree $t-s=18$, which is incompatible with the abutment. Thus the two d_2-differentials given above are correct, and this accounts for all the differentials from degree $t-s=18$.

(vii): The proofs in degrees $19 \le t-s \le 21$ are left as exercises for the reader who needs these results. □

3. The fiber of the cyclotomic trace map

When localized at $p=2$, the homotopy type of the spectrum $K(\mathbb{Z})$ is known. This involves the Bloch–Lichtenbaum spectral sequence relating motivic cohomology to algebraic K-theory, Voevodsky's proof of the Milnor conjecture, which relates motivic cohomology to étale cohomology, and knowledge of the étale cohomology of the rational 2-integers $\mathbb{Z}[\frac{1}{2}]$.

Similarly, the p-adic homotopy type of the spectrum $TC(\mathbb{Z},p)$ is known for each prime p. They were determined by Bökstedt and Madsen in [BM1] and [BM2] for p odd, and by the author in [R2], [R3], [R4] and [R5] for $p=2$. When $p=2$ the homomorphisms induced by $\mathrm{trc}_{\mathbb{Z}}\colon K(\mathbb{Z}) \to TC(\mathbb{Z},2)$ on homotopy groups are known after 2-adic completion. In this chapter we use this to describe the homotopy fiber of the cyclotomic trace map as a spectrum.

Let all spectra be implicitly completed at 2, throughout this chapter.

3.1. Some two-adic K-theory spectra. We say that a (-1)-connected spectrum is *connective*, and a 0-connected spectrum is *connected*. Let KO and KU denote the real and complex topological K-theory spectra, let ko and ku denote their connective covers, and let bo and bu denote their connected covers, respectively. Write bso and $bspin$ for the 1- and 3-connected covers of KO, and bsu for the 3-connected cover of KU, as usual.

Complex Bott periodicity provides a homotopy equivalence $\beta\colon \Sigma^2 KU \to KU$. There is a *complexification* map $c\colon KO \to KU$ and a *realification* map $r\colon KU \to$

KO. Smashing with the Hopf map $\eta\colon \Sigma^\infty S^1 \to \Sigma^\infty S^0$ yields a map also denoted $\eta\colon \Sigma KO \to KO$. We use the same notation for the various k-connected covers of these maps. There is a cofiber sequence of spectra

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{\Sigma^2 r\circ\beta^{-1}} \Sigma^2 ko\,. \tag{3.2}$$

This follows from R. Wood's theorem $KO \wedge \mathbb{C}P^2 \simeq KU$. See also [MQR, V.5.15]. Here we write $\Sigma^2 r \circ \beta^{-1}$ for a map $\partial\colon ku \to \Sigma^2 ko$ that satisfies $\partial \circ \beta = \Sigma^2 r$. This determines the map up to homotopy, even though $\beta\colon \Sigma^2 ku \to ku$ is not exactly invertible.

Theorem 3.3 (Quillen). *There is a cofiber sequence of spectra*

$$K(\mathbb{F}_3) \xrightarrow{i_3} ku \xrightarrow{\psi^3-1} bu \xrightarrow{\partial_3} \Sigma K(\mathbb{F}_3)\,. \quad \square$$

This is the spectrum level statement of Quillen's computation in [Q1].

The computation in [RW] by Weibel and the author of the 2-primary algebraic K-groups of rings of 2-integers in number fields relies on Suslin's motivic cohomology for fields [S2], Voevodsky's proof of the Milnor conjecture [Voe] and the Bloch–Lichtenbaum spectral sequence [BlLi]. In the case of the 2-integers $\mathbb{Z}[\frac{1}{2}]$ in $\mathbb{Q}$ the result implies that there is a 2-adic homotopy equivalence $K(\mathbb{Z}[\frac{1}{2}]) \simeq JK(\mathbb{Z}[\frac{1}{2}])$, where the latter spectrum was defined by Bökstedt in [B1]. This leads to the following statement:

Theorem 3.4 (Rognes–Weibel). *There is a cofiber sequence of spectra*

$$\Sigma ko \to K(\mathbb{Z}[\frac{1}{2}]) \xrightarrow{\pi_3} K(\mathbb{F}_3) \xrightarrow{\partial} \Sigma^2 ko$$

where π_3 is induced by the ring surjection $\pi_3\colon \mathbb{Z}[\frac{1}{2}] \to \mathbb{F}_3$. The connecting map ∂ is homotopic to the composite

$$K(\mathbb{F}_3) \xrightarrow{i_3} ku \xrightarrow{\Sigma^2 r\circ\beta^{-1}} \Sigma^2 ko\,.$$

Proof. Bökstedt's $JK(\mathbb{Z}[\frac{1}{2}])$ can be defined as the homotopy fiber of the composite

$$ko \xrightarrow{c} ku \xrightarrow{\psi^3-1} bu\,.$$

By [B1, Thm. 2] there is a map $\Phi\colon K(\mathbb{Z}[\frac{1}{2}]) \to JK(\mathbb{Z}[\frac{1}{2}])$ inducing a split surjection on homotopy. By [RW], [We] these spectra have isomorphic homotopy groups, hence Φ is a homotopy equivalence. There is a square of horizontal and vertical cofiber sequences:

$$\begin{tikzcd}
JK(\mathbb{Z}[\frac{1}{2}]) \arrow[r] \arrow[d] & ko \arrow[r] \arrow[d, "c"] & bu \arrow[d, equal] \\
K(\mathbb{F}_3) \arrow[r, "i_3"] \arrow[d, "\partial"] & ku \arrow[r, "\psi^3-1"] \arrow[d, "\Sigma^2 r\circ\beta^{-1}"] & bu \arrow[d] \\
\Sigma^2 ko \arrow[r, equal] & \Sigma^2 ko \arrow[r] & *\,.
\end{tikzcd}$$

The left hand vertical yields the asserted cofiber sequence. Bökstedt's cited construction of the map Φ identifies the composite $K(\mathbb{Z}[\frac{1}{2}]) \to JK(\mathbb{Z}[\frac{1}{2}]) \to K(\mathbb{F}_3)$ with that induced by the ring homomorphism π_3. $\square$

3.5. The reduction map. Let us recall the *Galois reduction map* from [DwMi, §13] and [R5, §3]. Let $\phi^3 \in \mathrm{Gal}(\bar{\mathbb{Q}}_2/\mathbb{Q}_2)$ be a Galois automorphism of the algebraic closure $\bar{\mathbb{Q}}_2$ of the field $\mathbb{Q}_2$ of 2-adic numbers, such that $\phi^3(\zeta) = \zeta^3$ when ζ is a 2-power root of unity, i.e., in $\mu_{2^\infty} \subset \bar{\mathbb{Q}}_2^\times$. We may further assume that $\phi^3(+\sqrt{3}) = +\sqrt{3}$. Then ϕ^3 induces a self-map of $K(\bar{\mathbb{Q}}_2)$ which is compatible up to homotopy with $\psi^3 \colon ku \to ku$ under Suslin's (implicitly 2-adic) homotopy equivalence $K(\bar{\mathbb{Q}}_2) \simeq ku$ from [S1]. Hence the inclusion $K(\mathbb{Q}_2) \to K(\bar{\mathbb{Q}}_2)^{h\phi^3}$ to the homotopy equalizer of ϕ^3 and the identity on $K(\bar{\mathbb{Q}}_2)$ yields a spectrum map $K(\mathbb{Q}_2) \to (ku)^{h\psi^3}$. The connective cover of the target is identified with $K(\mathbb{F}_3)$ by Quillen's theorem, which defines the Galois reduction map

$$\mathrm{red}\colon K(\mathbb{Q}_2) \to K(\mathbb{F}_3)\,.$$

Fix a choice of such a map.

Theorem 3.6 (Rognes). *There are cofiber sequences of spectra*

$$K^{\mathrm{red}}(\mathbb{Q}_2) \to K(\mathbb{Q}_2) \xrightarrow{\mathrm{red}} K(\mathbb{F}_3) \xrightarrow{\partial_2} \Sigma K^{\mathrm{red}}(\mathbb{Q}_2)$$

and

$$\Sigma K(\mathbb{F}_3) \xrightarrow{f_{\mathbb{C}}} K^{\mathrm{red}}(\mathbb{Q}_2) \to \Sigma ku \xrightarrow{\partial_1} \Sigma^2 K(\mathbb{F}_3)\,.$$

The former connecting map ∂_2 is determined by its composite with $\Sigma K^{\mathrm{red}}(\mathbb{Q}_2) \to \Sigma^2 ku$, which up to a two-adic unit is homotopic to the composite

$$K(\mathbb{F}_3) \xrightarrow{i_3} ku \xrightarrow{1-\psi^{-1}} bu \underset{\simeq}{\xrightarrow{\beta^{-1}}} \Sigma^2 ku\,.$$

The latter connecting map ∂_1 is homotopic to the composite

$$\Sigma ku \xrightarrow{\Sigma(1-\psi^{-1})} \Sigma bu \xrightarrow{\Sigma\partial_3} \Sigma^2 K(\mathbb{F}_3)\,.$$

Both connecting maps induce the zero map on homotopy, and the extensions

$$K_*^{\mathrm{red}}(\mathbb{Q}_2) \to K_*(\mathbb{Q}_2) \xrightarrow{\mathrm{red}_*} K_*(\mathbb{F}_3)$$

and

$$\pi_*(\Sigma K(\mathbb{F}_3)) \xrightarrow{f_{\mathbb{C}*}} K_*^{\mathrm{red}}(\mathbb{Q}_2) \to \pi_*(\Sigma ku)$$

are both split. □

This is the conclusion of [R5, 8.1]. To go on, we consider the ring homomorphisms $j\colon \mathbb{Z} \to \mathbb{Z}_2$ and $j'\colon \mathbb{Z}[\frac{1}{2}] \to \mathbb{Q}_2$.

Theorem 3.7 (Quillen). *There is a map of horizontal cofiber sequences of spectra*

$$\begin{tikzcd}
K(\mathbb{F}_2) \arrow[r] \arrow[d, equal] & K(\mathbb{Z}) \arrow[r] \arrow[d, "j"] & K(\mathbb{Z}[\frac{1}{2}]) \arrow[d, "j'"] \\
K(\mathbb{F}_2) \arrow[r] & K(\mathbb{Z}_2) \arrow[r] & K(\mathbb{Q}_2)
\end{tikzcd}$$

inducing a homotopy equivalence $\mathrm{hofib}(j) \xrightarrow{\simeq} \mathrm{hofib}(j')$. □

This is the spectrum level statement of the localization sequences in K-theory from [Q2].

Theorem 3.8 (Hesselholt–Madsen). *In the commutative square of spectra*

$$\begin{tikzcd} K(\mathbb{Z}) \arrow[r, "j"] \arrow[d, "\mathrm{trc}_{\mathbb{Z}}"] & K(\mathbb{Z}_2) \arrow[d, "\mathrm{trc}_{\mathbb{Z}_2}"] \\ TC(\mathbb{Z}) \arrow[r, "j", "\simeq"'] & TC(\mathbb{Z}_2) \end{tikzcd}$$

the right hand map induces a homotopy equivalence on connective covers. The lower map is a homotopy equivalence, and there is a cofiber sequence of spectra

$$\mathrm{hofib}(j) \to \mathrm{hofib}(\mathrm{trc}_{\mathbb{Z}}) \to \Sigma^{-2}H\mathbb{Z}\,. \quad \square$$

This is Theorem D of [HM], which uses McCarthy's theorem [Mc].

Theorem 3.9 (Rognes). *The natural map $j' \colon K(\mathbb{Z}[\frac{1}{2}]) \to K(\mathbb{Q}_2)$ induces an isomorphism of 2-adic homotopy groups modulo torsion, in each positive dimension $* \equiv 1 \bmod 4$.* $\square$

This is the content of [R5, 7.7]. By a homotopy group modulo torsion we here mean the quotient of the homotopy group by its torsion subgroup. Hence the assertion is stronger than just saying that j' induces a homomorphism whose kernel and cokernel are torsion groups.

Proposition 3.10. *There is a map of horizontal cofiber sequences of spectra*

$$\begin{tikzcd} \Sigma ko \arrow[r] \arrow[d, "j^{\mathrm{red}}"] & K(\mathbb{Z}[\frac{1}{2}]) \arrow[r, "\pi_3"] \arrow[d, "j'"] & K(\mathbb{F}_3) \arrow[d, "\bar{\jmath}", "\simeq"'] \\ K^{\mathrm{red}}(\mathbb{Q}_2) \arrow[r] & K(\mathbb{Q}_2) \arrow[r, "\mathrm{red}"] & K(\mathbb{F}_3) \end{tikzcd}$$

such that the right hand map $\bar{\jmath}$ is a homotopy equivalence. Hence there is a homotopy equivalence $\mathrm{hofib}(j^{\mathrm{red}}) \xrightarrow{\simeq} \mathrm{hofib}(j')$.

Proof. Suppose we have shown that the composite

$$\Sigma ko \to K(\mathbb{Z}[\frac{1}{2}]) \xrightarrow{j'} K(\mathbb{Q}_2) \xrightarrow{\mathrm{red}} K(\mathbb{F}_3)$$

is null homotopic. Then a choice of null homotopy defines an extension $\bar{\jmath} \colon K(\mathbb{F}_3) \to K(\mathbb{F}_3)$ of $\mathrm{red} \circ j'$ over π_3, as well as a lifting $j^{\mathrm{red}} \colon \Sigma ko \to K^{\mathrm{red}}(\mathbb{Q}_2)$. By the calculations of [R5, 4.2 and 4.4] the composite $\mathrm{red} \circ j'$ is surjective on homotopy in dimensions $0 \le * \le 7$, hence in all dimensions by v_1^4-periodicity. Thus $\bar{\jmath}$ induces surjections on homotopy in all dimensions, and must be a homotopy equivalence. This then completes the proof of the proposition.

To show that the composite map $\Sigma ko \to K(\mathbb{F}_3)$ is null homotopic, it suffices to show that precomposition with the connecting map $\partial \colon K(\mathbb{F}_3) \to \Sigma^2 ko$ in 3.4 induces an injection between the groups of homotopy classes of maps to $\Sigma K(\mathbb{F}_3)$:

$$[\Sigma^2 ko, \Sigma K(\mathbb{F}_3)] \xrightarrow{\partial^\#} [K(\mathbb{F}_3), \Sigma K(\mathbb{F}_3)]\,.$$

Here ∂ is the composite of $i_3\colon K(\mathbb{F}_3) \to ku$ with the map denoted $\Sigma^2 r \circ \beta^{-1}\colon ku \to \Sigma^2 ko$. Thus it suffices to show that both homomorphisms $i_3^\#$ and $(\Sigma^2 r \circ \beta^{-1})^\#$ are injective.

