École polytechnique fédérale de Lausanne

Semester project spring 2024 Bachelor in Mathematics

Introduction to Algebraic K-theory

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Acknowledgments

First and foremost, I wish to thank my advisor Prof. John Rognes for agreeing to help me with this project. This would have been impossible without his help, from selecting topics and clearing up the bigger picture, to answering my various technical questions. I wish to express my gratitude for more than advising my project directly, but also for cultivating my curiosity and passion for mathematics with many interesting discussions ranging from motivating spectra to mentioning in passing "topological modular forms".

I want to thank every professor who helped me get to where I am, in particular Prof. Jérôme Scherer and Prof. Kathryn Hess Bellwald for giving me my first taste of category theory and topology, which are instrumental to this project.

I want to thank my family as much as I can. Most of all my parents, Divvya and Michael Vorburger, who were responsible with igniting my curiosity at a young age. I would not be able to understand anything I wrote in this project if they had not taught me how to count while in the supermarket shopping for oranges or if they had not drawn fractions in the snow to help me understand how to group complete $(\mathbb{Z}\backslash\{0\}, *)$. Their unending support is what got me through my studies so far, I would never have been able to learn as much as I did if tea cups did not keep magically appearing at my desk while I was studying. I want to thank my sister, Anaya Vorburger for helping me laugh through tough days, for reminding me that some mathematics is about numbers when she asks me to help her with her homework and for being the best friend I could ever ask for. I want to thank my grandparents, in particular my grandmother Geeta Lakshmi for every single prayer she sends my way and my grandfather Joseph Vorburger for an unending supply of stimulating conversations.

I want to thank all of my friends from before university: Olivia Burnand, Témi Messmer, Aurélien Marclay, Aurélie Fornerod, Axel Brulé, Iliane Qorchi, Rebecca Xenakis, Leo Bosch, etc. I would not be who I am without the uncountable many core memories on the beach of "Lac Léman" with them.

I want to thank my "study buddies" from EPFL: Benoît Cuenot, Alexandre Pons, Abdelhamid Boumahdi, Dana Vauthey, Hamza Drissi Kamili, etc. Joking around is the perfect break between two exercise sheets. I am still wondering why the cardinality of a fish might be a sponge.

I want to thank the friends I made during my exchange in Oslo who could be the most literally by my side while I was working on my Bachelor project: Ariane Parkinson, Yui Takayanagi, Hanna Bertrand, Anouk Jullière, Luna Yokokawa, Akash Palghadmal, Fabian Tangen, Akimi Sugiyama, Mathilde Hadjoudj, Yuna Koizumi, etc.

If I could thank everyone I would, but at some point this open letter of gratitude needs to start being about mathematics.

Intorudction

Algebraic K-theory is a field which uses ideas from Algebraic Topology to construct a series of invariants of categories, the K-groups, and aims to organize the study of these groups. In this project we will not concern ourselves with application of Algebraic K-theory, but rather are motivated by the internal beauty of the subject.

There are several areas of mathematics which we will treat as acquired, summarized as follows. As Algebraic K-theory studies categories, comfort with this language will be assumed at least up to (co)limits, adjunctions and how these concepts interact (see approximately the first four chapters of [1] or of [2]). The main tools we will use to construct and study K-theory will either come from or be very inspired by ideas in Algebraic Topology, as such we treat any material contained in Hatcher's classic book ([3]) on the subject as acquired. The second tool we will use to study categories is the language of simplicial objects, as their combinatorial nature make them a natural midway point between categories and spaces when attempting to study categories via Algebraic Topology (see sections 8.1 through 8.3 of [4] or the first chapter of [5]). We will also assume that the reader is familiar with the basics of homological algebra (for example the content of the first three chapters of [4]), as this general framework will make it much easier to draw inspiration from Algebraic Topology. A good place to read about translating between categories and topological/simplicial spaces is section §3 of chapter IV of the K-book ([6]). We will use notions and notation from this chapter without further comment.
Finally, there are three subjects for which we do not assume familiarity, but will need, so the reader should be ready to accept results from these areas and should not fear the basic language coming from these subjects. These are: stable homotopy theory (see [7]), homotopical algebra (see [8] or [9] for a more in depth look) and spectral sequences (see chapter 5 of [4]).

The goal of this project is to cover a selection of topics from chapters IV and V of Weibel's K-book ([6]), with the end goal of relating the K-theories of some of the most basic rings (though not every result we include is used in proving this statement).

In section §1, we define the space $BGL(R)^+$ which is the key step of our first model for the K-theory space of a ring. In subsection §1.1, we define acyclic maps and give a useful recognition criterion. In §1.2, we study the +-construction in general, we prove the following existence theorem which also provides a universal property.

Theorem 0.0.1. *(Theorem 5.1 and 5.2 [10]) Let X be a path connected topological space and let P be a perfect normal subgroup of $\pi_1(X)$. Then*
(i) There exists $f : X \to Y$, a +-construction relative to P.
(ii) Let $f : X \to Y$ be a +-construction relative to P which in addition is a cofibration. If $g : X \to Z$ is any map such that $P \leq ker(\pi_1(g))$, then there is a unique up to homotopy map $h : Y \to Z$ such that $g = hf$.

We are then equipped to define the K-theory space of a ring, and we provide a computation of $K_0(R), K_1(R)$ and $K_2(R)$. We end section §1 with a technical subsection which contains key results to prove that $BGL(R)^+$ is an H-space, though we defer the actual proof to subsection §2.3.

In section §2, we work towards defining a K-theory space for all symmetric monoidal categories. We start by the topological analog of the group completion functor, which we also call group completion. We do this in subsection §2.1, where after defining the relevant concepts, we prove the following universal property of group completions of H-spaces.

Proposition 0.0.2. *(Proposition 1.2 in [11]) Let X be a CW H-space such that $\pi_0(X)$ has a countable cofinal sequence. Let $g : X \to Y$ be a group completion. Then if $f : X \to Z$ is any weak H-map into a group-like H-space there exists a weak H-map $f' : Y \to Z$ unique up to weak homotopy such that $f'g$ is weakly homotopic to f.*

We then specialize in section §2.2, to the H-space structure on the geometric realization of a symmetric monoidal category, and construct an auxiliary category whose geometric realization is going to be the group completion of the classifying space of the original symmetric monoidal category. The key element to prove this fact is the construction of a map

$$\pi_0(S)^{-1}H_q(X) \to H_q(S^{-1}X)$$

which we show is an isomorphism in the following theorem.

Theorem 0.0.3. *(Theorem 4.8 in [6] chapter IV), (page 221 of [12])(Theorem 7.2 of [13]) If every map in S is an isomorphism and translations are faithful (i.e. $\forall s, t \in S, Aut(t) \to Aut(s\Box t)$ is an injection), then the above map is an isomorphism.*

This is one of the more theoretically demanding results in this project, in particular as it requires constructing a spectral sequence. This result allows us to understand the homology of the base point component of the K-theory space of a symmetric monoidal category which in turn allows us to compute $K_1(S)$ for a symmetric monoidal category S. We then show in §2.3 that taking S to be the category of finitely generated projective R-modules, the group completion construction yields a K-theory space homotopy equivalent to the model constructed in subsection §1.3. The set-up of the proof aims to show that we can use the results from subsection §1.4 to prove the equivalence of models. The goal of this subsection (§2.3) is accomplished by the following result.

Theorem 0.0.4. *(Theorem 4.10 in [6] chapter IV) Let S be a symmetric monoidal category such that $S = iso(S)$, with faithful translations and having a cofinal subsequence of the form $\{s^n\}_{n=1}^{\infty}$ for some $s \in S$. Then*

$$K(S)^{\Box} \simeq K_0(S) \times BAut(S)^+,$$

with the plus construction relative to $E = [Aut(S), Aut(S)]$.

We can use this comparison to give a description of K_2 of a symmetric monoidal category.

In section §3, we define the K-theory space in general for quasi-exact categories. Subsection §3.1 is simply the necessary build up defining the Q-construction and thus the K-theory of quasi-exact categories. We also compute the 0th K-group of quasi-exact categories. We then prove a handful of elementary properties in section §3.2. As quasi-exact categories are naturally symmetric monoidal categories, we then prove that under appropriate conditions, the Q-construction and group completion construction provide homotopy equivalent K-theory spaces. This is done in subsection §3.3 and is accomplished by the following theorem.

Theorem 0.0.5. *(Theorem 7.1 of [6] chapter IV) Let $\mathcal{A}$ be a split quasi-exact category, and let $S = iso(\mathcal{A})$ be seen as a symmetric monoidal category. Then*

$$\Omega BQ\mathcal{A} \simeq BS^{-1}S.$$

The proof of this statement is the longest one of this bachelor project, and we will spend the rest of this section working on it.

In section §4, we define the final and most general K-theory space that we will deal with in this project: the $S.$ construction for Waldhausen categories. These are categories equipped with a notion of weak equivalence and with a notion of cofibration. This section requires a lot of set up, which we do in subsection §4.1. Before moving on to the next subsection, we will also define different properties a Waldhausen category might have, which will be relevant when we start proving theorems in section §5 that do not apply in full generality. In §4.2, we prove our final theorem which compares two different models for K-theory spaces. We show that the quasi-exact categories can naturally be seen as Waldhausen categories, and that applying the $S.$ construction or the Q-construction yields homotopy equivalent spaces. This is the content of the following theorem.

Theorem 0.0.6. *(Exercise 8.5, 8.6 in chapter IV of [6])(section 1.9 of [14]) Let $\mathcal{A}$ be a quasi-exact category, we have a homotopy equivalence*

$$BQA \simeq |iS.\mathcal{A}|.$$

The notation iS. serves to indicate that we are taking the weak equivalences to be the isomorphisms.

This concludes, what could philosophically be considered, the first half of the project dealing with defining and equating different constructions for K-theory. We then go on to prove the additivity theorem in §4.3, a fundamental result which helps in understanding what K-theory does to functors. This is accomplished by the following result.

Theorem 0.0.7. *(Theorem from [15]) Let $\mathcal{C}$ be a Waldhausen category and consider the exact functor $F : \mathcal{EC} \to \mathcal{C} \times \mathcal{C}$ which sends $A \rightarrowtail B \twoheadrightarrow C$ to (A, C). Then $S.F$ is a homotopy equivalence.*

In this section, we state a couple corollaries of this result which have also been called the additivity theorem. It is in the following subsection §4.4, that we start using the language of spectra. First, we prove that the K-theory space has the structure of an infinite loop space, and thus the functor sending a Waldhausen category to its K-theory space can naturally be made to land in the category of spectra, which has the added advantage of being stable. We accomplish this with the following result, which will be a key ingredient in multiple proofs of the next section.

Proposition 0.0.8. *(Proposition 1.5.5 in [14]) The sequence $wS.\mathcal{D} \to wS.S.F \to wS.S.\mathcal{C}$ is a homotopy fibration.*

Second, we also sketch how a bi-exact functor descends to a pairing in K-theory.

In section §5, we collect some important theorems in K-theory, the selection is motivated mainly by the results that are required to prove that we have a homotopy fibration that relates the K-theory of: the fields of prime order, the integers and the rationals. We start by proving Waldhausen localization in §5.1.

Theorem 0.0.9. *(2.1 in [6] chapter V) Let $\mathcal{A}$ be a category with cofibrations made into a Waldhausen category with two different classes $v(\mathcal{A}) \subset w(\mathcal{A})$ of weak equivalences. Suppose that $(\mathcal{A}, w)$ is: saturated, satisfies the cylinder axiom and the extension axiom. Denote by $\mathcal{A}^w$ the Waldhausen subcategory of $(\mathcal{A}, v)$ of objects such that $0 \to A$ is a w-weak equivalence. Then we have a homotopy fibration*

$$K(\mathcal{A}^w) \to K(\mathcal{A}, v) \to K(\mathcal{A}, w).$$

This result allows us to change the class of weak equivalences we are working with when studying the K-theory of Waldhausen categories. The hypotheses of this result are in general not satisfied by the crucial example of the category of finitely generated projective R-modules. The following result, called the Gillet-Waldhausen theorem fixes this issue because categories of bounded chain complexes satisfies the required hypotheses.

Proposition 0.0.10. *(2.2 in [6] chapter V)(1.11.7 of [16]) Let $\mathcal{C}$ be an exact category in some ambient abelian category $\mathcal{A}$. Suppose further, that whenever $f \in mor(\mathcal{C})$ is a surjection in $\mathcal{A}$ then $ker(f)$ is in $\mathcal{C}$. By considering complexes concentrated in degree 0 we get an exact inclusion $\mathcal{C} \to Ch^b(\mathcal{C})$ which induces a homotopy equivalence on K-theory spaces.*

To prove this result we require some basic familiarity with the language of spectra. We then prove the approximation theorem in §5.2 which is the key result used to prove that a functor which is not necessarily an inclusion is a homotopy equivalence.

Theorem 0.0.11. *(Theorem 2.4 in [6] chapter V)(Theorem 10 in [17]) Suppose $F : \mathcal{C} \to \mathcal{D}$ is an exact functor of saturated Waldhausen categories which satisfies the following requirements.*

(i) A morphism in $\mathcal{C}$ is a weak equivalence if and only if its image is a weak equivalence.

(ii) Every morphism in $\mathcal{C}$ can be factored as a cofibration followed by a weak equivalence (this is, in particular, true if $\mathcal{C}$ has a cylinder functor satisfying the cylinder axiom).

(iii) F satisfies the approximate lifting property which states that for every map $\beta : F(C) \to D$ there is a cofibration $\alpha : C \to C'$ and a weak equivalence $\beta' : F(C') \to D$ such that $\beta' \circ F(\alpha) = \beta$.

Then, the induced map on K-theory spaces $F : K(\mathcal{C}) \to K(\mathcal{D})$ is a homotopy equivalence.

We then prove two results which apply only in the case where F is an inclusion of a subcategory. The first of the two is the resolution theorem in §5.3.

Theorem 0.0.12. *(Theorem 3.1 in [18]) Suppose $\mathcal{A}$ is a full exact subcategory of an exact category $\mathcal{B}$ such that: a sequence of three objects in $\mathcal{A}$ which is exact in $\mathcal{B}$ is exact in $\mathcal{A}$; $\mathcal{A}$ is closed under extension and cokernels in $\mathcal{B}$. Assume further that every object B in $\mathcal{B}$ has a resolution, i.e. an exact sequence, $0 \to B \to A \to A' \to 0$ with $A, A' \in \mathcal{A}$.*
Then the inclusion $\mathcal{A} \to \mathcal{B}$ induces a homotopy equivalence $K(\mathcal{A}) \to K(\mathcal{B})$ on K-theory spaces.

We then prove the devissage theorem in §5.4.

Theorem 0.0.13. *(Theorem 4.1 in [18]) Let $\mathcal{A} \subset \mathcal{B}$ be an inclusion of abelian categories such that $\mathcal{A}$ is closed under direct sum, subobject and quotient in $\mathcal{B}$. Suppose every object in $\mathcal{B}$ has a finite filtration by monics such that all the filtration quotients are in $\mathcal{A}$. Then the inclusion $\mathcal{A} \to \mathcal{B}$ induces a homotopy equivalence $K(\mathcal{A}) \to K(\mathcal{B})$ of K-theory spaces.*

As promised, the final section §6 of this project relates the K-theory of some of the most fundamental rings. This is done by the following result.

Theorem 0.0.14. *Denote by $K(R)$ the K-theory space of the category of finitely generated projective R-modules with weak equivalences being the isomorphisms and the cofibration being monics. Then we have a homotopy fibration*

$$\bigvee_p K(\mathbb{F}_p) \to K(\mathbb{Z}) \to K(\mathbb{Q}).$$

The wedge is taken over all primes.

Conventions

We adopt a couple conventions for convenience and ease of comprehension.

For spaces, we assume all spaces to be nice enough so that the connected components have a universal cover. For a space X we denote its universal cover by X^u. We call a space simple if π_1 is abelian and the natural actions on all higher homotopy groups are each trivial, in particular H-spaces are simple. All the relevant spaces will be of the homotopy type of CW-complexes, thus even when this is not explicitly stated we make this assumption. Unless necessary for clarity, we will not be bothering ourselves with explicitly mentioning base points, for instance when computing homotopy groups.

Encouraged by the appendix of [19], the reader is free to assume that whenever we mention spectra, we are working with the category of symmetric spectra.

Rings are assumed to be associative with unit, but may fail to be commutative.

The notation $[n]$ refers to the finite set $\{1, ..., n\}$.

There are several constructions for categories, notably geometric realization, for which we need our categories to be small, we will not repeat this assumption throughout the text. Any category with a 0 object is assumed to be pointed, i.e. there is a distinguished 0 object and functors between categories with 0 objects preserve the distinguished choice of such an object.

This project contains a lot of diagrams, it will be clear from context (and clarified if not) whether we are considering commutative, homotopy commutative or general diagrams. We do not name every morphism in a diagram to avoid clutter. We adopt the philosophy that an oversaturated diagram contains more information than no diagram at all, thus we have allowed ourselves some rather large or intricate diagrams. What we mean by the decorated arrows $\hookrightarrow$, $\twoheadrightarrow$, $\xrightarrow{\simeq}$ will be clear from context, and will match the intuition that $\hookrightarrow$ means a morphism which is "like a monic", $\twoheadrightarrow$ "like an epic" and $\xrightarrow{\simeq}$ "like an isomorphism".

When a detail in a proof eluded me, I have decided to admit this in a remark at the end of the proof. In particular, during a proof, we will allow ourselves to overlook any unclear steps. We make this choice to not interrupt the flow while reading a proof.

Contents

1 The +-construction for K-theory of rings

In this section, we will give the necessary definitions to define the K-theory of a ring using the +-construction. First, we recall what an acyclic space is. Then we define what an acyclic map is, as we will need this to define what a +-construction is.

1.1 Acyclic maps

Definition 1.1.1. (Definition 1.3 and 1.4 in [6] chapter IV) We call a topological space X acyclic if it has the homology of a point. And we call a map $f : X \to Y$ of path-connected topological spaces acyclic if the homotopy fiber of the map is acyclic.

The goal of a +-construction is to simplify the fundamental group of a space by quotienting out by a perfect subgroup, while keeping the homology of the space unaffected. Studying the homotopy long exact sequence for the homotopy fibration $F_f \to X \xrightarrow{f} Y$ shows that if f is acyclic we have $\pi_1(Y) \cong \pi_1(X)/P$ (P is a normal subgroup) with $P \cong \pi_1(F_f)$ a perfect group because $\pi_1(F)^{ab} = H_1(F_f) = 0$. The following result (which we will call "acyclic recognition") shows that acyclic maps induce isomorphisms on homology.

Proposition 1.1.2. *(lemma 1.6 in [6] chapter IV) A map $f : X \to Y$ between path connected spaces is acyclic if and only if $f : H_*(X; M) \to H_*(Y; M)$ is an isomorphism for every $\pi_1(Y)$-module M.*

Proof. ($\Rightarrow$) The map $f : X \to Y$ being acyclic means that the homotopy fibration $F_f \to X \to Y$ has an "easy" to understand homology. Homology of a fibration is best understood using the Serre spectral sequence, which we can use as the necessary path connectedness is satisfied by assumption. Notice that $\pi_1(F_f) \to \pi_1(Y)$ is trivial, and so if M is a $\pi_1(Y)$-module, the $\pi_1(F_f)$ structure on M is trivial. This means that we can compute $H_*(F_f; M)$ using the universal coefficient theorem for homology. Namely, $H_i(F_f; M) = M$ if $i = 0$ and is 0 otherwise. Recall that the E^2 page of the Serre spectral sequence is

$$E^2_{pq} = H_p(Y; \mathcal{H}_q(F_f; M)).$$

And so, we see that the spectral sequence collapses on the E^2 page, because $E^2_{pq} = 0$ for $q \neq 0$. The sequence converges to $H_{p+q}(X; M)$, and clearly this implies that $H_p(Y; M) \cong H_q(X; M)$. In order to promote this abstract isomorphism to the desired result, we use that the edge homomorphism $H_p(X; M) \to E^\infty_{p,0} \subset E^2_{p,0} = H_p(Y; M)$ is the map induced by f.

($\Leftarrow$) We cannot deal with the general case immediately, so we first treat the case that Y is simply connected, and we will then deduce the full result by passing to universal covers. We also note that for the twisted coefficients of $H_q(X; M)$ the $\pi_1(X)$-module structure on M is induced by the $\pi_1(Y)$-module structure and the map $\pi_1(f)$.

For the simply connected case, we need the so-called comparison theorem for the Serre spectral sequence (see theorem A.0.1 to recall the statement). We then apply it to the map of fibrations

$$\begin{tikzcd} F_f \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, "Id_Y"] \\ * \arrow[r] & Y \arrow[r, "Id_Y"] & Y \end{tikzcd}.$$

If we take homology with untwisted $\mathbb{Z}$ coefficients; then the rightmost map obviously induces an isomorphism on homology and the middle one is an isomorphism by assumption. Thus, using the comparison theorem F_f has the homology of a point; in other words f is acyclic.

Now for the general case, let $f : X \to Y$ be a map that induces an isomorphism on homology with arbitrary twisted coefficients. We can consider the universal cover of Y, call it Y^u; then in order to translate to the above case we take the following pullback

$$\begin{tikzcd} X' \cong X \times_Y Y^u \arrow[r, "f'"] \arrow[d] & Y^u \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}.$$

We claim that $f' : X' \to Y^u$ is an isomorphism on homology with $\mathbb{Z}$ coefficients, which by the proof in the simply connected case will show that f' is acyclic. Consider the map $f'_* : H_q(X';\mathbb{Z}) \to H(Y^u;\mathbb{Z})$, it turns out that the untwisted homology of covering spaces can be realized as some twisted homology of the base space, as can be seen in [3] example 3H.2. In order to apply this idea, we need to understand to which subgroup the cover $X' \to X$ corresponds to (it is obviously a covering space by basic properties of pullback bundles). A prerequisite to understanding to which group X' corresponds to; we need to prove that it is connected, as it is only in that case that it corresponds to a subgroup. For this we use the equivalence of categories $Cov(X) \cong Set^{\pi_1(X)}$, with connected covers corresponding to $\pi_1(X)$-sets with a single orbit. The $\pi_1(X)$-set corresponding to X' is $\pi_1(Y)$ by properties of pullback bundles and the action is the one induced by the map $\pi_1(f)$. We can study this $\pi_1(X)$-set by using the isomorphism $f_* : H_0(X;\mathbb{Z}[\pi_1(Y)]) \to H_0(Y;\mathbb{Z}[\pi_1(Y)])$, which yields an isomorphism $\mathbb{Z}[\pi_1(Y)]_{\pi_1(X)} \cong \mathbb{Z}[\pi_1(Y)]_{\pi_1(Y)} \cong \mathbb{Z}$ and this implies that there is a single $\pi_1(X)$-orbit in $\pi_1(Y)$.
We will show that P, the subgroup corresponding to $X' \to X$ corresponds to $ker(\pi_1(f)) \leq \pi_1(X)$. Clearly $P \leq ker(f)$, which can be seen by applying π_1 to the above pullback. For the reverse inclusion, let $\gamma : S^1 \to X$ represent a loop in the kernel of $\pi_1(f)$. Because $f \circ \gamma : S^1 \to Y$ is trivial in $\pi_1(Y)$ we can lift it to Y^u, then we can use the universal property of the pullback to get a map to X' which proves the reverse inclusion. The above discussion is summarized in the following commutative diagram

$$\begin{tikzcd}
S^1 \arrow[dr, dotted, "\exists!"] \arrow[drr, dashed, bend left, "\exists(f\circ\gamma)^u"] \arrow[ddr, bend right, "\gamma"'] & & \\
& X' \arrow[r, "f'"] \arrow[d] & Y^u \arrow[d] \\
& X \arrow[r, "f"] & Y
\end{tikzcd} .$$

Now that we know to what group the cover corresponds to, we can use [3] 3H.2 to get $H(X';\mathbb{Z}) \cong H(X;\mathbb{Z}[\pi_1(X)/ker(f)]$ and $H(Y^u;\mathbb{Z}) \cong H(Y;\mathbb{Z}[\pi_1(Y)])$. So now to show $f'_* : H_*(X';\mathbb{Z}) \to H_*(Y^u;\mathbb{Z})$ is an isomorphism, we can consider the following diagram

$$\begin{tikzcd}
H_q(X';\mathbb{Z}) \arrow[r, "f'"] \arrow[d, no head, "\cong"] & H_q(Y^u;\mathbb{Z}) \arrow[d, no head, "\cong"] \\
H_q(X;\mathbb{Z}[\pi_1(X)/ker(f)]) \arrow[r, "\phi"] & H_q(Y;\mathbb{Z}[\pi_1(Y)])
\end{tikzcd} ;$$

where the bottom map is just defined as the natural composition of the other three maps. One can observe that this diagram comes from the following diagram of chain complexes after applying the homology functor

$$\begin{tikzcd}
C_*(X';\mathbb{Z}) \arrow[r, "f'"] \arrow[d, no head, "\cong"] & C_*(Y^u;\mathbb{Z}) \arrow[d, no head, "\cong"] \\
C_*(X^u;\mathbb{Z}) \otimes_{\pi_1(X)} \mathbb{Z}[\pi_1(X)/ker(f)] \arrow[r, "f^u"] & C_*(Y^u;\mathbb{Z}) \otimes_{\pi_1(Y)} \mathbb{Z}[\pi_1(Y)]
\end{tikzcd} ,$$

where f^u is the lift of $f \circ p : X^u \to Y$ to the universal cover $Y^u \to Y$ of Y and $p : X^u \to X$ is the universal cover of X. We claim $\mathbb{Z}[\pi_1(X)/ker(f)] \cong \mathbb{Z}[\pi_1(Y)]$ as $\pi_1(X)$ modules. If $\pi_1(f)$ is surjective, then this follows at once from the first isomorphism theorem. Because the homotopy fiber of f having the homology of a point, it is path connected, and so by studying the homotopy long exact sequence of $F_f \to X \to Y$ we get the desired surjectivity. And so $\phi : H_q(X;\mathbb{Z}[\pi_1(X)/ker(f)]) \to H_q(Y;\mathbb{Z}[\pi_1(Y)])$ is an isomorphism by assumption, as once we replace the domain using the isomorphism just discussed it is, in fact, the map $f : H_q(X;\mathbb{Z}[\pi_1(Y)]) \to H_q(Y;\mathbb{Z}[\pi_1(Y)])$.
Therefor, by the simply connected case f' is acyclic, all that remains to show is that this implies that f is acyclic as well. We do this by showing that the spaces $F_{f'}$ and F_f are homeomorphic, which we do by showing that $F_{f'}$ satisfies the universal property of F_f. For this we need to make the base points explicit for the three spaces X, Y and Y^u; x_0, y_0 and u_0 respectively. And we also will no

longer write X' for the space $X \times_Y Y^u$, as it will be easier if its pullback property is clear. We also explicitly mention the fact that our pathspaces are pointed, for example the space Y^I is the space of maps $I \to Y$ sending 0 to y_0.
To see that $F_{f'}$ satisfies the desired universal property, one needs to understand the following commutative diagram, which warrants some explaining.

$$\begin{tikzcd}
 & Z \arrow[dddl, bend right, "\alpha"'] \arrow[ddl, dashed, "(i)\exists! g"'] \arrow[dr, dashed, "(ii)\exists! h"] \arrow[ddrr, bend left, "\beta"] \arrow[drr, bend left, "l\circ\beta"] & & \\
 & & F_{f'} \arrow[r] \arrow[dll] & (Y^u)^I \arrow[dll, "ev_1"'] \arrow[d, shift right, "p_*"'] \arrow[loop right, "Id"] \\
X \times_Y Y^u \arrow[r, "f'"] \arrow[d] & Y^u \arrow[d, "p"] & F_f \arrow[r] \arrow[dll] & Y^I \arrow[u, shift right, "l"'] \arrow[dll, "ev_1"] \arrow[loop right, "Id"] \\
X \arrow[r, "f"] & Y & &
\end{tikzcd} .$$

The first point which needs clarifying are the maps $p_* : (Y^u)^I \to Y^I$ and $l : Y^I \to (Y^u)^I$, the former is post-composition by $p : Y^u \to Y$, and the second is the map which sends a path $\gamma : (I, 0) \to (Y, y_0)$ to the unique lift starting at u_0. These maps are obviously mutually inverse, continuity follows easily from the continuity of p, the fact that p is open and the definition of the compact open topology. To see that for all $\beta : Z \to Y^I, \alpha : Z \to X$ there is a unique pullback $Z \to F_{f'}$ (which is what we want to show) we proceed as follows:
First, for existence, we can pull back $\alpha : Z \to X$ and $ev_1 \circ l \circ \beta$ to define the map $g : Z \to X \times_Y Y^u$. Now pull back this g and $l \circ \beta$ to define h. For uniqueness suppose $\tilde{h}$ is another lift, then because p, l are mutually inverse, the composition $Z \xrightarrow{\tilde{h}} F_{f'} \to (Y^u)^I$ must be the map $l \circ \beta$. Similarly, the composition $Z \xrightarrow{\tilde{h}} F_{f'} \to X \times_Y Y^u$ must be the pullback g and so $\tilde{h}$ must be the pullback of g and $l \circ \beta$ which must be h. Which concludes the proof $F_f \cong F_{f'}$, and therefor the entire proof as well. $\square$

1.2 The general +-construction

With the previous section completed, we can see that the following definition accomplishes what we wanted a +-construction to do.

Definition 1.2.1. (Definition 1.4.1 in [6] chapter IV) Let X be a connected topological space with fundamental group G and let P be a perfect normal subgroup of G. A +-construction relative to P is an acyclic map $f : X \to Y$ such that $ker(\pi_1(f)) = P$.

Often we will not specify with respect to which subgroup we are performing a +-construction, in which case we are implicitly performing a +-construction with respect to the largest perfect subgroup of the fundamental group, called the perfect radical. This is justified by the following result.

Proposition 1.2.2. *(Exercise 1.5 in [6] chapter IV) Let G be a group, then the union of all perfect subgroups is a perfect normal subgroup.*

Proof. Let H, F be two perfect subgroups of G, and let $P = \langle H, F \rangle$ be the subgroup generated by all elements of H and F, then because H and F are perfect, we have $H \leq [P, P]$ and $F \leq [P, P]$, and so $[P, P]$ containing every generator of P must contain P, so by double inclusion we have that P is perfect. This implies that the union of all perfect subgroups is a perfect subgroup. Indeed, denoting by R the union of all perfect subgroups, and letting h_1, h_2 be elements of R we have perfect subgroups H_1, H_2 containing h_1, h_2 respectively. Thus, by the above reasoning, we also have that $\langle H_1, H_2 \rangle \leq R$ and so both the product and the commutator of h_1 and h_2 are in R. Thus, we have that

$$\bigcup_{i \in I} H_i = \langle H_i \rangle_{i \in I},$$

where the indices run over all perfect subgroups.
We proceed to demonstrate that R is normal. For this, notice that if we let $g \in G$, then conjugation by g is an isomorphism from P to gPg^{-1}, being perfect is an isomorphism invariant, so the conjugates of a perfect subgroup are perfect. This implies that the union of all perfect subgroups must be normal. $\square$

We have established now that any acyclic map is a +-construction for some perfect subgroup of the fundamental group. It turns out this construction can be performed for any perfect subgroup; this is the content of the following theorem due to Quillen, for which we follow the book by Berrick [10].

Theorem 1.2.3. *(Theorem 5.1 and 5.2 [10]) Let X be a path connected topological space and let P be a perfect normal subgroup of $\pi_1(X)$. Then*
(i) There exists $f : X \to Y$, a +-construction relative to P.
(ii) Let $f : X \to Y$ be a +-construction relative to P which in addition is a cofibration. If $g : X \to Z$ is any map such that $P \leq ker(\pi_1(g))$, then there is a unique up to homotopy map $h : Y \to Z$ such that $g = hf$.

The second result contained in this theorem immediately implies that a +-construction is unique up to homotopy equivalence under X.

Proof. (i) We fix a space X and a perfect normal subgroup P of $\pi_1(X)$. For the remainder of this proof, when we say +-construction, we implicitly mean relative to P. There are several ways to perform the +-construction; we present a rather straightforward method from [10] 5.1. The general case will be deduced from the case $P = \pi_1(X)$ by passing to the appropriate covering space. For now, assume $P = \pi_1(X)$.
We can use the usual method of eliminating the fundamental group by attaching two cells to a representative of each element of $\pi_1(X)$, constructing a simply connected space W, of which X is a subspace. We can summarize this by the following pushout

$$\begin{tikzcd} \vee_{\pi_1(X)} S^1 \arrow[r] \arrow[d] & X \arrow[d] \\ \vee_{\pi_1(X)} D^2 \arrow[r] & W \end{tikzcd}.$$

We can, without loss of generality, assume the top map to be a cofibration, which allows us to use excision to see $H_q(W, X; M) \cong H_q(\vee_{\pi_1(X)} D^2, \vee_{\pi_1(X)} S^1; M) \cong \tilde{H}_q(\vee_{\pi_1(X)} S^2; M)$. So we see that $\tilde{H}_q(W; M) \cong \tilde{H}_q(X; M) \oplus \tilde{H}_q(\vee_{\pi_1(X)} S^2; M)$, with M some $\pi_1(W)$ module. So we can easily attach 3 cells in such a way to kill the homology added by the 2 cells, summarized by the following diagram

$$\begin{tikzcd} \vee_{\pi_1(X)} S^2 \arrow[r] \arrow[d] & W \arrow[d] \\ \vee_{\pi_1(X)} D^3 \arrow[r] & Y \end{tikzcd}.$$

We claim that Y is the desired space. We work with homology with twisted coefficients N some $\pi_1(Y)$ module, which we will not specify explicitly. To show it is the desired space, note that by the homology exact sequence for (Y, X) and the acyclic recognition lemma, it suffices to show that $H_q(Y, X) = 0$ for all q. We will show this by studying the map of triples $(D^3, S^2, *) \to (Y, W, X)$ defined by the above diagram. On homology, this gives a map between long exact sequences of triples

$$\begin{tikzcd} {} \arrow[r] & H_{q+1}(Y, W) \arrow[r] \arrow[d] & H_q(Y, X) \arrow[r] \arrow[d] & H_q(W, X) \arrow[r] \arrow[d] & {} \\ {} \arrow[r] & H_{q+1}(\vee_{\pi_1(X)} D^3, \vee_{\pi_1(X)} S^2) \arrow[r] & H_q(\vee_{\pi_1(X)} D^3, *) \arrow[r] & H_q(\vee_{\pi_1(X)} S^2, *) \arrow[r] & {} \end{tikzcd}.$$

We can use excision on the two outer vertical maps followed by the 5 lemma to deduce the desired result $H_q(Y, X) = 0$.

So we know we can perform +-constructions when $P = \pi_1(X)$, now let $P \leq \pi_1(X)$ be any perfect normal subgroup. Let $X' \to X$ be the covering space associated to P, and Y' the +-construction relative to $\pi_1(X')$, which exists by the above reasoning. We can also assume without loss of generality that the map $X' \to Y'$ is a cofibration. We construct Y by a pushout

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} .$$

Now, because we turned the top map into a cofibration, so is the bottom map, so we can use excision to get $H_q(Y', X') \cong H_q(Y, X)$, but $X' \to Y'$ being acyclic means that $H_q(Y', X') = 0$, and so the same holds for $H_q(Y, X)$. So we know that $X \to Y$ is a +-construction, we just need it to be a +-construction relative to the subgroup P. Recall that by the Seifert-Van Kampen theorem, $\pi_1(-)$ preserves pushouts, in our case this result turns the above diagram into the following pushout

$$\begin{tikzcd} P \arrow[r] \arrow[d] & 1 \arrow[d] \\ \pi_1(X) \arrow[r] & \pi_1(Y) \end{tikzcd} ,$$

and so $\pi_1(Y) \cong \pi_1(X)/P$, which is the desired result.

(ii) Let $f : X \to Y$ be a cofibration which is a +-construction relative to $P \leq \pi_1(X)$ and $g : X \to Z$ be such that $\pi_1(g)(P) = 0$. Our goal is to construct a map $h : Y \to Z$ such that $h \circ f = g$. For this we shall need the following lemma

Lemma 1.2.4. *(See the introduction to chapter 5 of [10]) Suppose we have a triangle*

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ A \arrow[rr, "\gamma"'] & & B \end{tikzcd} ,$$

with α, β cofibrations, which commutes up to homotopy, then it can be made to commute strictly by replacing γ with γ' which is a homotopic map.