There is an exact sequence

$$[bu, \Sigma K(\mathbb{F}_3)] \xrightarrow{(\psi^3-1)^\#} [ku, \Sigma K(\mathbb{F}_3)] \xrightarrow{i_3^\#} [K(\mathbb{F}_3), \Sigma K(\mathbb{F}_3)] .$$

Any map $bu \to \Sigma K(\mathbb{F}_3)$ has the form $\partial_3 \circ \phi$, for some operation $\phi\colon bu \to bu$. Thus its precomposition with $(\psi^3 - 1)$ is null homotopic, because ϕ and (ψ^3-1) commute and $\partial_3 \circ (\psi^3 - 1) \simeq *$. Hence the left hand map is null and $i_3^\#$ is injective.

There is also an exact sequence

$$[\Sigma ko, \Sigma K(\mathbb{F}_3)] \xrightarrow{\eta^\#} [\Sigma^2 ko, \Sigma K(\mathbb{F}_3)] \xrightarrow{(\Sigma^2 r\circ\beta^{-1})^\#} [ku, \Sigma K(\mathbb{F}_3)] .$$

From 3.3 we see that $[\Sigma ko, \Sigma K(\mathbb{F}_3)]$ is zero, because postcomposition with (ψ^3-1) acts injectively on the homotopy classes of maps $ko \to ku$, see [MST]. Thus also $(\Sigma^2 r \circ \beta^{-1})^\#$ is injective, which completes the proof. $\square$

Fix choices of maps j^{red} and $\bar{j}$, as above.

Proposition 3.11. *There is a cofiber sequence of spectra*

$$K(\mathbb{F}_3) \to \mathrm{hofib}(j^{\mathrm{red}}) \to \Sigma^2 ko \xrightarrow{\partial} \Sigma K(\mathbb{F}_3) .$$

The connecting map ∂ is homotopic to the composite

$$\Sigma^2 ko \xrightarrow{\Sigma^2 c} \Sigma^2 ku \underset{\simeq}{\xrightarrow{\beta}} bu \xrightarrow{\partial_3} \Sigma K(\mathbb{F}_3) .$$

Proof. The map $\Sigma ko \to K(\mathbb{Z}[\frac{1}{2}])$ induces an isomorphism on 2-adic homotopy modulo torsion in dimensions $* \equiv 1 \bmod 8$, and multiplication by 2 times a 2-adic unit in dimensions $* \equiv 5 \bmod 8$. By 3.9 the same holds for the composite map from Σko to $K(\mathbb{Q}_2)$, and by 3.6 the same holds for the lift $j^{\mathrm{red}}\colon \Sigma ko \to K^{\mathrm{red}}(\mathbb{Q}_2)$, as well as the composite map $\Sigma ko \to K^{\mathrm{red}}(\mathbb{Q}_2) \to \Sigma ku$. Any such map factors as a self-map ϕ of Σko followed by the suspended complexification map $\Sigma c\colon \Sigma ko \to \Sigma ku$. Since the suspended complexification map induces the identity in dimensions $* \equiv 1 \bmod 8$, and multiplication by 2 in dimensions $* \equiv 5 \bmod 8$, it follows that ϕ is a 2-adic homotopy equivalence. We obtain the following diagram of horizontal and vertical cofiber sequences:

$$\begin{tikzcd}
K(\mathbb{F}_3) \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \Sigma K(\mathbb{F}_3) \arrow[d, "f_{\mathbb{C}}"] \\
\mathrm{hofib}(j^{\mathrm{red}}) \arrow[r] \arrow[d] & \Sigma ko \arrow[r, "j^{\mathrm{red}}"] \arrow[d, "\phi", "\simeq"'] & K^{\mathrm{red}}(\mathbb{Q}_2) \arrow[d] \\
\Sigma^2 ko \arrow[r, "\Sigma\eta"] & \Sigma ko \arrow[r, "\Sigma c"] & \Sigma ku .
\end{tikzcd}$$

The connecting map $\partial\colon \Sigma^2 ko \to \Sigma K(\mathbb{F}_3)$ is detected by its precomposition with $\Sigma^2 r \circ \beta^{-1}\colon ku \to \Sigma^2 ko$, because $[\Sigma ko, \Sigma K(\mathbb{F}_3)] = 0$. By the diagram above, the composite $\partial \circ \Sigma^2 r \circ \beta^{-1}$ is the desuspended connecting map $\Sigma^{-1}\partial_1 = \partial_3 \circ (1 - \psi^{-1})$ from 3.6. Thus $\partial = \partial_3 \circ \beta \circ \Sigma^2 c$ in the stable category, by the calculation

$$\partial_3 \circ \beta \circ \Sigma^2 c \circ \Sigma^2 r \circ \beta^{-1} = \partial_3 \circ (1 - \psi^{-1})$$

which uses $c \circ r = 1 + \psi^{-1}$, and $\psi^k \circ \beta = \beta \circ \Sigma^2(k\psi^k)$. $\square$

Proposition 3.12. *There is a cofiber sequence of spectra*

$$\Sigma^3 ko \to \mathrm{hofib}(j^{\mathrm{red}}) \to ku \xrightarrow{\partial} \Sigma^4 ko\,.$$

The connecting map ∂ is homotopic to the composite

$$ku \xrightarrow{\psi^3-1} bu \underset{\simeq}{\xrightarrow{\beta^{-1}}} \Sigma^2 ku \xrightarrow{\Sigma^2(\Sigma^2 r\circ\beta^{-1})} \Sigma^4 ko$$

with the same notation as in (3.2). It is characterized by the following homotopy commutative diagram

$$\begin{tikzcd}
ku \arrow[rrr, "\partial"] \arrow[d, "cov"] & & & \Sigma^4 ko \arrow[d, "cov"] \\
KU \arrow[r, "\psi^3-1"] & KU & \Sigma^4 KU \arrow[l, "\beta^2", "\simeq"'] \arrow[r, "\Sigma^4 r"] & \Sigma^4 KO\,.
\end{tikzcd}$$

The maps labeled cov are k-connective covering maps, for suitable k.

Proof. We use the factorization of the connecting map in 3.11 to form the following diagram of horizontal and vertical cofiber sequences:

$$\begin{tikzcd}
\Sigma^3 ko \arrow[r] \arrow[d, equal] & \mathrm{hofib}(j^{\mathrm{red}}) \arrow[r] \arrow[d] & ku \arrow[d, "\beta^{-1}\circ(\psi^3-1)"] \\
\Sigma^3 ko \arrow[r, "\Sigma^2\eta"] \arrow[d] & \Sigma^2 ko \arrow[r, "\Sigma^2 c"] \arrow[d, "\partial"] & \Sigma^2 ku \arrow[d, "\partial_3\circ\beta"] \\
* \arrow[r] & \Sigma K(\mathbb{F}_3) \arrow[r, equal] & \Sigma K(\mathbb{F}_3)\,.
\end{tikzcd}$$

The right hand column is a variant of the sequence in 3.3. It follows that the connecting map $ku \to \Sigma^4 ko$ for the top row is the composite of

$$ku \xrightarrow{\psi^3-1} bu \underset{\simeq}{\xrightarrow{\beta^{-1}}} \Sigma^2 ku$$

and the connecting map for the middle row, i.e., the double suspension of the connecting map $\Sigma^2 r \circ \beta^{-1} = \partial\colon ku \to \Sigma^2 ko$ of (3.2).

The covering maps induce an injection $cov_\#\colon [ku, \Sigma^4 ko] \to [ku, \Sigma^4 KO]$ and a bijection $cov^\#\colon [KU, \Sigma^4 KO] \cong [ku, \Sigma^4 KO]$, and the connecting map ∂ corresponds to $\Sigma^4 r \circ \beta^{-2} \circ (\psi^3 - 1)$ in $[KU, \Sigma^4 KO]$. Hence ∂ is characterized by the given diagram. $\square$

The following theorem is the main result of this chapter.

Theorem 3.13. *There is a cofiber sequence of spectra*

$$\Sigma^3 ko \to \mathrm{hofib}(\mathrm{trc}) \to \Sigma^{-2} ku \xrightarrow{\delta} \Sigma^4 ko\,.$$

The connecting map δ is characterized by the following homotopy commutative diagram

$$\begin{tikzcd}
\Sigma^{-2}ku \arrow[rrrr, "\delta"] \arrow[d, "cov"] & & & & \Sigma^4 ko \arrow[d, "cov"] \\
\Sigma^{-2}KU & KU \arrow[l, "\beta", "\simeq"'] \arrow[r, "\psi^3-1"] & KU & \Sigma^4 KU \arrow[l, "\beta^2", "\simeq"'] \arrow[r, "\Sigma^4 r"] & \Sigma^4 KO\,.
\end{tikzcd}$$

The maps labeled cov are suitable covering maps.

Proof. Consider the following diagram of horizontal and vertical cofiber sequences of spectra, obtained by combining 3.7, 3.8, 3.10 and 3.12:

$$\begin{tikzcd}
\Sigma^3 ko \arrow[r] \arrow[d, equal] & \mathrm{hofib}(j) \arrow[r] \arrow[d] & ku \arrow[d] \\
\Sigma^3 ko \arrow[r] \arrow[d] & \mathrm{hofib}(\mathrm{trc}) \arrow[r] \arrow[d] & X \arrow[d] \\
* \arrow[r] & \Sigma^{-2}H\mathbb{Z} \arrow[r, equal] & \Sigma^{-2}H\mathbb{Z}\,.
\end{tikzcd}$$

Here X is the cofiber of the composite map $\Sigma^3 ko \to \mathrm{hofib}(j) \to \mathrm{hofib}(\mathrm{trc})$. It is classified as an extension of $\Sigma^{-2}H\mathbb{Z}$ by ku by an element in $[\Sigma^{-2}H\mathbb{Z}, \Sigma ku] \cong \mathbb{Z}_2$, whose mod 2 reduction is detected by the composite $k\colon \Sigma^{-2}H\mathbb{Z} \to \Sigma ku \to \Sigma H\mathbb{Z}$ in $[\Sigma^{-2}H\mathbb{Z}, \Sigma H\mathbb{Z}] \cong \mathbb{Z}/2$. (Here $\Sigma ku \to \Sigma H\mathbb{Z}$ is the map inducing an isomorphism on π_1.) This composite k is the k-invariant of X relating the homotopy groups in dimensions -2 and 0.

Since $\Sigma^3 ko$ is 2-connected, this lowest k-invariant is the same for X as for $\mathrm{hofib}(\mathrm{trc})$. By combining 1.11 with 1.21 we obtain a cofiber sequence

$$\mathbb{C}P^\infty_{-1} \xrightarrow{i} \mathrm{hofib}(\mathrm{trc}) \xrightarrow{j} \mathrm{Wh}^{\mathrm{Diff}}(*) \tag{3.14}$$

whose connecting map is identified with $\widetilde{\mathrm{trc}}$. Since $\mathrm{Wh}^{\mathrm{Diff}}(*)$ is connected, it follows that the lowest k-invariants for $\mathrm{hofib}(\mathrm{trc})$ and $\mathbb{C}P^\infty_{-1}$ are equal. By 2.16 the latter is nonzero. Hence k is the essential map.

It follows that X is classified by a map $u \cdot \partial$ where $\partial\colon \Sigma^{-2}H\mathbb{Z} \to \Sigma ku$ classifies $\Sigma^{-2}ku$ and u is a 2-adic unit. We get a homotopy equivalence of cofiber sequences

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\simeq"] & \Sigma^{-2}H\mathbb{Z} \arrow[r, "u\cdot\partial"] \arrow[d, "u", "\simeq"'] & \Sigma ku \arrow[d, equal] \\
\Sigma^{-2}ku \arrow[r] & \Sigma^{-2}H\mathbb{Z} \arrow[r, "\partial"] & \Sigma ku\,.
\end{tikzcd}$$

Hence $X \simeq \Sigma^{-2}ku$, as claimed.

To characterize δ, we compare with the connecting map ∂ of 3.12. Precomposition with $\beta\colon ku \to \Sigma^{-2}ku$, or its K-localization, induces the vertical maps in the following commutative diagram:

$$\begin{tikzcd}
{[\Sigma^{-2}ku, \Sigma^4 ko]} \arrow[r, "cov_\#"] \arrow[d, "\beta^\#"] & {[\Sigma^{-2}ku, \Sigma^4 KO]} \arrow[d, "\beta^\#", "\cong"'] & {[\Sigma^{-2}KU, \Sigma^4 KO]} \arrow[l, "cov^\#", "\cong"'] \arrow[d, "\beta^\#", "\cong"'] \\
{[ku, \Sigma^4 ko]} \arrow[r, "cov_\#"] & {[ku, \Sigma^4 KO]} & {[KU, \Sigma^4 KO]}\,. \arrow[l, "cov^\#", "\cong"']
\end{tikzcd}$$

Here the maps labeled $cov_\#$ are injective, and the maps labeled $cov^\#$ are bijective. The class δ in $[\Sigma^{-2}ku, \Sigma^4 ko]$ maps to ∂ under $\beta^\#$, which in turn maps to $\Sigma^4 r \circ \beta^{-2} \circ (\psi^3 - 1)$ in $[KU, \Sigma^4 KO]$ by 3.12. The right hand $\beta^\#$ is bijective, so this characterizes the image of δ in $[\Sigma^{-2}KU, \Sigma^4 KO]$ as $\Sigma^4 r \circ \beta^{-2} \circ (\psi^3 - 1) \circ \beta^{-1}$. This characterizes δ up to homotopy, by the injectivity claims above. □

Remark 3.15. By [6A] or 1.8 this theorem also determines the homotopy fiber of the cyclotomic trace map $\mathrm{trc}_X \colon A(X) \to TC(X)$ completed at 2 for any 1-connected space X, since the natural map

$$\mathrm{hofib}(\mathrm{trc}_X) \xrightarrow{\simeq} \mathrm{hofib}(\mathrm{trc})$$

is a homotopy equivalence.

Let $v_2(k)$ be the 2-adic valuation of k.