Proof. Denote by H_t the homotopy from $H_0 = \gamma \circ \alpha$ to $H_1 = \beta$, then simply applying the homotopy lifting property to the following diagram

$$\begin{tikzcd} X \arrow[r, "H_t"] \arrow[d, "\alpha"'] & B \arrow[d, "Id_B"] \\ A \arrow[r, "\gamma=\gamma_0"] & B \end{tikzcd} .$$

This yields a homotopy γ_t from $\gamma = \gamma_0$ to a map $\gamma_1 = \gamma'$ making the triangle commute strictly. □

That being said we may return to our situation. In order to use the lemma, we factor g as $X \to M_g \to Z$ with the first map a cofibration and the second a homotopy equivalence. This concludes the preliminary work. Ideally f would be invertible up to homotopy. We can fix this if we can make $\pi_1(f)$ injective. Indeed, suppose $\pi_1(f)$ was injective, we know it is surjective by studying the long exact sequence for $F_f \to X \to Y$ using that F_f is path connected. So f would then be a homology (with local coefficients) isomorphism which is an isomorphism on π_1, which is enough to show that f is a homotopy equivalence (using that, then the induced map on universal covers is a homology isomorphism and exercise 4.2.12 [3]). We obviously will not be able to magically make f a homotopy equivalence, but this is enough to motivate the proof idea. Indeed, consider the following pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\iota"'] & Y \arrow[d, "\iota'"'] \\
M_g \arrow[r, "f'"] & M_g \cup_X Y
\end{tikzcd},$$

because π_1 preserves pushouts, and ι is up to homotopy the same as g, we can see that $\pi_1(f')$ is injective. We can also, by excision, see that $H_n(M_g \cup_X Y, M_g; M) \cong H_n(Y, X; M)$, which by acyclic recognition is enough to show that f' is an acyclic, thus by the above reasoning, f' is a homotopy equivalence. Call e' a representative of the unique homotopy class inverse to the class of f'. Now consider the following diagram, with the left triangle strictly commutative and the right triangle only up to homotopy

$$\begin{tikzcd}
 & X \arrow[d, "f"] \arrow[ddl, "\iota"'] \arrow[ddr, "\iota"] & \\
 & Y \arrow[d, "\iota'"] & \\
M_g \arrow[r, "f'"'] & M_g \cup_X Y \arrow[r, "e'"'] & M_g
\end{tikzcd}.$$

A pushout of a cofibration is a cofibration, so $\iota' \circ f$ and ι are both cofibration, so by the lemma we can make the diagram strictly commutative by replacing e' by some e. We can define $h = e \circ \iota'$ and we see this h is as desired after replacing M_g with Z again. We can see that any $\tilde{h}$ which is as desired must be constructed similarly using the universal property of $M_g \cup_X Y$ and the fact that we have a homotopy equivalence $w : Z \to M_g$ (call its inverse v). To see this, consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\iota"] & Y \arrow[d, "f'"] \arrow[ddrr, bend left, "\tilde{h}"] & & \\
M_g \arrow[r, "\iota'"'] \arrow[drr, bend right, "Id"'] & M_g \cup_X Y \arrow[dr, dashed, "\exists ! \tilde{e}"] & & \\
 & & M_g \arrow[r, bend left, "v"] & Z \arrow[l, bend left, "w"]
\end{tikzcd}.$$

From this diagram and the discussion above we can see that $w \circ \tilde{h} = \tilde{e} \circ f'$, then post compose with v, we lose strict commutativity, but get $\tilde{h} = v \circ \tilde{e} \circ f'$. This shows that any map $\tilde{h}$ satisfying the desired properties must be constructed in a way similar to h, and in this construction only the map e was a choice, but in any case needed to be homotopy inverse to ι' and so has a uniquely defined homotopy type. Which concludes the proof. □

1.3 The space $BGL(R)^+$ and K-theory of rings

With all of this preliminary work done, we can define a K-theory of rings as follows.

Definition 1.3.1. (Definition 1.1 in [6] chapter IV) We define the space $K(R) = K_0(R) \times BGL(R)^+$ and then $K_i(R) = \pi_i(K(R))$ (with respect to some unimportant base point).

Let's quickly unwind this definition, recall $GL(R) = \varinjlim GL_n(R)$ is a group, and so $BGL(R)$ is then the classifying space (a construction unique up to homotopy). As we established earlier, because we are not specifying the subgroup relative to which we are performing a plus construction, it will be relative to the perfect radical of $\pi_1(BGL(R)) = GL(R)$ which is $E(R)$ the commutator subgroup (this is the content of Whitehead's lemma, see [10] theorem 1.11, we prove it by more abstract means in §2.3). Taking the product with $K_0(R)$ which we view as a discrete topological space is an artificial addition so that the K-theory defined this way matches the classical K-theory of rings in degree 0. We can also quickly see that the K-theory defined this way matches classical K-theory in degree 1 as by construction $\pi_1(K(R)) = GL(R)/E(R)$.

In order to be sure that this is a satisfactory definition of the K-theory of a ring, we need to verify that it matches classical theory in degree 2. So we want to show $K_2(R) = H_2(E(R);\mathbb{Z})$, this is an immediate consequence of the following more general result.

Proposition 1.3.2. *(Exercise 1.8 in [6] chapter IV) Let P be a perfect normal subgroup of some group G. Let $f : BG \to BG^+$ be the corresponding $+$-construction and denote the homotopy fiber by F_f. Then $\pi_1(F_f)$ is the universal central extension of P and $\pi_2(BG^+) = H_2(P,\mathbb{Z})$.*

Proof. We consider the fibration $F_f \to BG \to BG^+$ and study its homotopy long exact sequence, which is

$$\cdots \to \pi_2(BG) \to \pi_2(BG^+) \to \pi_1(F_f) \to \pi_1(BG) \to \pi_1(BG^+) \to \pi_0(F_f) \to \cdots.$$

The homotopy of BG is well understood, and we know that F_f is path connected because it is acyclic by assumption, finally we also know that $\pi_1(BG^+) = G/P$ by construction, so we get the exact sequence

$$1 \to \pi_2(BG^+) \to \pi_1(F_f) \to G \to G/P \to 1.$$

By the exactness condition we know that $Im(\pi_1(F_f) \to G) = P$, so we get the short exact sequence

$$1 \to \pi_2(BG^+) \to \pi_1(F_f) \to P \to 1.$$

Now, by the theory of central extensions, for which we refer the reader to [4] (6.9), we will obtain the desired result if we show that the above short exact sequence is the universal central extension of P (which exists as P is perfect). By the recognition criterion (6.9.7 in [4]), in order for that to be the case we need to check

(i) $Im(\pi_2(BG^+) \to \pi_1(F_f)) \leq Z(\pi_1(F_f))$

(ii) $\pi_1(F_f)$ is perfect.

(iii) Every central extension of $\pi_1(F_f)$ splits as a product of an abelian group and $\pi_1(F_f)$.

The second condition is equivalent to $H_1(F_f;\mathbb{Z}) = 0$, and the third is implied if $H_2(F_f;\mathbb{Z}) = 0$, both of these are given by assumption that f is acyclic. So all we need to check is the first condition. To do this we use exactness to instead try and show $ker(\pi_1(F_f) \to \pi_1(BG)) \leq Z(\pi_1(F_f))$. This follows at once from the fact that the morphism $\pi_1(F_f) \to \pi_1(BG)$ is a crossed module, as stated in [4] 6.6.12. To see this, use the realization of the homotopy long exact sequence of a fibration as the homotopy exact sequence of a pair (see the proof of theorem 4.41 in [3]) and the standard action of $\pi_1(A)$ on $\pi_2(X, A)$ for a pair (X, A) (see lemma 4.39). This concludes the proof. $\square$

It is worth noting that the K_n are functors from the category of rings to the category of abelian groups. Indeed, passing from R to $BGL(R)$ is functorial; and although, with how we constructed it, the $+$-construction (relative to the perfect radical of the fundamental group) *a priori* is not functorial when viewed as a map into the category of topological spaces, it is when viewed as a functor into the homotopy category of topological spaces, and because π_n factors through the homotopy category, we get that each K_n is a functor.

1.4 Technical preparations to show that $BGL(R)^+$ is an H-space

I am aware of several directions to show that $BGL(R)^+$ is an H-space. For example we can follow the method which I believe is expected by exercise 1.11 of [6] chapter IV. That is to say extend the bloc sum operation on $BGL(R)$ to $BGL(R)^+$. Another option is to follow theorem 1.11 from [10]. Yet another possibility is to show that $BGL(R)^+$ is weakly simple (spaces whose fundamental group acts trivially on the homology of the covering space), and then compare it with a space which is known to be an H-space. We will do this by using lemma 6.2 of [20] (which is proposition 1.4.3 for us) and theorem 4.10 of [6] (which we prove in section 2.3). In this section we present the technical results which will allow us to deduce that $BGL(R)^+$ is an H-space by showing it is isomorphic to the base point component of a space which will clearly be an H-space (see §2.3).

The first result allows us to construct maps into simple spaces, in particular into H-spaces.

Proposition 1.4.1. *(Theorem 2.5 in [21]) Let $i : X \to X^+$ be the natural map, with X^+ the plus construction with respect to the perfect radical. Furthermore, assume that X^+ is weakly simple. Suppose $f : X \to Y$ is a map into a simple CW-complex, then there is a map $g : X^+ \to Y$ such that $f = g \circ i$. And furthermore, the map induced on homotopy groups is independent of the choice g.*

Proof. The construction of the map is not too hard. In fact, it is a direct application of corollary 4.73 of [3]. We recall the statement here for convenience.

Proposition 1.4.2. *If Y is a simple connected CW complex and (W, A) is a CW pair such that $H^{n+1}(W, A; \pi_n(Y)) = 0$ for all $n \geq 0$, then every map $A \to Y$ can be extended to a map $W \to Y$.*

In our situation $W = X^+, A = X, Y = Y$. By assumption our Y satisfies the assumptions needed to apply the above proposition, and we have $H^n(X^+, X; M) = 0$ for all n and for all M, so we can apply the theorem.
It remains to show the desired uniqueness property. Suppose g_0, g_1 are two such extensions. We will construct a homotopy between them by considering the following extension problem:

$$\begin{tikzcd} X^+ \times \partial I \cup X \times I \arrow[r, "g_0, g_1"] \arrow[d, "\iota"'] & Y \\ X^+ \times I \arrow[ur, dashed] & \end{tikzcd}.$$

Now the obstruction lies in $H^{n+1}(X^+ \times I, X^+ \times \partial I \cup X \times I; \pi_n(H))$. However, we can use the suspension isomorphism to get that these groups are isomorphic to $H^n(X^+, X; \pi_n(H))$, but as above, these groups are 0. And so any two extensions of f are homotopic.

□

The above will, in due time, construct the desired map, and we will use the following result to prove it is a homotopy equivalence. Keeping in mind when we will want to use the following result, we note that H-spaces are weakly simple, as they are simple.

Proposition 1.4.3. *(lemma 6.2 of [20]) Let $f : X \to Y$ be a map of weakly simple CW-spaces. Suppose that $H_n(f; \mathbb{Z})$ are isomorphisms for all $n \geq 0$ and $\pi_1(f)$ is an isomorphism for any choice of compatible base points. Then f is a homotopy equivalence.*

Proof. Let Z be any weakly simple CW-space. Described more explicitly, this implies if we take $\gamma \in \pi_1(Z)$ and $\sigma \in C_n(Z^u)$, we have that $[\gamma\sigma] = [\sigma]$ in $H_n(Z^u; \mathbb{Z})$. Letting $\epsilon : \mathbb{Z}[\pi_1(Z)] \to \mathbb{Z}$ be the augmentation map, this implies that the chain map $Id \otimes \epsilon : C_*(Z^u) \otimes_{\pi_1(Z)} \mathbb{Z}[\pi_1(Z)] \to C_*(Z^u) \otimes_{\pi_1(Z)} \mathbb{Z}$, equipping $\mathbb{Z}$ with the trivial action, is a weak equivalence of chain complexes. Which means the $\mathbb{Z}$ homology of Z is naturally isomorphic to the $\mathbb{Z}$ homology of Z^u.
We can deduce that our map $f : X \to Y$ is a π_1 isomorphism, such that the induced map on covering spaces is a homology isomorphism. We know this implies that f is a homotopy equivalence (see for example [3] exercise 4.2.12). □

The final technical result which we will need, in order to apply the above proposition, is to show that X^+ is, in fact, a weakly simple space. The case of interest to us is classifying spaces of direct-sum groups, i.e groups which come equipped with a homomorphism $\oplus : G \times G \to G$. We will see why care we about these kinds of groups in §2.3.

Proposition 1.4.4. *(Proposition 1.2 in [22]) Let $(G, \oplus)$ be a discrete direct sum group satisfying the following technical conditions:*
(i) $[G, G]$ is a perfect subgroup of G.
(ii) For any finite $\{g_i\}_{i=1}^n \subset G$ and any $g \in G$, there is $h \in [G, G]$ such that $gg_ig^{-1} = hg_ih^{-1}, \forall i \in [n]$.
(iii) For any finite $\{g_i\}_{i=1}^n \subset G$, there exists $c, d \in G$ such that $c(g_i \oplus e)c^{-1} = d(e \oplus g_i)d^{-1} = g_i \forall i \in [n]$.
Then BG^+, the plus construction with respect to $[G, G]$, is weakly simple.

Proof. We start with a lemma which studies the homotopy fibration associated the geometric realization of a short exact sequence of groups.

Lemma 1.4.5. *Let $1 \to H \to G \to G/H \to 1$ be a short exact sequence of groups. Then $BH \to BG \to BG/H$ is a homotopy fibration and the natural action $\pi_1(BG/H) \curvearrowright H_*(BH; \mathbb{Z})$ is induced by conjugation by a choice of representative.*

Proof. We work with the bar construction model for BG. To see that $BH \to BG \to BG/H$ is a homotopy fibration, consider the geometric realization of $G \xrightarrow{q} G/H$ and take the homotopy fiber F_{q_H}. Studying the homotopy long exact sequence shows that $\pi_n(F_{q_H})$ is H if $n = 1$ and 1 otherwise. The homotopy fiber of a map of CW-complexes has the homotopy type of CW-complex. It then follows from the homotopy uniqueness of spaces with the homotopy type of CW complex and a single non-trivial homotopy group that $F_{q_H} \simeq BH$. And so, taking the homotopy fiber over the base point $[] \in BG/H^0$ we see that our original sequence was indeed a homotopy fibration.
The action of the base space on the fiber is given by the following commutative diagram

$$\begin{tikzcd} & BH \arrow[r] \arrow[d, "\iota_0"] & BG \arrow[d] \\ BH \arrow[r, "\iota_1"] & BH \times I \arrow[r, "\gamma \circ \pi_I"] \arrow[ur, dashed, "\tilde{\gamma}"] & BG/H \end{tikzcd} .$$

This diagram warrants some explanation. The map γ is a loop in BG/H based at $[]$ the base point and $\tilde{\gamma}$ is any lift fitting into this diagram, we will see how to construct such a lift in the next paragraph. The important point is that $\tilde{\gamma} \circ \iota_1$ lands in the homotopy fiber over the base point, i.e. in BH. Thus defining a map $\tilde{\gamma} \circ \iota_1 : BH \to BH$. However, this depends on the choice of lift, we will show that on homology the action is independent of the choice of lift. In order to do this we will construct the above diagram in the category of small categories, where it is easier to understand the lift explicitly. We write the one object categories corresponding to a group the same way as we denote the group. The only other topological space which we need to see as a category is I, but this is just the geometric realization of the category $[1]$ with two objects and one morphism between them $0 \to 1$. The left, top and right map are obvious to translate to the world of categories. The bottom map is not much harder, but we make it explicit. The loop γ represents an element $gH \in G/H$; because we are working with the bar construction $\gamma \simeq [gH]$ where $[gH]$ is the one simplex of BG/H corresponding to the element $gH \in G/H$. So we can represent the map $\gamma \circ \pi_I$ by the map $H \times [1] \to [1] \to G/H$ where the second map sends the unique non identity morphism of $[1]$ to $[gH]$. So far we have translated all the maps except the lift to the category of small categories

$$\begin{tikzcd} & H \arrow[r] \arrow[d, "\iota_0"] & G \arrow[d] \\ H \arrow[r, "\iota_1"] & H \times [1] \arrow[r, "gH \circ \pi_{[1]}"] & G/H \end{tikzcd} .$$

The category $H \times [1]$ has two copies of the category H which we denote by H_0 and H_1 in the obvious way, and these are connected by a single morphism which we call $e : *_0 \to *_1$ with $*_i$ the unique object of the appropriate copy of H_i. In order for the diagram to commute, the lift has to send H_0 to H, seen as a subcategory of G, by the identity. Similarly, the map e has to be sent to a lift of gH, i.e. to an element of the form $gh' \in G$. Then in order for this lift to be a functor (i.e. to respect composition) we need $h \in H_1$ to be sent to $(gh')h(gh)^{-1}$, call the map constructed this way $\tilde{\Gamma}$. It is easy to see that up to homotopy, any diagram of the form

$$\begin{tikzcd}
 & BH \arrow[r] \arrow[d, "\iota_0"] & BG \arrow[d] \\
BH \arrow[r, "\iota_1"] & BH \times I \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma \circ \pi_I"] & BG/H
\end{tikzcd}$$

comes from a diagram of the form

$$\begin{tikzcd}
 & H \arrow[r] \arrow[d, "\iota_0"] & G \arrow[d] \\
H \arrow[r, "\iota_1"] & H \times [1] \arrow[ur, "\tilde{\Gamma}"] \arrow[r, "{[gH] \circ \pi_{[1]}}"] & G/H
\end{tikzcd} .$$

And so the map $\tilde{\gamma} \circ \iota_1$ corresponds to conjugation by gh' for some choice of h'. But on homology inner automorphism act trivially (see theorem 6.7.8 of [4]). And so the effect of conjugation by gh' on $H_*(H)$ is independent of the choice of h', which is what we wanted to show. $\square$

With this lemma in hand we can start to work on showing that BG^+ is weakly simple with G satisfying all the assumptions of the theorem. For this consider the homotopy fibration $BE \to BG \to BG/E$ where $E = [G, G]$. The action of $[gE] \in G/E$ on $H_*(E)$ is given by conjugation by some representative $g \in G$ of gE. Any element $x \in H_*(E)$ can by the bar resolution be represented as a formal sum of elements of the form $(e_1, ..., e_n), e_i \in E$. And the action on the cycle level is also given by conjugation. By property (i) of G, on a finite subset (notably the finite set of all e_i appearing in the representation of x as a cycle in the bar resolution), conjugation by an element g of G can be realized by e' in E. In other words $gxg^{-1} = e'xe'^{-1}$. But by theorem 6.7.8 in [4] inner automorphism act trivially on homology. In other words G/H acts trivially on $H_*(E)$. Next notice that $BE \to BG$ is a covering space and so G acts on BE via deck transformations. The construction of this action is similar enough to the construction of the action $G/E \curvearrowright H_*(E)$ to see that the action of deck transformations on homology factor through this action. In other words $\forall x \in H_*(E), gx = gEx$ with the action of g given by deck transformations and the action of gE is the one we just constructed.
So we have proven that BG acts trivially on the homology of its covering space BE; if taking the $+$-constructions preserve the property of being a covering space and the action of the fundamental group, we get the desired result. This is accomplished in part by the following lemma.

Lemma 1.4.6. *(Proposition 6.1 of [10]) Let X be a connected topological space with fundamental group G, denote the perfect radical of G by P and let H be an intermediate normal subgroup. Denote by $\tilde{X}$ the covering of X which corresponds to H. Then $\tilde{X}^+$ is the covering of X^+ corresponding to H/P.*

Proof. The covering space $\tilde{X}$ can be realized (up to homotopy equivalence) as the homotopy fiber of the unique map $X \to K(G/H, 1)$ corresponding on π_1 to the quotient by H. To see this, consider the following diagram

$$\begin{tikzcd}
 & \tilde{X} \arrow[d] & \\
F_q \arrow[r, "\iota"] & X \arrow[r, "q"] & K(G/H, 1)
\end{tikzcd} .$$

We can lift the map ι to a map into $\tilde{X}$ using the lifting properties of covering spaces. The map constructed this way can easily be examined to be a weak homotopy equivalence, as both of our spaces have the homotopy type of a CW-complex, we have that F_q is homotopy equivalent to $\tilde{X}$. For the remainder of this proof, we understand the word "covering space" to be up to homotopy equivalence, the reader will notice that this is acceptable as the properties we use are related to the action of the fundamental group on the covering space up to homotopy and the homotopy type of the covering space. As $P \leq H$ we can use theorem 1.2.3 to extend this map to the plus construction, and also take the homotopy fiber of that map. We will also want to assume that $X \to X^+$ is a fibration, which can always be done without loss of generality. Denoting fibrations by $\twoheadrightarrow$ and using that fibrations are stable under pullbacks we can summarize the situation in the following diagram

$$\begin{tikzcd}
& \tilde{X} \arrow[rr] \arrow[d, two heads] \arrow[ddl] & & PK(G/H,1) \arrow[d, two heads] \arrow[ddl, "Id"] \\
& X \arrow[rr] \arrow[ddl] & & K(G/H,1) \arrow[ddl, "Id"] \\
Y \arrow[rr] \arrow[d, two heads] & & PK(G/H,1) \arrow[d, two heads] & \\
X^{+} \arrow[rr] & & K(G/H,1) &
\end{tikzcd} .$$

The back and front face are pullbacks by construction, it is not hard to see that the right face is a pullback as well, and then it is simple to verify that the left face is a pullback as well. This implies that $\tilde{X} \to Y$ is a fibration and that $\tilde{X} \cong Y \times_{X^+} X$. Expressing $\tilde{X}$ this way, it is easy to notice that over each pair of compatible points $x \in \tilde{X}, y \in Y$ the homotopy fibers can be identified (and are thus homeomorphic). This shows that $\tilde{X} \to Y$ is acyclic, thus a plus construction. It remains to verify: with respect to which subgroup of $\pi_1(\tilde{X})$ it is a +-constructions, that $Y \to X^+$ is a covering and to which subgroup of $\pi_1(X^+)$ it corresponds to as a covering space.
The fact that Y is a cover of X^+ follows from the fact that the front face is a pullback square which can be rewritten as

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & PK((G/P)/(H/P),1) \arrow[d] \\
X^{+} \arrow[r] & K((G/P)/(H/P),1)
\end{tikzcd}$$

by the third isomorphism theorem. And so Y is the cover of X^+ associated to $H/P \leq G/P \cong \pi_1(X^+)$. Now we know that Y is a plus construction of X^+. By what we have done so far, we have the following diagram of fundamental groups

$$\begin{tikzcd}
\pi_1(\tilde{X}) \cong H \arrow[r] \arrow[d, hook] & H/P \arrow[d, hook] \\
\pi_1(X) \cong G \arrow[r] & \pi_1(X^+) \cong G/P
\end{tikzcd} .$$

The two vertical maps are inclusions and the bottom map is the evident quotient map, it is not hard to see that the top map must be the quotient map we want it to be. So we have that Y is a plus construction of $\tilde{X}$ with respect to $P \leq H \cong \pi_1(\tilde{X})$. This is the desired result. $\square$

All that remains to be seen is that the action of $\pi_1(BG^+)$ on the homology of its universal cover BE^+ is induced by the action of $\pi_1(BG)$ on $H_*(E)$. This follows from the fact that acyclic maps are homology isomorphism, that the action of $\pi_1(BG)$ on $H_*(BE)$ factors through G/E and that the map $BG \to BG^+$ is the quotient map $G \to G/E$ on fundamental groups. This proves that the fundamental group of BG^+ acts trivially on the homology of its universal cover BE^+. $\square$

Remark 1.4.7. The above proof is quite complicated, and so I apologize for any mistakes. In particular in lemma 1.4.6, I did not find a way to make this proof work with true covering spaces rather than covering spaces up to homotopy equivalence. I did not find any source on covering spaces treated only up to homotopy and thus am not entirely confident in what I have written.

2 K-theory for Symmetric monoidal categories

In this section we wish to develop a K-theory for symmetric monoidal categories, which we will denote by $K_n^\square$, but we might write K_n for aesthetic reasons if confusion is not likely. To do this we will define the group completion for general homotopy commutative H-spaces, and prove a uniqueness result to justify that the definition is not ad hoc. We will also construct, for a symmetric monoidal category S, a category $S^{-1}S$ which turns the monoid of isomorphism classes of objects in S into a group in the canonical way. Once this preparation is done we will show that $BS \to BS^{-1}S$ is a group completion of the H-space BS, a concept we will define in time. The last thing to do in this chapter is to relate the $+$-construction K-theory developed last chapter with the K-theory we will construct in this chapter.

2.1 Group completion of H-spaces

Definition 2.1.1. (Definition 4.4 in [6] chapter IV) Let X be a homotopy commutative H-space, we call a map $f : X \to Y$ a group completion of X if Y is an H-space, f is an H-space map, $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is the standard group completion of the monoid $\pi_0(X)$ and the homology rings with coefficients in k of the two spaces are related by requiring

$$\pi_0(X)^{-1}f_* : \pi_0(X)^{-1}H_*(X;k) \to H_*(Y;k)$$

to be an isomorphism. If X is a CW-complex we also require that Y is a CW-complex

In the case of main interest to us, where $X = BS$ our space is indeed a CW-complex. The requirement that Y must be a CW-complex ensures that Y is group-like, or in other words Y has homotopy inverses. This is a consequence of the following result.

Proposition 2.1.2. *(Theorem 2.2 in [23] chapter X) Let X be a CW complex and an H-space. Then X is group-like if and only if $\pi_0(X)$ is a group.*

Proof. ($\Rightarrow$) This direction is clear enough that I will not reprove it.
($\Leftarrow$) We will need the following lemma

Lemma 2.1.3. *(4.17 in [23] chapter III) An H-space X which is also a CW-complex is group-like if and only if the shear map $\phi : X \times X \to X \times X$ defined by $\phi(x, y) = (x, xy)$ is a homotopy equivalence.*

Proof. If X is group-like, it is easy to construct a homotopy inverse defined by $\psi(x, y) = (x, h(x)y)$ with $h(x)$ the inverse of x up to homotopy. Now assume the shear map is a homotopy equivalence, call a representative of the class of the homotopy inverse ψ, we want to show X is group-like. Define $h : X \to X$ by $\pi_2 \circ \psi \circ \iota_1$ where $\iota_i : X \to X \times X$ is the inclusion in the ith coordinate (and constant equal to e, the homotopy identity, in the other coordinate), and $\pi_i : X \times X \to X$ is projection onto the ith factor. We want to show $h : X \to X$ defines homotopy inverses of elements $x \in X$, i.e. we want to show

$$\mu \circ (1 \times h) \circ \Delta \simeq c_e$$

with $\mu : X \times X \to X$ the H-space multiplication, $\Delta : X \to X \times X$ the diagonal map and $c_e : X \to X$ the constant map sending everything to the homotopy identity $e \in X$. We also want the above homotopy to hold when replacing $1 \times h$ by $h \times 1$. We will content ourselves only with proving the first homotopy, the other case dealt with by analogy. We prove the above statement by the following chain of equalities/homotopies

$$\mu \circ (1 \times h) \circ \Delta \simeq \mu \circ (\pi_1 \circ \psi \circ \iota_1 \times \pi_2 \circ \psi \circ \iota_1) \circ \Delta.$$

The replacement of h is by definition and the replacement of 1 is by using that $p_1 \simeq p_1 \circ \phi \circ \psi = p_1 \circ \psi$.

$$\begin{aligned} \mu \circ (\pi_1 \circ \psi \circ \iota_1 \times \pi_2 \circ \psi \circ \iota_1) \circ \Delta &= \mu \circ (\pi_1 \times \pi_2) \circ (\psi \circ \iota_1 \times \psi \circ \iota_1) \circ \Delta \\ &= \mu \circ (\pi_1 \times \pi_2) \circ \Delta \circ \psi \circ \iota_1 \\ &= \mu \circ \psi \circ \iota_1 \simeq \pi_2 \circ \iota_1 = c_e. \end{aligned}$$

The last homotopy comes from $p_2 \simeq p_2 \circ \phi \circ \psi = \mu \circ \psi$. And so this proves the lemma. □

Now all that is left is showing that under the assumption that $\pi_0(X)$ is a group, the shear map $X \times X \to X \times X$ is a homotopy equivalence. To show this we use Whitehead's theorem (Hatcher's version does not explicitly deal with spaces which are not 0-connected, so we refer the reader to [23] V.3.5 and the discussion preceding said theorem to ensure that our usage of Whitehead's theorem is correct). Let (x_1, x_2) be an arbitrary base point of $X \times X$, and consider $\phi : (X \times X, (x_1, x_2)) \to (X \times X, (x_1, x_1x_2))$, we want to show it is an isomorphism on every homotopy group $\pi_i, i \geq 0$. We want to understand $\pi_n(\phi)$, to do this we just need to note that $\pi_n(X \times X) \cong \pi_n(X) \times \pi_n(X)$ and that the multiplication induced by $\mu : X \times X \to X$ induces a group structure on $\pi_n(X), \forall n \geq 0$ and this group structure agrees with the group structure given by concatenation by the Eckmann-Hilton argument, knowing this it can be seen that $\pi_n(\phi)(\sigma, \tau) = (\sigma, \sigma\tau)$ and this for all $n \geq 0$, this being clearly an isomorphism in each case, with the case $n = 0$ being true because $\pi_0(X \times X)$ is a group as it is a product of two groups. And so the shear map is a weak homotopy equivalence, which because X is a CW-complex proves our claim. □

We need a couple definitions to state (and appreciate) the uniqueness result for group completions.

Definition 2.1.4. (Remark preceding theorem 4.10 in [6] chapter IV)(Remark preceding corollary 1.2 in [6] chapter II) Let M be a commutative monoid, and $S \leq M$ a submonoid. Then S is called cofinal if $\forall m \in M, \exists n \in M$ such that $mn \in S$. S is called a cofinal sequence if $\forall m \in M, \exists n \in M$ such that $mn \in S$ and $S = \{a_i\}_{i=1}^{\infty}$ such that $\forall i, \exists b_i \in M$, such that $b_i a_i = a_{i+1}$.

To appreciate how this definition might be useful in measuring failure to be a group, notice that if M happens to be a group, then $S = \{1_M\}$ is cofinal. The reason for introducing both cofinal submonoids and cofinal sequences is that the statement of the main theorem of this section is different in [6] and in [11]. I have decided to follow the latter as Weibel refers to them (and not the other way around), and to write this section, [11] was considerably more useful.

Definition 2.1.5. (Remark preceding theorem 4.4.3 in [6] chapter IV) Let $\phi : X \to Y$ be a map of topological spaces, we call it a phantom map if the induced map $\phi_* : [A, X] \to [A, Y]$ is the trivial map for all finite CW complexes A.

The point of this definition is to notice that if $f : X \to Y$ is a group completion of an H-space and $\phi : X \to Y$ is a phantom H-space map, then $f + \phi : X \to Y$ is also a group completion, and so in so far as non-trivial phantom maps exist, we cannot hope for uniqueness of group completions up to homotopy equivalence. This is why we introduce the following weaker notion.

Definition 2.1.6. (Remark preceding theorem 4.4.3 in [6] chapter IV) We call two maps $f, g : X \to Y$ weakly homotopic, denoted $f \simeq_w g$, if they induce the same map after applying the functor $[A, -]$ for all finite CW complexes A. In the case of group-like H-spaces this amounts to calling f and g weakly homotopic if their difference is a phantom map.
The set of all maps under the equivalence relation of weak homotopy is denoted $[X, Y]_w$.

We regularly referred to [11] to prepare the proofs of this section. We are almost ready to state the universal property of the group completion which will allow us to show the uniqueness result, we just state and prove a technical lemma first.

Lemma 2.1.7. *(lemma 1.1 in [11]) Suppose $f : X \to X'$ is an integral homology isomorphism and Y a group-like H-space, with all spaces CW-complexes. Then*
(i) $f^* : [X', Y] \to [X, Y]$ is an isomorphism.
(ii) $f^* : [X', Y]_w \to [X, Y]_w$ is an isomorphism.

Proof. (i) We can assume Y is 0 connected by working connected component by connected component. We suspend the situation, in order to momentarily shift to the 1-connected situation. We want to prove that $\Sigma f^* : [\Sigma X', \Sigma Y] \to [\Sigma X, \Sigma Y]$ is an isomorphism. By the suspension isomorphism in homology $\Sigma f : \Sigma X \to \Sigma X'$ is an integral homology isomorphism between simply connected CW-complexes, and so is a homotopy equivalence. So Σf^* is an isomorphism. Now use the suspension loop adjunction and get an isomorphism $[X', \Omega\Sigma Y] \to [X, \Omega\Sigma Y]$. Because we know $\Omega\Sigma Y$ to be (assuming the result that the loop space of a CW complex to again have the homotopy type of a CW complex, see [24]) homotopy equivalent to the James reduced product $J(Y)$ by theorem 4J.1 in [3], we get an isomorphism $[X', J(Y)] \to [X, J(Y)]$. Call λ the homotopy equivalence $J(Y) \to \Omega\Sigma Y$, then by observing the following diagram (which because λ_* is an isomorphism defines the isomorphism $[X', J(Y)] \to [X, J(Y)]$)

$$\begin{tikzcd} {[X', J(Y)]} \arrow[r] \arrow[d, "\lambda_*"] & {[X, J(Y)]} \arrow[d, "\lambda_*"] \\ {[X', \Omega\Sigma Y]} \arrow[r, "f^*"] & {[X, \Omega\Sigma Y]} \end{tikzcd}$$

we can also notice that the isomorphism is given by pulling back via f.
Now because Y is group-like there is a retraction up to homotopy of $Y \xrightarrow{\iota} J(Y) \xrightarrow{r} Y$ determined by choosing some way to take n-products (i.e. choosing where to put parenthesis when multiplying an n-tuple from $J(Y)$). Our work so far summarizes nicely in the following diagram

$$\begin{tikzcd} {[X', Y]} \arrow[r, "f^*"] \arrow[d, "\iota"] & {[X, Y]} \arrow[d, "\iota"] \\ {[X', J(Y)]} \arrow[r, "f^*"] \arrow[d, "r"] & {[X, J(Y)]} \arrow[d, "r"] \\ {[X', Y]} \arrow[r, "f^*"] & {[X, Y]} \end{tikzcd} .$$

Commutativity of the diagram comes from the fact that we pulling back horizontally, but pushing forward vertically. It is easy to see that pushforwards and pullbacks commute. Now using that the composed vertical maps are the identity, the proof of the desired result is reduced to a quick diagram chase.
(ii) As above we may assume Y to be connected. We get surjectivity for free from the above case. Indeed, let $[g]_w$ be a weak homotopy class of maps $X \to Y$ represented by g. By the above case there is a map $g_0 : X' \to Y$ such that $f^*[g_0] = [g]$. Homotopic maps are obviously weakly so, and thus we get $f^*[g_0]_w = [g]_w$. For injectivity, because we are dealing with groups, it suffices to show that if $f^*[\phi]_w = [*]_w$ then $[\phi]_w = [*]_w$. In other words if $\phi \circ f$ restricted to any finite CW subcomplex is nullhomotopic, then ϕ restricted to any finite CW complex is nullhomotopic.
Without loss of generality we assume f to be cellular, and then replace it with a cofibration up to homotopy equivalence. We first make f cellular so that the map into the mapping cylinder is a cellular inclusion, so that we may assume (X', X) to be a CW-pair, and the inclusion to be a homology isomorphism. From the long exact sequence in homology of the pair (X', X) we get that $H_n(X', X; \pi) = 0$ for any coefficient group π. Let A' be some finite CW subcomplex of X', with the goal of applying excision in mind, we add to A' all the cells in X' not in X, call the new subcomplex A''. Let $A = A'' \cap X$, which is a finite CW complex as it is equal to $A' \cap X$. We get by excision that $H_n(A'', A; \pi) = 0, \forall n \geq 0$. Let $\phi : X' \to Y$ be such that $\phi \circ f$ is null homotopic when restricted to a finite CW subcomplex, in particular $f \circ \phi|_A \simeq *$. Because Y is simple and connected, and every space in our discussion is a CW complex, we can imitate the reasoning done in §1.4.2 to see that the obstruction to extending this homotopy to A'' lies in $H^n(A'', A; \pi_n(Y)) = 0$. And so the extension exists, in particular we can extend the nullhomotopy to A', in other words $\phi : X' \to Y$ is weakly nullhomotopic, as desired. □

Proposition 2.1.8. *(Proposition 1.2 in [11]) Let X be a CW H-space such that $\pi_0(X)$ has a countable cofinal sequence. Let $g : X \to Y$ be a group completion. Then if $f : X \to Z$ is any weak H-map into a group-like H-space there exists a weak H-map $f' : Y \to Z$ unique up to weak homotopy such that $f'g$ is weakly homotopy equivalent to f.*

Proof. Let $\{a_i\}_{i=0}^{\infty}$ be a countable cofinal sequence in $\pi_0(X)$, which means $\forall i, \exists b_i$ such that $b_i a_i = a_{i+1}$. We can take the mapping telescopes of the sequence of maps $b_i \cdot : X \to X$, call this space TX. We allowed ourselves the mild abuse of notation of writing b_i both for the class in $\pi_0(X)$ and an actual

element in X representing this class. Perform a similar construction for Y and Z, this time using right translation by $g(b_i)$ and $f(b_i)$ respectively. Name the associated mapping telescopes TY and TZ. Notice f and g are weak H-maps, so by replacing by weakly homotopic maps, there are maps $Tg : TX \to TY$ and $Tf : TX \to TZ$. Next notice that the multiplications by $g(b_i)$ and by $f(b_i)$ are homotopy equivalences for each i because Y and Z are group-like, so the inclusions $Y \to TY$ and $Z \to TZ$ are homotopy equivalences. The last thing to notice before moving on in the argument is that Tf, Tg restricted to $X \subset TX$ are just f, g respectively.
Computing the homology of TX (for example with [3] 3F.2) shows that $Tg : TX \to TY$ is a homology isomorphism. Now using the above lemma we get an isomorphism $[TY, TZ] \to [TX, TZ]$. By the homotopy equivalences we mentioned above this yields an isomorphism $[Y, Z] \to [TX, TZ]$, in particular this yields a map $Tf' : TY \to TZ$, or equivalently $f' : Y \to Z$, such that precomposing with $Tg : TX \to TY \simeq Y$ we get $Tf : TX \to TZ$, but by restricting to X, we get $f'g \simeq f$.
Now we still want to show that f' is unique up to weak homotopy and that it is a weak H-map. We first prove the former, suppose $k : Y \to Z$ is another weak H-map such that $kg \simeq_w f$, using the homotopy equivalences $TY \simeq Y$ and $TZ \simeq Z$, this is the same as $Tk \circ Tg \simeq_w Tf$. Then by the second part of the above lemma, we have $TkTg \simeq_w Tf \simeq_w Tf'Tg$, which by injectivity of the isomorphism $Tg_* : [TY, TZ]_w \to [TX, TZ]_w$ yields $Tk \simeq_w Tf'$, which after appropriate restriction is the desired result. To show that f' is a weak H-map, first notice that the product $\mu : X \times X \to X$ yields a map $T\mu : TX \times TX \to TX$ using the commutativity up to homotopy, and this product is compatible up to weak homotopy with the products of $TY \simeq Y$ and $TZ \simeq T$, with the maps Tf, Tg, Tf' and their appropriate restriction. Notice that $Tg \times Tg : TX \times TX \to TY \times TY$ is also a homology isomorphism, and so the above lemma applies as well to precomposition by $Tg \times Tg$. Now we just chain some equalities and homotopies, which will allow us to conclude by injectivity of $Tg \times Tg$. We allow ourselves the abuse of notation of using $T\mu$ for the multiplication of TX, TY and TZ.

$$T\mu(Tf' \times Tf')(Tg \times Tg) \simeq T\mu(Tf \times Tf) \simeq_w TfT\mu \simeq Tf'TgT\mu \simeq_w Tf'T\mu(Tg \times Tg);$$

so we get $T\mu(Tf' \times Tf') \simeq_w Tf'T\mu$, which by appropriate restriction is the desired result. □

Remark 2.1.9. I have a suspicion that this proof goes through all the same if we just remove every occurrence of the word "weak", but the insistence of our main source to use weak homotopies leads me to believe a technicality has eluded me.

We can "upgrade" the above results to a uniqueness result by using the following lemmas

Lemma 2.1.10. *(lemma 4.4.1 in [6] chapter IV) (i) Weakly homotopic maps induce the same map on integral homology.*
(ii) Let $f : X \to Y$ be an H-space map with X and Y group-like. Then if f is a homology isomorphism (or equivalently a group completion), f is a homotopy equivalence.

Proof. (i) Let $f, g : X \to Y$ be weakly homotopic maps, we want to show $H_n(f) = H_n(g)$, for this let $\alpha : \Delta^n \to X$ represent some homology class $[\alpha] \in H_n(X)$. Then $f_*([\alpha]) = [f \circ \alpha] = [g \circ \alpha] = g_*([\alpha])$ which because α was arbitrary proves the claim.
(ii) We may get this result following lemma 4.4.1 [6] chapter IV. The result is then a direct corollary to proposition 1.4.3; however, I like this proof a lot, so wanted to write it out. By direct application of lemma 2.1.7 we get that the functors $[-, X]$ and $[-, Y]$ turn f into an isomorphism. Start by applying $[-, X]$. This yields the isomorphism $f^* : [Y, X] \to [X, X]$, in particular a preimage g of the identity, or concretely a map such that $g \circ f \simeq Id_X$. Now taking homology we see that $g_* = (f_*)^{-1}$ so g is also an isomorphism on integral homology. By repeating the above reasoning we get $h : X \to Y$ such that $h \circ g \simeq Id_Y$, but by uniqueness up to homotopy of homotopy inverses we get that $h \simeq f$, and so f, g are homotopy inverse homotopy equivalences. □

And so we get the group completion uniqueness result.

Theorem 2.1.11. *(Theorem 4.4.3 in [6] chapter IV) Let X be an H-space such that $\pi_0(X)$ admits a countable cofinal sequence. Let $f' : X \to X'$ and $f'' : X \to X''$ be group completions. Then there exists a homotopy equivalence $g : X' \to X''$ which is a weak H-map unique up to weak homotopy such that gf' and f'' are weakly homotopy equivalent.*

Proof. We apply the universal property of group completion of H-spaces both for $X \to X'$ and $X \to X''$. We can summarize the situation we obtain by the following diagram, commutative up to weak homotopy:

$$\begin{tikzcd} & X' \arrow[d, "\exists!_w g'"] \arrow[dd, bend left=60, "\exists!_w Id_{X'}"] \\ X \arrow[ur, "f'"] \arrow[r, "f''"] \arrow[dr, "f'"'] & X'' \arrow[d, "\exists!_w g''"] \\ & X' \end{tikzcd},$$

where $\exists!_w$ means there exists a unique up to weak homotopy, we get an analogous diagram exchanging the roles of X' and X''. By the above pair of lemmas, the diagrams commute strictly after applying integral homology, and so g', g'' are integral homology isomorphisms, and so are homotopy equivalences. The rest of the desired result is a direct consequence of the universal property. $\square$

2.2 The group completion $BS^{-1}S$ of BS

Let $(S, \square, e)$ be a symmetric monoidal category, (which we from here on out might abbreviate to "an SM category") then the geometric realization BS is an H-space. To see this one just has to recall/notice that geometric realization preserves products, sends functors to maps of topological spaces and natural transformations to homotopies. In the same way as it is natural to study monoids via group completion, it is natural to turn BS into an H-space with homotopy inverses in the "minimal" way. We could do this by simply working with the space BS, but it is interesting to see if we can add inverses (up to natural transformation at least) before taking the geometric realization. The most natural way to do this is to proceed by studying actions of a symmetric monoidal category S on some category X (some of our definitions work for general monoidal categories, but we will not be needing that).

Definition 2.2.1. (Definition 4.7 in [6] chapter IV) A symmetric monoidal category S is said to act upon X by a functor $\square : S \times X \to X$ if there are natural transformation $s\square(t\square x) \cong (s\square t)\square x$ and $e\square x \cong x$ satisfying the "expected" coherence conditions.

Given an action of S an SM category on a category X, we can form a translation category analogous to the definition of $G \int Y$ where G is some group and Y a G-set.

Definition 2.2.2. (Definition 4.7.1 in [6] chapter IV) We denote the translation category by $\langle S, X\rangle$. The objects of this category are the same as X, but morphisms are equivalence classes of pairs $(s, s\square x \xrightarrow{\phi} y), s \in Ob(S), \phi \in Mor(X)$. Two pairs $(s, \phi), (s'\phi')$ are equivalent if there is an isomorphism $s \xrightarrow{\sigma} s'$ such that

$$\begin{tikzcd} s\square x \arrow[r, "\phi"] \arrow[d, "\sigma\square x"] & y \arrow[d, "Id_y"] \\ s'\square x \arrow[r, "\phi'"] & y \end{tikzcd}$$

commutes.

We can formally invert the action of S on X by letting S act on $S \times X$ on both factors at a time, then considering the category $S^{-1}X = \langle S, S \times X\rangle$. We let S act on $S^{-1}X$ by multiplication on the second coordinate. In what sense does this invert the action S on X?

Definition 2.2.3. (remark preceding theorem 4.8 in [6] chapter IV) Let S be a symmetric monoidal category acting on a category X, we say that the action is invertible, or that S acts invertibly, if each translation functor $s\square - : X \to X$ is a homotopy equivalence.

For example in the case of the category $S^{-1}X$, we have that $s\square - : S^{-1}X \to S^{-1}X, (t, x) \mapsto (t, s\square x)$, has as homotopy inverse the functor sending (t, x) to $(s\square t, x)$ because there exists a natural transformation from the identity to the functor defined by $(t, x) \mapsto (s\square t, s\square x)$.
Notice that S acts upon itself by translation, and so we have a category $S^{-1}S$, on which S acts

invertibly. It is worth noticing that in this case the translation category is also symmetric monoidal with the product given by $(s_1, t_1)\Box(s_2, t_2) = (s_1\Box s_2, t_1\Box t_2)$. This symmetric monoidal structure category also admits "inverses", indeed

$$(s_1, s_2)\Box(s_2, s_1) = (s_1\Box s_2, s_2\Box s_1) \cong (s_1\Box s_2, s_1\Box s_2).$$

Which because there is a map $(e, e) \to (s_1\Box s_2, s_1\Box s_2)$ shows that $\pi_0(S^{-1}S)$ admits inverses. However, according to remark 4.2.2 of [6] chapter IV, this association is not a natural transformation. Exercise 4.3 of [6] chapter IV tells us that the map $(s_1, s_2) \mapsto (s_2, s_1)$ does in fact define a homotopy inverse for $BS^{-1}S$, however we will not pursue this question.

The final thing we wish to notice about this construction before returning to the more general case is that the construction $S \mapsto S^{-1}S$ can be extended to a functor $Sym \to Cat$ from the category of small symmetric monoidal categories (with maps strict monoidal functors) to the category of small categories. The reason for this observation is for it to be clear that K-theory of symmetric monoidal categories, which we define now, is a functorial construction

Definition 2.2.4. (Definition 4.3 in [6] chapter IV) Let S be a symmetric monoidal category where every map is an isomorphism (if this is not the case just replace S by $iso(S)$), then the K-theory of S are the functors

$$K_n^{\Box}(S) = \pi_n(K^{\Box}(S)), K^{\Box}(S) = BS^{-1}S.$$

The base point of $BS^{-1}S$ is the 0-cell corresponding to (e, e).

Let's return to our general case of an SM category S acting on X. Then we have a natural S equivariant functor $\iota : X \to S^{-1}X$ which on objects maps x to (e, x), which induces a map $H_q(X) \to H_q(S^{-1}X)$, both of these groups are $\mathbb{Z}[\pi_0(S)]$ modules, and so we can localize the above map at the multiplicatively closed subset $\pi_0(S)$ (S is symmetric monoidal, thus $\mathbb{Z}[\pi_0(S)]$ is commutative, and so there are no subtleties with localization). Because $\pi_0(S)$ already acts invertibly on $H_q(S^{-1}X)$, we get a map

$$\pi_0(S)^{-1}H_q(X) \to H_q(S^{-1}X).$$

We have the following result due to Quillen, which will show that $S \to S^{-1}S$ accomplishes our goal of group completing S before geometric realization. To supplement the K-book, we used [12] to understand the proof.

Theorem 2.2.5. *(Theorem 4.8 in [6] chapter IV), (page 221 of [12])(Theorem 7.2 of [13]) If every map in S is an isomorphism and translations are faithful (i.e. $\forall s, t \in S, Aut(t) \to Aut(s\Box t)$ is an injection), then the above map is an isomorphism.*

Proof. A motivation for this proof comes from the fact that on objects, the map fits nicely into the sequence $X \to S \times X \to S$, i.e. a trivial fibration. We can hope that when considering morphisms, and taking geometric realization, this sequence kind of looks like a fibration, and so we will be able to analyze the homology of these categories using a Serre spectral sequence. This sadly will not actually happen, but motivates what happens.
The appropriate categorical notion which imitates the idea of being a fibration will be to show that $P : S^{-1}X \to \langle S, S\rangle$, which on objects is projection on the first coordinate, is cofibered with cofiber X. For future reference we state this as a lemma.

Lemma 2.2.6. *(Exercise 4.5 in [6] chapter IV), (page 220 of [12]) Let S be a symmetric monoidal category acting on a category X. Furthermore, assume translations are faithful in S and that every arrow in S is monic. Then, the natural projection functor $P : S^{-1}X \to \langle S, S\rangle$ is cofibered with cofiber X.*

Proof. On objects we obviously have the desired $P^{-1}(s) = X$ for all $s \in S$. To understand the morphisms, let's recall that the identity map of $s \in \langle S, S\rangle$ is the equivalence class of $(e, e\Box s \xrightarrow{\eta} s)$ with $\eta : e\Box s \to s$ the canonical isomorphism. We also from here on out identify $P^{-1}(s)$ interchangeably with the full subcategory of P/s where the objects correspond to the identity map in $\langle S, S\rangle$. And so an arbitrary element in the preimage is a map of the form

$$(t, \phi : t\Box s \to s, f : t\Box x \to x') : (s, x) \to (s, x'),$$

with $\sigma : t \to e$ an isomorphism such that the following diagram commutes

$$\begin{tikzcd} t\Box s \arrow[r, "\phi"] \arrow[d, "\sigma\Box s"] & s \arrow[d, "Id_s"] \\ e\Box s \arrow[r, "\eta"] & s \end{tikzcd}.$$

We would like to show that σ is uniquely determined. Indeed, let τ be another isomorphism such that $\tau\Box s$ fits in the above diagram. The fact that η is monic implies that $\sigma\Box s = \tau\Box s$, which because translations are faithful implies the desired uniqueness.
And so in particular, fixing $\sigma_{t,\phi} : t \to e$ to be the unique isomorphism fitting in the above diagram, for a map $(t, \phi : t\Box s \to s, f : t\Box x \to x')$ in the preimage of s, we can send it to $f \circ (\sigma_{t,\phi}^{-1}\Box x) \circ \eta^{-1} : x \to x'$. One can verify that this construction is functorial, and is an inverse to $I : X \to P^{-1}(s)$ sending $x \mapsto P(s, x) \xrightarrow{(e,\eta:e\Box s\to s)} s$ and $f : x \to x'$ to:

$$\begin{tikzcd} P(s,x) \arrow[rr, "P((e,\eta,f\circ\eta))"] \arrow[dr] & & P(s,x') \arrow[dl] \\ & s & \end{tikzcd},$$

where the η in the second coordinate is the natural isomorphism $e\Box s \to s$ in S and the other η is the natural isomorphism $e\Box x \to x$ in X. The fact that these two mapping are mutually inverse functors is seen using the uniqueness of $\sigma_{t,\phi}$ and the equivalence relation for maps in $P^{-1}(s)$ inherited from the definition of maps in $S^{-1}X$. Though we will not explicitly do this here. From here on out we may identify $P^{-1}(s)$ with X without further comment.
Now we want to show that the projection is pre-cofibered. Our goal is to find a left adjoint to the inclusion $I : X \to P/s$ given by the above identifications $X \cong P^{-1}(s)$ and $P^{-1}(s)$ as a full subcategory of P/s. Define $L(P(s', x) \xrightarrow{(t,\phi:t\Box s'\to s)}) = t\Box x$ and let L send a map

$$P((t, \phi : t\Box s' \to s'', f : t\Box x \to x')) : (P(s', x) \xrightarrow{(t',\phi':t'\Box s'\to s)}) \to (P(s'', x') \xrightarrow{(t'',\phi'':t''\Box s''\to s)} s)$$

to $(t''\Box f) \circ (\sigma\Box x) : t'\Box x \to t''\Box x'$. This $\sigma : t' \to t''\Box t$ is the unique isomorphism making the following diagram commute

$$\begin{tikzcd} t'\Box s' \arrow[r, "\phi"] \arrow[d, "\sigma\Box s'"] & s \arrow[d, "Id_s"] \\ t''\Box t\Box s' \arrow[r, "\phi''(t''\Box\phi)"] & s \end{tikzcd},$$

with existence and uniqueness established as above using the definition of maps in $\langle S, S\rangle$, the commutativity of the diagram in $\langle S, S\rangle$ determined by the map in P/s, the fact that all maps in S are monic and the faithfulness of translations. Now the fact that this construction is functorial is best proven by pen and paper, using the uniqueness of the σ used in the definition. We will not make this proof explicit. Now the adjunction isomorphism we are hoping for is between the sets

$$\alpha : Hom_X(t\Box x, x') \cong Hom_{P/s}(P(s', x) \xrightarrow{(t,\phi:t\Box s'\to s)} s, P(s, x') \xrightarrow{(e,\eta:e\Box s\to s)} s) : \beta.$$

We will state the maps giving this isomorphism, but will not prove that they are in fact mutually inverse or that the isomorphisms are natural. Define $\alpha(f) = (t, \phi : t\Box s' \to s, f : t\Box x \to x')$ and $\beta(r, \psi : r\Box s' \to s, g : r\Box x \to x') = g \circ (\sigma\Box x)$ where $\sigma : t \to e\Box r$ is the unique isomorphism making the following diagram commute

$$\begin{tikzcd} t\Box s' \arrow[r, "\phi"] \arrow[d, "\sigma\Box s'"] & s \arrow[d, "Id_s"] \\ e\Box r\Box s' \arrow[r, "\eta\circ(e\Box\psi)"] & s \end{tikzcd}.$$

Now we wish to show that the functor is cofibered, i.e. $P^{-1} : \langle S, S\rangle \to Cat$ is functorial. To do this we compute the cobase functor associated to a map $(t, \phi : t\Box s \to s') : s \to s'$ in $\langle S, S\rangle$. We apply successively the functors $I : P^{-1}(s) = X \to P/s$ then $(t, \phi : t\Box s \to s')_* : P/s \to P/s'$ and then $L : P/s' \to X = P^{-1}(s')$. This yields

$$x \mapsto P(s, x) \xrightarrow{(e,\eta)} s \mapsto P(s, x) \xrightarrow{t,\phi:t\Box s\to s'} s' \mapsto t\Box x,$$

so the cobase change functor of $(t, \phi : t\Box s \to s')$ is just translation by t on X, which clearly satisfies the desired property to make $P : S^{-1}X \to \langle S, S\rangle$ cofibered. $\Box$

Having shown that the functor of interest is indeed cofibered, we need to show that cofibered functors satisfy some kind of Serre spectral sequence, this is given by the following lemma.

Lemma 2.2.7. *(Exercise 3.7 in [6] chapter IV)(Proof of theorem 7.2 of [13]) Let $F : C \to D$ be a cofibered functor (between small categories), in particular $F^{-1} : D \to Cat$ is a functor. Then we have a spectral sequence*

$$E^2_{pq} = H_p(D; H_q F^{-1}) \Rightarrow H_{p+q}(C; \mathbb{Z}).$$

For the definition of homology of a category with a functor into abelian groups as coefficients, see (3.5) of [6] chapter IV.

Proof. We prove this by applying the spectral sequences for a double complex (see definition 5.6.1 and 5.6.2 of [4] and the rest of chapter 5). We need to construct a double complex on which we will use the spectral sequence machinery. Let $E^0_{pq}, p, q \geq 0$ be the free abelian group on the set of pairs

$$(F(c_q) \to d_0 \to \cdots \to d_p, c_0 \to \cdots \to c_q),$$

with the $d_i \in D$ and $c_i \in C$, if either p or q is strictly less than 0 $E^0_{pq} = 0$. We define the differentials by defining vertical $\partial^v_i, 0 \leq i \leq q$ and horizontal face $\partial^h_i, 0 \leq i \leq p$ maps, and then using the vertical differentials $\partial^v_{pq} = (-1)^p \sum_{i=0}^q (-1)^i \partial^v_i$ and horizontal differentials $\partial^h_{pq} = \sum_{i=0}^p (-1)^i \partial^h_i$. The fact that this yields a double complex follows from standard computation. The horizontal face maps are given, when $i \neq 0, p$, by composing $d_{i-1} \to d_i \to d_{i+1}$, when $i = 0$ replace d_{i-1} by $F(c_q)$ for this to make sense and when $i = p$ simply delete d_p. The vertical face maps, when $i \neq 0, q$, are given by composing $c_{i-1} \to c_i \to c_{i+1}$. When $i = 0$ instead simply delete c_0 and when $i = q$, delete c_q and replace $F(c - q) \to d_0$ with the composition $F(c_{q-1}) \to F(c_q) \to d_0$.
Now that we have a double complex we can apply the double complex spectral sequences in hopes of obtaining the desired spectral sequence. Notice because we are working with a first quadrant double complex, both the row and column filtration converge to the same homology $H_*(Tot(E^0_{\bullet\bullet}))$. We first use the column filtration to compute $H_*(Tot(E^0_{\bullet\bullet}))$. It is useful to write the 0th page of our spectral sequence in different ways to compute the E^1 and E^2 pages both with the row and column filtration. First, notice we can write

$$E^0_{pq} = \coprod_{c_0 \to \cdots \to c_q \in N_q(C)} \coprod_{F(c_q) \to d_0 \to \cdots \to d_p \in N_p(F(c_q)/D)} \mathbb{Z},$$

where $N_q(C)$ denotes the q simplicies of the nerve of C. We will abbreviate this to $E^0_{pq} = \coprod_{N_q(C)} \coprod_{N_p(F(c_q)/D)}$. Written this way it is easy to compute the homology in the p-direction using that homology commutes with coproducts. So we get $E^1_{pq} = \coprod_{N_q(C)} H_p(F(c_q)/D; \mathbb{Z})$. Because $F(c_q)/D$ has $F(c_q) \xrightarrow{Id} F(c_q)$ as an initial object we have that the category is contractible and thus the first page of our spectral sequence is

$$E^1_{pq} = \coprod_{N_q(C)} \delta_{p=0}\mathbb{Z}.$$

To be clear, by $\delta_{p=0}\mathbb{Z}$ means the 0 group if $p \neq 0$ and $\mathbb{Z}$ otherwise. From this it is easy to see that $E^2_{pq} = \delta_{p=0} H_q(C; \mathbb{Z})$ and that there is no room for differentials left. So the homology of the total complex is the homology of C.
Now to compute the E^2 page of the spectral sequence by first taking homology in the q direction we

need to present the 0th page differently. To do this we use that coproducts commute with each other to write

$$E^0_{pq} = \coprod_{d_0 \to \cdots d_p \in N_p(D)} \; \coprod_{F(c_0) \to \cdots \to F(c_q) \to d_0 \in N_q(F/d_0)} \mathbb{Z}.$$

We do not explicitly verify that these are two equivalent ways of writing our double complex. Also, we abbreviate similarly as above. Now we use that the functor is cofibered so $F/d_0 \simeq F^{-1}(d_0)$ to see that taking the homology in the q direction yields

$$E^1_{pq} = \coprod_{N_p(D)} H_q(F^{-1}(d_0); \mathbb{Z}).$$

Now taking the homology in the p direction we get by definition

$$E^2_{pq} = H_p(D; H_q(F^{-1})).$$

This converges to the homology of the total complex, which we computed to be $H_*(C; \mathbb{Z})$. This concludes the proof. $\square$

The following lemma along with theorem 4.8 of [23] chapter VI (which we admit) allows us to conclude that homology of a category with coefficients in a morphism inverting functor into abelian groups is the homology of the geometric realization with the associated local coefficients.

Lemma 2.2.8. *(Exercise 3.1 in [6] chapter IV)(Lemma 6.1 of [13]) Call a functor $F : C \to D$ morphism inverting if the image of every map is an isomorphism. If C is a category we have an equivalence of categories*

$$MI(Set^C) \leftrightarrows Cov(BC).$$

We denote by $MI(Set^C)$ the full subcategory of Set^C of morphism inverting functors and $Cov(BC)$ is the category of covering spaces of BC.
In particular morphism inverting functors $C \to Ab$ correspond to local coefficient systems on BC.

Proof. We first define the two functors which will define the equivalence. Let $F : C \to Set$ be a morphism inverting functor, there is a category $C \int F$ defined in example 3.3.2 of [6] chapter IV. This category comes equipped with an obvious forgetful functor $U_F : C \int F \to C$. We will show the geometric realization of this functor is a covering map, thus defining on objects $BU_\bullet : MI(Set^C) \to Cov(BC)$. On morphisms, it is not hard to see that a natural transformation $\tau : F \to G$ defines a functor $C \int \tau : C \int F \to C \int G$ which is equivariant with respect to the forgetful functors. From this it follows naturally that it defines a map of covering spaces, once we prove BU_F actually is a covering space.
To see that $BU_F : B(C \int F) \to BC$ is a covering of topological space, we will instead show that it is a cover of simplicial sets and refer to the following result without proof.

Proposition 2.2.9. *(Corollary A.49 of [13]) Let $p : E \to B$ be a simplicial covering. I.e. it is a map of simplicial sets, which is surjective on the 0-simplices and such that for any diagram of the form*

$$\begin{array}{ccc} \Delta(0) & \longrightarrow & E \\ \big\downarrow & & \big\downarrow p \\ \Delta(n) & \longrightarrow & B \end{array},$$

there is a lift $\Delta(n) \to E$ which makes the diagram commute. Then the geometric realization of p is a topological covering.

So we want to show that $NU_F : N(C \int F) \to NC$ is a simplicial covering. The surjectivity on 0 simplices follows from the evident surjectivity of U_F. Now consider a diagram of the form

$$\begin{array}{ccc} \Delta(0) & \longrightarrow & N(C \int F) \\ \big\downarrow & & \big\downarrow p \\ \Delta(n) & \longrightarrow & NC \end{array}.$$

The top horizontal map corresponds to a choice $c_0 \in C$ and an element $x_0 \in F(c_0)$. The bottom horizontal map corresponds to a choice of n composable maps starting at c_0, i.e. a sequence

$$c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n.$$

We can apply the functor F to this sequence and view it as a sequence of maps in C/F as follows

$$(c_0, x_0) \to (c_1, f_0(x_0)) \to \cdots (c_n, f_{n-1}(f_{n-2}(\cdots (f_0(x_0))))).$$

This is the desired lift, proving the desired claim. Now we define the map the other way.
It is well known how to define $\pi^{-1}(\gamma) : \pi^{-1}(x) \to \pi^{-1}(y)$ whenever $\gamma : x \to y$ is a path in some topological space X and $\pi : E \to X$ is a covering space. It is also easy to see using the path $\bar{\gamma}(t) = \gamma(1-t)$ that $\pi^{-1}(\gamma)$ is an isomorphism and that this construction depends only on the homotopy class of γ. Applying this to some covering $\pi : E \to BC$, we can define a morphism inverting functor $\pi^{-1} : C \to Set$ by considering objects of C as 0 cells of BC and morphisms as 1 cells. This indeed preserves composition because π^{-1} only detects homotopy class, and the path corresponding to $f \circ g$ is homotopic to the concatenation of the paths corresponding to f and to g. To see that it preserves identities it suffices to recall that the loop corresponding to $Id_c : c \to c$ in BC is contractible. It is not hard to see that a map of covering spaces yields a natural transformation of the corresponding functors.
One can see that $(-)^{-1} \circ BU_\bullet$ is the identity map. On objects this can be seen as the fiber over $c \in BC^0$ of BU_F is $F(c)$ and a similar analysis of morphisms, which we do not make explicit, yields the claim on morphisms. This implies that $(-)^{-1}$ is essentially surjective and full, thus if we show it is faithful we will have shown that it is an equivalence of categories. So suppose we are given two distinct morphisms of covering spaces

$$\begin{tikzcd} E_1 \arrow[rr, "\sigma"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & BC & \end{tikzcd} \qquad \begin{tikzcd} E_1 \arrow[rr, "\tau"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & BC & \end{tikzcd}.$$

Covering spaces of CW complexes are themselves CW complexes by lifting cells. The cell structures of E_1 and E_2 are compatible in such a way that if σ and τ agree on the n-skeleton of E_1, they will agree on the $n+1$ skeleton by commutativity of the above diagrams and because restricted to the interior of each cell π_1 and π_2 are homeomorphisms. So by induction, if $\sigma|_{E_1^0} = \tau|_{E_1^0}$, then $\sigma = \tau$. Now assume by way of contradiction that the natural transformations $\sigma_*, \tau_* : \pi_1^{-1} \to \pi_2^{-1}$ are equal, then by construction this says they agree on the 0-skeleton of E_1, which is a contradiction. Thus, the functor $(-)^{-1} : Cov(BC) \to MI(Set^C)$ is a fully faithful and essentially surjective functor, which shows it is an equivalence of categories. $\square$

Now we return from general theory to our case of interest. The spectral sequence for the functor $S^{-1}X \to \langle S, S\rangle$ is

$$E^2_{pq} = H_p(\langle S, S\rangle; H_q(X)) \Rightarrow H_{p+q}(S^{-1}X).$$

One can observe that the S action on the double complex which leads to this spectral sequence commutes with the differentials and after taking homology the action is independent of a choice of representative within an isomorphism class. Using exactness of localization, this means that we can localize the spectral sequence at $\pi_0(S)$ and still get a spectral sequence. Because S acts invertibly on $S^{-1}X$, the localized spectral sequence is

$$E^2_{pq} = H_p(\langle S, S\rangle, \pi_0(S)^{-1} H_q(X)) \Rightarrow H_{p+q}(S^{-1}X).$$

The functor $\pi_0(S)^{-1}H_q(X) : \langle S, S\rangle \to Ab$ is constant on objects and on morphisms is the map induced by the appropriate translation, which because of the localization is an isomorphism. So this is a morphism inverting functor. This means we can use topological results, namely the following lemma, to simplify E^2_{pq}.

Lemma 2.2.10. *(remark preceding theorem 4.8 in [6] chapter IV) Let $S = iso(S)$ be a symmetric monoidal category with faithful translations. Then $\langle S, S\rangle$ is contractible.*

Proof. Because every map in S is an isomorphism, we have that e is an initial object of $\langle S, S\rangle$. Indeed, let s be some object, then any map $e \to s$ is of the form $(t, \phi : t\Box e \to s)$. Suppose $(t', \phi' : t'\Box e \to s)$ is a different map $e \to s$, then we have a diagram

$$\begin{array}{ccc} t\Box e & \xrightarrow{\phi} & s \\ {\scriptstyle \phi'^{-1}\circ\phi}\downarrow & & \downarrow{\scriptstyle Id_s} \\ t'\Box e & \xrightarrow{\phi'} & s \end{array}.$$

This diagram shows that the two maps are in the same equivalence class, i.e. are the same map in $\langle S, S\rangle$. To prove that e is initial it suffices to notice that for any s, $(s, \eta : s\Box e \to s)$ is a map $e \to s$. This proves that e is initial, and so $B\langle S, S\rangle$ is contractible, ☐

This implies that our spectral sequence becomes

$$E^2_{pq} = \delta_{p=0}\pi_0(S)^{-1}H_q(X) \Rightarrow H_{p+q}(S^{-1}X).$$

There clearly is no room for differentials, so the spectral sequence collapses, and we obtain $\pi_0(S)^{-1}H_q(X) \cong H_q(S^{-1}X)$.
To obtain that the isomorphism is the desired isomorphism we use the comparison theorem (theorem 5.2.12 in [4]) and construct the maps on which we use this result via the following diagram which comes from [13]

$$\begin{array}{ccc} X & \longrightarrow & S^{-1}X \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \langle S, S\rangle \end{array}.$$

We do not detail this step of the proof, but point out to the interested reader that one has to localize the spectral sequences at $\pi_0(S)$ before applying the comparison theorem. ☐

Remark 2.2.11. As I am not entirely comfortable with the notions of spectral sequences, I am not certain that this final argument is indeed the one that shows that the abstract isomorphism is realized by the desired map. I have still included it in the proof as [13] heavily implies that this is in fact the desired method.

Combining this result with the following lemma shows that the map $BS \to BS^{-1}S$ is a group completion.

Lemma 2.2.12. *(lemma 4.3.1 in [6] chapter IV) Under the same assumptions as above, the map $S \to S^{-1}S$ induces a map $\pi_0(S) \to \pi_0(S^{-1}S)$, and this map is the standard group completion of the monoid $\pi_0(S)$.*

Proof. We construct the isomorphism explicitly, let A be the group completion of $\pi_0(S)$. We construct a map $\pi_0(S^{-1}S) \to A$ by defining a mapping $\alpha : S^{-1}S \to A$ which sends $(m, n) \to [m] - [n]$. Given how we defined the maps of $S^{-1}X$, we have $\alpha(s\Box m, s\Box n) = \alpha(m, n)$, and if we have maps $f : m \to m'$ and $f : n \to n'$ we see $\alpha(m, n) = [m] - [n] = [m'] - [n'] = \alpha(m', n')$. This is enough to see that α defines a map $\pi_0(S^{-1}S) \to A$ which is by construction an inverse to the map $\beta : A \to \pi_0(S^{-1}S)$ given by the universal property of A. Indeed, let $[m] \in \pi_0(S)$, then $\alpha(\beta([m])) = \alpha([(m, e)]) = [m] - [e] = [m]$ and similarly $\beta(\alpha([(m, n)])) = \beta([m] - [n]) = \beta([m]) + \beta(-[n]) = [(m, e)] + [(e, n)] = [(m, n)]$. ☐

The above lemma is a computation of $K_0^{\Box}(S)$, we can hope to also be able to compute some more of the lower $K_i^{\Box}$. We compute K_1 in the following proposition. We will also be able to compute K_2 by the end of the next section.

Proposition 2.2.13. *(Corollary 4.8.1 of [6] chapter IV) Let $S = iso(S)$ be such that translations are faithful and denote by Y_S the base point component of $BS^{-1}S$. Then*

$$H_q(Y_S) \cong \varinjlim_{s \in \pi_0(S) \int \pi_0(S)} H_q(Aut(s)).$$

In particular, because $K^{\Box}(S)$ is an H-space

$$K_1(S) \cong \varinjlim_{s \in \pi_0(S) \int \pi_0(S)} H_1(Aut(s)).$$

Proof. When we write $s \in \pi_0(S)$ we are implicitly choosing a representative of an isomorphism class. Recall $S \cong \bigsqcup_{s \in \pi_0(S)} Aut(s)$ where $Aut(s)$ are one object categories. We have for all $q \geq 0$ $H_q(BS^{-1}S) \cong \pi_0(S)^{-1} H_q(BS) \cong \pi_0(S)^{-1} \oplus_{s \in \pi_0(S)} H_q(Aut(s); \mathbb{Z})$. To localize at $\pi_0(S)$ we can form the direct limit over the translation category $\pi_0(S) \int \pi_0(S)$, similarly to forming $\mathbb{Z}_2$ as the direct limit of the system $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \cdots$.
So we have

$$H_*(BS^{-1}S) = \varinjlim_{t \in \pi_0(S) \int \pi_0(S)} H_*(\bigsqcup_{s \in \pi_0(S)} Aut(t \Box s)).$$

However, we are more interested in the homology of the base point component. Recall that the base point of $BS^{-1}S$ is the 0-cell corresponding to the object (e, e), i.e. the image of the 0-cell $e \in BS$ by the map inducing the isomorphism of the previous theorem. And so the component of (e, e) is given by $\varinjlim_{s \in \pi_0(S) \int \pi_0(S)} BAut(s \Box e) \cong \varinjlim_{s \in \pi_0(S) \int \pi_0(S)} BAut(s)$, and so denoting by Y_S the base point component of $BS^{-1}S$ we get that

$$H_q(Y_S) = \varinjlim_{s \in \pi_0(S) \int \pi_0(S)} H_q(Aut(s)).$$

Which is the desired result. $\Box$

2.3 Relating $K_n^{\Box}(P(R))$ to $K_n(R)$

It turns out there is a space homotopy equivalent to $K^{\Box}(S) = BS^{-1}S$ of a symmetric monoidal category, which is very similar to $K(R) = K_0(R) \times BGLR(R)^+$, under a certain cofinality condition. Before relating the +-construction to the group completion construction, we state a cofinality theorem which we need to prove the main result of this section.