Corollary 3.16. *In positive dimensions ($n > 0$) the homotopy groups of* hofib(trc) *are*

$$\pi_n(\mathrm{hofib}(\mathrm{trc})) \cong \begin{cases} 0 & \text{for } n \equiv 0, 1 \bmod 8, \\ \mathbb{Z} & \text{for } n \equiv 2, 6 \bmod 8, \\ \mathbb{Z}/16 & \text{for } n \equiv 3 \bmod 8, \\ \mathbb{Z}/2 & \text{for } n \equiv 4, 5 \bmod 8, \text{ and} \\ \mathbb{Z}/2^{v_2(k)+4} & \text{for } n = 8k - 1. \end{cases}$$

Also $\pi_n(\mathrm{hofib}(\mathrm{trc})) \cong \mathbb{Z}$ for $n = -2$ and $n = 0$. The remaining homotopy groups are zero.

Proof. This is a routine calculation, given the action of $\psi^3 - 1$ and $\Sigma^4 r$ on homotopy. □

The spectrum cohomology of hofib(trc) is given in 4.4 below. The Adams E_2-term

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(\mathrm{hofib}(\mathrm{trc})), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathrm{hofib}(\mathrm{trc}))^\wedge_2 \,. \tag{3.17}$$

is then easily deduced from the E_2-terms in the Adams spectral sequences for $\pi_*(ko)^\wedge_2$ and $\pi_*(ku)^\wedge_2$. Furthermore only one pattern of differentials is compatible with 3.16: There is an infinite h_0-tower of nonzero d^r-differentials from column $t - s = 8k$ for all $k \geq 1$, with $r = v_2(k) + 2$, and no other differentials. The spectral sequence is displayed in Tables 3.18(a) and (b) below.

4. Cohomology of the smooth Whitehead spectrum

We now determine the mod two spectrum cohomology of the smooth Whitehead spectrum of a point, as a module over the Steenrod algebra.

Consider the following diagram:

$$\begin{tikzcd} & \mathbb{C}P^\infty_{-1} \arrow[r, equal] \arrow[d, "i"] & \mathbb{C}P^\infty_{-1} \arrow[d, "\epsilon"] & \\ \Sigma^3 ko \arrow[r] \arrow[d, equal] & \mathrm{hofib}(\mathrm{trc}) \arrow[r] \arrow[d, "j"] & \Sigma^{-2} ku \arrow[r, "\delta"] \arrow[d] & \Sigma^4 ko \arrow[d, equal] \\ \Sigma^3 ko \arrow[r] & \mathrm{Wh}^{\mathrm{Diff}}(*) \arrow[r] & \Sigma\, \mathrm{hofib}(\epsilon) \arrow[r] & \Sigma^4 ko . \end{tikzcd} \tag{4.1}$$

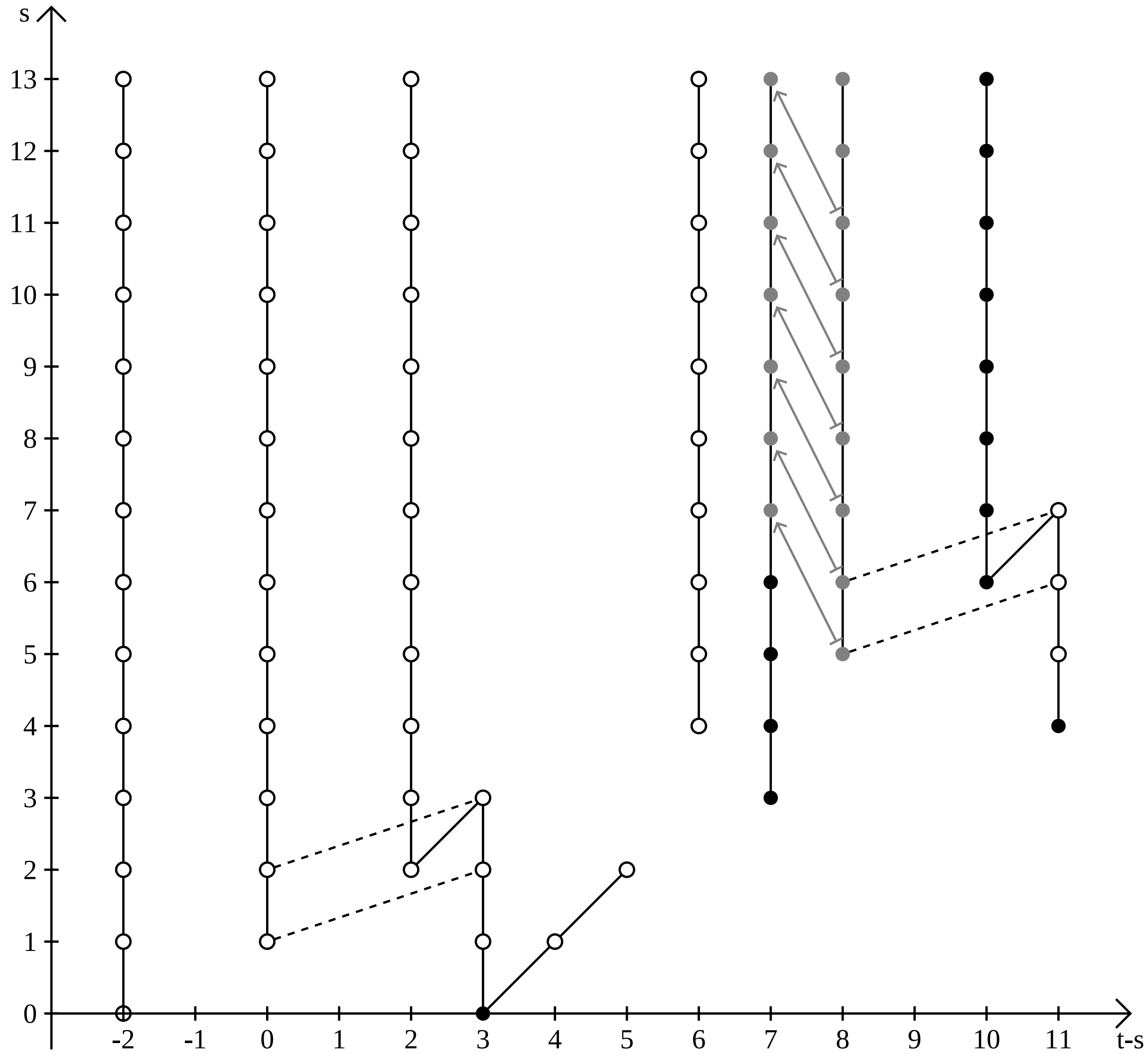

TABLE 3.18(A). The Adams E_2-term for hofib(trc)

The middle row is the cofiber sequence from Theorem 3.13, and the left column is (3.14). We let ϵ be the composite map $\mathbb{C}P^\infty_{-1} \to \mathrm{hofib}(\mathrm{trf}) \to \Sigma^{-2}ku$. Then the right column and bottom row are also cofiber sequences.

Proposition 4.2. *The map ϵ induces the unique surjection of A-modules*

$$\Sigma^{-2}A/A(Sq^1, Sq^3) \cong H^*(\Sigma^{-2}ku) \xrightarrow{\epsilon^*} H^*(\mathbb{C}P^\infty_{-1}) \cong \Sigma^{-2}A/C\,.$$

Hence

$$H^*(\Sigma\,\mathrm{hofib}(\epsilon)) \cong \Sigma^{-2}C/A(Sq^1, Sq^3)$$

as an A-module.

Proof. We use that $\Sigma^4 ko$ and $\mathrm{Wh}^{\mathrm{Diff}}(*)$ are connective spectra. Hence $\Sigma\,\mathrm{hofib}(\epsilon)$ is connective, and so ϵ induces an isomorphism in dimension -2. This determines ϵ^* since $H^*(\Sigma^{-2}ku)$ is a cyclic A-module, and ϵ^* is surjective because $H^*(\mathbb{C}P^\infty_{-1})$ is a cyclic A-module. We identify $H^*(\Sigma\,\mathrm{hofib}(\epsilon))$ with $\ker(\epsilon^*)$. $\square$

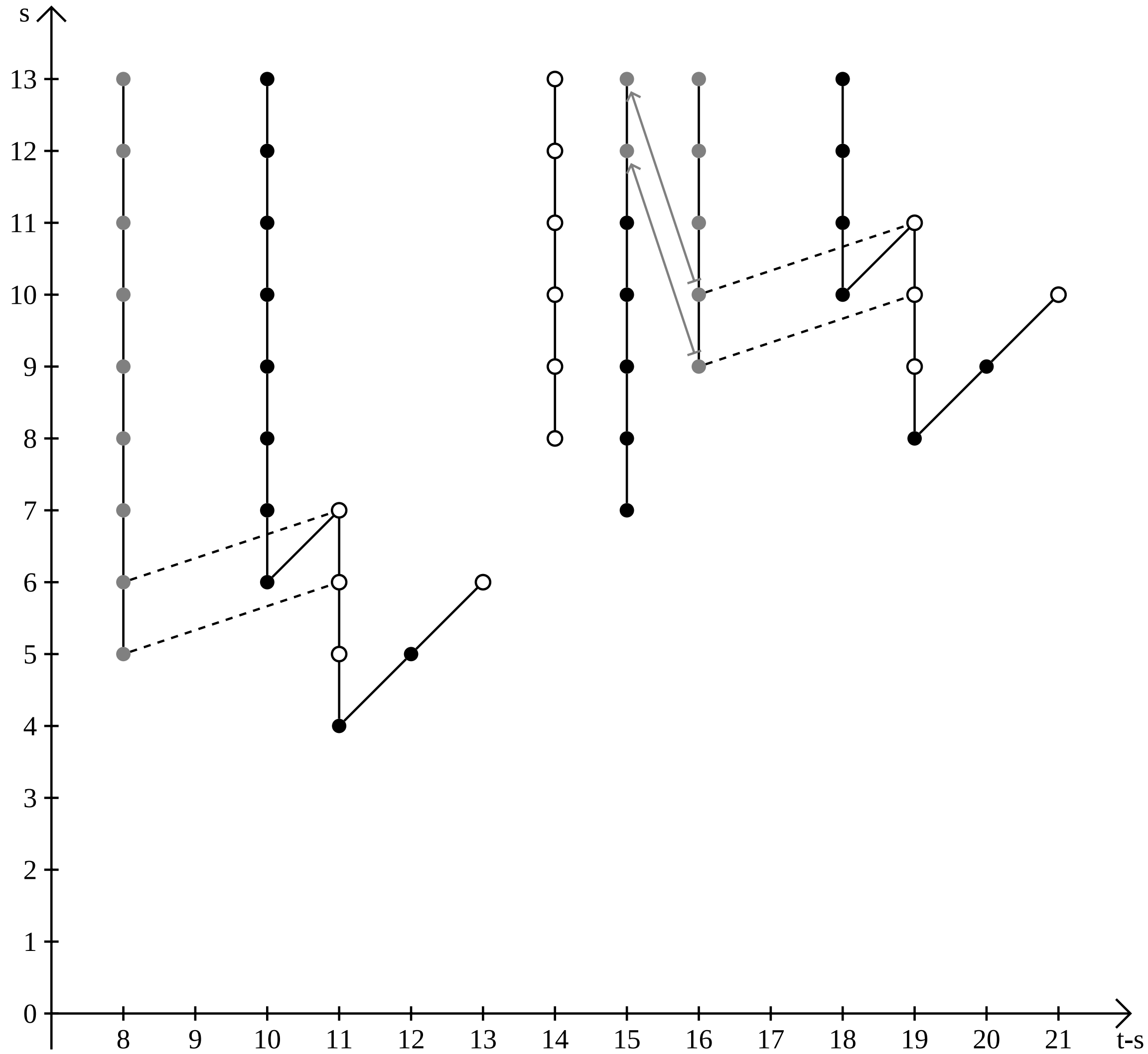

TABLE 3.18(B). The Adams E_2-term for hofib(trc)

Proposition 4.3. *The connecting map δ induces the zero homomorphism on cohomology.*

Proof. In fact, the group of A-module homomorphisms

$$H^*(\Sigma^4 ko) \cong \Sigma^4 A/A(Sq^1, Sq^2) \to \Sigma^{-2} A/A(Sq^1, Sq^3) \cong H^*(\Sigma^{-2} ku)$$

is zero. For $A/A(Sq^1, Sq^3)$ is $\mathbb{F}_2\{Sq^6, Sq^4 Sq^2\}$ in dimension 6, while $Sq^1 \circ Sq^6 \neq 0$ and $Sq^2 \circ Sq^4 Sq^2 \neq 0$ in this A-module. $\square$

Theorem 4.4. *The mod two spectrum cohomology of* hofib(trc) *is the unique non-trivial extension of A-modules*

$$\Sigma^{-2} A/A(Sq^1, Sq^3) \to H^*(\mathrm{hofib}(\mathrm{trc})) \to \Sigma^3 A/A(Sq^1, Sq^2)\,.$$

Theorem 4.5. *The mod two spectrum cohomology of* $\mathrm{Wh}^{\mathrm{Diff}}(*)$ *is the unique non-trivial extension of A-modules*

$$\Sigma^{-2} C/A(Sq^1, Sq^3) \to H^*(\mathrm{Wh}^{\mathrm{Diff}}(*)) \to \Sigma^3 A/A(Sq^1, Sq^2)\,.$$

The mod two spectrum cohomology of $A(*)$ is given by the splitting of A-modules

$$H^*(A(*)) \cong H^*(\mathrm{Wh}^{\mathrm{Diff}}(*)) \oplus \mathbb{F}_2 \,.$$

Here $\mathbb{F}_2 = H^*(S^0)$ denotes the trivial A-module concentrated in degree zero. We prove these two theorems together.