Definition 2.3.1. (remark preceding theorem 4.11 in [6] chapter IV) Let $F : S \to T$ be a monoidal functor between symmetric monoidal categories. We call F cofinal if $\forall t \in T, \exists t' \in T, s \in S$ such that $t \Box_T t' \cong f(s)$.

In practice the functor F will often be the inclusion of a subcategory, the following theorem allows us to use cofinal functors as a powerful replacement tool.

Theorem 2.3.2. *(Theorem 4.11 in [6] chapter IV) Let $F : S \to T$ be a cofinal strictly monoidal functor between symmetric monoidal categories such that every map is an isomorphisms and translations are faithful. Then*
(i) If T acts on a category X, then so does S via F, and $S^{-1}X \simeq T^{-1}X$.
(ii) If $F : Aut_S(s) \to Aut_T(Fs)$ is an isomorphism for all $s \in S$, then the base point components of $K^{\Box}(S)$ and $K^{\Box}(T)$ are homotopic.

Proof. (i) The fact that S acts on X via F does not warrant any further explanation. The homotopy equivalence $S^{-1}X \simeq T^{-1}X$ however does. This will be a corollary of the following lemma.

Lemma 2.3.3. *(Exercise 4.6 in [6] chapter IV) Let $S = iso(S)$ be a symmetric monoidal category with faithful translations, which acts invertibly on a category X. Then $S^{-1}X \simeq X$.*

Proof. Using lemma 2.2.6 we have that $S^{-1}X \to \langle S, S\rangle$ is cofibered with fiber X and cobase change maps the translations. And furthermore we showed in lemma 2.2.10 that $\langle S, S\rangle$ is contractible.
The condition that S acts invertibly on X says that the translations are homotopy equivalences, and so the cobase change maps of the cofibered functor $\rho : S^{-1}X \to \langle S, S\rangle$ are homotopy equivalences. So we can use Quillen's theorem B (A.0.3)and get a long exact sequence in homotopy

$$\cdots \to \pi_{n+1}(\langle S, S\rangle) \to \pi_n(X) \to \pi_n(S^{-1}X) \to \pi_n(\langle S, S\rangle) \to \cdots .$$

As we said $\langle S, S\rangle$ is contractible, which implies that the inclusion $X \to S^{-1}X$ is a weak homotopy equivalence, and so by Whitehead's theorem is a homotopy equivalence as geometric realization lands in the category of CW spaces. $\square$

Now we claim that S acts invertibly on X if and only if T does. If T acts invertibly, then it is clear that S does. Now suppose S acts invertibly on X, let $t \in T$, we want to show $t\square : X \to X$ is a homotopy equivalence. Let $t' \in T, s \in S$ be such that $t'\square t \cong F(s)$, and let ϕ be the homotopy inverse of $F(s)\square : X \to X$. Then translation by t has a left homotopy inverse $\phi \circ t'\square$ and right homotopy inverse $t'\square \circ \phi$, which implies that translation by t is a π_n isomorphism for each n, so is a homotopy equivalence.
We remarked in §2.2 when defining the category $S^{-1}X$ that S always acted invertibly on this category. And so we can chain the following homotopy equivalences

$$S^{-1}X \simeq T^{-1}(S^{-1}X) \cong S^{-1}(T^{-1}X) \simeq T^{-1}X,$$

which yields the desired homotopy equivalence. The middle isomorphism comes from explicitly working out the definition of both categories to notice they agree.
(ii) To do this we compare the homology of the base point components Y_S, Y_T of $BS^{-1}S$ and $BT^{-1}T$. We use the homology computation of Y_S in proposition 2.2.13 to compare these spaces.

$$H_*(Y_S) \cong \varinjlim_{\pi_0(S)\int\pi_0(S)} H_*(Aut(s)) \xrightarrow{F} \varinjlim_{\pi_0(S)\int\pi_0(S)} H_*(Aut(Fs)).$$

The map F commutes with the colimit over the translation category by strict monadicity, and is an isomorphism by assumption. By cofinality of F we have

$$\varinjlim_{\pi_0(S)\int\pi_0(S)} H_*(Aut(Fs)) \cong \varinjlim_{\pi_0(T)\int\pi_0(T)} H_*(Aut(t)).$$

As this colimit can be identified with $H_*(Y_T)$ we get that the map induced by F between the base point components of our spaces, Y_S and Y_T is an H-space map which is a homology isomorphism, which by lemma 1.4.3 shows that these spaces are homotopy equivalent. This finishes the proof. $\square$

Before stating the theorem which will establish this connection we need the following definition

Definition 2.3.4. (remark preceding theorem 4.10 in [6]) Let S be a symmetric monoidal category such that there is an object $s \in S$ such that the sequence $\{s^n\}_{n=1}^{\infty}$ is cofinal (note that s^n is the n-fold product $\square : S \times S \to S$ of s with itself, the notation unambiguous up to unique natural isomorphism). Then define $Aut(S) = \varinjlim_{\mathbb{N}} Aut_S(s^n)$.

This definition of $Aut(S)$ is more restrictive than the one used by Weibel, who accepts any cofinal sequence in S to define $Aut(S)$, which is even more restrictive than Bass's definition in [25] where the automorphism group is the colimit of automorphism groups over the translation category of the underlying monoid. We work with our assumption to simplify the proof of the proposition below, and not get too side tracked into chapter VII of Bass's book [25], which we use alongside [26] when proving $[Aut(S), Aut(S)]$ is perfect. Note that our definition is not too restrictive, as major categories of interest satisfy it, notably $FinSet$ with $s = \{*\}$ and $P(R)$ with $s = R$.
We begin by studying this group a bit, to apply lemma 1.4.4. The first thing we need to do is to show that $Aut(S)$ is a direct sum group. For this we use the pairing $-\square- : Aut(s^n) \times Aut(s^m) \to Aut(s^{n+m})$,

the problem is if we use this map, we will not get a map $Aut(S) \times Aut(S) \to Aut(S)$ as these maps do not act well with the stabilization maps. This can be fixed if we postcompose these maps with some Ad_{σ_p}, conjugation by $\sigma \in S_p \hookrightarrow Aut(s^p)$. This can easily be seen to always be doable, by induction. We summarize this idea in the following diagram

$$\begin{tikzcd}
Aut(s) \times Aut(s) \arrow[rr, "\square"] \arrow[d, "(-\square s, -\square s)"] & & Aut(s^2) \arrow[d, "-\square s^2"] \\
Aut(s^2) \times Aut(s^2) \arrow[r, "\square"] \arrow[d, "(-\square s, -\square s)"] & Aut(s^4) \arrow[r, "Ad_{\sigma_2}"] & Aut(s^4) \arrow[d, "-\square s^2"] \\
\vdots \arrow[r] \arrow[d, "(-\square s, -\square s)"] & \cdots \arrow[r] & \vdots \arrow[d, "-\square s^2"] \\
Aut(s^n) \times Aut(s^n) \arrow[r, "\square"] \arrow[d] & Aut(s^{2n}) \arrow[r, "Ad_{\sigma_n}"'] & Aut(s^{2n}) \arrow[d] \\
\vdots & & \vdots
\end{tikzcd}.$$

Now it can be seen that this gives a well-defined map $-\square- : Aut(S) \times Aut(S) \to Aut(S)$ in the colimit.

Proposition 2.3.5. *(Proposition 3 in [26]) Let S be a symmetric monoidal category such that $S = iso(S)$, with faithful translations and having a cofinal subsequence of the form $\{s^n\}_{n=1}^{\infty}$ for some $s \in S$. Then $E = [Aut(S), Aut(S)]$ is the perfect radical of $Aut(S)$.*

Proof. Notice that if the commutator subgroup is perfect, then it must be the perfect radical, as any perfect subgroup is contained in the commutator subgroup.
So we show that E is perfect. To see this we will use a particular case of the abstract Whitehead lemma.

Lemma 2.3.6. *(1.7, 1.8 in [25] chapter VII) Let $\{s^n\}_{n=0}^{\infty} \subset S$ be cofinal. Let $\alpha \in Aut(s^n)$ be of the form $\alpha_1 \square \alpha_2 \square ... \square \alpha_n$, with $\alpha_i \in Aut(s)$, $\sigma \in S_n \leq Aut(s^n)$ be the cycle $\sigma(i) = i - 1(mod\ n)$ and $\beta = Id_s \square \alpha_1^{-1} \square (\alpha_2\alpha_1)^{-1} \square ... \square (\alpha_{n-1}...\alpha_1)^{-1}$.*
Then

$$\beta\sigma\alpha\beta^{-1} = \sigma(Id_s \square Id_s \square ... \square (\alpha_n...\alpha_1)).$$

In particular, if $\alpha_n...\alpha_1 = Id_s$ we have that $\alpha = [\sigma^{-1}, \beta^{-1}]$

This lemma is proven by computing the LHS and RHS and comparing, which we do not bother doing. With this in hand, we can now show that E is perfect. Let $[\alpha, \beta] \in E$, then because $Aut(S) = \varinjlim Aut(s^n)$, there must exist some n such that $\exists \alpha_n, \beta_n \in Aut(s^n)$ such that $\iota_n(\alpha_n) = \alpha, \iota_n(\beta_n) = \beta$ with $\iota_n : Aut(s^n) \to Aut(S)$ the canonical map. Consider now the elements $\alpha_n \square \alpha_n^{-1} \square 1, \beta_n \square 1 \square \beta_n \in Aut(s^{3n})$. By the above lemma, these are both commutators, and so their images $\alpha \square \alpha^{-1} \square 1, \beta \square 1 \square \beta^{-1}$ in $Aut(S)$ are in E. Now computing

$$[\alpha \square \alpha^{-1} \square 1, \beta \square 1 \square \beta^{-1}]$$

yields $[\alpha, \beta]$, which shows that $[E, E] = E$, which is as desired. (note this also shows the non-abstract Whitehead lemma, which we used without proving in the last section). ☐

Lemma 2.3.7. *(Remark preceding proposition 1.2 in [22]) Let S be a symmetric monoidal category such that $S = iso(S)$, with faithful translations and having a cofinal subsequence of the form $\{s^n\}_{n=1}^{\infty}$ for some $s \in S$. Then $G = Aut(S)$ satisfies the following technical properties:*
(i) $[G, G]$ is a perfect subgroup of G.
(ii) For any finite $\{g_i\}_{i=1}^{n} \subset G$ and any $g \in G$, there is $h \in [G, G]$ such that $gg_ig^{-1} = hg_ih^{-1}, \forall i \in [n]$.
(iii) For any finite $\{g_i\}_{i=1}^{n} \subset G$, there exists $a, b \in G$ such that $a(g_i \oplus e)a^{-1} = b(e \oplus g_i)b^{-1} = g_i \forall i \in [n]$.

Proof. We have already proven (i) above. Now assume we are given $\{g_i\}_{i=1}^n \in Aut(S)$ and $g \in Aut(S)$. We can find N large enough so that all the g_i and g come from $Aut(s^N)$, denote the element of this group sent to g_i in the colimit by γ_i and denote the preimage of g in $Aut(s^N)$ by γ. Now we place ourselves in $Aut(s^{2N})$, where all the γ_i are sent to $\gamma_i \Box Id_{s^N}$. Consider $\delta = \gamma \Box \gamma^{-1}$, direct computation shows that

$$(\gamma \Box Id_{s^N})(\gamma_i \Box Id_{s^N})(\gamma^{-1} \Box Id_{s^N}) = \delta(\gamma_i \Box_{s^N})\delta^{-1}.$$

And this equation still holds in the colimit, so in order for this to prove that $Aut(S)$ satisfies property (ii), we need $\delta \in [Aut(s^{2N}), Aut(s^{2N})]$ or that this holds after some amount of stabilizing. But this inclusion is a direct application of the above abstract Whitehead lemma.

It will be useful to distinguish the pairing $Aut(s^n) \times Aut(s^m) \to Aut(s^{n+m})$ for different n, m and also from $Aut(S) \times Aut(S) \to Aut(S)$. So for the remainder of this proof we specify our notation by $\Box_{n,m}$, omitting subscripts if we are talking about the pairing $Aut(S) \times Aut(S) \to Aut(S)$. Note that $\Box : Aut(S) \times Aut(S) \to Aut(S)$ is the stabilization of the maps $Ad_{\sigma_n} \circ \Box_{n,n}$, not of $\Box_{n,n}$. Now we show that $\exists c, d \in Aut(S)$ such that $a(g_i \Box Id)a^{-1} = b(Id \Box g_i)b^{-1} = g_i$. Once again we consider the problem in $Aut(s^N)$, with N as above. And so the equation we want to solve is

$$\alpha(Ad_{\sigma_{2N}}(\gamma_i \Box_{N,N} Id_{s^N}))\alpha^{-1} = \beta(Ad_{\sigma_{2N}}(Id_{s^N} \Box_{N,N} \gamma_i))\beta^{-1} = g_i,$$

with $\alpha, \beta \in Aut(s^{2N})$. By definition of conjugation, this equation is solved by $\alpha = \beta = \sigma_{2N}^{-1}$. $\Box$

We are now ready to prove the big theorem of this section.

Theorem 2.3.8. *(Theorem 4.10 in [6] chapter IV) Let S be a symmetric monoidal category such that $S = iso(S)$, with faithful translations and having a cofinal subsequence of the form $\{s^n\}_{n=1}^\infty$ for some $s \in S$. Then*

$$K(S)^\Box \simeq K_0(S) \times BAut(S)^+,$$

with the plus construction relative to $E = [Aut(S), Aut(S)]$.

Proof. Note that to prove the homotopy equivalence, it will suffice to prove that the connected components are homotopic, as both spaces are of the form $K_0(S) \times B$ with B the appropriate connected component.

Because we are now only comparing connected components we can use the cofinality theorem and replace S by the full subcategory $\{s^n\}_{n=0}^\infty$ (with $s^0 = e$ added in order for the resulting category to indeed be symmetric monoidal). Now by the above lemmas and the lemmas from §1.4 we have that $BAut(S)^+$ is weakly simple, and so to prove the homotopy equivalences $Y_S \simeq BAut(S)^+$ it suffices to construct a homology isomorphism between the two spaces (recall Y_S is also weakly simple because it is an H-space).

We construct this map in three steps.

(i) We first construct maps $BAut(s^n) \to BS^{-1}S$ for each n. We do this by taking the geometric realization of $Aut(s^n) \to S \to S^{-1}S$ with all of the maps the natural inclusions.

(ii) We now need these maps to be compatible, at least up to homotopy, to get a map $BAut(S) \to BS^{-1}S$ by taking the mapping telescope model for $BAut(S)$. The following diagram

$$\begin{tikzcd} Aut(s^n) \arrow[r] \arrow[d, "s\Box-"'] & S \arrow[r] \arrow[d, "s\Box-"'] & S^{-1}S \arrow[d, "s\Box-"'] \\ Aut(s^{n+1}) \arrow[r] & S \arrow[r] & S^{-1}S \end{tikzcd}$$

is commutative. The right most map is given by the action of S on $S^{-1}S$, and we know this action is invertible, i.e. the geometric realization is a homotopy equivalence. Which means taking mapping telescopes we get a well defined map $BAut(S) \to BS^{-1}S$.

(iii) We use the result from §1.4 which extends maps out of a space to the plus construction of that space, assuming that one to be weakly simple. This allows us to extend the map we constructed $BAut(S) \to BS^{-1}S$ to a map $BAut(S)^+ \to BS^{-1}S$. Finally, we get the desired map by noticing that $BAut(S)^+$ is connected, so we can corestrict to Y_S.

Now all that remains to show that the map constructed this way is a homology isomorphism, we can use proposition 2.2.13 to understand that on homology, the map constructed in (i) is the canonical inclusion, in (ii) it is the identity map, and so in (iii) it is an isomorphism. To see this, in the following diagram the +-construction and the map from (ii) are homology isomorphisms, so the third one is as well

$$\begin{tikzcd} BAut(S) \arrow[r] \arrow[d] & Y_S \\ BAut(S)^+ \arrow[ur] & \end{tikzcd}.$$

So this concludes the proof. □

The above theorem has in particular the interesting corollary that the K-theory of a ring R, either as constructed in §1.3 or using the symmetric monoidal category of finitely generated projective modules over R yield the same result.
We can use the above theorem to compute $K_2^{\square}$ in some cases. The result below can be improved if we work with a more general definition of $Aut(S)$ and improve the above theorem consequently.

Proposition 2.3.9. *(Corollary 4.8.1 and exercise 4.10 in [6] chapter IV)(Theorem 4 in [26]) Let $S = iso(S)$ be such that $\exists s \in S$ such that $\{s^n\}_{n=0}^{\infty}$ is cofinal and such that translations are faithful, then*

$$K_2^{\square}(S) = \varinjlim_{n\in\mathbb{N}} H_2([Aut(s^n), Aut(s^n)]; \mathbb{Z}).$$

Proof. We can apply the above theorem and compute $K_2^{\square}$ by computing $\pi_2(BAut(S)^+)$. As $\pi_1(BAut(S)^+) = K_1(S)$ there exists by representability of cohomology a natural map $f : BAut(S)^+ \to K(K_1(S), 1)$, corresponding to the identity map on fundamental groups. We can try to compute $\pi_2(BAut(S)^+)$ by studying the homotopy long exact sequence $F_f \to BAut(S)^+ \to K(K_1(S), 1)$. We can immediately see that F_f is simply connected, and combining Hurewicz with the homotopy long exact sequence yields $K_2(S) \cong H_2(F_f; \mathbb{Z})$. The short exact sequence of groups $E \to Aut(S) \to K_1(S)$ (recall $E = [Aut(S), Aut(S)]$), yields a corresponding homotopy fibration, which we can try to compare to our other homotopy fibration, as in the following diagram

$$\begin{tikzcd} BE \arrow[r] & BAut(S) \arrow[r, "q"] \arrow[d] & BK_1(S) \arrow[d, "Id"] \\ F_f \arrow[r] & BAut(S)^+ \arrow[r, "f"] & BK_1(S) \end{tikzcd}.$$

Where $q : BAut(S) \to K_1(S)$ is the map corresponding to the quotient $Aut(S) \to K_1(S)$ on π_1. The diagram is indeed commutative as we can construct f by extending q. Using the commutativity, we see that the map $BE \to BAut(S)^+$ lands in the homotopy fiber of some point, so we can consider the following map of fibrations

$$\begin{tikzcd} BE \arrow[r] \arrow[d] & BAut(S) \arrow[r, "q"] \arrow[d] & BK_1(S) \arrow[d, "Id"] \\ F_f \arrow[r] & BAut(S)^+ \arrow[r, "f"] & BK_1(S) \end{tikzcd}.$$

Where the right most map is clearly a homology isomorphism, and the middle map is so by construction, so if we show that the base space acts trivially on the homology of the homotopy fiber for both fibrations, we get $H_*(BE; \mathbb{Z}) \cong H_*(F_f; \mathbb{Z})$ by theorem A.0.1. To show this we follow [26].

First we show that $K_1(S) \cong \pi_1(BK_1(S))$ acts trivially on $H_*(BE;\mathbb{Z})$. By lemma 1.4.5 we know that $K_1(S)$ acts on $H_*(BE;\mathbb{Z}) \cong H_*(E;\mathbb{Z})$ via conjugation. For this to make sense use lemma 2.2.13, theorem 2.3.2 and the fact that $H_1(G) = G^{ab}$ is a left adjoint, thus preserves colimits, to notice that $K_1(S) \cong Aut(S)/E$. To see that this action is trivial we can use that proposition 1.4.4 as $Aut(S)$ satisfies the assumptions of the proposition.
The final step of the proof is to show that $\pi_1(BK_1(S))$ acts trivially on $H_*(F_f)$. Notice that the map $f : BAut(S)^+ \to BK_1(S)$ is an H-space map. Now we follow the proof of the lemma on page 16-09 of [27] to show the desired claim. First notice that $BAut(S)^+$ and $BK_1(S)$ are connected H-spaces. We can let F_f be the fiber over the neutral element of $BK_1(S)$, and we can use the neutral elements as the base points for the rest of this proof. Let $[\gamma] \in \pi_1(BK_1(S))$, which because f is a π_1 isomorphism can be represented by $f \circ \gamma_A$ with $\gamma_A : I \to A$ a loop. For each $a_f \in F_f$ we have that $a_f\gamma_A$ (with the product being the H-space product of A) is a lift of $f \circ \gamma_A$. Fixing a simplex $\sigma : \Delta^n \to F_f$ representing some homology class $[\sigma] \in H_n(F_f;\mathbb{Z})$ we see that $[\gamma][\sigma] = [\sigma\gamma_A(1)] = [\sigma e_A]$ with e_A the neutral element of A. But because homology turns homotopy into equality and that e_A is a homotopy unit we have that $[\sigma e_A] = [\sigma]$, which shows the desired trivial action. This in turn allows the application of the spectral sequence comparison theorem which shows $K_2(S) = \pi_2(BAut(S^+)) \cong H_2(F_f;\mathbb{Z}) \cong H_2(BE;\mathbb{Z})$ which yields the desired result after noticing that: $E = \varinjlim_{n\in\mathbb{N}}[Aut(s^n), Aut(s^n)]$, B commutes with sequential colimits and so does homology. □

Remark 2.3.10. I was not able to convince myself that f is in fact an H-space map. My main source [26] claims this to be a consequence of the universality of the map into the +-construction with respect to maps into H-spaces. This presumably is a reference to theorem 1.8 in chapter I of [6]. However, although this result can be used to provide the map f, it says nothing about the map constructed this way being an H-space map.

3 The Q-construction

We introduce the Q-construction of K-theory which works for something called quasi-exact categories. This construction is somewhat more abstract than the preceding two construction, its value lies in its theoretical power to prove theorems. Many elementary properties of K-theory, such as sending products to products, are easier to prove for the Q-construction than with the preceding two constructions. We will end with the $+ = Q$ theorem, which will show that in some case we can enjoy both the relative concreteness of the $+$-construction and the abstract advantages of the Q construction.

3.1 K-theory of quasi-exact categories

In this section we discuss Quillen's Q-construction to define K-theory for exact categories. We will follow [6] chapter IV section 6, and we will adapt the material to the situation of quasi-exact categories; whose definition we take from exercise 14 from the corresponding section of the K-book. I believe Weibel got this notion from [28], as he cites Deitmar, though this paper does not appear in the K-book's bibliography. We do this namely in order for our constructions to cover the category of finite pointed sets.
We also note that in this section the implicit restriction of dealing with small categories enters before taking geometric realization, as it is already necessary when we define the Q-construction.

Definition 3.1.1. (Exercise 6.14 in [6] chapter IV) Consider a category $\mathcal{C}$ with a distinguished zero object 0 and admitting all finite coproducts $\sqcup$. We specify a family $\mathcal{E}$ of sequences of composable maps, which we call exact sequences, in $\mathcal{C}$ of the form

$$0 \to A \to B \to C \to 0.$$

We call $\mathcal{E}$ admissible and the pair $(\mathcal{C}, \mathcal{E})$ a quasi-exact category if $\mathcal{E}$ satisfies

(i) $\mathcal{E}$ is closed under isomorphisms of sequences.

(ii) For any sequence $0 \to A \to B \to C \to 0$ in $\mathcal{E}$ the map $A \to B$ is a kernel of $B \to C$ and $B \to C$ is a cokernel of $A \to B$. So in an exact sequence the maps $A \to B$ are monic and the maps $B \to C$ are epic. We call the monics and epics appearing in an exact sequence admissible.

(iii) All sequences of the form $0 \to A \to A \sqcup B \to B \to 0$ are in $\mathcal{E}$.

(iv) The class of admissible epics is closed under composition and pullback along monics.

(v) The class of admissible monics is closed under composition and pullback along admissible epics.

Two small remarks on notation: we allow ourselves to not explicitly mention the family $\mathcal{E}$ of exact sequences; we will denote admissible monics by $\rightarrowtail$ and admissible epics by $\twoheadrightarrow$. Weibel instead develops the theory for exact categories, defined as follows

Definition 3.1.2. (Definition 7.0 in [6] chapter II) An exact category is a pair $(\mathcal{C}, \mathcal{E})$ with $\mathcal{C}$ an additive category and $\mathcal{E}$ a collection of sequences of the form $0 \to A \to B \to C \to 0$. We require that the pair $(\mathcal{C}, \mathcal{E})$ satisfies

(i) There is an embedding of $\mathcal{C}$ as a full subcategory of an abelian category $\mathcal{A}$.

(ii) $\mathcal{E}$ is the collection of sequences in $\mathcal{C}$ which are exact in $\mathcal{A}$.

(iii) $\mathcal{C}$ is closed under extension in $\mathcal{A}$. This means if $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{A}$ such that A, C are in $\mathcal{C}$, then there is an object isomorphic to B in C.

The claims in [29] §2 reassure us that exact categories are quasi-exact, we will not prove this here as it is simply a matter of chasing diagrams. We however note that to show this claim the first three conditions are easily verified; however, condition four and five require a little more investment.

We can now define the Q-construction on a quasi-exact category $\mathcal{C}$.

Definition 3.1.3. (Definition 6.1 in [6] chapter IV) Let $(\mathcal{C}, \mathcal{E})$ be a quasi-exact category. We define a new category $Q\mathcal{C}$ which has the same objects as $\mathcal{C}$ but has as morphisms from A to B equivalence classes of pairs $A \twoheadleftarrow B_2$ and $B_2 \rightarrowtail B$; with two morphisms equivalent if they fit in a diagram of the form

$$\begin{tikzcd} A & B_2 \arrow[l, two heads] \arrow[r, tail] \arrow[d, "\cong"] & B \arrow[d, "Id"] \\ A \arrow[u, leftarrow, "Id"] & B_3 \arrow[l, two heads] \arrow[r, tail] & B \end{tikzcd} .$$

Composition of morphisms $A \twoheadleftarrow B_2 \rightarrowtail B$ and $B \twoheadleftarrow C_2 \rightarrowtail C$ is $A \twoheadleftarrow C_1 \rightarrowtail C$; where $C_1 = B_2 \times_B C_2$ and the maps are defined by the following diagram

$$\begin{tikzcd} & C_1 \arrow[r, tail] \arrow[d, two heads] & C_2 \arrow[r, tail] \arrow[d, two heads] & C \\ A & B_2 \arrow[l, two heads] \arrow[r, tail] & B & \end{tikzcd} .$$

The fact that the maps $A \twoheadleftarrow C_1$ and $C_1 \rightarrowtail C$ are indeed epic/monic as claimed follows from axioms (iv) and (v) in the definition of a quasi-exact category.

It is worth spending some time to understand this construction better, for this recall that in an arbitrary category a subobject of an object B is an equivalence class of monics $B_2 \rightarrowtail B$, with two monics equivalent if they factor through each other ([6] chapter IV 6.1.2). In a quasi-exact category $\mathcal{C}$, we call a subobject admissible if any choice of representative is an admissible monic. And so we see that morphism in $Q\mathcal{C}$ from A to B uniquely define a subobject of B; picking a representative, say B_2, to finish defining our morphism in $Q\mathcal{C}$ we just need an epic map from B_2 to A. This perspective allows us to see that morphisms from 0 to B in $Q\mathcal{C}$ correspond to subobjects of B.
Notice also that there are two distinguished classes of morphisms in $Q\mathcal{C}$ of particular importance. There are the morphisms of the form $A = A \rightarrowtail B$ and $A \twoheadleftarrow B = B$, and it is easy to see that every morphism $A \twoheadleftarrow B_2 \rightarrowtail B$ factors uniquely as the composition of $A \twoheadleftarrow B_2$ and $B_2 = B_2 \rightarrowtail B$.
We can combine these two observation to obtain a 1-1 correspondence between isomorphisms in $\mathcal{C}$ and $Q\mathcal{C}$. Indeed, if $i : A \cong B$ is an isomorphism in $\mathcal{C}$ we get an isomorphism in $Q\mathcal{C}$ which we can represent either as $A \twoheadleftarrow B = B$, because i^{-1} is an epimorphism, or as $A = A \rightarrowtail B$ because i is a monomorphism. Now given any isomorphism from A to B in $Q\mathcal{C}$, we see that the subobject defined by the given isomorphism is maximal, and so is isomorphic to B, which implies that every isomorphism comes from an isomorphism in $\mathcal{C}$ by choosing an appropriate representative.

Having developed some intuition with the Q-construction, we can define the associated K-theory space.

Definition 3.1.4. (Definition 6.3 in [6] chapter IV) Let $\mathcal{C}$ be a quasi-exact category, then we define $K\mathcal{C} = \Omega BQ\mathcal{C}$ and we define the K-groups of $\mathcal{C}$ by

$$K_i(\mathcal{C}) = \pi_i(K(\mathcal{C})).$$

In order for this definition of $K_i(\mathcal{C})$ to make unambiguous sense when considering base points we need $BQ\mathcal{C}$ to be connected. This can be seen by noticing that the map $0 \to A$ in C defines a morphism in $Q\mathcal{C}$ from 0 to A. Thus, we have a path between the corresponding vertices of $BQ\mathcal{C}$, which shows that the 1-skeleton of this space is connected, implying that the space is connected.
Note also that this is a functorial construction from the category whose objects are quasi exact categories and morphisms are functors preserving finite limits and finite colimits. We call such functors exact, as they send exact sequences to exact sequences in quasi-exact categories. In exact categories, an additive functor is exact if and only if it preserves exact sequences, further justifying the definition (see [1] chapter VIII specifically page 201). The only point which needs discussing is that an exact functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $QF : Q\mathcal{C} \to Q\mathcal{D}$. We discuss this in the next section §3.2.
It is perhaps strange that we take functors preserving all finite limits and colimits as our definition of

an exact functor, instead of simply a functor sending exact sequences to exact sequences. We do this in order for Q to be functorial, and we justify it by the following example of a functor which preserves exact sequences, but not coproducts.

Example 3.1.5. Consider the forgetful functor $FinAb \to FinSet_*$ from the category of finite abelian groups to the category of pointed finite sets, the distinguished point being the identity. $FinAb$ has a clear exact structure, and for $FinSet$ take those sequence $A \xrightarrow{f} B \xrightarrow{g} C$ such that $Im(f) = g^{-1}(*_C)$ with $*_C$ the distinguished element of C. The forgetful functor clearly sends exact sequences to exact sequences, but does not preserve the coproduct, as in $FinAb$ the cardinality is multiplicative with respect to the coproduct, but it is additive with respect to the coproduct of $FinSet$.

We now compute $K_0(\mathcal{C})$.

Proposition 3.1.6. *(Proposition 6.2 in [6] chapter IV)(Theorem 1 in [29] §2) Let $\mathcal{C}$ be a quasi-exact category. The group $K_0(\mathcal{C})$ is generated by the objects $[A]$ of $\mathcal{C}$ subject to the relations $[B] = [A] + [C]$ whenever $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{C}$.*

Proof. Call G the group generated by the objects of $\mathcal{C}$ subject to the relations $[B] = [A]+[C]$ whenever there is an exact sequence $0 \to A \rightarrowtail B \twoheadrightarrow C \to 0$. Note that the additive notation is justified as for any two objects $[A], [B]$ we have the two exact sequence

$$0 \to A \to A \sqcup B \to B \to 0$$

and

$$0 \to B \to B \sqcup A \to A \to 0.$$

We present only the proof from [6], there is another very interesting proof in [29]. Notice it is an easy consequence of the Seifert-Van Kampen theorem that for any small category C if we find a maximal tree T of the graph underlying C, then $\pi_1(C)$ admits a presentation with generators all morphisms of C and relations given by

(i) $[f] = 1$ if $f \in T$

(ii) $[Id_c] = 1$ for all $c \in C$

(iii) $[f \circ g] = [f][g]$.

In our case we can take T to be the collection of all morphisms $0 = 0 \rightarrowtail A$, which is both clearly a tree and maximal. From the composition $(0 = 0 \rightarrowtail A) \circ (A = A \rightarrowtail B) = (0 = 0 \rightarrowtail B)$ we see that every morphism $A \twoheadleftarrow B_1 \rightarrowtail B$ in $Q\mathcal{C}$ defines an element of $\pi_1(BQ\mathcal{C})$ independent of the monic in $\mathcal{C}$ defining the morphism. Next we see that the composition $(0 \twoheadleftarrow A = A) \circ (A \twoheadleftarrow B = B) = (0 \twoheadleftarrow B = B)$ shows that we can restrict our set of generators to morphisms of the form $0 \twoheadleftarrow A$, which we from here on out abbreviate to (A). Now consider $0 \to A \rightarrowtail B \twoheadrightarrow C \to 0$ an exact sequence of $\mathcal{C}$. We consider the composition in $Q\mathcal{C}$ defined by the following diagram

$$\begin{tikzcd} & P \cong ker(B \twoheadrightarrow C) \cong A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, tail] \arrow[d, two heads] & B \\ 0 & 0 \arrow[l, two heads] \arrow[r, tail] & C & \end{tikzcd}.$$

Which by the observations we have already made shows that for an exact sequence $0 \to A \rightarrowtail B \twoheadrightarrow C \to 0$ we have $(C \twoheadleftarrow B) = (A)$ in $\pi_1(BQ\mathcal{C})$. This has two consequences. The first is obtained by multiplying by (C) on the left yields $(B) = (C)(A)$, which by the same reasoning as for G implies our fundamental group to be abelian. The second is that two admissible epics with the same kernel yield the same element in $\pi_1(BQ\mathcal{C})$. We now have a clear map $\pi_1(BQ\mathcal{C}) \to G$ sending (A) to $[A]$. This map is an isomorphism if every relationship in $\pi_1(BQ\mathcal{C})$ can be obtained from what we have already discussed. To do this, consider an arbitrary composition of morphisms in $Q\mathcal{C}$

$$\begin{tikzcd} & B_1 \times_B C_1 \arrow[r, tail] \arrow[d, two heads] & C_1 \arrow[r, tail] \arrow[d, two heads] & C \\ A & B_1 \arrow[l, two heads] \arrow[r, tail] & B & \end{tikzcd}.$$

This yields the relationship $(B \twoheadleftarrow C_1)(A \twoheadleftarrow B_1) = (A \twoheadleftarrow B_1 \times_B C_1)$. We can multiply by (A) (no need to specify left or right as we have shown that the group is abelian), yielding the equation $(B \twoheadleftarrow C_1)(B_1) = (B_1 \times_B C_1)$. We can decompose $(B_1 \times_B C_1)$ as $(B_1)(B_1 \twoheadleftarrow B_1 \times_B C_1)$; now we may cancel out (B_1) to get the equation $(B \twoheadleftarrow C_1) = (B_1 \twoheadleftarrow B_1 \times C_1)$ which is equivalent to our original equation. These two elements are in fact the same in $\pi_1(BQ\mathcal{C})$ as they have the same kernel by abstract nonsense, which by above reasoning is enough to show that they define the same element. This completes the proof.

□

3.2 Elementary properties

In this section we prove some simple computational tools for the K-theory of quasi-exact categories.

Proposition 3.2.1. *(Mentioned in definition 6.3 and exercise 6.2 of chapter IV of [6]) The Q-construction is a functor from the category of small quasi-exact categories qE to the category of small categories Cat. Furthermore, for any pair of quasi-exact categories $\mathcal{C}, \mathcal{D}$ it induces a functor, which we also call Q, from $Fun_{qE}(\mathcal{C}, \mathcal{D})$ to $Fun_{Cat}(Q\mathcal{C}, Q\mathcal{D})$.*

Proof. We first define $Q : qE \to Cat$ on exact functors. Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between quasi-exact categories. On objects we naturally will define $QFA = FA$ as $Q\mathcal{C}$ and $\mathcal{C}$ have the same objects. On morphisms, because F preserves admissible monics and admissible epics, we can simply define $F(A \twoheadleftarrow B_1 \rightarrowtail B) = FA \twoheadleftarrow FB_1 \rightarrowtail FB$. Now we need to show that QF respects composition. This follows from the fact that F preserves exact sequences and finite limits, in particular pullbacks. Now that we have defined QF for any exact functor of quasi-exact categories, we can work on the second part of the proposition. Fix quasi-exact categories $\mathcal{C}, \mathcal{D}$, we aim to define $Q : Fun_{qE}(\mathcal{C}, \mathcal{D}) \to Fun_{Cat}(Q\mathcal{C}, Q\mathcal{D})$. To do this we need to define Q on natural transformations. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors and $\tau : F \to G$ a natural transformation, then we can define $Q\tau : QF \to QG$ by $Q\tau_A = \tau_A$, as justified by the following diagram

$$\begin{tikzcd}
F(A) \arrow[d, "\tau_A"] & F(B_1) \arrow[l, two heads] \arrow[r, tail] \arrow[d, "\tau_{B_1}"] & F(B) \arrow[d, "\tau_B"] \\
G(A) & G(B_1) \arrow[l, two heads] \arrow[r, tail] & G(B)
\end{tikzcd} .$$

The map in the middle is not actually important, as it disappears" into the equivalence relation. But it serves to show that $Q\tau$ defined this way is indeed a natural transformation. With all of our constructions the preservation of identities is clear. □

This result has the following corollary.

Corollary 3.2.2. *(Exercise 6.2 in [6]) Let $\mathcal{C} \cong \mathcal{C}'$ be equivalent quasi-exact categories, with the equivalence established by exact functors. Then $Q\mathcal{C} \cong Q\mathcal{C}'$.*

This is namely useful to show that if we work with categories which are only skeletally small, its K-theory can be defined by choosing an equivalent small subcategory. And the K-theory defined this way is independent of the choice of equivalent small subcategory.

The product of two quasi-exact categories can easily be seen to be quasi-exact. The geometric realization is known to commute with products, and it is easy to see that the Q-construction commutes with products as well.

Proposition 3.2.3. *(6.4 in [6] chapter IV) Let $\mathcal{C}, \mathcal{D}$ be quasi-exact categories, then $K_i(\mathcal{C} \times \mathcal{D}) = K_i(\mathcal{C}) \times K_i(\mathcal{D})$.*

This proposition is enough to show that $K(\mathcal{C})$ is an H-space as the coproduct is always exact, so defines a map $Q\mathcal{C} \times Q\mathcal{C} \to Q\mathcal{C}$ which is associative up to natural isomorphism and admits a unit up to natural isomorphism (the vertex corresponding to 0).

Proposition 3.2.4. *(Example 7.1.7 in [6] chapter II) Let I be a small filtering category, and $\mathcal{C}_\bullet : I \to qE$ be an I-shaped diagram of small quasi-exact categories. Then the colimit $\mathcal{C} = \varinjlim_{i\in I} \mathcal{C}_i$ is a quasi-exact category and $K_n(\mathcal{C}) = \varinjlim_{i\in I} K_n(\mathcal{C}_i)$*

Proof. We need to show the colimit admits a natural quasi-exact structure. Because geometric realization and homotopy groups commute with filtered colimits, we just need to show that the Q construction preserve filtered colimits in order to have the desired claim.
Let $\mathcal{C} = \{\mathcal{C}_i\}_{i\in I}$ be a small filtered family of small quasi-exact categories. It will be useful to understand (co)limits in the category $\varinjlim_I \mathcal{C}_i$ better, to do this we use a realization I got while reading [30]. Denote the set of functors between two small categories A, B by $Fun(A, B)$. Then we have $Fun(J, \mathcal{C}) = \varinjlim_I Fun(J, \mathcal{C}_i)$. This means J-shaped diagrams in $\mathcal{C}$ always come from some $\mathcal{C}_i$. This also means that the natural functors $\iota_i : \mathcal{C}_i \to \mathcal{C}$ are cocontinuous as colimits commute with colimits. Furthermore, I claim that we can similarly create finite limits: Suppose J is a finite category, and let $F : J \to \mathcal{C}$ be a J-shaped diagram in $\mathcal{C}$. This diagram comes from some $\mathcal{C}_i$. To account for this, we call $F : J \to \mathcal{C}_i$, and refer to our original F by the appropriate post composition $\iota_i F : J \to \mathcal{C}$. Assume further that in some $\mathcal{C}_k$ for $k \geq i$ $\iota_{ik} F : J \to \mathcal{C}_k$ admits a limit. Let $d \in \mathcal{C}$ be some object with some cone over $\iota_i F : J \to \mathcal{C}$. We can use the fact that I is filtered and J to find some k' such that the cone with apex d comes from $\mathcal{C}_{k'}$ and $k' \geq k$. Because the functors $\iota_{kk'} : \mathcal{C}_k \to \mathcal{C}_{k'}$ are exact, we get $\iota_{kk'}(\varprojlim_J \iota_{ik} F) = \varprojlim_J \iota_{ik'} F$, and so the cone in $\mathcal{C}_{k'}$ defines a map $d \to \varinjlim_J \iota_i F$ in $\mathcal{C}$. Uniqueness of this map is proven similarly, assuming we have another one, finding some $\mathcal{C}_{i'}$ which contains both the map we constructed and the preimage of this hypothetical other map. In this category these two maps must be the same by construction of the limit, and so they are in fact the same map in $\mathcal{C}$.
In order to discuss a quasi-exact structure on $\mathcal{C}$ we need to specify a family of quasi exact sequences. An exact sequence is a $\bullet \to \bullet \to \bullet$ shaped diagram, these all come from some $\mathcal{C}_i$, so call a sequence in $\mathcal{C}$ exact if it is exact in some $\mathcal{C}_i$. Notice that the sequence is then also exact in each $k \geq i$. For this to make sense we need to be sure if $A_i \to B_i \to C_i$ is exact in $\mathcal{C}_i$, then, denoting the image in $\mathcal{C}$ by $A \to B \to C$, we need the first map to be the kernel of the second, and the second the cokernel of the first. But, by what we have shown, the functors $\iota_i : \mathcal{C}_i \to \mathcal{C}$ preserve finite limits and colimits, so the desired property is satisfied. This implies the inclusion functors $\iota_i : \mathcal{C}_i \to \mathcal{C}$ sends exact sequences to exact sequences, and so in particular admissible monomorphisms to admissible epimorphisms. In light of our discussion about colimits and finite limits in $\mathcal{C}$, it is easy to see that $\mathcal{C}$ with the specified family of exact sequences is a quasi-exact category.
The fact that the Q-construction preserves colimits is not hard to see, and writing it out does not teach much, so we omit the proof. □

We remark that the mapping $R \to P(R)$ sending a ring to the category of finitely generated projective modules is not a functor out of the category of rings. In the next section we will relate the K-theory of $P(R)$ via the Q-construction to the K-theory developed in §1.3. Assuming this, our preceding remark implies that if $R = \varinjlim_{\alpha\in I} R_\alpha$, we do not immediately have $K_i(R) = \varinjlim_I K_i(R_\alpha)$. It turns out we actually do have this result. We can see this either by replacing $P(R)$ with the equivalent category of idempotent matrices over R (with maps induced by the maps $R^n \to R^m$) (see 6.4 [6] chapter IV) or to use a device called Kleisli rectification (see exercise 6.5 [6] chapter IV).

3.3 The $+ = Q$ theorem

Following §7 of [6] chapter IV, we prove the key theorem in relating the different K-theories developed thus far. Consider a quasi-exact category $(\mathcal{A}, \vee, 0, \mathcal{E})$ there is a natural underlying symmetric monoidal category obtained simply by ignoring $\mathcal{E}$. Heuristically, we would expect the loss of information in doing so to correspond to the exact sequences which are not split, i.e. of the form $0 \to A \to A \vee B \to B \to 0$, as these exact sequences can in a sense be detected by the symmetric monoidal structure. To state this idea precisely we call a quasi-exact category split if every exact sequence is of the form. This intuition turns out to be true as given by the following theorem.

Theorem 3.3.1. *(Theorem 7.1 of [6] chapter IV) Let $\mathcal{A}$ be a split quasi-exact category, and let*

$S = iso(\mathcal{A})$ *be seen as a symmetric monoidal category. Then*

$$\Omega BQ\mathcal{A} \simeq BS^{-1}S.$$

In the case that $\mathcal{A} = P(R)$ is the category of finitely generated projective modules, it is not hard to check this category is split exact, and further because this category admits the cofinal sequence $\{R^n\}_{n=1}^{\infty}$ we can use the above theorem and theorem 2.3.8 to show that the Q-construction on $P(R)$ yields the same K-theory as the plus construction §1.3. So in particular we can transfer the results from §3.2 to the case of rings (and any future results we might prove).

The main idea of the proof is to construct a fibration $BS^{-1}S \to BC \to BQ\mathcal{A}$, with C contractible, as taking the homotopy fiber will yield a fibration $\Omega BQ\mathcal{A} \to BS^{-1}S \to BC$, the first map will be the desired weak equivalence by the homotopy long exact sequence and thus a homotopy equivalence because both spaces are CW complexes. To construct the category C we need the following definition.

Definition 3.3.2. (7.3 in [6] chapter IV) Given a quasi-exact category $\mathcal{A}$, we define the category $\mathcal{E}\mathcal{A}$ as follows. The objects of $\mathcal{E}\mathcal{A}$ are the admissible short exact sequences of $\mathcal{A}$ and morphisms between two such sequences $A \rightarrowtail B \twoheadrightarrow C, A' \rightarrowtail B' \twoheadrightarrow C'$ are equivalence classes of diagrams of the form

$$\begin{tikzcd}
A \arrow[r, tail] & B \arrow[r, two heads] & C \\
A'' \arrow[r, tail] \arrow[u, tail, "\alpha"] \arrow[d, "Id"'] & B \arrow[r, two heads] \arrow[u, "Id"'] \arrow[d, tail, "\beta"'] & C'' \arrow[u, two heads] \arrow[d, tail] \\
A' \arrow[r, tail] & B' \arrow[r, two heads] & C'
\end{tikzcd},$$

with the middle row also an admissible exact sequence. Two such diagrams are equivalent if there is an isomorphism between them which is the identity at every vertex except potentially at the middle right vertex.

Because of how morphisms are defined, there is an obvious functor $t : \mathcal{E}\mathcal{A} \to Q\mathcal{A}$ sending $A \rightarrowtail B \twoheadrightarrow C$ to C. We denote the preimage of a certain $C \in Q\mathcal{A}$ by $\mathcal{E}_C$ (instead of just $t^{-1}(C)$). Recall this is the subcategory of $\mathcal{E}\mathcal{A}$ whose objects are those sent to C by t and morphisms are those sent to Id_C by t. To be explicit, morphisms are the (equivalence class of) diagrams of the form

$$\begin{tikzcd}
A \arrow[r, tail] & B \arrow[r, two heads] & C \\
A'' \arrow[r, tail] \arrow[u, tail, "\alpha"] \arrow[d, "Id"'] & B \arrow[r, two heads] \arrow[u, "Id"'] \arrow[d, tail, "\beta"'] & C \arrow[u, "Id"'] \arrow[d, "Id"'] \\
A' \arrow[r, tail] & B' \arrow[r, two heads] & C
\end{tikzcd},$$

with the rows exact sequences. We can see (7.4 [6] chapter IV) that α is an isomorphism as $A \rightarrowtail B$ and $A'' \rightarrowtail B$ are both kernels of the same map, so are isomorphic, and it is not hard to see α is this isomorphism. Because we are working in a split quasi-exact category both B and B' are isomorphic to $A \vee C$ and that β is an isomorphism between them, coming from the universal property as the coproduct. There is an inclusion $S \to \mathcal{E}\mathcal{A}$, sending A to $A \xrightarrow{Id_A} A \to 0$ and sending an isomorphism $\sigma : A \to A'$

$$\begin{tikzcd}
A \arrow[r, "Id"] & A \arrow[r] & 0 \\
A' \arrow[r, "\sigma^{-1}"] \arrow[u, "\sigma^{-1}"] \arrow[d, "Id"'] & A \arrow[r] \arrow[u, "Id"] \arrow[d, "\sigma"'] & 0 \arrow[u] \arrow[d] \\
A' \arrow[r, "Id"] & A' \arrow[r] & 0
\end{tikzcd}.$$

As $\mathcal{EA}$ is symmetric monoidal under the operation of pointwise direct sum of exact sequences (we are allowed to do these constructions as all our exact sequences split); we have that the category S acts on $\mathcal{EA}$ by $A \vee (A' \rightarrowtail B' \twoheadrightarrow C') = A \vee A' \rightarrowtail A \vee B' \twoheadrightarrow C'$. Notice that for an exact sequence E in $\mathcal{EA}$ we have $t(A \square E) = t(E)$, we say that the action is fiberwise (with respect to t).

The proof of the theorem is presented just as in [6] as a chain of lemmas. Our first two lemmas serve to define the fibered functor which we will, with Quillen's theorem B (A.0.3), use to obtain the desired homotopy fibration.

Lemma 3.3.3. *(Exercise 7.2 and lemma 7.7 of [6] chapter IV) With the same notation as for the main theorem. The functor $t : \mathcal{EA} \to Q\mathcal{A}$ is fibered. The base change maps are detailed in the course of the proof.*

Proof. The category $\mathcal{E}_C$ embeds in C/t as the full subcategory of pairs of the form $(C = t(E), E)$ where $E = A \rightarrowtail B \twoheadrightarrow C$ is an exact sequence in $\mathcal{E}_C$. This way we have an inclusion $I : \mathcal{E}_C \to C/t$. Our goal is to find a functor $I \dashv R$ which is a right adjoint to I, thus showing t is pre-fibered. Let $(C \xrightarrow{\phi} t(E'), E')$ with $E' = A' \rightarrowtail B' \twoheadrightarrow C'$ an exact sequence in $\mathcal{EA}$. More explicitly, E' and ϕ fit into the diagram

$$\begin{tikzcd}
 & & C \\
 & & C_1 \arrow[u, two heads, "\sigma"'] \arrow[d, tail, "\tau"] \\
A' \arrow[r, tail, "f'"] & B' \arrow[r, two heads, "g'"] & C'
\end{tikzcd} .$$

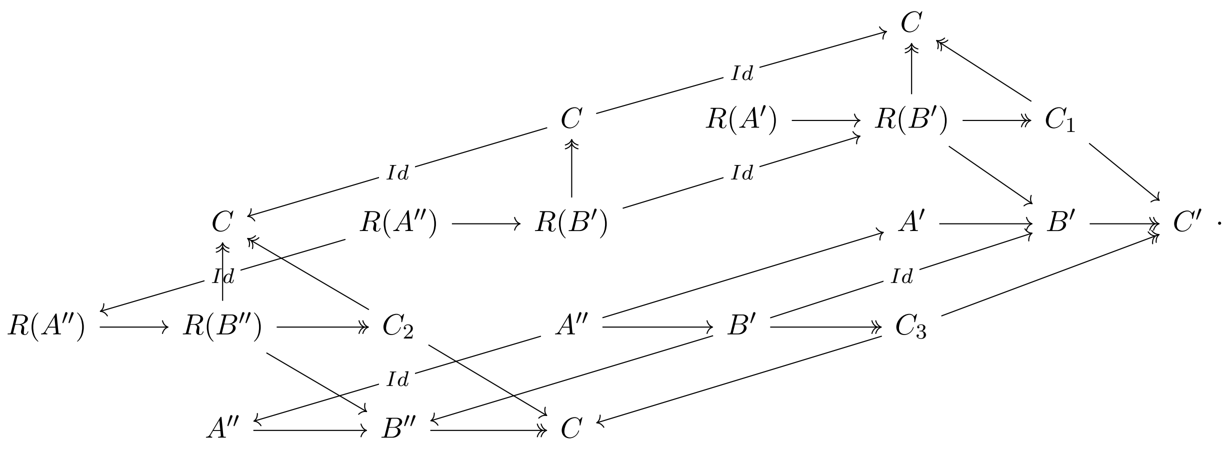
We can complete this diagram as

$$\begin{tikzcd}
 & & C \\
A^* \cong ker(\sigma \circ g^*) \arrow[r, tail] & B^* \cong B' \times_{C'} C_1 \arrow[r, "g^*"] \arrow[d, tail, "\tau^*"] & C_1 \arrow[u, two heads, "\sigma"'] \arrow[d, tail, "\tau"] \\
A' \arrow[r, tail, "f'"] & B' \arrow[r, two heads, "g'"] & C'
\end{tikzcd} .$$

Fixing a choice B^*, A^* of pullback and kernel, we can define $R((\phi : C \to t(E'), E')) = A^* \rightarrowtail B^* \twoheadrightarrow C$. Let $(C \to t(E''), E'')$ be another element in C/t, we want to define R on morphisms; to do that we need a morphism in C/t which, by choosing a representative, is defined by a diagram of the following form (which we purposefully contort somewhat to have space to add more elements)

$$\begin{tikzcd}
 & & & & C & & \\
 & & C \arrow[urr, "Id"] \arrow[dll, "Id"'] & & & C_1 \arrow[ul, two heads] \arrow[dr, tail] & \\
C & & & & A' \arrow[r] & B' \arrow[r, two heads] & C' \\
 & C_2 \arrow[ul, two heads] \arrow[dr] & A'' \arrow[urr] \arrow[r] \arrow[dll, "Id"] & B' \arrow[r, two heads] \arrow[urr, "Id"] \arrow[dll] & C_3 \arrow[urr, two heads] \arrow[dll] & & \\
A'' \arrow[r] & B'' \arrow[r, two heads] & C & & & &
\end{tikzcd} .$$

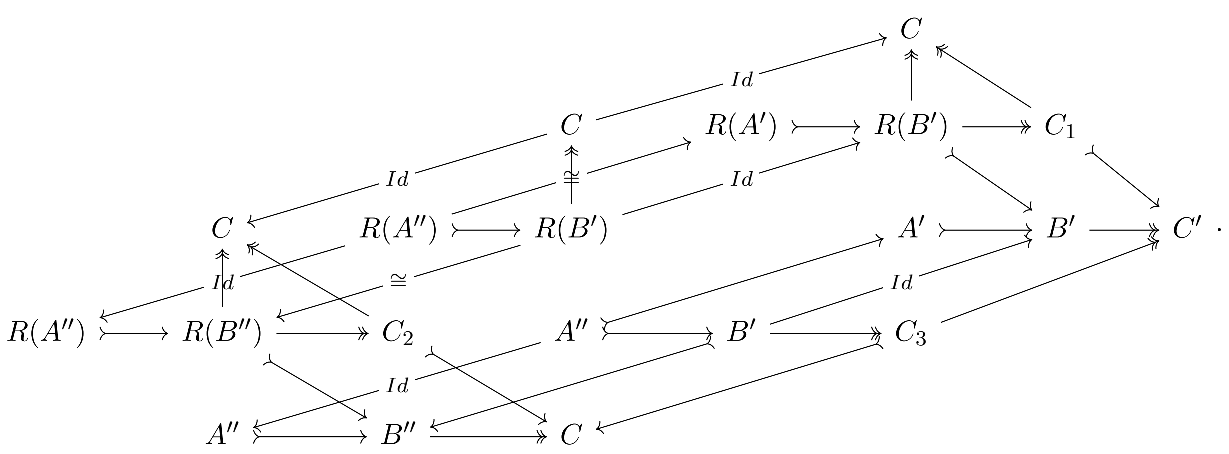
We abuse notation slightly and write $R(A' \rightarrowtail B' \twoheadrightarrow C', C \to t(E'))$ as $R(A') \rightarrowtail R(B') \twoheadrightarrow C$. We can fill the above diagram with $R(A''), R(B''), R(A')$ and $R(B')$



The middle exact sequence $R(A'') \to R(B') \to C$ and the identity maps have been added in anticipation of the morphism we wish to define between $R(E')$ and $R(E'')$ in $\mathcal{E}_C$. To define the map $R(A'') \to R(A')$ use that these are both kernels of $R(B') \to C$. To define the map $R(B') \to R(B'')$ we need to define maps $R(B') \to B''$ and $R(B') \to C_2$. The first of these maps already is in our diagram as can be seen by recalling that the identity map can be traversed "in reverse". To construct a map $R(B') \to C_2$ we can use that the slanted face commutes, i.e. the following triangle commutes in $Q\mathcal{A}$

$$\begin{tikzcd}
 & C & \\
C_1 \arrow[ur, twoheadrightarrow] \arrow[d, leftarrowtail] & & C_2 \arrow[ul, twoheadrightarrow] \arrow[d, rightarrowtail] \\
C' & C_3 \arrow[l, twoheadrightarrow] \arrow[r, rightarrowtail] & C''
\end{tikzcd} .$$

Indeed, this implies that C_2 is the pullback of C_1 and C_3 along C', and so to define the map $R(B') \to C_2$ it suffices to use the maps which already exist to C_1 and C_3. All of this can be summarized in the following diagram

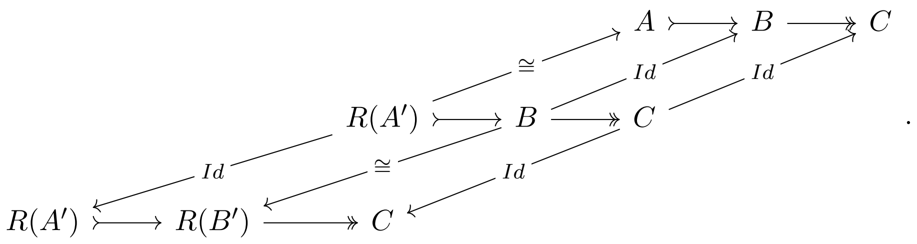


The fact that the maps in $\mathcal{E}_C$ are isomorphisms as already been discussed following the definition of $\mathcal{E}\mathcal{A}$. This defines R on morphisms.

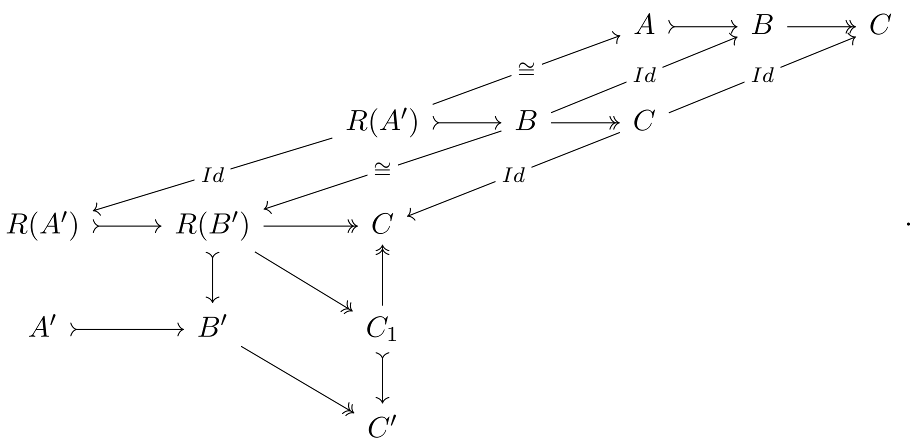
Now having defined a candidate right adjoint, we need to show it is actually a right adjoint. Fixing $E = A \rightarrowtail B \twoheadrightarrow C$ and $E' = A' \rightarrowtail B' \twoheadrightarrow C'$, we want to define mutually inverse isomorphisms.

$$\psi : Hom_{\mathcal{E}_C}(E, R((C \to t(E'), E'))) \cong Hom_{C/t}(I(E), (C \to t(E'), E')) : \varphi.$$

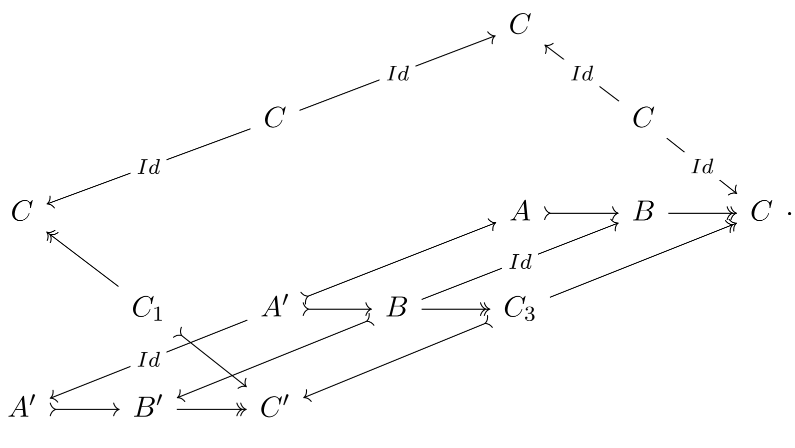
We will content ourselves with defining ψ, φ. A morphism E to $R(E')$ in $\mathcal{E}_C$ is given by the following diagram



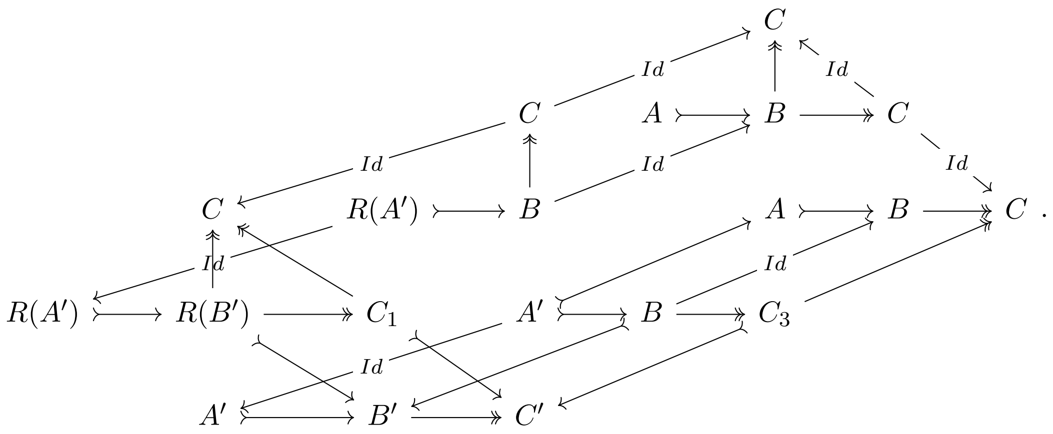
In order to define a corresponding map $I(E) \to (C \to t(E'), E')$ in C/t it is useful to complete the above graph with the information "lost" by R



We want to define a morphism under C between E' and E in C/t. For this we need a map $A' \to A$, a map $B \to B'$ and a map $C \to C'$ (though this last one in $Q\mathcal{A}$, not in $\mathcal{A}$ like the other two). There is a map from B to B' and a natural choice of map from $C \to C'$ is given by $C \twoheadleftarrow C_1 \rightarrowtail C'$. The only map which needs constructing is a map $A' \to A$. For this notice that adding a 0 map $A' \to C_1$ defines a cone over the diagram of which $R(B')$ is a limit, thus defines a map $A' \to R(B')$. From this it is not hard to see that this map factors through $R(A')$, which we recall is the kernel of $R(B') \to C$. Once this is done it is clear how to use this map to define a map from $A' \to A$. This defines ϕ.
We now assume we have a map $I(E) \to (C \to t(E'), E')$, which can be described by the following diagram



And so we need to define a map $E \to R((C \to t(E'), E'))$. To do this we want to fill the following diagram with maps $R(A') \to A$ and $B \to R(B')$



The map $R(A') \to B$ clearly factors through A. To define the map $B \to R(B')$ we use a reasoning analogous to when we defined R on morphisms. Because of the commutativity of the slanted face (as a diagram in $Q\mathcal{A}$) we see that C_1 is a pullback of C and C_3. We have obvious maps from B to C and C_3, thus defining a natural map to C_1. Because we have a natural map $B \to B'$ this defines via pullback a natural map $B \to R(B')$. This defines the map ψ. This completes the part of the proof that $t : \mathcal{EA} \to Q\mathcal{A}$ is fibered which we have set out to detail. We have mainly omitted explicitly checking that: R is a functor, ϕ and ψ are mutually inverse and that they are natural in both variables. $\square$

Lemma 3.3.4. *(Exercise 4.11 and 7.2 in [6] chapter IV) Let S be a symmetric monoidal category which acts on a category X and let Y be another category. Let $F : X \to Y$ be a functor. We call the action fiberwise (with respect to F) if*

$$\begin{tikzcd} & X \arrow[dr, "F"] & \\ S \times X \arrow[ur, "\pi_X"] \arrow[r, "\square"] & X \arrow[r, "F"] & Y \end{tikzcd}$$

commutes. Then for all $y \in Y$ the action of S on X restricts to the fibers $F^{-1}(y)$, and so induces a functor $S^{-1}F : S^{-1}X \to Y$ whose fibers are $(S^{-1}F)^{-1}(y) = S^{-1}F^{-1}(y)$.
If in addition F is fibered and the action of S on the fibers commute with the base change functors, we call such functors cartesian, then we also have that the functor $S^{-1}F$ is fibered.

Proof. The fact that the action restricts to the fibers is clear. Now define $S^{-1}F : S^{-1}X \to Y$ on objects by $(s, x) \to F(x)$. To define $S^{-1}F$ on maps, consider a diagram

$$\begin{tikzcd} (t\square s, t\square x) \arrow[r, "(\phi,f)"] \arrow[d, "\sigma\square Id_{(s,x)}"'] & (s', y) \arrow[d, "Id"] \\ (t'\square s, t'\square x) \arrow[r, "(\phi',f')"] & (s', y) \end{tikzcd},$$

with $\sigma : t \to t'$ an isomorphism. We can view this as a diagram in $S \times X$ and apply the functor F

$$\begin{tikzcd} F(x) \arrow[r, "f"] \arrow[d, "Id_{F(x)}"] & F(y) \arrow[d, "Id_{F(y)}"] \\ F(x) \arrow[r, "f'"] & F(y) \end{tikzcd}.$$

Where all the simplification comes from the fact that the action of S is fiberwise. So we see that $F(f) = F(f')$ whenever $(\phi, f) = (\phi', f')$, thus defining a functor $S^{-1}F : S^{-1}X \to Y$. It is a simple comparison to see that the fibers of this functor are $S^{-1}(F^{-1}(y))$.

Now assume furthermore that F is fibered and that the action of S on X is cartesian. F being fibered means we have an adjunction

$$\begin{tikzcd} F^{-1}(y) \arrow[rr, bend left, "I"] \arrow[rr, phantom, "\perp"] & & y/F \arrow[ll, bend left, "R"] \end{tikzcd} .$$

We want to define an adjunction

$$\begin{tikzcd} (S^{-1}F)^{-1}(y) \arrow[rr, bend left, "I"] \arrow[rr, phantom, "\perp"] & & y/S^{-1}F \arrow[ll, bend left, "S^{-1}R"] \end{tikzcd} .$$

We do not change the notation of I as this functor is not within our control, and is always the same inclusion. We introduce the abuse of notation $R(y \to F(x)) = R(x)$ and similarly for morphisms. We do this to lighten the notation when defining $S^{-1}R$. Now let $y \to F(s, x)$ be an object in $y/S^{-1}F$, we set $S^{-1}R(y \to F(s, x)) = (s, R(x))$. To define $S^{-1}R$ consider the following diagram

$$\begin{tikzcd} & y \arrow[dl] \arrow[dr] & \\ F(s,x) \arrow[rr, "F(t,\phi)"] & & F(s',x') \end{tikzcd} ,$$

with $\phi : (t\Box s, t\Box x) \xrightarrow{(\sigma,f)} (s', x')$. We can send this to the map $(t, S^{-1}R(\phi))$, with $S^{-1}R(\phi) : (t\Box s, t\Box R(x)) \xrightarrow{(\sigma,R(f))} (s', R(x'))$. It is an easy verification that the codomain of $S^{-1}R$ is indeed $(S^{-1}F)^{-1}(y)$ and that on morphisms this definition is independent of the choice of representative. To see that $I \dashv S^{-1}R$ we define the unit $S^{-1}\eta$ and counit $S^{-1}\epsilon$ similarly to our definition of $S^{-1}R$, using the unit and counit of $I \dashv R$; and we get the commutativity of the desired diagrams for free. Indeed, we only need to verify the commutativity in the X coordinate, as in the S coordinate we can let every map be the identity. And on the X coordinate commutativity comes from the corresponding commutative diagram for $I \dashv R$, up to verifying compatibility with choice of representative. We do not delve into the details.
All that remains to show is that the base change maps act as they must for the functor $S^{-1}F$ to be fibered. This is the stage where we need the action of S on X to be cartesian with respect to F. Let $f : y \to y'$ be a map in Y, recall the base change map f^* is defined by the composition

$$F^{-1}(y') \xrightarrow{I} y'/F \xrightarrow{f/F} y/F \xrightarrow{R} F^{-1}(y) .$$

Define $S^{-1}f^*$ to be the base change map with respect to $S^{-1}F$, i.e. the composition

$$(S^{-1}F)^{-1}(y') \xrightarrow{I} y'/S^{-1}F \xrightarrow{f/S^{-1}F} y/S^{-1}F \xrightarrow{S^{-1}R} (S^{-1}F)^{-1}(y) .$$

We can compute $S^{-1}f^*(s, x)$, doing so step by step we get

$$(s, x) \mapsto y' \xrightarrow{Id_{y'}} F(s, x) \mapsto y \xrightarrow{f} F(s, x) \mapsto S^{-1}R(y \xrightarrow{f} F(s, x)) = (s, R(y \xrightarrow{f} F(x)) = (s, f^*(x)).$$

Having done this, we similarly get on morphisms

$$\begin{tikzcd} (t\Box s, t\Box x) \arrow[r, "(\phi,g)"] \arrow[d, "(\sigma\Box s,\sigma\Box x)"] & (s', x') \arrow[d, "Id"] & & (t\Box s, f^*(t\Box x)) \arrow[r, "(\phi,f^*(g))"] \arrow[d, "(\sigma\Box s,f^*(\sigma\Box x))"] & (s', f^*(x')) \arrow[d, "Id"] \\ (t'\Box s, t'\Box x) \arrow[r, "(\phi',g')"] & (s', x') & \mapsto & (t'\Box s, f^*(t'\Box x)) \arrow[r, "(\phi',f^*(g'))"] & (s', f^*(x')) \end{tikzcd} .$$

But by the cartesian assumption, we can take the t, t' and σ out of the f^*, which shows that $S^{-1}f^*$ is independent of a choice of a representative of a morphism in $(S^{-1}F)^{-1}(y)$. With this computation completed it is easy to see that F^{-1} being a functor implies that $(S^{-1}F)^{-1}$ is a functor. This concludes the proof that $S^{-1}F$ is fibered. $\square$

We already know that the action of S on $\mathcal{E}\mathcal{A}$ is fiberwise, and it is not hard to see it is cartesian as well. So this defines a functor $S^{-1}t : S^{-1}\mathcal{E}\mathcal{A} \to Q\mathcal{A}$ (we will allow ourselves the abuse of notation of calling this functor t, denoting it by $S^{-1}t$ only if it makes things significantly clearer). So the fibers of the functor $S^{-1}t$ are $S^{-1}\mathcal{E}_C$, in a perfect world, the fibers would have been $S^{-1}S$ (as then the application of Quillen's theorem B would be our only step left). The next three lemmas serve to console ourselves from this disappointment.