Proof of 4.4 and 4.5. We apply mod 2 spectrum cohomology to (4.1). By 4.2 the map ϵ induces a surjection in each dimension, so $\Sigma^{-2}ku \to \Sigma\,\mathrm{hofib}(\epsilon)$ induces an injection in each dimension. By 4.3 the map δ induces the zero homomorphism in each dimension, and combining these facts we see that $\Sigma\,\mathrm{hofib}(\epsilon) \to \Sigma^4 ko$ also induces the zero homomorphism in cohomology. Thus the long exact sequences in cohomology associated to the middle and lower horizontal cofiber sequences in (4.1) break up into short exact sequences. These express $H^*(\mathrm{hofib}(\mathrm{trc}))$ and $H^*(\mathrm{Wh}^{\mathrm{Diff}}(*))$ as extensions of A-modules, as claimed.

$$\begin{tikzcd}
 & \Sigma^{-2}A/C \arrow[r, equal] & \Sigma^{-2}A/C \\
\Sigma^3 A/A(Sq^1, Sq^2) \arrow[d, equal] & H^*(\mathrm{hofib}(\mathrm{trc})) \arrow[l] \arrow[u, "i^*"] & \Sigma^{-2}A/A(Sq^1, Sq^3) \arrow[l] \arrow[u, "\epsilon^*"'] \\
\Sigma^3 A/A(Sq^1, Sq^2) & H^*(\mathrm{Wh}^{\mathrm{Diff}}(*)) \arrow[l] \arrow[u, "j^*"] & \Sigma^{-2}C/A(Sq^1, Sq^3) \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.6}$$

It remains to characterize the extensions, which are represented by elements of Ext^1_A. Recall that $H^*(ko) = A/A(Sq^1, Sq^2) = A//A_1$ where $A_1 \subset A$ is the sub-Hopf algebra generated by Sq^1 and Sq^2. Hence there are change-of-rings isomorphisms

$$\mathrm{Ext}^1_A(\Sigma^3 A//A_1, \Sigma^{-2}A/A(Sq^1, Sq^3)) \cong \mathrm{Ext}^1_{A_1}(\Sigma^3\mathbb{F}_2, \Sigma^{-2}A/A(Sq^1, Sq^3))$$

and

$$\mathrm{Ext}^1_A(\Sigma^3 A//A_1, \Sigma^{-2}C/A(Sq^1, Sq^3)) \cong \mathrm{Ext}^1_{A_1}(\Sigma^3\mathbb{F}_2, \Sigma^{-2}C/A(Sq^1, Sq^3))\,.$$

An A_1-module extension of $\Sigma^{-2}A/A(Sq^1, Sq^3)$ by $\Sigma^3\mathbb{F}_2$ is determined by the values of Sq^1 and Sq^2 on the nonzero element of $\Sigma^3\mathbb{F}_2$, and these are connected by the Adem relation $Sq^1Sq^2Sq^1 = Sq^2Sq^2$.

By inspection of $\Sigma^{-2}A/A(Sq^1, Sq^3)$ and $\Sigma^{-2}C/A(Sq^1, Sq^3)$ as A_1-modules, there are precisely two such A_1-module extensions in both cases; one trivial (split) and one nontrivial (not split). Furthermore the map of extensions induced by (4.1) induces an isomorphism

$$\mathbb{Z}/2 \cong \mathrm{Ext}^1_{A_1}(\Sigma^3\mathbb{F}_2, \Sigma^{-2}C/A(Sq^1, Sq^3)) \xrightarrow{\cong} \\ \mathrm{Ext}^1_{A_1}(\Sigma^3\mathbb{F}_2, \Sigma^{-2}A/A(Sq^1, Sq^3)) \cong \mathbb{Z}/2\,.$$

Thus to prove that each extension is the unique nontrivial extension of its kind, it suffices to show that $H^*(\mathrm{Wh}^{\mathrm{Diff}}(*))$ does not split as the sum of $\Sigma^3 A/A(Sq^1, Sq^2)$ and $\Sigma^{-2}C/A(Sq^1, Sq^3)$.

Now $\Sigma^{-2}C/A(Sq^1, Sq^3)$ is 3-connected, and by [BW, 1.3] the bottom homotopy group of $\mathrm{Wh}^{\mathrm{Diff}}(*)$ is $\pi_3(\mathrm{Wh}^{\mathrm{Diff}}(*)) \cong \mathbb{Z}/2$. Hence there is a nontrivial Sq^1 acting on the nonzero class in $H^3(\mathrm{Wh}^{\mathrm{Diff}}(*))$, which tells us that $\Sigma^3 A/A(Sq^1, Sq^2)$ does not split off from $H^*(\mathrm{Wh}^{\mathrm{Diff}}(*))$.

This proves that both extensions are nontrivial, and completes the proofs. □

Remark 4.7. By (4.6), we see that the lifted cyclotomic trace map

$$\widetilde{\mathrm{trc}} \colon \mathrm{Wh}^{\mathrm{Diff}}(*) \to \widetilde{TC}(*) \simeq \Sigma\mathbb{C}P^\infty_{-1}$$

induces the zero homomorphism on mod 2 spectrum cohomology. The map is nevertheless very useful.

Question 4.8. The map ϵ lives in the group

$$[\mathbb{C}P^\infty_{-1}, \Sigma^{-2}ku] \cong [\mathbb{C}P^\infty_+, ku] = KU^0(\mathbb{C}P^\infty) \cong \mathbb{Z}[[\gamma^1]]$$

where the first isomorphism is the Thom isomorphism in complex topological K-theory for the virtual complex bundle $-\gamma^1$ over $\mathbb{C}P^\infty_+$. To which power series in γ^1 does ϵ correspond ?

Proposition 4.9. *The linearization map $L\colon TC(*) \to TC(\mathbb{Z})$ and the suspended map $\Sigma\epsilon\colon \Sigma\mathbb{C}P^\infty_{-1} \to \Sigma^{-1}ku$ induce the same homomorphisms up to 2-adic units, on homotopy groups modulo torsion in dimensions $* \equiv 3 \bmod 4$.*

Proof. The suspended map $\Sigma\epsilon$ is the composite

$$\Sigma\mathbb{C}P^\infty_{-1} \to TC(*) \xrightarrow{L} TC(\mathbb{Z}) \to \Sigma\,\mathrm{hofib}(\mathrm{trc}) \to \Sigma^{-1}ku\,.$$

The first map splits $TC(*)$, the second is the linearization map, the third is the connecting map in the cofiber sequence generated by $\mathrm{trc}_{\mathbb{Z}}$, and the fourth suspends a map that appears in 3.13. The first map induces an isomorphism on homotopy groups modulo torsion in all positive dimensions, since the other summand $\Sigma^\infty S^0$ has finite homotopy groups in positive dimensions. The third and fourth maps also induce an isomorphism on homotopy groups modulo torsion in dimensions $* \equiv 3 \bmod 4$, by the calculation of $\mathrm{trc}_{\mathbb{Z}}$ in [R5, 9.1], and the description of δ in 3.13 □

5. Two-primary homotopy of $\mathrm{Wh}^{\mathrm{Diff}}(*)$

Let ι_{-2} be the generator in dimension -2 of $H^*(\Sigma^{-2}ku) \cong \Sigma^{-2}A/A(Sq^1, Sq^3)$, and let ι_3 be the generator in dimension 3 of $H^*(\Sigma^3 ko) \cong \Sigma^3 A/A(Sq^1, Sq^2)$. By 4.2 the map $\epsilon\colon \mathbb{C}P^\infty_{-1} \to \Sigma^{-2}ku$ of (4.1) induces a surjection on cohomology, and we regard

$$\ker(\epsilon^*) = \Sigma^{-2}C/A(Sq^1, Sq^3) \subset \Sigma^{-2}A/A(Sq^1, Sq^3) = H^*(\Sigma^{-2}ku)$$

as a submodule of $H^*(\Sigma^{-2}ku)$. It is thus spanned by suitable monomials $Sq^I\iota_{-2}$ taken modulo $A(Sq^1, Sq^3)\iota_{-2}$. By inspection $\ker(\epsilon^*)$ is 3-connected. The bottom cofiber sequence in (4.1) induces the nontrivial extension

$$0 \to \ker(\epsilon^*) \to H^*(\mathrm{Wh}^{\mathrm{Diff}}(*)) \to H^*(\Sigma^3 ko) \to 0\,.$$

dim	x	$Sq^1(x)$	$Sq^2(x)$	$Sq^4(x)$	$Sq^8(x)$
3	ι_3	$Sq^4Sq^2\iota_{-2}$	$Sq^7\iota_{-2}$	$Sq^4\iota_3$	$Sq^8\iota_3$
4	$Sq^4Sq^2\iota_{-2}$	0	$Sq^6Sq^2\iota_{-2}$	0	$Sq^8Sq^4Sq^2\iota_{-2}$
5	$Sq^7\iota_{-2}$	0	$Sq^9\iota_{-2}$ $\equiv Sq^7Sq^2\iota_{-2}$	$Sq^{11}\iota_{-2}$	$Sq^{13}Sq^2\iota_{-2}$ $+Sq^{11}Sq^4\iota_{-2}$
6	$Sq^6Sq^2\iota_{-2}$	$Sq^9\iota_{-2}$ $\equiv Sq^7Sq^2\iota_{-2}$	0	$Sq^{10}Sq^2\iota_{-2}$	$Sq^{10}Sq^4Sq^2\iota_{-2}$
7	$Sq^9\iota_{-2}$ $\equiv Sq^7Sq^2\iota_{-2}$ $Sq^4\iota_3$	0 0	0 $Sq^6\iota_3$	$Sq^{11}Sq^2\iota_{-2}$ $Sq^{13}\iota_{-2}$	
8	$Sq^8Sq^2\iota_{-2}$	0	$Sq^{10}Sq^2\iota_{-2}$	$Sq^{12}Sq^2\iota_{-2}$	
9	$Sq^{11}\iota_{-2}$ $Sq^6\iota_3$	0 $Sq^7\iota_3$	$Sq^{13}\iota_{-2}$ 0	$Sq^{13}Sq^2\iota_{-2}$ $Sq^{13}Sq^2\iota_{-2}$ $+Sq^{11}Sq^4\iota_{-2}$ $+Sq^{10}\iota_3$	
10	$Sq^{10}Sq^2\iota_{-2}$ $Sq^8Sq^4\iota_{-2}$ $Sq^7\iota_3$	$Sq^{11}Sq^2\iota_{-2}$ 0 0	0 $Sq^{10}Sq^4\iota_{-2}$ 0	0 $Sq^{12}Sq^4\iota_{-2}$ $Sq^{11}\iota_3$	
11	$Sq^{13}\iota_{-2}$ $Sq^{11}Sq^2\iota_{-2}$ $Sq^8\iota_3$	0 0 $Sq^8Sq^4Sq^2\iota_{-2}$	0 $Sq^{13}Sq^2\iota_{-2}$ $Sq^{10}\iota_3$		
12	$Sq^{12}Sq^2\iota_{-2}$ $Sq^{10}Sq^4\iota_{-2}$ $Sq^8Sq^4Sq^2\iota_{-2}$	$Sq^{13}Sq^2\iota_{-2}$ $Sq^{11}Sq^4\iota_{-2}$ 0	$Sq^{14}Sq^2\iota_{-2}$ 0 $Sq^{10}Sq^4Sq^2\iota_{-2}$		
13	$Sq^{15}\iota_{-2}$ $Sq^{13}Sq^2\iota_{-2}$ $Sq^{11}Sq^4\iota_{-2}$ $Sq^{10}\iota_3$	0 0 0 $Sq^{11}\iota_3$			
14	$Sq^{14}Sq^2\iota_{-2}$ $Sq^{12}Sq^4\iota_{-2}$ $Sq^{10}Sq^4Sq^2\iota_{-2}$ $Sq^{11}\iota_3$				

TABLE 5.1. $H^*(\mathrm{Wh}^{\mathrm{Diff}}(*))$ in dimensions ≤ 14.

We let $\iota_3 \in H^3(\mathrm{Wh}^{\mathrm{Diff}}(*))$ denote the unique lift of $\iota_3 \in H^3(\Sigma^3 ko)$. With these notations we list a basis for $H^*(\mathrm{Wh}^{\mathrm{Diff}}(*))$ in dimensions $* \leq 14$ in Table 5.1, together with generators for the A-module structure.

We now consider the Adams spectral sequences associated with the spectra in the cofiber sequence of spectra

$$\mathbb{C}P^{\infty}_{-1} \xrightarrow{i} \mathrm{hofib}(\mathrm{trc}) \xrightarrow{j} \mathrm{Wh}^{\mathrm{Diff}}(*) \tag{5.2}$$

appearing vertically in (4.1). They are

$$ {}_cE_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(\mathbb{C}P^\infty_{-1}), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathbb{C}P^\infty_{-1})^\wedge_2 \tag{5.3} $$

$$ {}_fE_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(\mathrm{hofib}(\mathrm{trc})), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathrm{hofib}(\mathrm{trc}))^\wedge_2 \tag{5.4} $$

$$ {}_wE_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(\mathrm{Wh}^{\mathrm{Diff}}(*)), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathrm{Wh}^{\mathrm{Diff}}(*))^\wedge_2 \,. \tag{5.5} $$

The prefix 'c' refers to the truncated complex projective space, 'f' refers to the homotopy fiber of the cyclotomic trace map, and 'w' refers to the Whitehead spectrum. The spectral sequence ${}_cE_*$ was already studied in (2.17), 2.18 and 2.21, while the spectral sequence ${}_fE_*$ appeared in (3.17) and 3.18. The spectral sequence ${}_wE_*$ is displayed below, in Tables 5.7(a) and (b).

The diagram (5.2) induces a short exact sequence of A-modules in cohomology, by (4.6), and thus a long exact sequence of Ext-groups

$$ \dots \to {}_cE_2^{s,t} \xrightarrow{i_*} {}_fE_2^{s,t} \xrightarrow{j_*} {}_wE_2^{s,t} \xrightarrow{\partial} {}_cE_2^{s+1,t} \to \dots\,. \tag{5.6} $$

By the geometric boundary theorem [Ra, 2.3.4], the connecting map ∂ is induced by the spectrum map $\widetilde{\mathrm{trc}}\colon \mathrm{Wh}^{\mathrm{Diff}}(*) \to \Sigma\mathbb{C}P^\infty_{-1}$ extending (5.2), and so each map in the long exact sequence is part of a map of spectral sequences. Furthermore these maps are compatible with the maps in the long exact sequence in 2-completed homotopy induced by (5.2).

The E_2-term of the Adams spectral sequence (5.5) for $\mathrm{Wh}^{\mathrm{Diff}}(*)$ is displayed in dimensions $t-s \le 21$ in Tables 5.7(a) and (b). This was obtained by hand from a minimal resolution of $H^*(\mathrm{Wh}^{\mathrm{Diff}}(*))$ in internal degrees $t \le 14$, using Table 5.1, and using Bruner's Ext-calculator program [Br] in higher dimensions. The notation in these tables is that the maps in (5.6) take a class denoted '$\bullet$' in one spectral sequence to a class denoted '$\circ$' in the following spectral sequence, i.e., $\bullet \mapsto \circ$. As before, the classes hit by or supporting differentials are shaded gray. In bidegree $(2,14)$ a d_2-differential hits the sum of the two given generators, which are therefore both half-shaded.