Lemma 3.3.5. *(Exercise 7.1 in [6] chapter IV) Let $\mathcal{A}$ be a quasi-exact category and let $S = iso(A)$ be the corresponding symmetric monoidal category. Translations are faithful in S. In particular $BS^{-1}S$ is the group completion of $\coprod_{A\in\mathcal{A}} Aut(A)$.*

Proof. The second part of the lemma follows from the first. Let $A, B \in S$ be two objects, we define a map $Aut(A) \to Aut(A \vee B)$ which sends an automorphism $\sigma : A \to A$ to $\sigma \vee Id_B : A \vee B \to A \vee B$. The following diagram shows we can obtain σ back from $\sigma \vee Id_B$

$$\begin{tikzcd}
 & B \arrow[r, "Id_B"] \arrow[d] & B \arrow[d] \arrow[dr, "0"] & \\
A \arrow[r] \arrow[dr, "Id_A"'] & A \vee B \arrow[r, "\sigma\vee Id_B"] & A \vee B \arrow[r, "\exists!"] & A \\
 & A \arrow[r, "\sigma"] \arrow[u] & A \arrow[u] \arrow[ur, "Id_A"] &
\end{tikzcd} .$$

This concludes the proof. $\square$

Lemma 3.3.6. *(Exercise 4.7 in [6] chapter IV and theorem on page 223 of [12]) Let S be a symmetric monoidal category, X be a category on which S acts with every arrow monic. Suppose further that the translations $Aut(s) \to Aut(s\Box x)$ are all injective. Then the sequence $S^{-1}S \xrightarrow{-\Box x} S^{-1}X \xrightarrow{\pi} \langle S, X\rangle$ is a homotopy fibration (π is the natural projection).*

Proof. The proof that the cofibration $S^{-1}X \to \langle S, S\rangle$ is cofibered with fiber X and cobase change given by translation in lemma 2.2.6 can be adapted easily to showing that under the above assumption the projection $S^{-1}X \to \langle S, X\rangle$ is cofibered with fiber S and cobase given by translation. We have a fiberwise action of S on $S^{-1}X$ given by translation on the first coordinate, whose restriction to the fibers is the translation action of S on itself. This action commutes with the cobase change maps up to natural isomorphism because S is symmetric monoidal. And so we can use lemma 3.3.4 to obtain a cofibered functor $S^{-1}S^{-1}X \to \langle S, X\rangle$ with cofiber $S^{-1}S$. The cobase change maps are given by translation in the second coordinate, and so are homotopy equivalences as S acts invertibly on $S^{-1}S$. This is the necessary set up to apply Quillen's theorem B (A.0.3), so we have a homotopy fibration $S^{-1}S \to S^{-1}S^{-1}X \to \langle S, X\rangle$. To prove that the sequence $S^{-1}S \xrightarrow{-\Box x} S^{-1}X \to \langle S, X\rangle$ is a homotopy fibration we consider the following diagram from [12] page 223

$$\begin{tikzcd}
 & S^{-1}S \arrow[rr, "(A,B)\mapsto(A,(B,x_0))"] & & S^{-1}S^{-1}X \arrow[rr, "(A,(B,x))\mapsto x"] & & \langle S, X\rangle \\
S^{-1}S \arrow[ur, "(A,B)\mapsto(B,A)"] \arrow[rr, "(A,B)\mapsto(A,B\Box x_0)"] & & S^{-1}X \arrow[ur, "(A,x)\mapsto(e,(A,x))"] \arrow[rr, "(A,x)\mapsto x"] & & \langle S, X\rangle \arrow[ur, "Id_{\langle S,X\rangle}"] &
\end{tikzcd} ,$$

with x_0 some fixed element of X. The first and last map from front to back are obviously homotopy equivalences, the middle map being a homotopy equivalence follows from lemma 2.3.3. The second square obviously commutes but the first does not. However, it does commute up to homotopy. To show this we define natural transformations ω, ν fitting into the following diagram

$$\begin{tikzcd}
(B,(A,x_0)) \arrow[r, "\omega_{A,B}"] \arrow[d, "{(g,(f,Id_{x_0}))}"] & (B,(B\Box A, B\Box x_0)) \arrow[d, "{(g,(g\Box f, g\Box Id_{x_0}))}"] & (e,(A,B\Box x_0)) \arrow[l, "\nu_{A,B}"'] \arrow[d, "{(Id_e,(f,g\Box Id_{x_0}))}"] \\
(D,(C,x_0)) \arrow[r, "\omega_{C,D}"] & (D,(D\Box C, D\Box x_0)) & (e,(C,D\Box x_0)) \arrow[l, "\nu_{C,D}"']
\end{tikzcd},$$

with $f : A \to C, g : B \to D$ maps in S. The map $\omega_{A,B}$ is in $S^{-1}S^{-1}X$ and so is defined by a triple (s, ϕ, ψ) with $s \in S, \phi : s\Box B \to B, \psi : s\Box(A, x_0) \to (B\Box A, B\Box x_0)$. But ψ is a map in $S^{-1}X$ so is itself given by a triple $t \in S, \alpha : t\Box s\Box A \to B\Box A, \beta : t\Box x_0 \to B\Box x_0$. We can take $s = e$ and $t = B$, letting $\phi = \eta_B, \alpha = B\Box\eta_A, \beta = Id_{B\Box x_0}$, with η the natural transformation $e\Box- \cong Id_S$. We define $\nu_{A,B}$ in a very similar manner. This establishes a weak equivalence with a homotopy fibration, to obtain the desired result please consult the following remark. ☐

Remark 3.3.7. This proof in fact does not show the desired claim, indeed homotopy fibrations are not closed under pointwise weak equivalence. In order to obtain the desired result we would have to compare the null homotopies associated with these homotopy fibrations and how these interact with the pointwise weak equivalence. Working out the details of this process eluded me and thus was omitted from the proof.

Remark 3.3.8. To apply the above lemma to the case of interest $S = iso(\mathcal{A})$ and $X = \mathcal{E}_C$, we need to verify they satisfy the assumptions. Every arrow in $\mathcal{E}_C$ is indeed monic, which follows from the fact that arrows used to define the morphisms are all monic in $\mathcal{A}$. The desired injectivity of translations follows from a reasoning quite similar to the proof of lemma 3.3.5 showing that translations are faithful in S.

Lemma 3.3.9. *(Proposition 7.6 in [6] chapter IV) With the same notation as in the main theorem, and letting $E = A \to B \to C$ be an exact sequence in $\mathcal{E}_C$, we have that each $S^{-1}S \xrightarrow{-\Box E} S^{-1}\mathcal{E}_C$ is a homotopy equivalence.*

Proof. By the previous lemma, we have a homotopy long exact sequence for the maps $S^{-1}S \xrightarrow{-\Box E} S^{-1}\mathcal{E}_C \to \langle S, \mathcal{E}_C\rangle$. By Whitehead's theorem, it suffices to show that $\mathcal{E}_C$ is contractible. First notice that $\mathcal{E}_C$ admits a symmetric monoidal structure, given by

$$(A \to A \vee C \to C)\Box(A' \to A' \vee C \to C) = A \vee A' \to A \vee A' \vee C \to C.$$

This is indeed an admissible split exact sequence as every exact sequence in $\mathcal{E}\mathcal{A}$ splits and every split sequence is exact. The unit for this symmetric monoidal structure is $e = 0 \to C \to C$. It is not hard to show that this monoidal structure induces a monoidal structure on $\langle S, \mathcal{E}_C\rangle$ and so in particular this space is an H-space. Notice that because every exact sequence is split, for any sequence $E = A \to A \vee C \to C$ in $\langle S, \mathcal{E}_C\rangle$ we have a morphism $(A, Id : A\Box e \to E)$ which defines a path from e to E in $B\langle S, \mathcal{E}_C\rangle$ showing that the 1-skeleton of this space is connected, which shows that this space is connected. Using lemma 2.1.2 we see that the space $B\langle S, \mathcal{E}_C\rangle$ is a group like H-space, in particular has homotopy inverses.
Now notice that we have a natural transformation $E \mapsto E\Box E$, which if $E = A \to B \to C$ is given by the maps $(A, Id : A\Box E \to E\Box E)$. For this to be clear we specify that the first $\Box$ is the one coming from the action of S whereas the second comes from the symmetric monoidal structure of $\mathcal{E}_C$. Writing the H-space structure of $B\langle S, \mathcal{E}_C\rangle$ additively, the above natural transformation defines a homotopy between the identity map and multiplication by 2. This means that on every homotopy group the identity equals multiplication by 2, which can only be the case if all the homotopy groups are 0, which implies that our space is contractible by Whitehead's theorem. ☐

The next lemma allows us to pass from category theory to topology via Quillen's theorem B.

Lemma 3.3.10. *(Theorem 7.8 in [6] chapter IV) With the same notation as for the main theorem, the sequence $S^{-1}\mathcal{E}_C \to S^{-1}\mathcal{E}\mathcal{A} \to Q\mathcal{A}$ is a homotopy fibration.*

Proof. The only thing we need to show is that the base change maps are homotopy equivalences, as then the result follows from Quillen's theorem B (A.0.3). We use that every morphism factors to reduce to showing that the base change maps associated to $C' = C' \rightarrowtail C$ and $C' \twoheadleftarrow C = C$ are homotopy equivalences. We also use that homotopy equivalences satisfy 2 out of 3 to reduce further to the maps to and from the 0 object. We use the following diagrams to accomplish this

$$\begin{tikzcd} & 0 \arrow[dl, tail] \arrow[dr, tail] & \\ C' \arrow[rr, tail] & & C \end{tikzcd} \qquad \begin{tikzcd} C' \arrow[rr, two heads] \arrow[dr, two heads] & & C \arrow[dl, two heads] \\ & 0 & \end{tikzcd}.$$

Notice that because every map in $\mathcal{E}_C$ is an isomorphism, we have that $\mathcal{E}_0 \cong S$ by sending $A \xrightarrow{\cong} A' \to 0$ to A'. And so in particular $S^{-1}\mathcal{E}_0 \cong S^{-1}S$.
Let $E \in \mathcal{E}_C$ be the exact sequence $0 \to C \to C$, we know from lemma 3.3.9 that $-\Box E : S^{-1}\mathcal{E}_0 \to S^{-1}\mathcal{E}_C$ is a homotopy equivalence. Recall that for a map $\phi : C' \to C$, the base change map sends $A \to B \to C$ to $A' \to B' \to C'$ defined by in the following diagram

$$\begin{tikzcd} & & C' \\ A' \cong ker(B' \to C') \arrow[r, tail] & B' \cong B \times_C C'' \arrow[r, two heads] \arrow[ur, two heads] \arrow[d, tail] & C'' \arrow[u, two heads] \arrow[d, tail] \\ A \arrow[r, tail] & B \arrow[r, two heads] & C \end{tikzcd}.$$

Now consider the map $0 = 0 \rightarrowtail C$, the base change map sends an exact sequence $A \to B \to C$ to $A \to A \to 0$ (with $A \to A$ an isomorphism because it is the kernel of the unique map to the 0 object). Now apply the functor $-\Box E$, which recalling that every exact sequence splits, yields $A \to B \to C$. So the base change map is a left inverse to $-\Box E$, thus is the homotopy inverse of a homotopy equivalence and is in particular a homotopy equivalence.
Now consider the map $0 \twoheadleftarrow C = C$, the base change map sends $A \to B \to C$ to $B \to B \to 0$ (with $B \to B$ an isomorphism as above). Post composing with $-\Box E$ yields, because every exact sequence splits, the exact sequence $B \to B \vee C \to C$, or more explicitly $A \vee C \to A \vee C \vee C \to C$. And so we see the composition of the base change map with $-\Box E$ is equal to $C\Box-$ with the action of S on $S^{-1}\mathcal{E}_C$ given by the inclusion $S \to \mathcal{EA}$. But we know S acts invertibly on $S^{-1}\mathcal{E}_C$, so be $C\Box-$ is a homotopy equivalence. We conclude that the base change map is a homotopy equivalence by the fact the homotopy equivalences satisfy two out of three. This concludes the proof. $\Box$

As we remarked before starting the proof, this homotopy fibration implies we have a homotopy fibration $\Omega BQ\mathcal{A} \to BS^{-1}S \to BS^{-1}\mathcal{EA}$ (Because $S^{-1}S \simeq S^{-1}\mathcal{E}_C$). In order to obtain the desired result by Whitehead's theorem, it suffices to show that $S^{-1}\mathcal{EA}$ is contractible, which is achieved by the following lemma.

Lemma 3.3.11. *(Proof of theorem 7.1 and exercise 7.3 in [6] chapter 4)(Lemma 7.10 of [13]) With notation as above the category $\mathcal{EA}$ is contractible, and thus so is $S^{-1}\mathcal{EA}$.*

Proof. For this we define for a category $\mathcal{C}$ the category $Sub(\mathcal{C})$ whose objects are the morphisms of $\mathcal{C}$ and a morphism from $A \to B$ to $C \to D$ is given by a commutative diagram

$$\begin{tikzcd} A \arrow[r] & B \arrow[d] \\ C \arrow[u] \arrow[r] & D \end{tikzcd}.$$

Denote by $iQ\mathcal{A}$ the subcategory of $Q\mathcal{A}$ generated by monics $C = C \rightarrowtail C'$, we will (in time) show that $Sub(iQ\mathcal{A}) \cong \mathcal{EA}$ are equivalent categories. Why will we do this? Because $iQ\mathcal{A}$ has an initial object 0 and because we have the following lemma.

Lemma 3.3.12. *(Exercise 3.9 of [6] chapter IV) The functor cod : $Sub(\mathcal{C}) \to \mathcal{C}$ is a homotopy equivalence*

Proof. The functor $cod : Sub(\mathcal{C}) \to \mathcal{C}$ is explicitly given on objects by $cod(A \to B) = B$. We will show that this functor is a homotopy equivalence using Quillen's theorem A (A.0.2). We first show that this functor is pre-cofibered, i.e. the natural inclusion $cod^{-1}(B) \to Cod/B$ has a left adjoint. An element of cod/B can be represented by a pair of composable morphisms $A' \to B' \to B$ and a morphism by a commutative diagram

$$\begin{tikzcd}
A' \arrow[r] & B' \arrow[r] \arrow[d] & B \arrow[d, "Id_B"] \\
A'' \arrow[r] \arrow[u] & B'' \arrow[r] & B
\end{tikzcd}.$$

Define $L : cod/B \to cod^{-1}(B)$ by sending $A' \to B' \to B$ to the composite $A' \to B$ and define L on morphisms in the obvious way. To see that this construction is a left adjoint to the inclusion $\iota : cod^{-1}(B) \to cod/B$ which sends $A \to B$ to $A \to B \xrightarrow{Id_B} B$ compare the two following diagrams

$$\begin{tikzcd}
A' \arrow[r, "g' \circ f'"] & B \arrow[d, "Id_B"] \\
A \arrow[r, "f"] \arrow[u, "\alpha"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] \arrow[d, "g'"] & B \arrow[d, "Id_B"] \\
A \arrow[r, "f"] \arrow[u, "\alpha"] & B \arrow[r, "Id_B"] & B
\end{tikzcd}.$$

It is not hard to see this establishes the isomorphism which shows $L \dashv \iota$. Now we notice that $cod^{-1}(B)$ is contractible as it has a terminal object $id_B : B \to B$. This proves the desired result by Quillen's theorem A. $\square$

So all that remains to show that $\mathcal{EA}$ is contractible is that $Sub(iQ\mathcal{A}) \cong \mathcal{EA}$. The objects of $Sub(iQ\mathcal{A})$ are $A = A \rightarrowtail B$ and morphisms are commutative diagrams of the form

$$\begin{tikzcd}
A & A \arrow[l, "Id_A"'] \arrow[r, tail] & B \\
A' \arrow[u, tail] \arrow[d, "Id_{A'}"] & & B \arrow[u, "Id_B"] \arrow[d, tail] \\
A' & A' \arrow[l, "Id_{A'}"] \arrow[r, tail] & B'
\end{tikzcd}.$$

Admissible monics fit into exact sequences by definition. We use this and the fact that all exact sequences split to extend the above diagram into a diagram of the form

$$\begin{tikzcd}
A & A \arrow[l, "Id_A"'] \arrow[r, tail] & B \arrow[r, two heads] & C \\
A' \arrow[u, tail] \arrow[d, "Id_{A'}"] \arrow[rr, tail] & & B \arrow[u, "Id_B"] \arrow[d, tail] \arrow[r, two heads] & C'' \arrow[u, two heads] \arrow[d, tail] \\
A' & A' \arrow[l, "Id_{A'}"] \arrow[r, tail] & B' \arrow[r, two heads] & C'
\end{tikzcd}.$$

This defines a functor $Sub(iQ\mathcal{A}) \to \mathcal{EA}$, and it is easy to see that this functor is essentially surjective and fully faithful, thus is an equivalence of categories. This proves that $\mathcal{EA}$ is contractible.
Now we want to show this implies that $S^{-1}\mathcal{EA}$ is contractible. Because $\mathcal{EA}$ is contractible we have that $[B\mathcal{EA}, B\mathcal{EA}] = \{*\}$ and so in particular S acts invertibly on $\mathcal{EA}$. Thus, by 2.3.3 $S^{-1}\mathcal{EA} \simeq \mathcal{EA}$ and so is contractible which concludes the proof. $\square$

4 Waldhausen's $S.$ construction

In this section we define Waldhausen's definition of K-theory for categories that now carry his name. We will relate this construction to the Q construction. We then prove the additivity theorem, one of the most important results in Algebraic K-theory. And finally we discuss how the $S.$ construction allows us to see that we have a K-theory spectra, and we introduce a potential multiplicative structure which can appear in K-theory when we have the data of a "biexact functor".

4.1 Definition

We first define the categories of interest, which is done by defining separately a category with cofibrations and a category with weak equivalences. A Waldhausen category will be a category which has both cofibrations and weak equivalences.

Definition 4.1.1. (Definition 9.1 of [6] chapter II) A category with cofibrations is a category $\mathcal{C}$, with a 0 object, equipped with a specified class of morphisms $co(\mathcal{C})$ which we call cofibrations and denote by $\rightarrowtail$. This class of morphisms is required to satisfy the following axioms:

(i) Every isomorphism is a cofibration;

(ii) The unique maps $0 \to A$ are all cofibrations;

(iii) Cofibrations are closed under composition:

(iv) Cofibrations are closed under pushouts.

Two immediate consequences are noteworthy. First, the coproduct of any two objects exist by taking the pushout of $A \leftarrowtail 0 \rightarrowtail B$ and the inclusion of factors are cofibrations. Second, by taking the pushout of $0 \leftarrow A \rightarrowtail B$ we see that cofibrations admit cokernel, which we call quotients and denote by $B \twoheadrightarrow B/A$. A sequence $A \rightarrowtail B \twoheadrightarrow B/A$ is called a cofibration sequence.

Definition 4.1.2. (Definition 9.1.1 of [6] chapter II) A category with weak equivalences is a category $\mathcal{C}$ equipped with a specified class of morphisms $w(\mathcal{C})$ which we call weak equivalences and denote by $\xrightarrow{\sim}$. This class of morphisms is required to satisfy the following axioms:

(i) All isomorphisms are weak equivalences;

(ii) Weak equivalences are required to be closed under composition

(iii) A pushout of weak equivalences is a weak equivalence in the sense that if we have a diagram

$$\begin{tikzcd} A \arrow[d, "\sim"] & B \arrow[l] \arrow[r] \arrow[d, "\sim"] & C \arrow[d, "\sim"] \\ A' & B' \arrow[l] \arrow[r] & C' \end{tikzcd}$$

such that both rows admit a pushout, then the natural map $A \cup_B C \to A' \cup_{B'} C'$ is a weak equivalence.

The third axiom for categories with weak equivalences is called the "gluing axiom". Combining the above two definitions yields:

Definition 4.1.3. (Definition 9.1.1 of [6] chapter II) A Waldhausen category is a category with cofibrations and weak equivalences.

We also introduce the notion of exact functors and Waldhausen subcategories.

Definition 4.1.4. (Remark following definition 8.1 of [6] chapter IV)

(i) A functor between Waldhausen categories which preserve the 0 object, cofibrations, weak equivalences and pushouts along

(ii) A Waldhausen subcategory $\mathcal{A}$ of a Waldhausen category $\mathcal{C}$ is a subcategory such that the inclusion is exact, the cofibrations are exactly the maps which are cofibrations in $\mathcal{C}$ with cokernel in $\mathcal{A}$ and the weak equivalences are exactly the maps which are weak equivalences in $\mathcal{C}$.

Having defined the appropriate notion of a map between Waldhausen categories, we now have a category $Wald$ of (small) Waldhausen categories. And so we get a notion of simplicial Waldhausen category. We spend most of the rest of this section defining a particular simplicial Waldhausen category $S.\mathcal{C}$ for any Waldhausen category $\mathcal{C}$.

Definition 4.1.5. (Definition 8.3 of [6] chapter IV) Let $\mathcal{C}$ be a category with cofibrations. For $n \geq 0$, let $S_n\mathcal{C}$ be the category whose objects are cofibrations sequences of length n

$$A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$$

together with a choice of every subquotient $A_{ij} = A_j/A_i$ which we require to be compatible in the sense that they fit in a commutative diagram

$$\begin{tikzcd}
 & & & & A_{n-1,n} \\
 & & & & \vdots \arrow[u, two heads] \\
 & & A_{32} \arrow[r, tail] & \cdots \arrow[r, tail] & A_{n2} \arrow[u, two heads] \\
 & A_{21} \arrow[r, tail] & A_{31} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] \arrow[u, two heads] & A_{n1} \arrow[u, two heads] \\
A_1 \arrow[r, tail] & A_2 \arrow[r, tail] \arrow[u, two heads] & A_3 \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] \arrow[u, two heads] & A_n \arrow[u, two heads]
\end{tikzcd}.$$

It is useful to add the convention that the bottom row is the 0th row, so we note A_{0i} for A_i and to write $A_{ii} = 0$. It will also be useful to view the above diagram as an $(n+1)\times(n+1)$ square by adding a 0th column which is all 0 and an nth row which is also all 0 and filling all the remaining empty positions with the 0 object.
A morphism in this category is simply a natural transformation of cofibration of sequences.
This construction is functorial in an obvious way.

It is interesting to note that the explicit choice of subquotients is a matter of notational convenience more so than of philosophical importance. What we mean by this is that because it is always possible to make such a choice, we obtain equivalent categories whether we include the choice of subquotients in the definition of objects. In light of this, we will not feel guilty if we omit explicitly mentioning the choice of subquotients to make the notation lighter.
It turns out that the categories $S_n\mathcal{C}$ inherit a Waldhausen structure from $\mathcal{C}$. The weak equivalences are easy to define simply as natural transformations which are pointwise weak equivalences. Cofibration are slightly harder to define. For $n = 0$ we have that $S_0\mathcal{C}$ is the trivial one object category and for $n = 1$ we have $S_1\mathcal{C} \cong \mathcal{C}$, so in both of these case it is clear how to define cofibrations. For $n = 2$ we call a map a cofibration if in the induced diagram

$$\begin{tikzcd}
A_1 \arrow[r, tail] \arrow[d] & A_2 \arrow[r, two heads] \arrow[d] & A_{12} \arrow[d] \\
B_1 \arrow[r, tail] & B_2 \arrow[r, two heads] & B_{12}
\end{tikzcd}$$

the outer vertical maps and the natural map $A_2 \cup_{A_1} B_1 \to B_2$ are cofibrations. For a general n we call a natural transformation a cofibration if for all $0 \leq i < j < k \leq n$ the induced diagram

$$\begin{tikzcd}
A_{ij} \arrow[r, tail] \arrow[d] & A_{ki} \arrow[r, two heads] \arrow[d] & A_{kj} \arrow[d] \\
B_{ij} \arrow[r, tail] & B_{ki} \arrow[r, two heads] & B_{kj}
\end{tikzcd}$$

is a cofibration in $S_2\mathcal{C}$. We do not verify that this indeed defines a Waldhausen structure on the categories $S_n\mathcal{C}$. We also note that this is enough to show that cofibration in $S_n\mathcal{C}$ are in particular pointwise cofibrations (though this is in general not sufficient to be a cofibration).
We want the $S_n\mathcal{C}$ to fit together into a simplicial Waldhausen category, i.e. we want to define face and degeneracy functors. Viewing the objects of $S_n\mathcal{C}$ as an $(n+1)\times(n+1)$ grid of objects in the way indicated in the above definition there is a (somewhat) natural way to do this. Let $0 \leq i \leq n$ and define $\partial_i : S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ by deleting the ith row and ith column in the obvious way, for convenience one might want to reindex the A_{ij} as needed. Note that for ∂_0, in order for the resulting $n \times n$ square to start at the 0-column we need to quotient out by A_1. As for the degeneracy functors let $\sigma_i : S_n\mathcal{C} \to S_{n+1}\mathcal{C}$ be the map duplicating the ith column and the ith row by inserting an identity map and naturally fix $A_{i,i+1}$ to be the distinguished 0 object. The fact that the ∂_i and σ_i satisfy the necessary relation for $S.\mathcal{C}$ to be a simplicial object with $S_n\mathcal{C}$ as n simplices is shown by routine verification. However for this to be a simplicial Waldhausen category, we need all of these functors to be exact, this is the content of the next proposition.

Proposition 4.1.6. *(Exercise 8.2 of [6] chapter IV) Let $\mathcal{C}$ be a Waldhausen category. The functors $\partial_i : S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ and $\sigma_i : S_n\mathcal{C} \to S_{n+1}\mathcal{C}$ are all exact functors. In particular the S. construction maps Waldhausen categories to simplicial Waldhausen categories, furthermore this construction naturally defines a functor $Wald \to Wald^{\Delta^{op}}$.*

Proof. Due to the pointwise definitions of the 0 object and weak equivalences, it is easy to see that both of these structures are preserved by the face and degeneracy functors. To see that cofibrations are preserved, notice that a choice of $0 \leq i < j < k \leq n$ to test whether the induced map on the associated triple of subquotients in the codomain $S_{n+1}\mathcal{C}$ or $S_{n-1}\mathcal{C}$ (depending on whether we want to show the face or degeneracy maps preserve cofibrations) is a cofibration in $S_2\mathcal{C}$ is, via reindexing, just a corresponding choice of a triple subquotient in the domain $S_n\mathcal{C}$. But the morphism in $S_2\mathcal{C}$ defined by the reindexed triple is a cofibration by assumption of studying a cofibration in $S_n\mathcal{C}$. Once we know cofibrations are preserved it is not hard to see that pushouts along cofibrations are preserved as the pushout is computed pointwise. This concludes the proof. □

It is not hard to see that if we denote $wS_n\mathcal{C}$ the subcategory of $S_n\mathcal{C}$ of weak equivalences, these also fit into a simplicial category. We can view this as a functor $w : Wald \to Cat$ which by post composition defines a functor $Wald^{\Delta^{op}} \to Cat^{\Delta^{op}}$. With this in hand we can define the K-theory space of a Waldhausen category.

Definition 4.1.7. Let $\mathcal{C}$ be a Waldhausen category, denote by $|wS.\mathcal{C}|$ the following composition of functors

$$Wald \xrightarrow{S.\bullet} Wald^{\Delta^{op}} \xrightarrow{w\circ -} Cat^{\Delta^{op}} \xrightarrow{B\circ -} Top^{\Delta^{op}} \xrightarrow{|-|} Top.$$

To clarify $B \circ -$ is postcomposition by geometric realization of categories and $|-|$ is geometric realization of simplicial topological spaces. The K-theory space $K(\mathcal{C})$ of a Waldhausen category is given by

$$\Omega|wS.\mathcal{C}|.$$

We define the K-groups of $\mathcal{C}$ to be the homotopy groups of $K\mathcal{C}$.

Notice that there is no base point related ambiguity when taking the loop space of $|wS.\mathcal{C}|$ as this space is connected. To see that this space is connected just notice that every point is path connected to a 0 simplex, of which there is only one, as $S_0\mathcal{C} = [1]$ is the trivial category. Also notice that the coproduct, which always exists between two objects of a Waldhausen category, induces an H-space structure on $K(\mathcal{C})$.

Proposition 4.1.8. *(Proposition 8.4 of [6] chapter IV) Let $\mathcal{C}$ be a Waldhausen category, then $K_0(\mathcal{C})$ is generated by the set of weak equivalence classes of objects of $\mathcal{C}$ under the relation $[B] = [A] + [B/A]$ for every cofibration $A \rightarrowtail B$.*

Proof. Because the loop space functor shifts homotopy groups we have that $K_0(\mathcal{C}) = \pi_1(|wS.\mathcal{C}|)$. Recall that $wS_0\mathcal{C}$ is a point. Thus the fundamental group of a simplicial topological space $X_\bullet$ with X_0 a point is presented by

$$\Big\langle [x], \forall x \in \pi_0(X_1) \Bigm| \partial_1([y]) = \partial_0([y])\partial_2([y]), \forall [y] \in \pi_0(X_2) \Big\rangle.$$

To see this in general, one can consult the following MSE answer to a question I asked [31]. In our case we have another way, which I learned from lemma 8.4.6 in [32]. Indeed, instead of viewing this as a simplicial space, we view it as a bisimplicial set $N_p wS_q\mathcal{C}$. We can study explicitly the $(1,0)$ simplices to be trivial, the $(0,1)$ simplicies are the objects of $\mathcal{C}$, the $(1,1)$ simplicies are the weak equivalences of $\mathcal{C}$ and the $(0,2)$ simplicies correspond to cofibrations in $\mathcal{C}$. With this in hand, we can understand π_1 by successive application of Seifert-Van Kampen (viewing the geometric realization as constructed one cell at a time via pushouts) and recalling what the face maps for our bisimplicial set are, in order to understand how the 2-cells glue to the 1-cells. It is not hard for the presentation we obtain this way to be rewritten as the presentation we claimed above.
Independently of how, once we accept this presentation, it suffices to understand $\pi_0(BwS_1\mathcal{C}), \pi_0(BwS_2\mathcal{C})$ and the maps $\partial_0, \partial_1, \partial_2 : wS_2\mathcal{C} \to wS_1\mathcal{C}$. The connected components of $wS_1\mathcal{C}$ are the weak equivalence classes of objects, $wS_2\mathcal{C}$ are the weak equivalence classes of cofibrations and $\partial_i, 0 \leq i \leq 2$ sends a cofibration sequence to the ith element. Thus, it is clear that $K_0(\mathcal{C})$ has the claimed presentation. □

We mention here three additional axioms a Waldhausen category can satisfy, which often come up in nature and as hypotheses in theorems.

Definition 4.1.9. (Definition 9.1.1 in [6] chapter II) A Waldhausen category is called saturated if the weak equivalences satisfy two out of three.

Definition 4.1.10. (8.2.1 in [6] chapter IV) We say that a Waldhausen category satisfies the extension axiom if it satisfies the following 5-lemma. In a diagram of cofibration sequences where the outer maps are weak equivalences,

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\simeq"] & B \arrow[r, two heads] \arrow[d] & C \arrow[d, "\simeq"] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & C' \end{tikzcd},$$

then the middle map must also be a weak equivalence.

Definition 4.1.11. (Definition 8.8 and axiom 8.8.1 in [6] chapter IV) Let $\mathcal{C}$ be a Waldhausen category. Recall $Ar(\mathcal{C})$ is the category of arrows in $\mathcal{C}$ with morphisms from $a \to b$ to $c \to d$ being a pair $a \to c$ and $b \to d$ making the obvious square commute. We say $T : Ar(\mathcal{C}) \to \mathcal{C}$ is a cylinder functor if there are natural transformation $j_1 : dom \to T, j_2 : cod \to T$ and $p : T \to cod$ (where dom, cod are the functors $Ar(\mathcal{C}) \to \mathcal{C}$ sending an arrow to the domain or codomain respectively). We require that all of these maps satisfy some condition which we list here.

(i) They must fit into a diagram

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] & B \arrow[l, "j_2"'] \arrow[dl, "Id_B"] \\ & B & \end{tikzcd}.$$

(ii) We must have $T(0 \rightarrowtail A) = A$ and that the associated maps j_2, p are both the identity.

(iii) The map $j_1 \sqcup j_2 : A \sqcup B \to T(f)$ is a cofibration for all $f : A \to B$.

(iv) If a morphism $(a, b) : f \to f'$ in $Ar(\mathcal{C})$ is pointwise a weak equivalence, then so is $T(a, b) : T(f) \to T(f')$.

(v) If a morphism $(a, b) : f \to f'$ in $Ar(\mathcal{C})$ is pointwise a cofibration then so is $T(a, b) : T(f) \to T(f')$. Furthermore, the map (whose existence is ensured by (iii))

$$dom(f') \sqcup_{dom(f)} T(f) \sqcup_{cod(f)} cod(f') \to T(f')$$

is also a cofibration.

We say that $\mathcal{C}$ satisfies the cylinder axiom if it has a cylinder functor T and every component map of the natural transformation $p : T \to cod$ is a weak equivalence.

To round of this subsection off we define the relative K-theory space for an exact functor between Waldhausen categories. Though we will only see why this space deserves this name with proposition 4.4.4.

Definition 4.1.12. (Definition 8.5.3 of [6] chapter IV) Let $F : \mathcal{B} \to \mathcal{C}$ be an exact functor between Waldhausen categories. The category of small categories is complete, so define S_nF as the pullback (there is a mild abuse of notation as S_nF is also the functor induced by F from $S_n\mathcal{B}$ to $S_n\mathcal{C}$)

$$\begin{tikzcd} S_nF \arrow[r] \arrow[d] & S_{n+1}\mathcal{C} \arrow[d, "\partial_0"] \\ S_n\mathcal{B} \arrow[r, "S_nF"'] & S_n\mathcal{C} \end{tikzcd} .$$

A product of Waldhausen categories can easily be seen to be a Waldhausen category, and so it is not hard to give S_nF a natural Waldhausen category structure making it a Waldhausen subcategory of $S_n\mathcal{B} \times S_{n+1}\mathcal{C}$.
The S_nF fit into a simplicial Waldhausen category. We can apply the $wS.$ construction degreewise yielding a bisimplicial Waldhausen category $wS.S.F$. Now we can use geometric realization (three times actually) to get a topological space which we denote by $|wS.S.F|$. The relative K-theory space of F is

$$K(F) = \Omega^2|wS.S.F|.$$

It will be interesting to notice that the functors $S_n\mathcal{B} \leftarrow S_nF \to S_{n+1}\mathcal{C}$ are exact. And similarly using the cone $S_n\mathcal{B} \xleftarrow{S_nF\circ\iota_n} \mathcal{C} \xrightarrow{\iota_{n+1}} S_{n+1}\mathcal{C}$ we can see that there is an inclusion as a Waldhausen subcategory $\mathcal{C} \to S_nF$. The maps $\iota_n : \mathcal{C} \to S_n\mathcal{C}$ are given by inclusion as the constant cofibration sequence $C \rightarrowtail C \rightarrowtail \cdots \rightarrowtail C$.

4.2 Equivalence of the Q and $S.$ construction

Let $\mathcal{A}$ be a quasi-exact category, such an object can naturally be seen as Waldhausen category by declaring the cofibrations to be the admissible monics and the weak equivalences to be the isomorphisms. Upon doing this it is natural to ask whether the Q-construction and $S.$ construction yield the same K-theory, this is accomplished by the following result

Theorem 4.2.1. *(Exercise 8.5, 8.6 in chapter IV of [6])(section 1.9 of [14]) Let $\mathcal{A}$ be a quasi-exact category, we have a homotopy equivalence*

$$BQA \simeq |iS.\mathcal{A}|.$$

The notation iS. serves to indicate that we are taking the weak equivalences to be the isomorphisms.

Proof. In order to show this result we will construct the following commutative diagram

$$\begin{tikzcd} |s.\mathcal{A}| \arrow[d] & |sub(s.\mathcal{A})| \arrow[l] \arrow[r] \arrow[d] & BQ\mathcal{A} \arrow[d] \\ |iS.\mathcal{A}| & |sub(iS.\mathcal{A})| \arrow[l] \arrow[r] & |iQ.\mathcal{A}| \end{tikzcd} .$$

We will then show that each map is a homotopy equivalence, which will yield the desired result that $|iS.\mathcal{A}| \simeq BQ\mathcal{A}$. By the fact that homotopy equivalences satisfy two out of three we will not need to explicitly show that the middle vertical map and top right horizontal maps are homotopy equivalences.

We deal first with the left vertical map. For this we notice because we are working with small categories, we have a functor $Ob : Wald \to Set$ which sends a category to its set of objects. We can apply this degreewise to the $S.$ construction, yielding a simplicial set $s.\mathcal{C}$ associated to any simplicial Waldhausen category. Recall that sets can be viewed as categories with only identity morphisms, and thus there is a natural inclusion $s.\mathcal{C} \to iS.\mathcal{C}$. We claim this map is a homotopy equivalence. For this first recall/notice (see proof of lemma 1.4.1 in [14]) that we can describe homotopies of simplicial maps as follows. We have the over category $\Delta/[1]$, and so to any simplicial object in a category $\mathcal{C}$ we can correspond the functor $X^* : (\Delta/[1])^{op} \to \Delta^{op} \xrightarrow{X} \mathcal{C}^{\Delta^{op}}$. A natural transformation of two functors constructed this way $\tau : X^* \to Y^*$ corresponds to a homotopy of maps $X \to Y$. To make this somewhat clearer notice that the face maps $\partial_0, \partial_1 : [0] \to [1]$ and the obvious isomorphism $\Delta^{op} \cong (\Delta/[0])^{op}$ gives us two inclusions $\Delta^{op} \to (\Delta/[1])^{op}$, and so a diagram

$$\begin{tikzcd}
\Delta^{op} \arrow[d] & & \\
\Delta^{op}/[1] \arrow[rr, bend left, "X^*"] \arrow[rr, bend right, "Y^*"'] \arrow[rr, phantom, "\Downarrow \tau"] & & \mathcal{C}^{\Delta^{op}} . \\
\Delta^{op} \arrow[u] & &
\end{tikzcd}$$

Restricting τ to the first or second copy of Δ lying in $\Delta/[1]$ yields the two maps of simplicial sets between which τ gives a homotopy. We leave further details to the interested reader. With this in hand we can prove the following lemma.