Proposition 5.8. *The map $i\colon \mathbb{C}P^\infty_{-1} \to \mathrm{hofib}(\mathrm{trc})$ induces a map*

$$ i_*\colon {}_cE_2^{s,t} \to {}_fE_2^{s,t} $$

of Adams E_2-terms, which is surjective in dimensions $t-s \le 2$, $t-s = 4$ and $t-s \equiv 5, 6 \bmod 8$. In positive dimensions $t-s \equiv 3 \bmod 8$ its image equals the three h_0-divisible classes. In other dimensions the map is zero.

Proof. Note that ${}_fE_2$ has dimension 0 or 1 in each bidegree. In the range of bidegrees displayed in Tables 2.18, 3.18 and 5.7, the claim follows by a dimension count using exactness in (5.6). Since hofib(trc) agrees with its Bousfield K-localization in dimensions $* \ge 1$ by 3.13, the result is propagated to higher dimensions by applying suitable periodicity operators. □

Proposition 5.9. *In the Adams spectral sequence ${}_wE_*$ the nonzero differentials landing in homotopical dimension ≤ 21 are*

(i) $d_2^{s,s+8} \ne 0$ for $s \ge 0$.

(ii) $d_2^{s,s+11} \ne 0$ for $s \ge 1$, with image divisible by h_0^{s+2}.

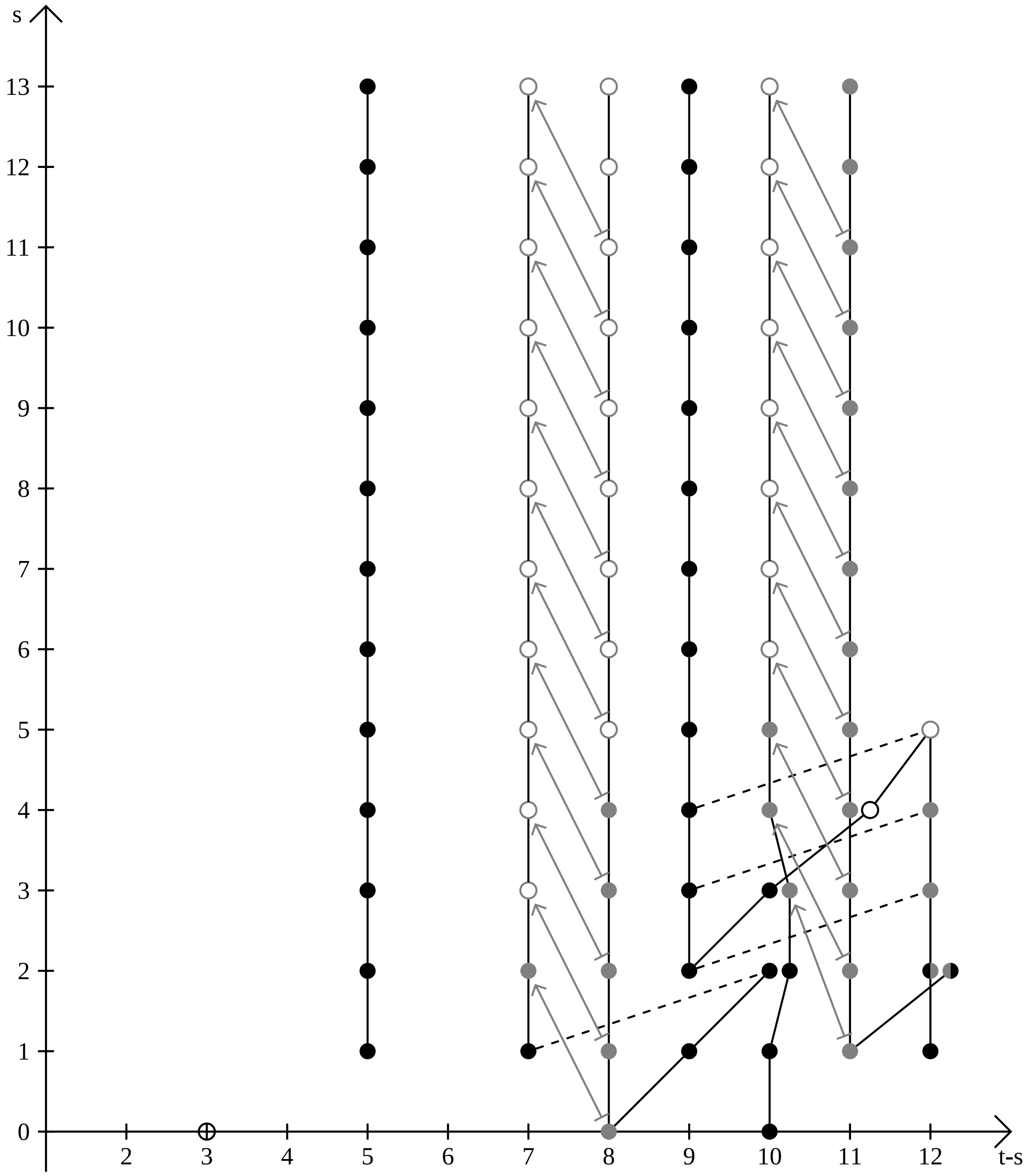

TABLE 5.7(A). The Adams E_2-term for $\mathrm{Wh}^{\mathrm{Diff}}(*)$

(iii) $d_2^{s,s+13} \neq 0$ *for* $0 \leq s \leq 3$. *The image of* $d_2^{0,13}$ *contains* $h_0 \cdot x + h_1 \cdot y$ *for nonzero classes* x, y. *The image of* $d_2^{s,13+s}$ *for* $1 \leq s \leq 3$ *is divisible by* h_0^{s+1}.

(iv) $d_2^{1,15} \neq 0$ *has image divisible by* h_1^2.

(v) $d_2^{0,16} \neq 0$ *has image divisible by* h_0.

(vi) $d_3^{s,s+16} \neq 0$ *for* $s \geq 1$ *is zero on the* h_0*-torsion classes.*

(vii) $d_2^{s,s+18} \neq 0$ *for* $s = 0, 1$.

(viii) $d_3^{2,20} \neq 0$ *is zero on the* h_1*-divisible classes.*

(ix) $d_2^{s,s+19} \neq 0$ *for* $s = 3, 4$ *is zero on the* h_1*-divisible classes, and takes* h_0*-torsion values.*

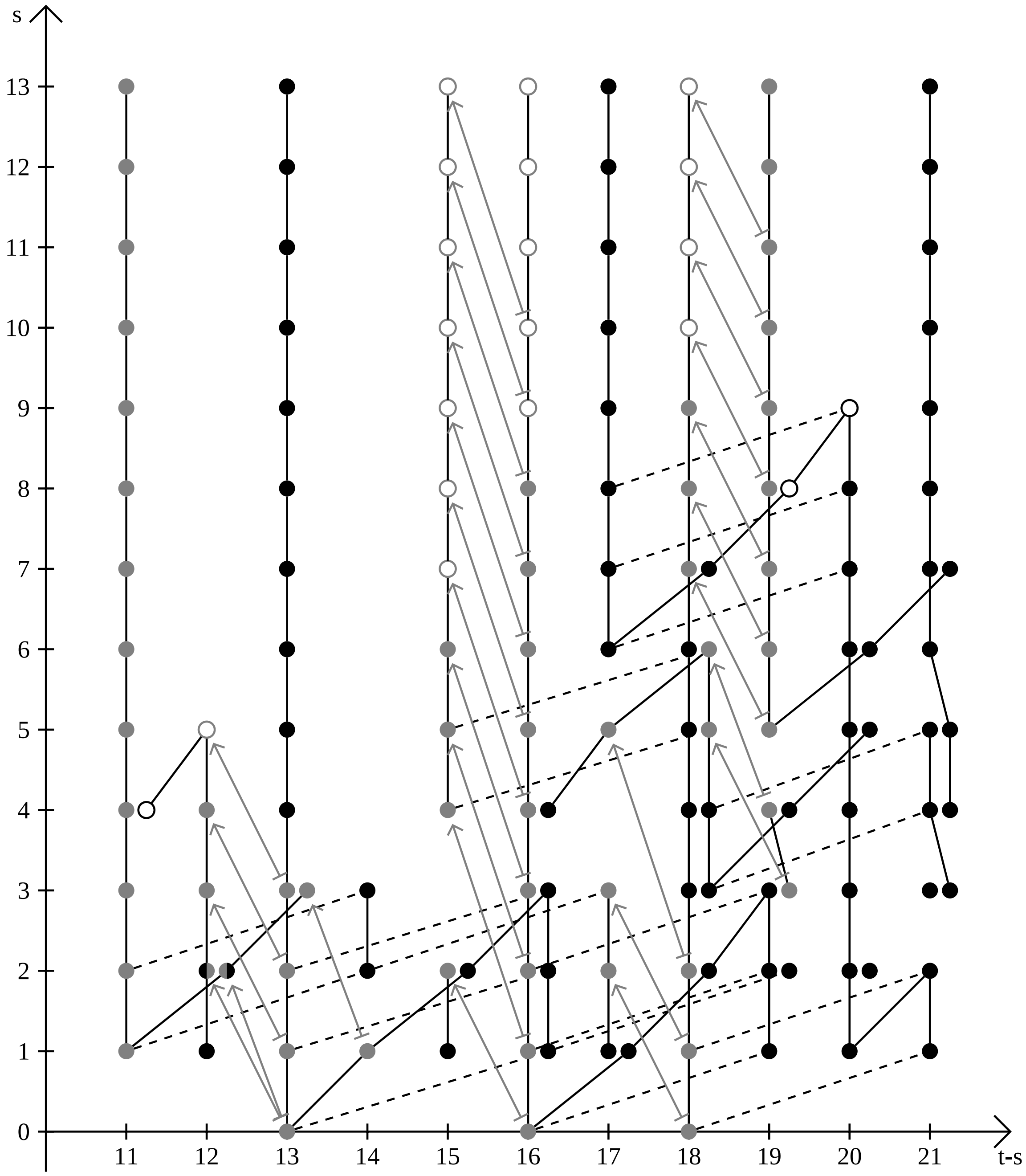

TABLE 5.7(B). The Adams E_2-term for $\mathrm{Wh}^{\mathrm{Diff}}(*)$

(x) $d_2^{s,s+19} \neq 0$ *for* $s \geq 5$*, with image divisible by* h_0^{s+2}*.*

(xi) $d_2^{3,24} \neq 0$*,* $d_2^{4,25} \neq 0$*,* $d_2^{5,26} \neq 0$*,* $d_2^{6,27} \neq 0$*,* $d_2^{7,28} \neq 0$*,* $d_2^{0,22} \neq 0$ *and* $d_2^{5,27} \neq 0$ *all have rank* 1*.*

Proof. The differentials in ${}_fE_*$ given in Tables 3.18(a) and (b) induce differentials in ${}_wE_*$ by naturality with respect to the spectral sequence map j_* in (5.6). Likewise the differentials in ${}_cE_*$ given in 2.21 lift by the connecting map ∂ in (5.6) to detect differentials in ${}_wE_*$. Taking the h_0-multiplications in ${}_wE_2$ into account, this gives rise to all the differentials listed above.

It remains to check that there are no further differentials in ${}_wE_*$. Any such would have to map from classes '•' detected by ∂ to classes '∘' in the image of j_*. For bidegree reasons the only possible targets are the h_1-divisible classes '∘' in bidegree $(s,t) = (4k, 12k+3)$ with $k \geq 1$. These classes represent the image of $\pi_{8k+3}(\mathrm{hofib}(\mathrm{trc}))$ in $\pi_{8k+3}(\mathrm{Wh}^{\mathrm{Diff}}(*))$. Now the generator of $\pi_{8k+3}(\mathrm{hofib}(\mathrm{trc})) \cong \mathbb{Z}/16$ maps to the order 2 class $\eta^2\mu_{8k+1}$ in $K_{8k+3}(\mathbb{Z}) \cong \mathbb{Z}/16$, which generates the kernel of the cyclotomic trace map $\mathrm{trc}_{\mathbb{Z}}$ to $\pi_{8k+3}(TC(\mathbb{Z})) \cong \mathbb{Z} \oplus \mathbb{Z}/8$ by [R5, 9.1]. Hence by the diagram in 1.11, the image of $\pi_{8k+3}(\mathrm{hofib}(\mathrm{trc}))$ in $\pi_{8k+3}(\mathrm{Wh}^{\mathrm{Diff}}(*))$ is nontrivial, and so the cited class in bidegree $(4k, 12k+3)$ must survive to the E_∞-term. Hence it is not hit by a differential. □

Theorem 5.10. *The 2-primary homotopy groups of* $\mathrm{Wh}^{\mathrm{Diff}}(*)$ *in dimensions* $* \leq 21$ *are as follows:*

$$
\begin{aligned}
\pi_n(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= 0 \qquad \textit{for } n \leq 2,\\
\pi_3(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= \mathbb{Z}/2\\
\pi_4(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= 0\\
\pi_5(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= \mathbb{Z}\\
\pi_6(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= 0\\
\pi_7(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= \mathbb{Z}/2\\
\pi_8(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= 0\\
\pi_9(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= \mathbb{Z}/2 \oplus \mathbb{Z}\\
\pi_{10}(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/8\\
\pi_{11}(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= \mathbb{Z}/2\\
\pi_{12}(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= \mathbb{Z}/4\\
\pi_{13}(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= \mathbb{Z}\\
\pi_{14}(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= \mathbb{Z}/4\\
\pi_{15}(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= (\mathbb{Z}/2)^2\\
\pi_{16}(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= \mathbb{Z}/2 \oplus \mathbb{Z}/8\\
\pi_{17}(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= (\mathbb{Z}/2)^2 \oplus \mathbb{Z}\\
\pi_{18}(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= (\mathbb{Z}/2)^3 \oplus \mathbb{Z}/32\\
\pi_{19}(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= \mathbb{Z}/2 \rtimes \mathbb{Z}/2 \rtimes \mathbb{Z}/8 \rtimes \mathbb{Z}/2\\
\pi_{20}(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= \#2^7\\
\pi_{21}(\mathrm{Wh}^{\mathrm{Diff}}(*)) &= \#2^4 \oplus \mathbb{Z}\,.
\end{aligned}
$$

In the long exact sequence in homotopy induced by the cofiber sequence

$$\mathbb{C}P^\infty_{-1} \xrightarrow{i} \mathrm{hofib}(\mathrm{trc}) \xrightarrow{j} \mathrm{Wh}^{\mathrm{Diff}}(*)$$

the image of j_* *is* $\mathbb{Z}/2$ *in dimensions* $n \equiv 3 \bmod 8$ *and zero otherwise, for* $n \leq 21$.

Proof. This follows by inspection of the E_∞-term of the Adams spectral sequence for $\mathrm{Wh}^{\mathrm{Diff}}(*)$, and the long exact sequence

$$\cdots \to \pi_n(\mathbb{C}P^\infty_{-1}) \xrightarrow{i_*} \pi_n(\mathrm{hofib}(\mathrm{trc})) \xrightarrow{j_*} \pi_n(\mathrm{Wh}^{\mathrm{Diff}}(*)) \xrightarrow{\widetilde{\mathrm{trc}}_*} \pi_{n-1}(\mathbb{C}P^\infty_{-1}) \to \dots\,.$$

The long exact sequence shows that $\pi_{18}(\mathrm{Wh}^{\mathrm{Diff}}(*)) \cong \pi_{17}(\mathbb{C}P^\infty_{-1})$, which was found in 2.13. Next $\pi_{19}(\mathrm{Wh}^{\mathrm{Diff}}(*))$ is an extension of the torsion in $\pi_{18}(\mathbb{C}P^\infty_{-1}) \cong \mathbb{Z}/2 \rtimes \mathbb{Z}/8 \rtimes \mathbb{Z}/2 \oplus \mathbb{Z}$ by $\mathbb{Z}/2$. Also $\pi_{20}(\mathrm{Wh}^{\mathrm{Diff}}(*))$ is the kernel of a homomorphism from $\pi_{19}(\mathbb{C}P^\infty_{-1}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/64$ with image $\mathbb{Z}/8$. This is some group of order 2^7, denoted $\#2^7$ in the statement of the theorem. Lastly $\pi_{21}(\mathrm{Wh}^{\mathrm{Diff}}(*))$ is the sum of a group of order 2^4 and an infinite cyclic group, as can be read off from the E_∞-term of ${}_wE_*$. $\square$

Regarding multiplicative structure, we have the following addendum.

Lemma 5.11. *The homomorphism $\eta_\# : \pi_n(\mathrm{Wh}^{\mathrm{Diff}}(*)) \to \pi_{n+1}(\mathrm{Wh}^{\mathrm{Diff}}(*))$ has image $(\mathbb{Z}/2)^2$ for $n = 9$, image $\mathbb{Z}/2$ for $n = 10$ and is zero for all other $n \leq 14$.*

The homomorphism $\nu_\#\colon \pi_n(\mathrm{Wh}^{\mathrm{Diff}}(*)) \to \pi_{n+3}(\mathrm{Wh}^{\mathrm{Diff}}(*))$ has image $\mathbb{Z}/2$ for $n = 7$ and is zero for all other $n \leq 14$.

The homomorphism $\sigma_\#\colon \pi_n(\mathrm{Wh}^{\mathrm{Diff}}(*)) \to \pi_{n+7}(\mathrm{Wh}^{\mathrm{Diff}}(*))$ has image $\mathbb{Z}/2$ for $n = 5$ and is zero for all other $n \leq 11$.

Proof. The nonzero multiplications listed are all detected by nontrivial h_1, h_2 or h_3-multiplications in the Adams spectral sequence (5.5) for $\mathrm{Wh}^{\mathrm{Diff}}(*)$. To see that there are no other nonzero multiplications in this range one can use Adams filtration arguments in this spectral sequence, combined with naturality with respect to the map $\widetilde{\mathrm{trc}}\colon \mathrm{Wh}^{\mathrm{Diff}}(*) \to \Sigma\mathbb{C}P^\infty_{-1}$. For example, $\pi_{14}(\mathrm{Wh}^{\mathrm{Diff}}(*))$ is detected in $\pi_{13}(\mathbb{C}P^\infty_{-1})$, but the image of $\pi_7(\mathrm{Wh}^{\mathrm{Diff}}(*))$ in $\pi_6(\mathbb{C}P^\infty_{-1})$ is divisible by ν and $\sigma\nu = 0$, so $\sigma_\# = 0$ for $n = 7$. $\square$

6. Cohomology of $K(\mathbb{Z})$ and the linearization map

We continue to implicitly complete all spectra at 2. Bökstedt's spectrum $JK(\mathbb{Z})$ is the homotopy fiber of the composite

$$ko \xrightarrow{\psi^3-1} bspin \xrightarrow{c} bsu\,.$$

It is also homotopy equivalent to the algebraic K-theory spectrum $K(\mathbb{Z})$, by [RW], [We]. Hence there is a diagram of horizontal and vertical cofiber sequences of spectra:

(6.1)
$$\begin{tikzcd}
bso \arrow[r, equal] \arrow[d, "\eta"] & bso \arrow[r] \arrow[d, "t"] & * \arrow[r] \arrow[d] & \Sigma bso \arrow[d, "\eta"] \\
spin \arrow[r, "\zeta"] \arrow[d, "c"] & j \arrow[r] \arrow[d, "i"] & ko \arrow[r, "\psi^3-1"] \arrow[d, equal] & bspin \arrow[d, "c"] \\
su \arrow[r] \arrow[d] & K(\mathbb{Z}) \arrow[r] \arrow[d] & ko \arrow[r] \arrow[d] & bsu \arrow[d] \\
\Sigma bso \arrow[r, equal] & \Sigma bso \arrow[r] & * \arrow[r] & \Sigma^2 bso\,.
\end{tikzcd}$$

The right hand column is a connected covering of (3.2), and the second row defines the connective real image of J spectrum j. We let $t = \zeta \circ \eta$ be the composite of the Bott map $\eta\colon bso \to spin$ and the connecting map $\zeta\colon spin \to j$.

Miller and Priddy [MP] define spectra $g/o_\oplus$ and ibo as the pullbacks in the following diagram:

$$\begin{tikzcd} g/o_\oplus \arrow[r] \arrow[d] & ibo \arrow[r] \arrow[d] & S^0 \arrow[d, "e"] \\ bso \arrow[r, "\eta"] & spin \arrow[r, "\zeta"] & j\,. \end{tikzcd} \tag{6.2}$$

(More precisely, they define the underlying infinite loop spaces $G/O_\oplus = \Omega^\infty g/o_\oplus$ and $IBO = \Omega^\infty ibo$.) Here $e\colon S^0 \to j$ is the map representing the real Adams e-invariant. Its fiber c is the cokernel of J spectrum, which is K-acyclic. Thus the unit map $i\colon S^0 \to K(\mathbb{Z})$ factors, uniquely up to homotopy, as e composed with $i\colon j \to K(\mathbb{Z})$. By (6.1) the cofiber of the lower composite map $\zeta\eta$ in (6.2) is $K(\mathbb{Z})$. Hence there is a cofiber sequence

$$g/o_\oplus \to S^0 \xrightarrow{i} K(\mathbb{Z}) \tag{6.3}$$

of 2-complete spectra. Thus there is a fiber sequence $G/O_\oplus \to QS^0 \to K(\mathbb{Z})$ of underlying infinite loop spaces, and we might write $G/O_\oplus = IK(\mathbb{Z})$ as the 'ideal' in $QS^0 = \Omega^\infty S^0$ defining $K(\mathbb{Z})$ (at the prime 2).

We compute the mod 2 spectrum cohomology $H^*(K(\mathbb{Z}))$ by means of the cofiber sequence $su \to K(\mathbb{Z}) \to ko$, where $su \simeq \Sigma^3 ku$. In view of (6.3) this also determines $H^*(g/o_\oplus)$. Miller and Priddy conjecture in [MP] that $G/O_\oplus \simeq G/O$ as infinite loop spaces. If confirmed, this would also lead to a calculation of the spectrum cohomology $H^*(g/o)$. It is known that $G/O \simeq G/O_\oplus$ as 2-complete spaces, and that $H_*(G/O;\mathbb{F}_2) \cong H_*(G/O_\oplus;\mathbb{F}_2)$ as Hopf algebras over the Steenrod– and Dyer–Lashof algebras, by unpublished calculations of J. Tornehave.

Theorem 6.4. *The mod 2 spectrum cohomology of $K(\mathbb{Z})$ is the unique nontrivial extension of A-modules*

$$A/A(Sq^1, Sq^2) \to H^*(K(\mathbb{Z})) \to \Sigma^3 A/A(Sq^1, Sq^3)\,.$$

The A-module $H^*(\Sigma g/o_\oplus)$ is the connected cover of $H^*(K(\mathbb{Z}))$, i.e., the kernel of the augmentation $H^*(K(\mathbb{Z})) \to \mathbb{F}_2$.

Proof. We use the cofiber sequence $K(\mathbb{Z}) \to ko \to bsu$ where the right hand map is the composite of $\psi^3 - 1\colon ko \to bspin$ and $c\colon bspin \to bsu$. The induced map

$$\Sigma^4 A/A(Sq^1, Sq^3) \cong H^*(bsu) \to H^*(ko) \cong A/A(Sq^1, Sq^2)$$

is the zero homomorphism. For the complexification map c induces multiplication by 2 on π_4, and thus the zero map on H^4. Thus the long exact sequence in spectrum cohomology decomposes as the A-module extension above. The second claim follows from the cofiber sequence $S^0 \to K(\mathbb{Z}) \to \Sigma g/o_\oplus$.

It remains to characterize the extension. There are precisely two such A-module extensions, since

$$\mathrm{Ext}^1_A(\Sigma^3 A/A(Sq^1, Sq^3), A/A(Sq^1, Sq^2)) \cong \mathrm{Ext}^1_{E_1}(\Sigma^3\mathbb{F}_2, A/A(Sq^1, Sq^2)) \cong \mathbb{F}_2\,.$$

Here $E_1 \subset A$ is the exterior algebra generated by Sq^1 and $Q_1 = Sq^3 + Sq^2Sq^1$. We know that $H^*(K(\mathbb{Z}))$ is a nontrivial extension, because $H_3^{spec}(K(\mathbb{Z}); \mathbb{Z}_2) \cong \pi_2(g/o_\oplus) \cong \mathbb{Z}/2$ implies that there is a nonzero Sq^1 from dimension 3 to dimension 4 in $H^*(K(\mathbb{Z}))$. $\square$

We list a monomial basis for $H^*(K(\mathbb{Z}))$ in dimensions ≤ 14 in Table 6.5. It differs from $H^*(\Sigma g/o_\oplus)$ only in dimension 0. The notation is that $\iota_0 \in H^0(K(\mathbb{Z}))$ pulls back from the generator of $H^0(ko)$, while $\iota_3 \in H^3(K(\mathbb{Z}))$ is the unique lift of the generator in dimension 3 of $H^*(su) \cong \Sigma^3 A/A(Sq^1, Sq^3)$. We have chosen $Sq^9(\iota_3) = Sq^1Sq^8(\iota_3)$ as the lift in $H^*(K(\mathbb{Z}))$ of $Sq^9\iota_3 = Sq^7Sq^2\iota_3$ in $H^*(su)$.

The linearization map $L\colon A(*) \to K(\mathbb{Z})$ from [Wa1] and 1.8 is compatible with the unit maps from S^0. When combined with the pullback diagram (6.2) defining $g/o_\oplus$ it yields the following spectrum level diagram with horizontal cofiber sequences:

(6.6)
$$\begin{tikzcd}
S^0 \arrow[r, "i"] \arrow[d, equal] & A(*) \arrow[r] \arrow[d, "L"] & \mathrm{Wh}^{\mathrm{Diff}}(*) \arrow[d, "\bar{L}"] \\
S^0 \arrow[r, "i"] \arrow[d, "e"] & K(\mathbb{Z}) \arrow[r] \arrow[d, equal] & \Sigma g/o_\oplus \arrow[d] \\
j \arrow[r, "i"] & K(\mathbb{Z}) \arrow[r] & \Sigma bso\,.
\end{tikzcd}$$

Proposition 6.7. *The reduced linearization map $\bar{L}\colon \mathrm{Wh}^{\mathrm{Diff}}(*) \to \Sigma g/o_\oplus$ is a rational equivalence, but induces the zero homomorphism between the bottom homotopy groups $\pi_3(\mathrm{Wh}^{\mathrm{Diff}}(*)) \cong \mathbb{Z}/2$ and $\pi_3(\Sigma g/o_\oplus) \cong \mathbb{Z}/2$. The induced map on mod 2 spectrum cohomology*

$$\bar{L}^*\colon H^*(\Sigma g/o_\oplus) \to H^*(\mathrm{Wh}^{\mathrm{Diff}}(*))$$

is zero in all dimensions.

Proof. The linearization map $L\colon A(*) \to K(\mathbb{Z})$ is a rational equivalence between spectra of finite type, by [Wa1, 2.2], so its 2-adic completion is also a rational equivalence. Comparison with (6.6) shows that also $\bar{L}$ is a rational equivalence.

The homomorphism $\pi_3(\bar{L})$ is induced from the homomorphism

$$\pi_3(L)\colon \pi_3(A(*)) \cong \mathbb{Z}/24 \oplus \mathbb{Z}/2 \to K_3(\mathbb{Z}) \cong \mathbb{Z}/48$$

by passage to the quotient with respect to subgroups $\pi_3^S \cong \mathbb{Z}/24$ on both sides. Algebraically, the only possibility is that $\pi_3(\bar{L}) = 0$.