Lemma 4.2.2. *(Lemma 1.4.1 in [14]) A functor of Waldhausen categories $F : \mathcal{C} \to \mathcal{D}$ yields a map of simplicial sets $s.F : s.\mathcal{C} \to s.\mathcal{D}$. Furthermore, a natural transformation of functors yields a homotopy of induced maps.*

Proof. The fact that a functor of Waldhausen categories induces a map of simplicial sets is clear, to prove that natural transformations $\tau : F \to G$ of functors $\mathcal{C} \to \mathcal{D}$ induce simplicial homotopies we use the above discussion. So we wish to construct a natural transformation $\tau_* : s.\mathcal{C}^* \to s.\mathcal{D}^*$. So we need to associate to each map $[n] \to [1]$ a map $s_n\mathcal{C} \to s_n\mathcal{D}$. The elements of $s_n\mathcal{C}$ can be described as a certain subset of the set of functors $Ar([n]) \to \mathcal{C}$. The category $Ar([n])$ is the category of arrows in $[n]$ with morphisms from $a \to b$ to $c \to d$ being pairs of morphism $a \to c, b \to d$ fitting into a commutative square. We make this remark as it will be easier to define τ_* this way. Notice that the natural transformation $\tau : F \to G$ can equivalently be described as a functor $\mathcal{C} \times [1] \to \mathcal{D}$.
Now let $a : [n] \to [1]$ be a map, and let $A : Ar([n]) \to \mathcal{C}$ be a functor which corresponds to some element of $s.\mathcal{C}$. The $Ar(-)$ construction can easily be extended to a functor of small categories, so let $\tau_{*,a}(A)$ be the composition

$$Ar([n]) \xrightarrow{(A, Ar(a))} \mathcal{C} \times Ar([1]) \xrightarrow{(Id, p)} \mathcal{C} \times [1] \xrightarrow{\tau} \mathcal{D}.$$

The only element of the above definition which requires details is the map $p : Ar([1]) \to [1]$ which maps Id_0 to 0, Id_1 to 1 and $0 \to 1$ to 1, from this it is obvious how to define p on morphisms. From this it is simply a matter of verifying that the functor defined this way indeed corresponds to an element of $s_n\mathcal{D}$ and that the map constructed this way is indeed natural. We do not make these details explicit. $\square$

From this it follows that an equivalence of Waldhausen categories yields a homotopy equivalence of corresponding simplicial sets. Now we prove the corollary to lemma 1.4.1 of [14], but for the proof we follow the ideas implied in lecture 3 of [33] theorem 5.4.

Proposition 4.2.3. *(Corollary to lemma 1.4.1 of [14]) Let $\mathcal{C}$ be a Waldhausen category whose weak equivalences are the isomorphism. Denote by $s.\mathcal{C}$ the simplicial set of objects underlying the simplicial category $S.\mathcal{C}$. The natural bisimplicial set inclusion $s.\mathcal{C} \to iS.\mathcal{C}$ is a homotopy equivalence ($s.\mathcal{C}$ is constant in the second simplicial direction).*

Proof. There is a Waldhausen category $Iso_n(\mathcal{C})$ whose objects are the length n chain of composable isomorphisms, whose morphisms are defined pointwise making the obvious ladder commute and whose cofibrations and weak equivalences are defined pointwise. From this it is not hard to observe that we have $i_n S_m \mathcal{C} \cong s_m Iso_n(\mathcal{C})$, further this isomorphism is natural with respect the degeneracy and face operators, so induces an isomorphism of bisimplicial sets. Under this isomorphism the natural inclusion $s.\mathcal{C} \to iS.\mathcal{C}$ becomes the map $s.\mathcal{C} \to s.Iso.(\mathcal{C})$ induced in degree n (in the simplicial direction in which $s.\mathcal{C}$ is constant) by the inclusion as the length m identity chain $\iota_n : \mathcal{C} \to Iso_n(\mathcal{C})$.
Because the $s.$ construction preserves homotopy equivalences by the previous lemma, if we show that ι_n is a homotopy equivalence we are done. One can even show that it is an exact equivalence of categories, following exercise 3.7 of lecture 2 of [33], but we will content ourselves by noting that it is a homotopy equivalence with the help of Quillen's theorem A A.0.2. Indeed, it is not hard to see that a terminal object of $\iota_n/(C_1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} C_n)$ is given by the obvious map $(C_1 = \cdots = C_1) \to (C_1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} C_n)$. This proves the desired claim. □

Next we treat both left horizontal map, as they are homotopy equivalences for the same reason. We first need to define what we mean by $sub(X)$ for a simplicial object X.

Definition 4.2.4. (Appendix 1 of [34]) Consider $T : \Delta \to \Delta$ the functor which sends an ordered set $[n]$ to $[2n+1]$, where the effect of this functor on face and degeneracy maps are to be deduced from writing $T(x_0 < x_1 < \cdots < x_n)$ as $x_0 < x_1 < \cdots < x_n < x'_n < x'_{n-1} < \cdots < x'_0$. Then for a simplicial object X the simplicial set $sub(X)$ is the composition $X \circ T$. We are slightly abusing the equivalence of categories between Δ and the category of finite ordered sets, but this is not a problem.

We do not verify that the above definition of Segal subdivision applied to the nerve of a category produces the same simplicial set as first applying the subdivision defined in lemma 3.3.11 and then taking the nerve. One can show that in general $sub(X) \simeq X$ for any simplicial set, see for example proposition A.1 in [34] or lemma 5.2 in [33] lecture 3. We only need to consider the case where X has a single 0 simplex, this extra hypothesis allows for a (to my knowledge) new proof.

Lemma 4.2.5. *Let X be a simplicial set with a single 0 simplex, then $sub(X) \simeq X$.*

Proof. Let X be a simplicial set with a single 0 simplex. One can hope, for our common usage of the notation $sub(-)$ to not be too abusive that subdividing the category of simplices of X produces the same result as taking the category of simplices of the subdivision of X. In formulas, we pray that

$$Simp(sub(X)) \simeq sub(Simp(X)).$$

Assuming this, it is easy to obtain the desired result. Indeed, we have the following chain of homotopy equivalences by proposition A.0.7, lemma 3.3.12 and the assumption,

$$X \simeq N.Simp(X) \simeq N.sub(Simp(X)) \simeq N.Simp(sub(X)) \simeq sub(X).$$

Thus, if we find an equivalence of categories $F : Simp(sub(X)) \to sub(Simp(X))$ we will have proven the desired result. Denote by $*$ the unique 0 simplex of X, whenever we consider a map $* \to \Delta^p$ in $Simp(X)$ we mean the inclusion as the first vertex. We can now define F. On objects, let F send $\Delta^{2n+1} \to X$ to the following element of $sub(Simp(X))$

$$\begin{tikzcd} * \arrow[d] \arrow[dr] & \\ \Delta^{2n+1} \arrow[r] & X \end{tikzcd} .$$

Now given a morphism of simplices of $sub(X)$

$$\begin{tikzcd} \Delta^{2n+1} \arrow[r, "\alpha"] \arrow[dr] & \Delta^{2m+1} \arrow[d] \\ & X \end{tikzcd}$$

map it to the following map in $sub(Simp(X))$

$$\begin{tikzcd} * \arrow[d] & * \arrow[l, "Id"'] \arrow[d] \\ \Delta^{2n+1} \arrow[r, "\alpha"] \arrow[dr] & \Delta^{2m+1} \arrow[d] \\ & X \end{tikzcd}.$$

It is easy to verify that this is indeed a functor with the claimed domain and codomain, to show that it induces a homotopy equivalence we will use Quillen's theorem A A.0.2. The reason we require to have a single 0 simplex is so that we have a unique map $\Delta^p \to *$ for all $\Delta^p \in Ob(Simp(X))$. Let $B =$

$$\begin{tikzcd} \Delta^p \arrow[d] \arrow[dr] & \\ \Delta^q \arrow[r] & X \end{tikzcd}$$

be an object of $sub(Simp(X))$, we want to show that each F/B is contractible, we will do this by providing a final object. Viewing $Simp(sub(X))$ as a subcategory of $Simp(X)$ it is easy to notice that this category is equivalent to the category $Simp(sub(X))/\Delta^q$ (this is where we need the existence and uniqueness of morphisms $\Delta^p \to *$). Denote by $\sigma : [2q+1] \to [q]$ the morphism which maps i to itself when $i \leq q$ and to q if $i \geq q$. Denote by $d : [q] \to [2q+1]$ the natural inclusion, and notice that $\sigma \circ d = Id$. Then we can observe that

$$\begin{tikzcd} \sigma(\Delta^q) \arrow[d, "d"] \arrow[dr] & \\ \Delta^q \arrow[r] & X \end{tikzcd}$$

will be the desired final object. Indeed, given a diagram

$$\begin{tikzcd} \Delta^{2n+1} \arrow[dr, "\alpha"'] & \sigma(\Delta^q) \arrow[d, "d"] \arrow[dr] & \\ & \Delta^q \arrow[r] & X \end{tikzcd}$$

we can always add $\sigma \circ \alpha : \Delta^{2n+1} \to \sigma(\Delta^q)$ and given any map $\beta : \Delta^{2n+1} \to \sigma(\Delta^q)$ such that $d \circ \beta = \alpha$ we see by applying $\sigma \circ -$ on both sides of the equality and using the identity $\sigma \circ d = Id$ that $\beta = \sigma \circ \alpha$. This proves the desired result. □

Remark 4.2.6. I was not able to show that to promote the abstract homotopy equivalent map to the statement that the desired map is a homotopy equivalence. For this we refer the reader to lemma 5.2 of lecture 3 of [33].

This proves that the top horizontal map is a homotopy equivalence, to obtain the result for the bottom map, fix the simplicial direction of the nerve, i.e. consider $N_p iS.\mathcal{A}$ and its subdivision, and then use the realization lemma A.0.5.

Now we show that the right most vertical map is a homotopy equivalence. For this we need the following result called the swallowing lemma.

Lemma 4.2.7. *(Lemma 1.6.5 in [14]) Let $\mathcal{A}$ be a subcategory of a category $\mathcal{B}$ containing every object (but not necessarily every morphism). Consider the bicategory $\mathcal{AB}$ defined in example 3.10.2 of [6] chapter IV. The natural inclusion $\mathcal{B} \to \mathcal{AB}$ is a homotopy equivalence.*

Proof. We can fix the $\mathcal{A}$ direction, and focus on showing that $\mathcal{B} \to \mathcal{A}_n\mathcal{B}$ is a homotopy equivalence. This will yield the desired result by the realization lemma A.0.5. The m simplicies of the simplicial set $\mathcal{A}_n\mathcal{B}$ are $(n+1) \times (m+1)$ grids with vertical maps in $\mathcal{A}$ and horizontal maps in $\mathcal{B}$

$$\begin{array}{ccccccc}
A_{0n} & \longrightarrow & A_{1n} & \longrightarrow & \cdots & \longrightarrow & A_{mn} \\
\uparrow & & \uparrow & & & & \uparrow \\
\vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & & & \uparrow \\
A_{01} & \longrightarrow & A_{11} & \longrightarrow & \cdots & \longrightarrow & A_{m1} \\
\uparrow & & \uparrow & & & & \uparrow \\
A_{00} & \longrightarrow & A_{10} & \longrightarrow & \cdots & \longrightarrow & A_{m0}
\end{array} .$$

We can project on the bottom horizontal line, yielding a retract to the inclusion $\mathcal{B} \to \mathcal{A}_n\mathcal{B}$. The other composition sends a diagram as above to

$$\begin{array}{ccccccc}
A_{00} & \longrightarrow & A_{10} & \longrightarrow & \cdots & \longrightarrow & A_{m0} \\
\uparrow & & \uparrow & & & & \uparrow \\
\vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & & & \uparrow \\
A_{00} & \longrightarrow & A_{10} & \longrightarrow & \cdots & \longrightarrow & A_{m0} \\
\uparrow & & \uparrow & & & & \uparrow \\
A_{00} & \longrightarrow & A_{10} & \longrightarrow & \cdots & \longrightarrow & A_{m0}
\end{array} ,$$

with every vertical map the identity. It is not hard to see how the arrows in the original diagram allow us to define a natural transformation from this map to the identity, which shows that $\mathcal{B}$ is a deformation retract of $\mathcal{A}_n\mathcal{B}$. This proves the desired result. $\square$

Now let $iQ.\mathcal{A}$ be the simplicial category whose n simplices is the category whose objects degree n elements of the nerve of $Q\mathcal{A}$ and whose morphisms are pointwise isomorphisms making the obvious diagram commute. Applying the above lemma shows that the obvious inclusion $Q\mathcal{A}$ into $iQ.\mathcal{A}$ is a homotopy equivalence.

Now we show that the bottom right map is a homotopy equivalence. We define the map degreewise. So we need to define a map between the categories $sub(iS_n\mathcal{A}) \to iQ_n\mathcal{A}$. We also point out that the index n is somewhat misleading, as we are working in the subdivided simplex, this corresponds to

$i_0S_{2n+1}\mathcal{A}$. So we are searching for a way to correspond to a diagram

$$\begin{tikzcd}
A_{0n} \arrow[r, tail] \arrow[d, two heads] & A_{1n} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_{n-1,n} \arrow[r, tail] \arrow[d, two heads] & A_{nn} \\
A_{0,n-1} \arrow[r, tail] \arrow[d, two heads] & A_{1,n-1} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_{n-1,n-1} & \\
\vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] & & & \\
A_{01} \arrow[r, tail] \arrow[d, two heads] & A_{11} & & & \\
A_{00} & & & &
\end{tikzcd}$$

an object of $iQ_n\mathcal{A}$. Notice that the bottom "staircase" is a sequence of n composable morphisms in $Q\mathcal{A}$, which is exactly what we are looking for. The rest of the diagram indicates all the ways how to compose this staircase, as can be seen by recalling that composition is given by pullbacks. Recalling that morphisms in $Q\mathcal{A}$ are defined as an equivalence class of morphism we get the desired map $iS_n\mathcal{A} \to iQ_n\mathcal{A}$ by sending a diagram as above to the equivalence class of n the composable morphisms in $Q\mathcal{A}$ appearing in the staircase. To see that this is a homotopy equivalence notice that choosing a representative yields a map the other way. It is not hard to see that this is an equivalence of categories. Thus, the result map after geometric realization is a homotopy equivalence.

All that is left is to define the middle vertical map and the top right horizontal map. We will not explicitly check that the diagram commutes as making this reasoning explicit adds nothing of value. The middle vertical map comes from the functoriality of the $sub(-)$ construction applied to the left most vertical map. The top right map is defined in the same way as the bottom right maps on object, the only difference is that by forgetting the morphisms we cannot check as directly that it is a homotopy equivalence. Admitting that the diagram does in fact commute, this proves that the Q and $S.$ construction yield the same K-theory. $\square$

4.3 Additivity theorem

We now prove our first big theorem which does more than help define an Algebraic K-theory or equate two Algebraic K-theories, the additivity theorem. Our proof comes from [15]. For this we define a bisimplicial set for any exact functor $F : \mathcal{C} \to \mathcal{D}$.

Definition 4.3.1. (Definition from [15]) Let $F : \mathcal{C} \to \mathcal{D}$ define $S.F|\mathcal{D}$ in bi-degree m, n to be the set

$$(0 = C_0 \rightarrowtail \cdots \rightarrowtail C_m, F(C_m) \rightarrowtail D_0 \rightarrowtail \cdots \rightarrowtail D_n),$$

with $C_i \in \mathcal{C}, D_i \in \mathcal{D}$. The face and degeneracy operators are the natural deletion and insertion similar to the definition of $S_n\mathcal{C}$. These maps fit together to make $S.F|\mathcal{D}$ a bisimplicial set.

Notice there is a natural projection $S.F|\mathcal{D} \to S.\mathcal{C}$ if we "cheat" and view $S.\mathcal{C}$ as bisimplicial set which is constant in the second coordinate. For a simplicial set X. we denote $X.L$ the associated bisimplicial constant in the second coordinate. We define $X.R$, but constant in the first coordinate. There is also a natural projection $\rho : S.F|\mathcal{D} \to S.\mathcal{D}R$ sending a generic element as in the definition to $0 = D_0/D_0 \rightarrowtail D_1/D_0 \rightarrowtail \cdots \rightarrowtail D_n/D_0$. We use this definition to derive a technical lemma from the following proposition.

Proposition 4.3.2. *(Proposition from [15]) Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between Waldhausen categories. Then $S.F : S.\mathcal{C} \to S.\mathcal{D}$ is a homotopy equivalence if and only if $\rho : S.F|\mathcal{D} \to S.\mathcal{D}R$ is a homotopy equivalence.*

Proof. We have a commutative diagram

$$\begin{array}{ccccc} S.\mathcal{D}R & \xleftarrow{\rho} & S.F|\mathcal{D} & \xrightarrow{\pi} & S.\mathcal{C}L \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle S.F} \\ S.\mathcal{D}R & \xleftarrow{\rho} & S.Id_{\mathcal{D}}|\mathcal{D} & \xrightarrow{\pi} & S.\mathcal{D}L \end{array},$$

where we allowed ourselves a slight abuse of notation in the middle vertical map. The proposition follows if we show that both π appearing in the diagram and the bottom ρ are homotopy equivalences because homotopy equivalences satisfy two out of three.

Notice if we show that the top π is a homotopy equivalence then because F is arbitrary we will have shown that the bottom π is a homotopy equivalence. By the realization lemma (A.0.5) it suffices to show that $\pi_m : S.F|\mathcal{D}([m], -) \to S_m\mathcal{C}L$ is a homotopy equivalence. We can rewrite this as wanting to show that the natural map

$$\coprod_{0=C_0\rightarrowtail\cdots\rightarrowtail C_m} N(co(F(C_m)/\mathcal{D})) \to \coprod_{0=C_0\rightarrowtail\cdots\rightarrowtail C_m} *,$$

is a homotopy equivalence (by $co(F(C_m)/\mathcal{D})$ we mean the subcategory of $F(C_m)/\mathcal{D}$ with objects only the cofibrations and morphisms only those which are cofibration in $\mathcal{D}$). To prove this it suffices to prove that each $N(co(F(C_m)/\mathcal{D}))$ is contractible. This is clear because each $co(F(C_m)/\mathcal{D})$ has the identity map as an initial object.

It suffices to show that $\rho : S.Id_{\mathcal{D}}|\mathcal{D} \to S.\mathcal{D}R$ is a homotopy equivalence. For this we follow [35]. We use the realization lemma A.0.5 in order to reduce to showing that $\rho_n : S.Id_{\mathcal{D}}|\mathcal{D}(-, [n]) \to S_n\mathcal{D}R$ is a homotopy equivalence. We will construct an explicit homotopy inverse, motivated by the fact that it is not hard to find a map which is a strict one-sided inverse (and so we hope it is an inverse on the other side up to homotopy). Define $\nu_n : S_n\mathcal{D}R \to S.Id_{\mathcal{D}}|\mathcal{D}(-, [n])$ by sending $0 = F_0 \rightarrowtail F_1 \rightarrowtail \cdots \rightarrowtail F_n$ to $0 = 0 = \cdots 0 \rightarrowtail F_0 \rightarrowtail F_1 \rightarrowtail \cdots \rightarrowtail F_n$. As was announced it is easy to see that $\rho_n \circ \nu_n$ is the appropriate identity map, now we study the other composition. We get that $\nu_n \circ \rho_n$ maps $0 = D_0 \rightarrowtail D_1 \rightarrowtail \cdots \rightarrowtail D_m \rightarrowtail E_0 \rightarrowtail \cdots \rightarrowtail E_n$ to $0 = 0 = \cdots = 0 \rightarrowtail E_0/E_0 \rightarrowtail E_1/E_0 \rightarrowtail \cdots \rightarrowtail E_n/E_0$. We show this map is homotopic to the identity by providing an explicit homotopy, we leave it up to the reader to verify that this is a homotopy as this is routine verification. Let $0 \leq i \leq m$ and consider the map $h_i : S.Id_{\mathcal{D}}|\mathcal{D}([m], [n]) \to S.Id_{\mathcal{D}}|\mathcal{D}([m+1], [n])$ which sends a generic element $0 = D_0 \rightarrowtail D_1 \rightarrowtail \cdots \rightarrowtail D_m \rightarrowtail E_0 \rightarrowtail \cdots \rightarrowtail E_n$ to $0 = D_0 \rightarrowtail \cdots \rightarrowtail D_i \rightarrowtail E_0 = E_0 = \cdots = E_0 \rightarrowtail E_1 \rightarrowtail \cdots \rightarrowtail E_n$. The map $d_{m+1}h_m$ doubles E_0 and then collapses the identity map inserted this way, so is obviously the identity map. The map d_0h_0 replaces all the D_i by E_0 for $i > 0$, then deletes D_0 and finally quotients out by E_0, all the E_0 added thus become 0 and the part which "survives" becomes $E_0/E_0 \rightarrowtail E_1/E_0 \rightarrowtail \cdots \rightarrowtail E_n/E_0$. So up to routine verification, this shows the desired claim that $\nu_n \circ \rho_n \simeq Id_{S.Id_{\mathcal{D}}|\mathcal{D}(-,[n])}$. And this concludes the proof of this proposition ☐

From this we can prove the technical result which is the test for homotopy equivalence we will use in the proof of additivity.

Lemma 4.3.3. *(Corollary from [15]) Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor of Waldhausen categories. For each $n \in \mathbb{N}$ there is a simplicial map $E_n : S.F|\mathcal{D}(-, [n]) \to S.F|\mathcal{D}(-, [n])$ which sends a generic element*

$$(0 = C_0 \rightarrowtail \cdots \rightarrowtail C_m, F(C_m) \rightarrowtail D_0 \rightarrowtail \cdots \rightarrowtail D_n)$$

to

$$(0 = 0 \rightarrowtail \cdots \rightarrowtail 0, F(0) = 0 = D_0/D_0 \rightarrowtail D_1/D_0 \rightarrowtail \cdots \rightarrowtail D_n/D_0).$$

Suppose that each E_n is a homotopy equivalence, then so is $S.F$.

Proof. The map $\rho_n : S.F|\mathcal{D} \to S_n\mathcal{D}R(-)$ is split by an inclusion I_n sending $D_1 \rightarrowtail \cdots \rightarrowtail D_n$ to

$$(0 = 0 \rightarrowtail \cdots \rightarrowtail 0, F(0) = 0 = D_0/D_0 \rightarrowtail D_1/D_0 \rightarrowtail \cdots \rightarrowtail D_n/D_0).$$

It is not hard to see that $I_n \circ \rho = E_n$. We apply geometric realization and π_i. Because $\rho_n \circ I_n = Id$ and $I_n \circ \rho_n$ is an isomorphism this is enough to see that $\pi_i(|\rho_n|)$ is injective and surjective, thus an isomorphism. So ρ_n is a weak equivalence of CW complexes. Thus ρ is degreewise a homotopy equivalence, which by the realization lemma implies that ρ is a homotopy equivalence. We conclude the desired result by the previous proposition. $\square$

We are almost ready to prove the additivity theorem following Prof. McCarthy. We just need one more definition.

Definition 4.3.4. (9.3 in [6] chapter II) Let $\mathcal{C}$ be a Waldhausen category, and denote by $\mathcal{EC}$ the category whose objects are cofibrations sequences of $\mathcal{C}$ and with morphisms natural transformations of such sequences. This category can be made into a Waldhausen category. Call a map

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d] & C \arrow[d] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & C' \end{tikzcd}$$

a cofibration if the outer vertical and the natural map $A' \cup_A B \to B'$ are all cofibration. And call a map a weak equivalence if it is pointwise a weak equivalence.

Theorem 4.3.5. *(Theorem from [15]) Let $\mathcal{C}$ be a Waldhausen category and consider the exact functor $F : \mathcal{EC} \to \mathcal{C} \times \mathcal{C}$ which sends $A \rightarrowtail B \twoheadrightarrow C$ to (A, C). Then $S.F$ is a homotopy equivalence.*

Proof. To make things clearer, consider a generic element of $S.F|\mathcal{C}^2([m],[n])$

$$\begin{tikzcd} 0 = A_0 \arrow[r] & A_1 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] \arrow[d, tail] & A_m \arrow[d, tail] & & A_m \arrow[r, tail] & S_0 \arrow[r, tail] & \cdots \arrow[r, tail] & S_n \\ 0 = B_0 \arrow[r] & B_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] \arrow[d, two heads] & B_m \arrow[d, two heads] & & & & & \\ 0 = C_0 \arrow[r] & C_1 \arrow[r, tail] & \cdots \arrow[r, tail] & C_m & & C_m \arrow[r, tail] & T_0 \arrow[r, tail] & \cdots \arrow[r, tail] & T_n \end{tikzcd} .$$

The notation of the horizontal maps in the left part of the diagram is slightly misleading, as we require more than being a pointwise cofibration. In addition, these assemble into a cofibration in $\mathcal{EC}$
Let X_n be the subsimplex of $S.F|\mathcal{C}^2(-.[n])$ given by letting all the $A_i = 0$ and $S_0 = 0$ and thus $B_i = C_i$. We define $\Gamma : S.F|\mathcal{C}^2(-,[n]) \to S.F|\mathcal{C}^2(-,[n])$ by the natural map given by quotienting by $A_\bullet$ and S_0. Specifically, Γ sends a generic element as above to

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] \arrow[d, tail] & 0 \arrow[d, tail] & & 0 \arrow[r, tail] & 0 = S_0/S_0 \arrow[r, tail] & \cdots \arrow[r, tail] & S_n/S_0 \\ 0 = C_0 \arrow[r] & C_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] \arrow[d, two heads] & C_m \arrow[d, two heads] & & & & & \\ 0 = C_0 \arrow[r] & C_1 \arrow[r, tail] & \cdots \arrow[r, tail] & C_m & & C_m \arrow[r, tail] & T_0 \arrow[r, tail] & \cdots \arrow[r, tail] & T_n \end{tikzcd} .$$

So Γ is a retraction of the natural inclusion $X_n \to S.F|\mathcal{C}^2(-,[n])$. Suppose we show that $E_n|_{X_n}$ is a homotopy equivalence and $\Gamma \simeq Id_{S.F|\mathcal{C}^2(-,[n])}$. Then $E_n \simeq E_n \circ \Gamma = E_n|_{X_n} \circ \Gamma$, which would show that E_n is a homotopy equivalence, proving the claim by the above lemma.
We first show that $E_n|_{X_n}$ is a homotopy equivalence. We can write a general element of X_n as

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] \arrow[d, tail] & 0 \arrow[d, tail] & & 0 = S_0 \arrow[r, tail] & S_1 \arrow[r, tail] & \cdots \arrow[r, tail] & S_n \\ 0 = C_0 \arrow[r] & C_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] \arrow[d, two heads] & C_m \arrow[d, two heads] & & & & & \\ 0 = C_0 \arrow[r] & C_1 \arrow[r, tail] & \cdots \arrow[r, tail] & C_m & & C_m \arrow[r, tail] & T_0 \arrow[r, tail] & \cdots \arrow[r, tail] & T_n \end{tikzcd} .$$

The image of such an element under E_n can be seen to be

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] \arrow[d, tail] & 0 \arrow[d, tail] & 0 = S_0 \arrow[r, tail] & S_1 \arrow[r, tail] & \cdots \arrow[r, tail] & S_n \\
0 \arrow[r] & 0 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] \arrow[d, two heads] & 0 \arrow[d, two heads] & & & & . \\
0 \arrow[r] & 0 \arrow[r, tail] & \cdots \arrow[r, tail] & 0 & 0 \arrow[r, tail] & 0 = T_0/T_0 \arrow[r, tail] & \cdots \arrow[r, tail] & T_n/T_0
\end{tikzcd}$$

We construct an explicit homotopy from $E_n|_{X_n}$ to Id_{X_n} following [36]. Let $h_i : (X_n)_m \to (X_n)_{m+1}$ be the map which sends a generic element of X_n as above to

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & 0 \arrow[d] & 0 = S_0 \arrow[r] & S_1 \arrow[r] & \cdots \arrow[r] & S_n \\
0 = C_0 \arrow[r, tail] & C_1 \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] \arrow[d] & C_i \arrow[r, tail] \arrow[d] & T_0 \arrow[r, "Id"] \arrow[d] & \cdots \arrow[r, "Id"] \arrow[d] & T_0 \arrow[d] & & & & . \\
0 = C_0 \arrow[r, tail] & C_1 \arrow[r, tail] & \cdots \arrow[r, tail] & C_i \arrow[r, tail] & T_0 \arrow[r, "Id"] & \cdots \arrow[r, "Id"] & T_0 & T_0 \arrow[r] & T_0 \arrow[r] & \cdots \arrow[r] & T_n
\end{tikzcd}$$

We put enough T_0 so that the left hand side consists of $m+1$ composable morphisms, in particular h_m adds one T_0 at the very right. Then we can see that $d_m h_m = Id_{X_n}$ as d_m exactly removes the T_0 which h_m added. We also see that $d_0 h_0$ is the desired map as h_0 replaces every non-zero object on the left half of the diagram by T_0 and so d_0 quotients this by T_0, which is the desired map.

Next we show that Γ is in fact homotopic to the identity map, which will conclude the proof. We provide an explicit definition of the components h_i of a homotopy. Consider a generic element of $S.F|\mathcal{D}([m],[n])$, let $X_j = B_j \sqcup_{A_j} S_0$ and $0 \leq i \leq m$. Define h_i of this generic element to be

$$\begin{tikzcd}
0 = A_0 \arrow[r, tail] & A_1 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & A_i \arrow[r, tail] \arrow[d, tail] & S_0 \arrow[r, tail] \arrow[d, tail] & S_0 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & S_0 \arrow[d, tail] \\
0 = B_0 \arrow[r, tail] & B_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & B_i \arrow[r, tail] \arrow[d, two heads] & X_i \arrow[r, tail] \arrow[d, two heads] & X_{i+1} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & X_m \arrow[d, two heads] \\
0 = C_0 \arrow[r, tail] & C_0 \arrow[r, tail] & \cdots \arrow[r, tail] & C_i \arrow[r, tail] & C_i \arrow[r, tail] & C_{i+1} \arrow[r, tail] & \cdots \arrow[r, tail] & C_m
\end{tikzcd}$$

and

$$\begin{tikzcd}
S_0 \arrow[r] & S_0 \arrow[r] & \cdots \arrow[r] & S_m \\
\\
C_m \arrow[r] & T_0 \arrow[r] & \cdots \arrow[r] & T_m
\end{tikzcd}$$

(one should imagine these two diagrams next to each other as an element of $S.F|\mathcal{D}([m+1],[n])$). To check that these maps assemble into the desired homotopy is routine verification. This concludes the proof of the additivity theorem. $\square$

One can use the above homotopy level theorem to obtain a result more obviously warranting the name of additivity theorem. For this we need the following definition.

Definition 4.3.6. (Definition 1.1 in [6] chapter V) Suppose we have three exact functors $F, F', F'' : \mathcal{C} \to \mathcal{D}$ between Waldhausen categories, together with natural transformations which we denote $F \rightarrowtail F' \twoheadrightarrow F''$. We call such a sequence an exact or cofiber sequence of exact functors if for every object $A \in \mathcal{C}$ we have that $F(A) \rightarrowtail F'(A) \twoheadrightarrow F''(A)$ is a cofibration sequence and for every cofibration $A \rightarrowtail B$ the natural map $F(A) \cup_{F'(A)} F(B') \rightarrowtail F(B)$ is a cofibration.

With this definition in hand we can state the additivity theorem as stated in the K-book.

Corollary 4.3.7. *(Theorem 1.2 in [6] chapter V) Let* $F \rightarrowtail F' \twoheadrightarrow F'' : \mathcal{C} \to \mathcal{D}$ *be an exact sequence of exact functors between Waldhausen categories. Then* $F'_* \simeq F_* + F''_*$ *as maps between the H-spaces* $K(\mathcal{C}) \to K(\mathcal{D})$*. In particular by taking homotopy groups we have* $K_i(F') = K_i(F) + K_i(F'')$*.*

Proof. An exact sequence of functors can easily be seen to induce a functor $\mathcal{E}F : \mathcal{C} \to \mathcal{E}\mathcal{D}$. Now consider the exact functors $s, t, q : \mathcal{E}\mathcal{D} \to \mathcal{D}$ which send an exact sequence respectively to the source, target and quotient term of the exact sequence. Notice that t indeed preserves cofibrations (the rest of the verification of exactness are clear) as can be seen by the following diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[d] & \\ A' \arrow[r] & A' \cup_A B \arrow[dr] & \\ & & B' \end{tikzcd} .$$

Indeed, the map $B \to A' \cup_A B$ is a cofibration as it is a pushout of a cofibration and $A' \cup_A B \to B'$ is a cofibration by the Waldhausen structure of $\mathcal{E}\mathcal{D}$.
It suffices to show that $t_* \simeq (s \sqcup q)_*$ because the H-space structure of $K(\mathcal{C})$ comes from the coproduct. It is not hard to see that the following diagram commutes

$$\begin{tikzcd} \mathcal{D} \times \mathcal{D} \arrow[r, "\bigsqcup"] & \mathcal{E}\mathcal{D} \arrow[r, bend left, "t"] \arrow[r, bend right, "s \sqcup q"'] & \mathcal{D} \end{tikzcd} .$$

The functor $\bigsqcup : \mathcal{D} \times \mathcal{D} \to \mathcal{E}\mathcal{D}$ sending a pair (A, B) to the exact sequence $A \rightarrowtail A \sqcup B \twoheadrightarrow B$ is a one-sided inverse of the map we proved to be a homotopy equivalence in the above theorem 4.3.5, and in particular by 2 out of 3 is a homotopy equivalence. Thus, we see that $t_* \simeq s_* + q_*$ which proves the desired claim. □

Waldhausen [14] proves the equivalence of four different versions of the additivity theorem. We add to our discussion of additivity the following version of additivity.

Definition 4.3.8. (Definition following lemma 1.1.6 in [14]) Let $\mathcal{B}$ be a Waldhausen category, with two exact subcategories $\mathcal{A}, \mathcal{C}$. There is an obvious category of cofibrations $\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ whose objects are the cofibrations with first term in $\mathcal{A}$, middle term in $\mathcal{B}$ and final term in $\mathcal{C}$. It is easy to see how to make this a Waldhausen category, inspired by the definition of $\mathcal{E}\mathcal{C}$ and that the three natural projection functors are all exact.

Proposition 4.3.9. *(Proposition 1.3.2 (iv) implies (i) from [14]) Let* $\mathcal{A}, \mathcal{C}$ *be exact subcategories of a Waldhausen category* $\mathcal{B}$*, then the map* $wS.\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \to wS.\mathcal{A} \times wS.\mathcal{C}$ *sending* $A \rightarrowtail B \twoheadrightarrow C$ *to* (A, C) *is a homotopy equivalence.*

Proof. Call the functor we wish to show is a homotopy equivalence F, it splits by sending a pair (A, C) to the cofibration sequence $A \rightarrowtail A \sqcup C \twoheadrightarrow C$, call this map σ. Because we know $F\sigma$ is the identity, to prove the desired claim it suffices to show that the composition σF is homotopic to the identity. Define two exact endofunctors of $\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$, d which maps a cofibration $A \rightarrowtail B \twoheadrightarrow C$ to the trivial cofibration $A = A \twoheadrightarrow 0$ and c which maps a cofibration sequence to the trivial cofibration $0 \rightarrowtail B = B$. It is not hard to see that we have exact sequences of endofunctors $d \rightarrowtail Id \twoheadrightarrow c$ and $d \rightarrowtail \sigma F \twoheadrightarrow c$. By the corollary to the additivity theorem 4.3.7, this implies that $\sigma F \simeq c + d \simeq Id$ which proves the desired result. □

4.4 Infinite loop spaces and pairing in Waldhausen K-theory

In this section we wish to show that the definition of relative K-theory we gave warrants that name, in other words it fits into an exact sequence of K-groups. We will be able to do this at the space level, we shall then use this to deloop the K-theory space. For a functor $F : \mathcal{C} \to \mathcal{D}$ we already have a sequence of exact functors between simplicial Waldhausen categories ($\mathcal{D}$ is viewed as a constant simplicial Waldhausen category)

$$\mathcal{D} \to S.F \to S.\mathcal{C}.$$

We can apply the $wS.$ construction degreewise and the next proposition explains why we might want to do that.