In cohomology we have the following map of extensions of A-modules:

$$\begin{tikzcd}
H^*(\Sigma g/o_\oplus) \arrow[r] \arrow[d, "\bar{L}^*"] & H^*(K(\mathbb{Z})) \arrow[r, "i^*"] \arrow[d, "L^*"] & \mathbb{F}_2 \arrow[d, equal] \\
H^*(\mathrm{Wh}^{\mathrm{Diff}}(*)) \arrow[r] & H^*(A(*)) \arrow[r, "i^*"] & \mathbb{F}_2\,.
\end{tikzcd}$$

The lower extension is split, as in 4.5. Here $H^*(K(\mathbb{Z}))$ is generated as an A-module by classes ι_0 and ι_3, as in 6.4 and 6.5. The class ι_0 maps to the split summand $\mathbb{F}_2$ of $H^*(A(*))$, hence the submodule it generates maps to zero in positive degrees. Likewise ι_3 maps to zero by the π_3-calculation above and the Hurewicz theorem. Thus L^* is zero in positive degrees, and $\bar{L}^*$ is zero in all degrees. $\square$

dim	x	$Sq^1(x)$	$Sq^2(x)$	$Sq^4(x)$	$Sq^8(x)$
0	ι_0	0	0	$Sq^4\iota_0$	$Sq^8\iota_0$
1					
2					
3	ι_3	$Sq^4\iota_0$	$Sq^2\iota_3$	$Sq^4\iota_3$	$Sq^8\iota_3$
4	$Sq^4\iota_0$	0	$Sq^6\iota_0$	0	$Sq^8Sq^4\iota_0$
5	$Sq^2\iota_3$	0	$Sq^7\iota_0$	$Sq^4Sq^2\iota_3$	$Sq^8Sq^2\iota_3$
6	$Sq^6\iota_0$	$Sq^7\iota_0$	0	$Sq^{10}\iota_0$	$Sq^{10}Sq^4\iota_0$
7	$Sq^7\iota_0$	0	0	$Sq^{11}\iota_0$	
	$Sq^4\iota_3$	0	$Sq^6\iota_3$	$Sq^6Sq^2\iota_3$	
8	$Sq^8\iota_0$	0	$Sq^{10}\iota_0$	$Sq^{12}\iota_0$	
9	$Sq^6\iota_3$	$Sq^7\iota_3$	0	$Sq^{10}\iota_3$ $+Sq^8Sq^2\iota_3$	
	$Sq^4Sq^2\iota_3$	0	$Sq^6Sq^2\iota_3$	$Sq^{13}\iota_0$	
10	$Sq^{10}\iota_0$	$Sq^{11}\iota_0$	0	0	
	$Sq^7\iota_3$	0	$Sq^8Sq^4\iota_0$ $+Sq^9\iota_3$	$Sq^{11}\iota_3$	
11	$Sq^{11}\iota_0$	0	$Sq^{13}\iota_0$		
	$Sq^8\iota_3$	$Sq^9\iota_3$	$Sq^{10}\iota_3$		
	$Sq^6Sq^2\iota_3$	$Sq^8Sq^4\iota_0$ $+Sq^9\iota_3$	0		
12	$Sq^{12}\iota_0$	$Sq^{13}\iota_0$	$Sq^{14}\iota_0$		
	$Sq^8Sq^4\iota_0$	0	$Sq^{10}Sq^4\iota_0$		
	$Sq^9\iota_3$ $\equiv Sq^7Sq^2\iota_3$	0	$Sq^{10}Sq^4\iota_0$		
13	$Sq^{13}\iota_0$	0			
	$Sq^{10}\iota_3$	$Sq^{11}\iota_3$			
	$Sq^8Sq^2\iota_3$	0			
14	$Sq^{14}\iota_0$				
	$Sq^{10}Sq^4\iota_0$				
	$Sq^{11}\iota_3$				

TABLE 6.5. $H^*(K(\mathbb{Z}))$ in dimensions ≤ 14.

Corollary 6.8. *There is a long exact sequence in mod 2 spectrum cohomology*

$$\cdots \to H^*(TC(\mathbb{Z})) \xrightarrow{L^* \oplus \mathrm{trc}_{\mathbb{Z}}^*} H^*(TC(*)) \oplus H^*(K(\mathbb{Z})) \xrightarrow{\mathrm{trc}_*^* - L^*} H^*(A(*)) \xrightarrow{\partial} \dots .$$

Here $L\colon A(*) \to K(\mathbb{Z})$ and $\mathrm{trc}_*\colon A(*) \to TC(*)$ induce zero maps in positive dimensions, ∂ induces an injective map in positive dimensions, and $L\colon TC(*) \to TC(\mathbb{Z})$ and $\mathrm{trc}_{\mathbb{Z}}\colon K(\mathbb{Z}) \to TC(\mathbb{Z})$ both induce surjections in all dimensions.

Proof. The sequence arises by applying mod 2 spectrum cohomology to the homotopy cartesian square in 1.8 for $X = *$. The assertions for $L\colon A(*) \to K(\mathbb{Z})$ and

trc_* follow from 4.7 and 6.7. The rest follows by exactness. In fact $L^* \oplus \mathrm{trc}^*_{\mathbb{Z}}$ will be surjective in positive degrees, which is stronger than the stated conclusion. □

Remark 6.9. The 'rigid tube' map from [Wa3, §3] provides a space level map of horizontal fiber sequences

$$\begin{tikzcd} G/O \arrow[r] \arrow[d, "hw"] & BSO \arrow[r, "j"] \arrow[d, "s"] & BSG \arrow[d, "w"] \\ \Omega\mathrm{Wh}^{\mathsf{Diff}}(*) \arrow[r] & QS^0 \arrow[r, "i"] & A(*)\,. \end{tikzcd}$$

We call the left vertical map hw the Hatcher–Waldhausen map. It was proved in [R1] that this gives a diagram of infinite loop maps if one uses a *multiplicative* infinite loop space structure on each of the spaces in the lower row. However, these are generally different from the additive infinite loop space structures we have been considering in this paper. Let $\Omega\mathrm{Wh}^{\mathsf{Diff}}_{\otimes}(*)$ denote the spectrum with underlying infinite loop space given as the homotopy fiber of the unit map $i\colon SG = Q(S^0)_1 \to A(*)_1$ with the multiplicative infinite loop space structures.

It can be read off from Tables 5.1, 5.7 and 6.5 that the (space level) Hatcher–Waldhausen map $hw\colon G/O \to \Omega\mathrm{Wh}^{\mathsf{Diff}}(*)$ does not admit a four-fold delooping, when the target is given the additive infinite loop space structure. For by [Wa3], $\pi_2(hw)\colon \mathbb{Z}/2 \cong \mathbb{Z}/2$ is an isomorphism, and a k-invariant argument (see 7.5 below) shows that $\pi_4(hw)\colon \mathbb{Z} \to \mathbb{Z}$ is a 2-adic equivalence. If hw were to admit a four-fold delooping then $\sigma \cdot hw(x) = hw(\sigma \cdot x)$ for any $x \in \pi_4(G/O)$. But $\pi_{11}(G/O) = 0$, while the minimal resolution leading to Table 5.7 shows that there is a nonzero h_3-multiplication from the class representing the generator of $\pi_4(\Omega\mathrm{Wh}^{\mathsf{Diff}}(*))$ to the class representing the element of order 2 in $\pi_{11}(\Omega\mathrm{Wh}^{\mathsf{Diff}}(*))$. See also 5.11. This contradicts the existence of the four-fold delooping. Note that we did not specify a choice of four-fold delooping of G/O in this argument, so it applies to both $\Omega^\infty(\Sigma^4 g/o)$ and $\Omega^\infty(\Sigma^4 g/o_\oplus)$, in case they are different.

The spectrum map $g/o \to \Omega\mathrm{Wh}^{\mathsf{Diff}}_{\otimes}(*)$ constructed geometrically in [R1] thus shows that the spectra $\mathrm{Wh}^{\mathsf{Diff}}(*)$ and $\mathrm{Wh}^{\mathsf{Diff}}_{\otimes}(*)$ cannot be homotopy equivalent.

7. A spectrum map from $\mathrm{Wh}^{\mathsf{Diff}}(*)$ to $\Sigma g/o_\oplus$

Observe by inspection of Tables 5.1 and 6.5 that $H^*(\mathrm{Wh}^{\mathsf{Diff}}(*))$ and $H^*(\Sigma g/o_\oplus)$ are abstractly isomorphic as A-modules in dimensions $* \le 9$. In this chapter we construct a spectrum map

$$M\colon \mathrm{Wh}^{\mathsf{Diff}}(*) \to \Sigma g/o_\oplus$$

inducing an isomorphism in these dimensions. As before, all spectra are implicitly 2-completed in this chapter.

Lemma 7.1. *There is a spectrum map $m\colon \mathrm{hofib}(\mathrm{trc}) \to K(\mathbb{Z})$ making the following diagram of horizontal cofiber sequences commute:*

$$\begin{tikzcd} \mathrm{hofib}(\mathrm{trc}) \arrow[r] \arrow[d, "m"] & \Sigma^{-2} ku \arrow[r, "\delta"] \arrow[d, "r\beta^{-1}"] & \Sigma^4 ko \arrow[d, "\beta^2\Sigma^4 c"] \\ K(\mathbb{Z}) \arrow[r] & ko \arrow[r, "c(\psi^3-1)"] & bsu\,. \end{tikzcd}$$

Proof. The maps in the right hand square are characterized (up to homotopy) by their K-localizations, and after K-localization we can compute

$$\beta^2\Sigma^4 c \circ L_K\delta = \beta^2 \circ \Sigma^4 c \circ \Sigma^4 r \circ \beta^{-2} \circ (\psi^3 - 1) \circ \beta^{-1} = c(\psi^3 - 1) \circ r\beta^{-1} .$$

Hence the right hand square homotopy commutes. We let m be the induced map of horizontal homotopy fibers. $\square$

Lemma 7.2. *There is a spectrum map $M\colon \mathrm{Wh}^{\mathsf{Diff}}(*) \to \Sigma g/o_{\oplus}$ making the following diagram of horizontal cofiber sequences commute:*

$$\begin{tikzcd}
\mathbb{C}P^{\infty}_{-1} \arrow[r, "i"] \arrow[d] & \mathrm{hofib}(\mathrm{trc}) \arrow[r, "j"] \arrow[d, "m"] & \mathrm{Wh}^{\mathsf{Diff}}(*) \arrow[d, "M"] \\
S^0 \arrow[r, "i"] & K(\mathbb{Z}) \arrow[r] & \Sigma g/o_{\oplus} .
\end{tikzcd}$$

Proof. We must show that the composite map

$$\mathbb{C}P^{\infty}_{-1} \xrightarrow{i} \mathrm{hofib}(\mathrm{trc}) \xrightarrow{m} K(\mathbb{Z}) \to \Sigma g/o_{\oplus}$$

is null homotopic. Consider the following diagram of horizontal and vertical cofiber sequences:

$$\begin{tikzcd}
S^0 \arrow[r, equal] \arrow[d, "i"] & S^0 \arrow[d, "i"] & \\
K(\mathbb{Z}) \arrow[r] \arrow[d] & ko \arrow[r, "c(\psi^3-1)"] \arrow[d] & bsu \arrow[d, equal] \\
\Sigma g/o_{\oplus} \arrow[r] & ko/S^0 \arrow[r] & bsu .
\end{tikzcd}$$

We have $[\mathbb{C}P^{\infty}_{-1}, su] = 0$ by an application of the Atiyah–Hirzebruch spectral sequence, so we can identify $[\mathbb{C}P^{\infty}_{-1}, K(\mathbb{Z})]$ with the kernel of

$$c(\psi^3 - 1)_{\#}\colon [\mathbb{C}P^{\infty}_{-1}, ko] \to [\mathbb{C}P^{\infty}_{-1}, bsu] .$$

By another calculation with the Atiyah–Hirzebruch spectral sequence using [Ad] and [AW], this kernel is isomorphic to $\mathbb{Z}_2$, and is generated by the composite map

$$\mathbb{C}P^{\infty}_{-1} \to \mathbb{C}P^{\infty}_{+} \to S^0 \xrightarrow{i} K(\mathbb{Z}) .$$

The left hand map pinches the bottom cell to a point; the middle map retracts $\mathbb{C}P^{\infty}$ to a point. The composite maps to zero in $[\mathbb{C}P^{\infty}_{-1}, \Sigma g/o_{\oplus}]$, so m extends to a map M as claimed. $\square$

Lemma 7.3. *The map $M\colon \mathrm{Wh}^{\mathsf{Diff}}(*) \to \Sigma g/o_{\oplus}$ induces an isomorphism on π_3.*

Proof. Consider the maps of long exact sequences of homotopy groups induced by the diagrams in 7.1 and 7.2. The isomorphism $\pi_4(\beta^2\Sigma^4 c)\colon \mathbb{Z} \cong \mathbb{Z}$ passes to quotient isomorphisms $\pi_3(m)\colon \mathbb{Z}/16 \cong \mathbb{Z}/16$ and $\pi_3(M)\colon \mathbb{Z}/2 \cong \mathbb{Z}/2$. $\square$

Theorem 7.4. *There is a spectrum map*

$$M\colon \mathrm{Wh}^{\mathrm{Diff}}(*) \to \Sigma g/o_{\oplus}$$

inducing an isomorphism on mod 2 spectrum cohomology in dimensions $* \le 9$. So M is precisely 9-connected, and induces a map of spaces

$$\Omega M\colon \Omega\mathrm{Wh}^{\mathrm{Diff}}(*) \to G/O_{\oplus} \simeq G/O$$

such that $\pi_*(\Omega M)$ is an isomorphism for $* \le 8$.

Proof. The A-module homomorphism $M^*\colon H^*(\Sigma g/o_{\oplus}) \to H^*(\mathrm{Wh}^{\mathrm{Diff}}(*))$ is an isomorphism in degree 3 by 7.3. We can then compute M^* in dimensions $* \le 14$ from Tables 5.1 and 6.5, finding that $H^*(\mathrm{hofib}(M))$ is 9-connected, has rank 1 in each dimension $10 \le * \le 13$, and has rank ≥ 1 in dimension 14. Thus ΩM is 8-connected, and the surjection $\pi_8(\Omega M)$ is in fact an isomorphism, since both its source and target are isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$. $\square$

Theorem 7.5. *The Hatcher–Waldhausen map $hw\colon G/O \to \Omega\mathrm{Wh}^{\mathrm{Diff}}(*)$ induces an isomorphism on 2-primary homotopy in dimensions $* \le 8$, and an injection on 2-primary homotopy in dimensions $* \le 13$. Its 2-completion is thus precisely 8-connected.*

Proof. Let P^nX denote the nth Postnikov section of a (simple) space X. The map

$$P^2(hw)\colon P^2G/O \to P^2\Omega\mathrm{Wh}^{\mathrm{Diff}}(*)$$

is a homotopy equivalence by 7.3. The k-invariants of G/O and $\Omega\mathrm{Wh}^{\mathrm{Diff}}(*)$ all lift to spectrum cohomology, since these are infinite loop spaces, and are abstractly isomorphic for $n \le 8$ by 7.4. They can be partly read off from the minimal resolution for $H^*(\mathrm{Wh}^{\mathrm{Diff}}(*))$ that was used to generate Table 5.7, yielding the following facts: Let $\beta_1\colon K(\mathbb{Z}/2,n) \to K(\mathbb{Z},n+1)$ be the mod 2 Bockstein map, and let $i_1\colon K(\mathbb{Z},n) \to K(\mathbb{Z}/2,n)$ be the mod 2 reduction map. Then $i_1\beta_1 = Sq^1$. For $m \ge n$ let $p_n^m\colon P^mX \to P^nX$ be a projection in the Postnikov system. Then

$$k^5\colon K(\mathbb{Z}/2,2) \simeq P^2\Omega\mathrm{Wh}^{\mathrm{Diff}}(*) \to K(\mathbb{Z},5)$$

is $\beta_1 Sq^2$, while

$$k^7\colon P^4\Omega\mathrm{Wh}^{\mathrm{Diff}}(*) \to K(\mathbb{Z}/2,7)$$

factors as $Sq^5p_2^4$. The last k-invariant we consider is

$$k^9 = k_1^9 \times k_2^9\colon P^6\Omega\mathrm{Wh}^{\mathrm{Diff}}(*) \to K(\mathbb{Z}/2 \oplus \mathbb{Z},9) \simeq K(\mathbb{Z}/2,9) \times K(\mathbb{Z},9)\,.$$

Its projection k_2^9 to $K(\mathbb{Z},9)$ factors over p_4^6, and the composite

$$K(\mathbb{Z},4) \to P^4\Omega\mathrm{Wh}^{\mathrm{Diff}}(*) \xrightarrow{\bar{k}_2^9} K(\mathbb{Z},9)$$

is $\beta_1 Sq^4 i_1$. Here $k_2^9 = \bar{k}_2^9 \circ p_4^6$.

Considering the maps of Postnikov sections $P^n(hw)\colon P^nG/O \to P^n\Omega\mathrm{Wh}^{\mathrm{Diff}}(*)$ and comparing the k-invariants for G/O and $\Omega\mathrm{Wh}^{\mathrm{Diff}}(*)$, it follows that also $P^4(hw)$

and $P^6(hw)$ are homotopy equivalences, and that $P^8(hw)$ induces an isomorphism on π_8 modulo the torsion subgroups. Hence $\pi_*(hw)$ is an isomorphism for $* \leq 7$. In particular the image of $\nu^2 \in \pi_6(SG) \cong \pi_6^S$ in $\pi_6(G/O)$ maps to the generator of $\pi_6(\Omega \mathrm{Wh}^{\mathrm{Diff}}(*))$.

The 2-torsion in $\pi_8(G/O)$ is the image of $\bar\nu \in \pi_8(SG) \cong \pi_8^S$, satisfying $\eta \cdot \bar\nu = \nu \cdot \nu^2$. The image of $\bar\nu$ in $\pi_8(\Omega \mathrm{Wh}^{\mathrm{Diff}}(*))$ is nonzero, because $\eta \cdot hw(\bar\nu) = \nu \cdot hw(\nu^2)$ is nonzero, as can be seen from Table 5.7(a) or detected by ΩM. Hence $\pi_8(hw)$ is also an isomorphism on the torsion in dimension 8. So hw is 8-connected, but cannot be 9-connected because $\pi_9(G/O) = (\mathbb{Z}/2)^2$ cannot surject to $\pi_9(\Omega \mathrm{Wh}^{\mathrm{Diff}}(*)) = (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/8$.

The nonzero multiplications by η in $\pi_*(\Omega \mathrm{Wh}^{\mathrm{Diff}}(*))$ given in 5.11 then imply that $\pi_n(hw)$ is injective for $9 \leq n \leq 11$ and $n = 13$. Finally $\pi_{12}(hw)$ is injective since $\pi_{12}(G/O) = \mathbb{Z}$ and hw is a rational equivalence [B1]. □

References

[Ad] J.F. Adams, *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–632.

[AW] J.F. Adams and G. Walker, *On complex Stiefel manifolds*, Proc. Camb. Philos. Soc. **61** (1965), 81–103.

[At] M.F. Atiyah, *Thom complexes*, Proc. Lond. Math. Soc. **11** (1961), 291–310.

[BlLi] S. Bloch and S. Lichtenbaum, *A spectral sequence for motivic cohomology*, Invent. Math. (to appear).

[B1] M. Bökstedt, *The rational homotopy type of $\Omega \mathrm{Wh}^{\mathrm{Diff}}(*)$*, Proc. Algebraic Topology, Aarhus 1982 (I. Madsen and B. Oliver, eds.), Lect. Notes Math., vol. 1051, Springer-Verlag, 1984, pp. 25–37.

[B2] M. Bökstedt, *Topological Hochschild homology*, preprint, Bielefeld, Topology (to appear).

[6A] M. Bökstedt, G. Carlsson, R. Cohen, T. Goodwillie, W.-C. Hsiang and I. Madsen, *On the algebraic K-theory of simply connected spaces*, Duke Math. J. **84** (1996), 541–563.

[BHM] M. Bökstedt, W.-C. Hsiang and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **111** (1993), 865–940.

[BM1] M. Bökstedt and I. Madsen, *Topological cyclic homology of the integers*, Asterisque **226** (1994), 57–143.

[BM2] M. Bökstedt and I. Madsen, *Algebraic K-theory of local number fields: the unramified case*, Prospects in topology (Princeton, NJ, 1994), Ann. of Math. Studies, vol. 138, Princeton University Press, 1995, pp. 28–57.

[BW] M. Bökstedt and F. Waldhausen, *The map $BSG \to A(*) \to QS^0$*, Algebraic topology and algebraic K-theory, Proc. Conf., Princeton, NJ (USA), Ann. of Math. Studies, vol. 113, Princeton University Press, 1987, pp. 418–431.

[Br] R.R. Bruner, Ext-*calculator*, Computer code available from rrb@math.wayne.edu.

[BuLa] D. Burghelea and R. Lashof, *The homotopy type of the space of diffeomorphisms. I*, Trans. Am. Math. Soc. **196** (1974), 1–36.

[D1] B.I. Dundas, *Relative K-theory and topological cyclic homology*, Acta Math. **179** (1997), 223–242.

[D2] B.I. Dundas, *The cyclotomic trace for symmetric monoidal categories*, Geometry and Topology: Aarhus (K. Grove, I.H. Madsen and E.K. Pedersen, eds.), Contemp. Math., vol. 258, AMS, 2000, pp. 121–144.

[DuMc] B.I. Dundas and R. McCarthy, *Topological Hochschild homology of ring functors and exact categories*, J. Pure Appl. Algebra **109** (1996), 231–294.

[Dw] W.G. Dwyer, *Twisted homological stability for general linear groups*, Ann. of Math. **111** (1980), 239–251.

[DwMi] W.G. Dwyer and S.A. Mitchell, *On the K-theory spectrum of a ring of algebraic integers*, K-Theory **14** (1998), 201–263.

[EKMM] A.D. Elmendorf, I. Kriz, M.A. Mandell and J.P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, AMS, 1997.

[Go] T.G. Goodwillie, *Relative algebraic K-theory and cyclic homology*, Ann. of Math. **124** (1986), 347–402.

[H] A.E. Hatcher, *Concordance spaces, higher simple-homotopy theory, and applications*, Algebr. geom. Topol., Stanford/Calif. 1976, Proc. Symp. Pure Math., vol. 32, Part 1, 1978, pp. 3–21.

[HM] L. Hesselholt and I. Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology **36** (1997), 29–101.

[Ig] K. Igusa, *The stability theorem for smooth pseudoisotopies*, K-Theory **2** (1988), 1–355.

[KS] R.C. Kirby and L.C. Siebenmann, *Foundational essays on topological manifolds, smoothings and triangulations*, Ann. of Math. Studies, vol. 88, Princeton University Press, 1977.

[K] K. Knapp, *Some applications of K-theory to framed bordism: e-invariant and transfer*, Habilitationsschrift, Bonn (1979).

[LMS] L.G. Lewis, Jr., J.P. May and M. Steinberger, *Equivariant stable homotopy theory*, Lect. Notes Math., vol. 1213, Springer-Verlag, 1986.

[Ly] M. Lydakis, *Smash products and Γ-spaces*, Math. Proc. Camb. Philos. Soc. **126** (1999), 311–328.

[Ma] I. Madsen, *Algebraic K-theory and traces*, Current developments in mathematics, 1995 (Cambridge, MA), Internat. Press, Cambridge, MA, 1995, pp. 191–321.

[MST] I. Madsen, V. Snaith and J. Tornehave, *Infinite loop maps in geometric topology*, Math. Proc. Camb. Phil. Soc. **81** (1977), 399–429.

[MQR] J.P. May, F. Quinn and N. Ray, with contributions by J. Tornehave, *E_∞ ring spaces and E_∞ ring spectra*, Lect. Notes Math., vol. 577, Springer-Verlag, 1977.

[Mc] R. McCarthy, *Relative algebraic K-theory and topological cyclic homology*, Acta Math. **179** (1997), 197–222.

[MP] H.R. Miller and S.B. Priddy, *On G and the stable Adams conjecture*, Geom. Appl. Homotopy Theory II, Proc. Conf., Evanston 1977, Lect. Notes Math., vol. 658, Springer-Verlag, 1978, pp. 331–348.

[MT] M. Mimura and H. Toda, *The $(n+20)$-th homotopy groups of n-spheres*, J. Math. Kyoto Univ. **3** (1963), 37–58.

[Mo] R.E. Mosher, *Some stable homotopy of complex projective space*, Topology **7** (1968), 179–193.

[Mu1] J. Mukai, *The S^1-transfer map and homotopy groups of suspended complex projective spaces*, Math. J. Okayama Univ. **24** (1982), 179–200.

[Mu2] J. Mukai, *On stable homotopy of the complex projective space*, Jap. J. Math. **19** (1993), 191–216.

[Mu3] J. Mukai, *The element $\bar\kappa$ is not in the image of the S^1-transfer*, Math. J. Okayama Univ. **36** (1994), 133–143.

[Q1] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. **96** (1972), 552–586.

[Q2] D. Quillen, *Higher algebraic K-theory. I*, Algebraic K-Theory I, Proc. Conf. Battelle Inst. 1972, Lecture Notes Math., vol. 341, Springer-Verlag, 1973, pp. 85–147.

[Ra] D.C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press, 1986.

[R1] J. Rognes, *The Hatcher–Waldhausen map is a spectrum map*, Math. Ann. **299** (1994), 529–549.

[R2] J. Rognes, after M. Bökstedt, *Trace maps from the algebraic K-theory of the integers*, J. Pure Appl. Algebra **125** (1998), 277–286.

[R3] J. Rognes, *The product on topological Hochschild homology of the integers with mod four coefficients*, J. Pure Appl. Algebra **134** (1999), 210–217.

[R4] J. Rognes, *Topological cyclic homology of the integers at two*, J. Pure Appl. Algebra **134** (1999), 219–286.

[R5] J. Rognes, *Algebraic K-theory of the two-adic integers*, J. Pure Appl. Algebra **134** (1999), 287–326.

[RW] J. Rognes and C. Weibel, *Two–primary algebraic K-theory of rings of integers in number fields*, J. Am. Math. Soc. **13** (2000), 1-54.

[Se] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

[S1] A.A. Suslin, *On the K-theory of algebraically closed fields*, Invent. Math. **73** (1983), 241–245.

[S2] A.A. Suslin, *Algebraic K-theory and motivic cohomology*, Proc. Intern. Congress Math., Zürich, 1994, Birkhäuser, 1995, pp. 342–351.

[To] H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies, vol. 49, Princeton University Press, 1962.

[Voe] V. Voevodsky, *The Milnor Conjecture*, preprint, K-Theory Archive, no. 170 (1996).

[Vog] W. Vogell, *The involution in the algebraic K-theory of spaces*, Algebraic and geometric topology, Proc. Conf., New Brunswick/USA 1983, Lect. Notes Math., vol. 1126, Springer-Verlag, 1985, pp. 277–317.

[Wa1] F. Waldhausen, *Algebraic K-theory of topological spaces. I*, Algebr. geom. Topol., Stanford/Calif. 1976, Proc. Symp. Pure Math., vol. 22, Part 1, 1978, pp. 35–60.

[Wa2] F. Waldhausen, *Algebraic K-theory of topological spaces, II*, Algebraic topology, Proc. Symp., Aarhus 1978, Lect. Notes Math., vol. 763, Springer-Verlag, 1979, pp. 356–394.

[Wa3] F. Waldhausen, *Algebraic K-theory of spaces, a manifold approach*, Current trends in algebraic topology, Semin. London/Ont. 1981, CMS Conf. Proc. 2, Part 1, 1982, pp. 141–184.

[Wa4] F. Waldhausen, *Algebraic K-theory of spaces, localization, and the chromatic filtration of stable homotopy*, Algebraic topology, Proc. Conf., Aarhus 1982, Lect. Notes Math., vol. 1051, Springer-Verlag, 1984, pp. 173–195.

[Wa5] F. Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology, Proc. Conf., New Brunswick/USA 1983, Lect. Notes Math., vol. 1126, Springer-Verlag, 1985, pp. 318–419.

[Wa6] F. Waldhausen, *Algebraic K-theory of spaces, concordance, and stable homotopy theory*, Algebraic topology and algebraic K-theory, Proc. Conf., Princeton, NJ (USA), Ann. of Math. Studies, vol. 113, 1987, pp. 392–417.

[Wa7] F. Waldhausen, *On the construction of the Kan loop group*, Doc. Math. **1** (1996), 121–126.

[We] C. Weibel, *The* 2*-torsion in the K-theory of the integers*, C. R. Acad. Sci. Paris **324** (1997), 615–620.

[WW1] M. Weiss and B. Williams, *Automorphisms of manifolds and algebraic K-theory. I*, K-Theory **1** (1988), 575–626.

[WW2] M. Weiss and B. Williams, *Assembly*, Novikov conjectures, Index theorems and Rigidity, Cambridge University Press, 1995, pp. 332–352.

[WW3] M. Weiss and B. Williams, *Automorphisms of manifolds*, Surveys on Surgery Theory: Volume 2 (S. Cappell, A. Ranicki and J. Rosenberg, eds.), Ann. of Math. Study, vol. ?, Princeton University Press, 2000, pp. ?–?.

Department of Mathematics, University of Oslo, Norway
E-mail address: `rognes@math.uio.no`