Proposition 4.4.1. *(Proposition 1.5.5 in [14]) The sequence $wS.\mathcal{D} \to wS.S.F \to wS.S.\mathcal{C}$ is a homotopy fibration.*

Proof. We will prove this by showing that we can apply lemma 5.2 of [37]. We have repeated the statement of this result in the appendix as proposition A.0.6. The composition is clearly a constant map, as we project onto a different coordinate then the one we include in to. Now we fix some n and consider the sequence of simplicial sets

$$wS.\mathcal{D} \to wS.S_nF \to wS.S_n\mathcal{C}.$$

The base space is connected, so it suffices to show that this sequence of simplicial sets is a homotopy fibration. We consider two interesting subcategories of S_nF, on the one hand we have that $\mathcal{D}$ lives in S_nF by inclusion as the constant cofibration sequence, call $\mathcal{A}$ this subcategory isomorphic to $\mathcal{D}$. Now consider the subcategory $\mathcal{B}$ of pairs $(0 = C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_n, 0 \rightarrowtail D_0 \rightarrowtail \cdots \rightarrowtail D_n)$ with $D_0 = 0$. We see that such elements are uniquely determined by the first coordinate, and so this subcategory is isomorphic to $S_n\mathcal{C}$. Now there are exact functors $J_\mathcal{A}, J_\mathcal{B} : S_nF \to S_nF$ which map into the subcategory denoted by the subscript. Define $J_\mathcal{A}$ to send a generic element $(0 = C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_n, 0 \rightarrowtail D_0 \rightarrowtail \cdots \rightarrowtail D_n)$ to $(0 = \cdots = 0, D_0 = \cdots = D_0)$ and define $J_\mathcal{B}$ to send a generic element to $(0 = C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_n, 0 \rightarrowtail D_0/D_0 \rightarrowtail D_1/D_0 \rightarrowtail \cdots \rightarrowtail D_n/D_0)$. It is clear that these fit into an exact sequence of functors $\mathcal{J}_\mathcal{A} \rightarrowtail Id \twoheadrightarrow J_\mathcal{B}$, and so by additivity we have that the identity functor of $wS.S_nF$ is homotopic to $wS.J_\mathcal{A} + wS.J_\mathcal{B}$.
We have a map $G : \mathcal{D} \times S_n\mathcal{C} \to S_nF$ which maps a pair $(D, 0 = C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_n)$ to $(0 = C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_n, 0 \rightarrowtail D \rightarrowtail D \sqcup F(C_1) \rightarrowtail \cdots \rightarrowtail D \sqcup F(C_n))$. The natural projection $(J_\mathcal{A}, J_\mathcal{B}) : S_nF \to \mathcal{D} \times S_n\mathcal{C}$ splits this map, and by the above discussion the other composition is homotopic to the identity. Applying the $wS.$ construction yields vertical homotopy equivalences fitting into the following diagram

$$\begin{tikzcd}
wS.\mathcal{D} \arrow[r, tail] \arrow[d, "Id"] & wS.S_nF \arrow[r, two heads] \arrow[d, "wS.(J_\mathcal{A},J_\mathcal{B})"] & wS.S_n\mathcal{C} \arrow[d, "Id"] \\
wS.\mathcal{D} \arrow[r, tail] & wS.\mathcal{D} \times wS.S_n\mathcal{C} \arrow[r, two heads] & wS.S_n\mathcal{C}
\end{tikzcd}.$$

Because the composites of both the top and bottom rows are equal to a constant map, the pointwise weak equivalence is enough to deduce that the top row is a homotopy fiber sequence from the fact that the bottom row is. This verifies the last assumption needed to apply lemma A.0.6, yielding the desired result. □

With such a tool in hand, it is natural to study what happens when $F = Id_\mathcal{C}$. The above lemma yields something interesting in this case.

Lemma 4.4.2. *(Lemma 8.5.4 in [6] chapter IV) The simplicial set $wS.Id_\mathcal{C}$ is contractible.*

Proof. A generic element of $wS_nId_\mathcal{C}$ is entirely determined by its value in the second coordinate, which is free to be anything. Thus, this space is equivalence to $wS_{n+1}\mathcal{C}$ which means that as a simplicial set $wS.Id_\mathcal{C} = PwS.\mathcal{C}$ (see [4] 8.3.14). We know that the path space of a simplicial set is homotopy equivalent to the constant simplicial set of 0 simplices of the original simplicial set. In our case this means $wS.Id_\mathcal{C}$ is homotopy equivalent to a single point, i.e. contractible. □

We can use the above lemma in the fibration up to homotopy obtained in proposition 4.4.1 to obtain $\Omega|wS.S.\mathcal{C}| \simeq |wS.\mathcal{C}|$. In fact, we can do better.

Proposition 4.4.3. *(8.5.5 in [6] chapter IV) The space $K(\mathcal{C})$ is an infinite loop space, with $(n-1)$th delooping given by $K(\mathcal{C}) = \Omega^n|wS^n.\mathcal{C}|$, where $S^n.\mathcal{C}$ is the nth iterated application of the $S.$ construction.*

Proof. We proceed by induction on n, the 0th delooping is by definition and the case $n = 2$ follows from applying the loop space construction to the homotopy equivalence discussed above. Now suppose we have that $K(\mathcal{C}) = \Omega^n|wS^n.\mathcal{C}|$, we want to show the result for $n+1$. From the fibration obtained in proposition 4.4.1 applied to the identity functor, we can by repeated degreewise application of the $S.$ construction obtain a sequence of $n+1$-simplicial categories

$$wS^n.\mathcal{C} \to wS^{n+1}.Id_{\mathcal{C}} \to wS^{n+1}.\mathcal{C},$$

where $wS^n.\mathcal{C}$ is constant in the "final" simplicial degree. By repeated application of the realization lemma, lemma A.0.6 and geometric realization we can see on the one hand that $wS^{n+1}.Id_{\mathcal{C}}$ is contractible (by degreewise contractility and induction) and the above sequence of $n+1$ simplicial categories is a fibration up to homotopy (because it is degreewise so by induction, and then apply lemma A.0.6). This is enough to obtain the desired claim. ☐

This allows to work with the K-theory spectrum $\mathbf{K}\mathcal{C}$ which is $\Omega|wS.\mathcal{C}|$ at level 0 and $\{|wS^n.\mathcal{C}|\}_{n=1}^{\infty}$ beyond 0. Our work so far shows this is an Ω-spectrum. In fact one can even show (see appendix of [19]) that these spaces assemble into a symmetric spectrum, we will assume this result. It is easy to see this does not change the definition of the K-groups of a Waldhausen category. We will not systematically work with the K-theory spectrum, and will be alternating freely between this and the K-theory space.

By repeatedly taking the homotopy fiber of the fibration from proposition 4.4.1 and the discussion preceding the previous result, we get for any exact functor of Waldhausen categories $F : \mathcal{C} \to \mathcal{D}$ a fibration up to homotopy

$$\Omega^2|wS.S.F| \to \Omega|wS.\mathcal{C}| \to \Omega|wS.\mathcal{D}|.$$

As these are all K-theory spaces, we have almost justified calling $K(F) = \Omega^2|wS.S.F|$ a relative K-theory space. In order for this space to actually deserve this name, we need the following proposition.

Proposition 4.4.4. *(Exercise 1.7 in [6] chapter V) Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor of Waldhausen categories. In the long exact sequence coming from the up to homotopy fibration*

$$K(F) \to K(\mathcal{C}) \to K(\mathcal{D})$$

constructed above, the map $K_i(\mathcal{C}) \to K_i(\mathcal{D})$ is the one induced by F.

Proof. We have a diagram of bisimplicial categories, where the two bisimplicial categories on the left of the diagram are constant in the second simplicial direction

$$\begin{tikzcd} wS.\mathcal{C} \arrow[r] \arrow[d, "wS.F"] & wS.S_nId_{\mathcal{C}} \arrow[r] \arrow[d, "F"] & wS.S_n\mathcal{C} \arrow[d, "Id"] \\ wS.\mathcal{D} \arrow[r] & wS.S_nF \arrow[r] & wS.S_n\mathcal{C} \end{tikzcd}.$$

The middle vertical map applies the functor F on the second coordinate in the obvious way, this is well-defined as F is exact, so preserves quotients. We know that geometrically realizing this diagram yields a map of up to homotopy fibrations. At which point we can take homotopy fibers. This yields

$$\begin{tikzcd} \Omega|wS.S.\mathcal{C}| \arrow[r] \arrow[d, "Id"] & |wS.\mathcal{C}| \arrow[r] \arrow[d, "wS.F"] & |wS.S_nId_{\mathcal{C}}| \arrow[d, "F"] \\ \Omega|wS.S.\mathcal{C}| \arrow[r] & |wS.\mathcal{D}| \arrow[r] & |wS.S_nF| \end{tikzcd}.$$

Our goal is to use that the first top horizontal map is a homotopy equivalence to consider the up to homotopy fibration $|wS.\mathcal{C}| \to |wS.\mathcal{C}| \to |wS.S.F|$ and understand the first map, we can do this by adding a morphism to our diagram, to get

$$\begin{tikzcd}
 & \Omega|wS.S.\mathcal{C}| \arrow[r] \arrow[d, "Id"] & |wS.\mathcal{C}| \arrow[r] \arrow[d, "wS.F"] & |wS.S_n Id_{\mathcal{C}}| \arrow[d, "F"] \\
|wS.\mathcal{C}| \arrow[r] \arrow[urr, bend left, "Id"] & \Omega|wS.S.\mathcal{C}| \arrow[r] & |wS.\mathcal{D}| \arrow[r] & |wS.S_n F|
\end{tikzcd}.$$

Now the diagram only commutes up to homotopy, but this is enough to see, that up to homotopy, the map $wS.\mathcal{C} \to wS.\mathcal{D}$ is the map induced by F. Which is the desired result. $\square$

This officially finishes the discussion to support the claim that $K(\mathcal{F})$ is an appropriate construction for relative K-theory. We may now move on to discussing pairings in K theory. To do this we use the material at the end of section 8 in [6] chapter IV, the discussion on page 342 of [14] and the appendix of [19].

Definition 4.4.5. (8.11 in [6] chapter IV) Let $\mathcal{A},\mathcal{B},\mathcal{C}$ be Waldhausen categories, a functor $F : \mathcal{A}\times\mathcal{B} \to \mathcal{C}$ is called biexact if each functor $F(\mathcal{A}, -), F(-,\mathcal{B})$ is exact and for any pair of cofibration $(A \rightarrowtail A') \in co(\mathcal{A}).(B \rightarrowtail B') \in co(\mathcal{B})$ the natural map $F(A', B) \cup_{F(A,B)} F(A, B') \to F(A', B')$ is a cofibration.

Now suppose we have a biexact functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$. To show that this induces a pairing in K-theory $\mathbf{F} : \mathbf{K}\mathcal{A} \wedge \mathbf{K}\mathcal{B} \to \mathbf{K}\mathcal{C}$. We detail the pairing in the case $\mathbf{K}\mathcal{A}_1 \wedge \mathbf{K}\mathcal{B}_1 \to \mathbf{K}\mathcal{C}_2$ and refer the reader to the appendix of [19] for a full exposition in the language of symmetric spectra.

A biexact functor induces a functor $S_n\mathcal{A} \times S_m\mathcal{B} \to S_nS_m\mathcal{C}$ by sending a pair $(0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n, 0 \rightarrowtail B_1 \rightarrowtail B_m)$ to a diagram

$$\begin{tikzcd}
F(A_n,0) \arrow[r, tail] & F(A_n,B_1) \arrow[r, tail] & \cdots \arrow[r, tail] & F(A_n,B_m) \\
\vdots \arrow[u, tail] & \vdots \arrow[u, tail] & & \vdots \arrow[u, tail] \\
F(A_1,0) \arrow[r, tail] \arrow[u, tail] & F(A_1,B_1) \arrow[r, tail] \arrow[u, tail] & \cdots \arrow[r, tail] & F(A_1,B_m) \arrow[u, tail] \\
0 \arrow[r, tail] \arrow[u, tail] & F(0,B_1) \arrow[r, tail] \arrow[u, tail] & \cdots \arrow[r, tail] & F(0,B_m) \arrow[u, tail]
\end{tikzcd}.$$

It is not hard these fit into a bisimplicial map, and then to restrict/corestrict to get a map $wS.\mathcal{A} \times wS.\mathcal{B} \to wS.S.\mathcal{C}$. This corestriction works by bi-exactness. Notice, with notation as above, that if either $A_i = 0$ for all $1 \le i \le n$ or $B_i = 0$ for all $1 \le i \le m$ then because we point of all our categories with 0 objects, we have that the image is a diagram with all object the 0 objects and all maps the identity. What this means is that the geometric realization $|wS.\mathcal{A}| \times |wS.\mathcal{B}| \to |wS.S.\mathcal{C}|$ factors through the smash product. The map $|wS.\mathcal{A}| \wedge |wS.\mathcal{B}| \to |wS.S.\mathcal{C}|$ induces a map $\Omega|wS.\mathcal{A}| \wedge \Omega|wS.\mathcal{B}| \to \Omega^2|wS.S.\mathcal{C}|$. Indeed, the smash product is a functor from Top^2 to Top which means we can smash maps, in particular loops. This defines a map $\Omega|wS.\mathcal{A}| \wedge \Omega|wS.\mathcal{B}| \to \Omega^2(|wS.\mathcal{A}| \wedge |wS.\mathcal{B}|)$ and now we post compose with Ω^2 of the map constructed above to get a map $\Omega|wS.\mathcal{A}| \wedge \Omega|wS.\mathcal{B}| \to \Omega^2|wS.S.\mathcal{C}|$.

This shows that a biexact functor induces a pairing of K-theory spectra which by precomposition by $\Omega|wS.\mathcal{A}| \times \Omega|wS.\mathcal{B}| \to \Omega|wS.\mathcal{A}| \wedge \Omega|wS.\mathcal{B}|$ provides a pairing $K_i(\mathcal{A}) \times K_j(\mathcal{B}) \to K_{i+j}(\mathcal{C})$. This ends this section.

Remark 4.4.6. Both in [14] and [6] there is some subtlety in the fact that $S.S.\mathcal{C}$ is a bicategory, and thus we need to take week equivalences in two different directions and then we need to somehow show $wS.wS.\mathcal{C} \simeq wS.S.\mathcal{C}$ using the swallowing lemma. I was not able to apply the swallowing lemma to obtain this result. Further, other sources, notably [19] ignores this subtlety in the appendix. Which (somewhat) justifies ignoring this subtlety as well.

5 Selection of theorems in K-theory

We prove in this section a selection of theorems, which we will use to relate the K-theory of the fields of prime order, the rationals and the integers.

5.1 Waldhausen Localization

The key result of this section allows to change the class of weak equivalences in a Waldhausen category, which we would expect to be related to the original Waldhausen category. Further intuition would lead us to believe that in K-theory, the difference should be exactly those objects which are weakly equivalent to 0 with respect to the finer class of weak equivalences. This intuition is correct as formalized by the following statement.

Theorem 5.1.1. *(2.1 in [6] chapter V) Let $\mathcal{A}$ be a category with cofibrations made into a Waldhausen category with two different classes $v(\mathcal{A}) \subset w(\mathcal{A})$ of weak equivalences. Suppose that $(\mathcal{A}, w)$ is: saturated, satisfies the cylinder axiom and the extension axiom. Denote by $\mathcal{A}^w$ the Waldhausen subcategory of $(\mathcal{A}, v)$ of objects such that $0 \to A$ is a w-weak equivalence. Then we have a homotopy fibration*

$$K(\mathcal{A}^w) \to K(\mathcal{A}, v) \to K(\mathcal{A}, w).$$

Proof. We use the notation of the theorem without repeating it. The proof will consist in applying the homotopy fibration $\Omega|wS.S.\mathcal{C}| \to |wS.\mathcal{D}| \to |wS.S.F|$ for any exact functor $F : \mathcal{C} \to \mathcal{D}$ to the exact inclusion $\iota : (\mathcal{A}^w, v) \to (\mathcal{A}, v)$ and then identifying the relative K-theory space. We will on the one hand show that $v.S.S.\iota$ can be identified with $v.cow.S.\mathcal{A}$ (we will detail what this is in the course of the proof) and then fit this into a commutative diagram which will allow us to replace $v.cow.S.\mathcal{A}$ with $wS.\mathcal{A}$.

We as usual allow ourselves to ignore the choice of quotients in the various $S.$ constructions we use. Consider $A \rightarrowtail B$ a cofibration which is also a $w-$weak equivalence, also called a $w-$trivial cofibration. Applying the gluing axiom to

$$\begin{tikzcd} 0 \arrow[d, "Id"] & A \arrow[l] \arrow[r, tail, "\sim_w"] \arrow[d, tail, "\sim_w"] & B \arrow[d, "Id"] \\ 0 & B \arrow[l] \arrow[r, "Id"] & B \end{tikzcd}$$

shows that the map $A/B \to B/B \cong 0$ is a $w-$weak equivalence, which shows that a $w-$trivial cofibration has quotient in $\mathcal{A}^w$ by applying the saturation axiom to

$$\begin{tikzcd} A/B \arrow[dr] & \\ 0 \arrow[u] \arrow[r, "Id"'] & 0 \end{tikzcd} .$$

Conversely, suppose $A \rightarrowtail B$ has quotient in $\mathcal{A}^w$. We can apply the extension axiom to

$$\begin{tikzcd} A \arrow[r, "Id"] \arrow[d, "Id"] & A \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d] \\ A \arrow[r, tail] & B \arrow[r, two heads] & A/B \end{tikzcd}$$

to obtain that such a cofibration must be $w-$trivial. So the $w-$ trivial cofibrations are exactly those whose quotient lies in $\mathcal{A}^w$. What this shows is that projection onto the second coordinate of $S_n\iota$ yields an equivalence with $cow_n\mathcal{A}$ the category of all sequences of n composable $w-$trivial cofibrations, with morphisms ladders with vertical maps in $\mathcal{A}$. Indeed, it is easy to see that if $C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_n$ is an element in the image of the projection $S_n\iota \to S_n\mathcal{A}$, then each cofibration $C_0 \rightarrowtail C_i$ is a w-trivial cofibration, and because $w(\mathcal{A})$ is saturated this is enough to show that each $C_i \rightarrowtail C_j$ is a $w-$weak equivalence and so in particular a $w-$trivial cofibration. This yields a map $S_n\iota \to cow_n\mathcal{A}$, it is not

hard to construct the map the other way thus establishing the equivalence of categories.
And so by exactness of the equivalence we also get an equivalence of $vS_n\iota$ and $v.cow_n\mathcal{A}$ which by the realization lemma yields an equivalence of $vS.\iota$ and $v.cow.\mathcal{A}$. Now because of the level of generality with which we are working we can replace $\mathcal{A}$ by $S.\mathcal{A}$, which yields a homotopy equivalence $vS.S.\iota \to v.cow.S.\mathcal{A}$. Indeed, saturation, the extension axiom and cylinder axiom are all inherited by $S.$ construction. And so we can postcompose the homotopy fibration

$$\Omega|vS.S.\mathcal{A}^w| \to |v.S\mathcal{A}| \to |vS.S.\iota|$$

with the homotopy equivalence we just established, to obtain the homotopy fibration

$$\Omega|vS.S.\mathcal{A}^w| \to |v.S\mathcal{A}| \to |v.cow.S.\mathcal{A}|.$$

We can replace the left most space by $vS.\mathcal{A}^w$ using the loop space structure of Waldhausen K-theory shown in proposition 4.4.3. This allows us to fit the above homotopy fibration in the following diagram of simplicial sets

$$\begin{tikzcd} vS.\mathcal{A}^w \arrow[r] & vS.\mathcal{A} \arrow[r] \arrow[d] & v.cow.S.\mathcal{A} \arrow[d] \\ & wS.\mathcal{A} \arrow[r] & v.w.S.\mathcal{A} \end{tikzcd}.$$

The bottom map is a homotopy equivalence by an immediate application of the swallowing lemma. If we show that the vertical map is a homotopy equivalence we are done as replacing the base space with a homotopy equivalent space does not change the homotopy type of the homotopy fiber, thus we will get the desired result. The fact that the vertical map is a homotopy equivalence will follow at once from the following lemma

Lemma 5.1.2. *(Exercise 8.15 in [6] chapter IV)(Lemma 1.6.3 in [14]) Let $(\mathcal{C}, co, w)$ be a saturated Waldhausen category with a cylinder functor $T : \mathcal{C}^{[2]} \to \mathcal{C}$ satisfying the cylinder axiom. Then the inclusion of the category of trivial cofibrations into the category of weak equivalences is a homotopy equivalence.*

Proof. Let $w.\mathcal{C}$ be the category of weak equivalence in $\mathcal{C}$ and $cow.\mathcal{C}$ the subcategory of those weak equivalences which are also cofibrations. We show that the inclusion $\iota : cow.\mathcal{C} \to w.\mathcal{C}$ is a homotopy equivalence by appealing to Quillen's theorem A (A.0.2).
That means we need to show that for each object $B \in \mathcal{C}$ that the category ι/B is contractible. By the cylinder axiom, for every object $f : A \xrightarrow{\sim_w} B$ in ι/B the map $p : T(f) \to B$ is a weak equivalence, thus T defines an endofunctor of ι/B, which we call p_*. The saturation axiom is used to see that the maps $A \to T(f) \leftarrow B$ are weak equivalences by applying two out of three to the post composition with p. These are also cofibrations as they are the composite of two maps which are axiomatically cofibrations, for example in the case of A we have $A \rightarrowtail A \sqcup B \rightarrowtail T(f)$.
This means that we can extend the maps $A \to T(f) \leftarrow B$ to maps of functors, which can easily be checked to be natural, $Id \to p_* \leftarrow c_{Id_B}$, where c_{Id_B} is the constant map sending every object $A \xrightarrow{\sim_w} B$ to the identity of B. This shows that the identity of ι/B is homotopic to a constant map, in other words ι/B is contractible, which is the desired claim. □

□

The above result can be "upgraded" to a result about K-theory spectra, i.e. we have a homotopy fibration of spectra. This is interesting because spectra are a stable homotopy category and thus homotopy fibrations and homotopy cofibrations are the same thing (see [9] theorem 7.1.11). The upgrading into a result about spectra is done by the following result.

Proposition 5.1.3. *(From my advisor Prof. Rognes) Suppose we have a sequence of functors of Waldhausen categories $A \xrightarrow{F} B \xrightarrow{G} C$ such that $\boldsymbol{K}(G) \circ \boldsymbol{K}(F)$ is null homotopic (thus $K(G) \circ K(F)$ as well by applying Ω^∞). Then $K(A) \xrightarrow{K(F)} K(B) \xrightarrow{K(G)} K(C)$ is a homotopy fibration in the category of spaces with the additional property that $\pi_0(K(G))$ is surjective if and only if $\boldsymbol{K}(A) \xrightarrow{\boldsymbol{K}(F)} \boldsymbol{K}(B) \xrightarrow{\boldsymbol{K}(G)} \boldsymbol{K}(C)$ is a homotopy fibration in the category of spectra.*

Proof. ($\Leftarrow$) Saying that $\mathbf{K}(A) \xrightarrow{\mathbf{K}(F)} \mathbf{K}(B) \xrightarrow{\mathbf{K}(G)} \mathbf{K}(C)$ is a homotopy fibration is the same as saying that the natural map $\alpha : K(A) \to hofib(\mathbf{K}(G))$ is a weak equivalence. Now the spectra $hofib(\mathbf{K}(G))$ can be understood better one degree at a time. We have $hofib(\mathbf{K}(G))_n = hofib(\mathbf{K}(G)_n)$, in particular in degree 0 we have that this is equal to $hofib(K(G))$. The map α is also constructed degreewise, thus applying the functor $\Omega^\infty : Sp \to Top$ to all the data which is given to us yields a map $\alpha_0 : K(A) \to hofib(K(G))$ which can serve as the natural map between these two spaces.
We are done proving this direction if we show each $\pi_i(\alpha_0)$ is an isomorphism. Ideally, we want to show that $\pi_i(\alpha_0) = \pi_i(\alpha)$, with the first π_i^* meant to emphasize that this is the functor whose domain is the category of spectra. This (up to isomorphism) follows from the fact that both $\mathbf{K}(A)$ and $hofib(\mathbf{K}(G))$ are Ω-spectra, thus all the maps in the diagram over which $\pi_i^*(\alpha) : \pi_i^*(\mathbf{K}(A)) \to \pi_i^*(hofib(\mathbf{K}(G)))$ is a colimit are isomorphisms. And so in particular $\pi_i(\alpha_0)$ is an isomorphism if and only if $\pi_i^*(\alpha)$ is an isomorphism.
The fact that $\mathbf{K}(A)$ is an Ω-spectrum has been proven in proposition 4.4.3, so we just need to argue the same holds true for $hofib(\mathbf{K}(G))$. This follows at once by using the fact that Ω commutes with homotopy fibers. To see this one needs the definition of $\mathbf{K}(B)_n \xrightarrow{\mathbf{K}(G)_n} \mathbf{K}(C)_n$ and a small strengthening of proposition 4.4.3 to generalize to morphisms. The desired surjectivity follows from the connectivity of the spectra and the homotopy long exact sequence.
($\Rightarrow$) We want to show that $\mathbf{K}(A) \xrightarrow{\mathbf{K}(F)} \mathbf{K}(B) \xrightarrow{\mathbf{K}(G)} \mathbf{K}(C)$ is a homotopy fibration, i.e. we want to show that the natural map $\alpha : \mathbf{K}(A) \to hofib(\mathbf{K}(G))$ is a weak equivalence. Note that both of these spectra are connective, the first by study of i-cells of the nth level space for $i < n$ and the second by the homotopy long exact sequence and surjectivity of $\pi_0(K(G))$. Thus, we only need to concern ourselves with the positive homotopy groups. For this, same as above, we can use the fact that we are working with Ω-spectra to see that $\pi_i^* = \pi_i \circ \Omega^\infty$. Now by assumption, the natural map $K(A) \to hofib(K(G))$ is a weak equivalence. And similarly as above, using the level-wise construction of α we can see that the natural map $K(A) \to hofib(K(G))$ can be taken to be α_0, thus proving that $\pi_i^*(\alpha) = \pi_i(\Omega^\infty(\alpha))$ is an isomorphism. $\square$

This allows us to upgrade Waldhausen localization as the nullhomotopy of the composite is defined at the category level. Indeed, the composite functor $\mathcal{A}^w \to \mathcal{A}$ is the inclusion, and for every object $A \in \mathcal{A}^w$ there is a unique morphism $0 \to A$ which fit together into a natural transformation from the 0 map to the composite. A perfectly analogous reasoning can be done after any number of finite iterations of the $S.$ construction essentially because weak equivalences are defined pointwise in $S.\mathcal{A}$. This in particular means that the induced map in K-theory spectra is nullhomotopic, and applying Ω^∞ yields the nullhomotopy of the homotopy fibration given by Waldhausen localization.

We present an application of the above theorem which allows us to compute the K-theory of an exact category by instead computing K-theory of a Waldhausen category satisfying the cylinder axiom, which is technically convenient. To state the theorem, we need to define $Ch^b(\mathcal{C})$ for an exact category $\mathcal{C}$ embedded into an abelian category $\mathcal{A}$. This category is the category of bounded chain complexes in $\mathcal{C}$ turned into a Waldhausen category by letting the cofibrations be the pointwise admissible monics and the weak equivalences the maps which are quasi-isomorphisms in $Ch^b(\mathcal{A})$. We do not verify that this indeed, defines a Waldhausen category.

Proposition 5.1.4. *(2.2 in [6] chapter V)(1.11.7 of [16]) Let $\mathcal{C}$ be an exact category in some ambient abelian category $\mathcal{A}$. Suppose further, that whenever $f \in mor(\mathcal{C})$ is a surjection in $\mathcal{A}$ then $ker(f)$ is in $\mathcal{C}$. By considering complexes concentrated in degree 0 we get an exact inclusion $\mathcal{C} \to Ch^b(\mathcal{C})$ which induces a homotopy equivalence on K-theory spaces.*

Proof. Denote by $Ch^{[a,b]}$ the category of chain complexes in $\mathcal{C}$ such that the term indexed by i is 0 when $i \notin [a, b]$ and by $\mathcal{C}^{[a,b]}_{exact}$ the subcategory of those complexes which are acyclic viewed as elements of $Ch(\mathcal{A})$. We can take the colimit as a and b tend to infinity to obtain the category $Ch^b(\mathcal{C})$ of bounded chain complexes and the subcategory $\mathcal{C}^b_{exact}$ of those chain complexes which are acyclic in $\mathcal{A}$. We equip $Ch^{[a,b]}$ with two different class of weak equivalences $i \subset w$. The former is the class of pointwise isomorphisms and the latter are those maps which are weak equivalences in $Ch(\mathcal{A})$. It is easy to see that the Waldhausen structure defined above on the colimit agrees with the colimit of

$(Ch^{[a,b]}, w)$ taken in the category of Waldhausen categories. In other words we have the colimit in the category of Waldhausen categories

$$\varinjlim(Ch^{[a,b]}, w) \cong (Ch^b(\mathcal{C}), w).$$

We similarly get a Waldhausen structure $(Ch^b(\mathcal{C}), i)$ and $(\mathcal{C}^b_{exact}, i)$. The category $Ch^b(\mathcal{C})$ is saturated with respect to both classes of weak equivalences and admits a cylinder functor satisfying the cylinder axiom given by the classical cylinder functor on chain complexes (see 1.5.5 in [4]). Thus, we can apply the Waldhausen localization theorem to obtain a homotopy fibration

$$K((Ch^b)^w, i) \to K(Ch^b, i) \to K(Ch^b, w).$$

One can observe that $(Ch^b)^w$ is the category of acyclic complexes, i.e. the category $\mathcal{C}^b_{exact}$ because $\mathcal{C}$ is closed under kernels of surjections and so by induction we can see that the images of the differentials of bounded chain complexes are all in $\mathcal{C}$, and thus a bounded chain complex in $\mathcal{C}$ is exact if and only if it is exact in the ambient abelian category.
If we work with K-theory spectra, because we can upgrade Waldhausen localization to a homotopy fibration of spectra, the above sequence is also a homotopy cofibration of spectra. Thus, from here on out we work with an appropriate category of spectra. We are going to compute the homotopy cofiber using a different method than Waldhausen localization, the spectra we will obtain this way is going to be weakly homotopy equivalent to $K(Ch^b, w)$ as the homotopy cofiber is uniquely defined up to weak homotopy equivalence. The method we use is to write the category Ch^b as $\varinjlim_{(a,b)\to(-\infty,\infty)} Ch^{[a,b]}$. It is clear how the inclusion $(Ch^b)^w \to Ch^b$ restricts and corestricts to an inclusion $(Ch^{[a,b]})^w \to Ch^{[a,b]}$ and that the colimit of these is the original map. We will compute the cofiber of the restricted/corestricted maps and then pass to the colimit. While we go through with this line of thought we will assume that the weak equivalences are the isomorphisms.
We have a "forget differential functor" $U : Ch^{[a,b]}(\mathcal{C}) \to \prod_{i=a}^b \mathcal{C}$, we will show this induces a homotopy equivalence on K-theory. It is not hard to see, assuming that $a < b$, that this is the same map as the composite

$$Ch^{[a,b]}(\mathcal{C}) \to \mathcal{C} \times Ch^{[a+1,b]} \xrightarrow{Id\times U} \prod_a^b \mathcal{C}.$$

The first map sends a chain complex $C_a \to C_{a+1} \to \cdots \to C_b$ to $(C_a, C_{a+1} \to \cdots \to C_b)$ and the second map is the identity on the fist coordinate and the appropriate "forget differentials" functor on the second coordinate. This opens the way for an induction on $b - a$ starting at $b = a + 1$, at which point the "forget differential map" is just the identity so obviously a homotopy equivalence. Now assume we know $U : Ch^{[a+1,b]} \to \prod_{a+1}^b \mathcal{C}$ induces a homotopy equivalence on K-theory we want to prove the result for $Ch^{[a,b]}$. Consider the obvious inclusion of $Ch^{[a+1,b]}$ into $Ch^{[a,b]}$ and the inclusion as chain complexes concentrated in degree a of $\mathcal{C}$ in $Ch^{[a,b]}$ to apply proposition 4.3.9 to obtain $K(\mathcal{E}(\mathcal{C}, Ch^{[a,b]}, Ch^{[a+1,b]})) \simeq K(\mathcal{C}) \times K(Ch^{[a+1,b]})$. The left-hand side of this equivalence can easily be seen to be homotopy equivalent to $Ch^{[a,b]}$ as the rest of the data defining an object of the extension category is determined by choosing the middle term of the exact sequence. The right hand side is homotopy equivalent to $\prod_a^b K(\mathcal{C})$ by induction hypothesis. To compute the cofiber of the map $(Ch^{[a,b]})^w \to Ch^{[a,b]}$ using the K-theory homotopy equivalence we just proved, we need to understand this inclusion better.
We do this by showing that $K((Ch^{[a,b]})^w)$ is homotopy equivalent to $\prod_{a+1}^b K(\mathcal{C})$ and figuring out under this and the above homotopy equivalences if we can see to what map the inclusion corresponds to. In other words we wish to fill the following diagram into a square

$$\begin{tikzcd} K((Ch^{[a,b]})^w) \arrow[r] \arrow[d, "\simeq"] & K(Ch^{[a,b]}) \arrow[d, "\simeq"] \\ \prod_{a+1}^b K(\mathcal{C}) & \prod_a^b K(\mathcal{C}) \end{tikzcd}.$$

When $a = b + 1$ we have that the map sending $C_a \xrightarrow{\partial_a} C_{a+1}$ to C_a is an equivalence of categories as it is fully faithful and essentially surjective because ∂_a must be an isomorphism as this chain complex must be weakly equivalent to the 0 complex. In particular this map induces a homotopy equivalence on K-theory spaces. This opens the way for an induction. Suppose we have the result for $(Ch^{[a,b]})^w$ we want to show we have (and understand) a homotopy equivalence $(Ch^{[a,b+1]})^w \simeq \prod_{a+1}^{b+1} \mathcal{C}$. Now using the natural inclusions $(Ch^{[a,b]})^w \hookrightarrow (Ch^{[a,b+1]})^w$ and $\mathcal{C} \hookrightarrow (Ch^{[a,b+1]})^w$, we will show that $(Ch^{[a,b+1]})^w$ is equivalent to the extension category $\mathcal{E}((Ch^{[a,b]})^w, (Ch^{[a,b+1]})^w, (Ch^{[b,b+1]})^w)$ and so by induction and proposition 4.3.9 we will obtain that $K((Ch^{[a,b]})^w)$ and $\prod_{a+1}^{b} K(\mathcal{C})$ are homotopy equivalence. We will construct the equivalence on objects. Extension to morphisms, explicit verification that they are inverse (up to natural isomorphism) and exactness are left out as the details do not add anything. Given an exact complex $C_a \xrightarrow{\partial_a} \cdots \xrightarrow{\partial_b} C_{b+1}$ we want to associate an extension. To obtain an exact complex of one length less, we naturally delete C_{b+1}, but this is not necessarily surjective, so to fix this we replace C_b by the image of ∂_{b-1} (or rather by exactness, the kernel of the surjective map ∂_b, which is in $\mathcal{C}$ by assumption). For similar reasons, to obtain an element of $(Ch^{[b,b+1]})^w$, we keep only the last two terms, but for this to be exact we need to replace C_b by $Im(\partial_b)$, which is isomorphic to C_{b+1} as exactness forces ∂_b to be surjective (in order not to abuse closure properties we may assume $Im(\partial_b) = C_{b+1}$). This shows how to an exact complex $C_a \xrightarrow{\partial_a} \cdots \xrightarrow{\partial_b} C_{b+1}$ we associate the extension

$$\begin{tikzcd}
C_a \arrow[r, "\partial_a"] \arrow[d] & \cdots \arrow[r, "\partial_{b-2}"] & C_{b-1} \arrow[r, "\partial_{b-1}"] \arrow[d] & Im(\partial_{b-1}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
C_a \arrow[r, "\partial_a"] \arrow[d] & \cdots \arrow[r, "\partial_{b-2}"] & C_{b-1} \arrow[r, "\partial_{b-1}"] \arrow[d] & C_b \arrow[r, "\partial_b"] \arrow[d] & C_{b+1} \arrow[d] \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & Im(\partial_b) \arrow[r, "\cong"] & C_{b+1}
\end{tikzcd} .$$

The second to last column is a cofibration because by exactness $im(\partial_{b-1}) \cong ker(\partial_b)$, and we have that the quotient by the kernel is isomorphic to the image. The rest of the conditions to verify that this is a cofibration are obvious. The inverse functor is given by sending an extension to the middle term. This proof also allows us to see that the homotopy equivalence $K((Ch^{[a,b]})^w) \to \prod_{a+1}^{b} \mathcal{C}$ sends a chain complex $C_a \to \cdots \to C_b$ to $(B^{a+1}, B^{a+2}, ..., B^b)$ where $B^i = Im(\partial_{i-1})$ (which we recall all exist in $\mathcal{C}$ by induction because this category is closed under kernels of surjections). We also record here that it is not hard to see that the functors $Im(\partial_k)$ are exact. Now we can finally fill in the square

$$\begin{tikzcd}
K((Ch^{[a,b]})^w) \arrow[r] \arrow[d, "\simeq"] & K(Ch^{[a,b]}) \arrow[d, "\simeq"] \\
\prod_{a+1}^{b} K(\mathcal{C}) & \prod_{a}^{b} K(\mathcal{C})
\end{tikzcd}$$

as we wished. To do this we need to understand the map from the top left to the bottom right only in function of the data the left vertical map remembers. For this notice that for every functor $\bullet_k : (Ch^{[a,b]})^w \to \mathcal{C}$ which maps $C_a \to \cdots \to C_b$ to C_k we have an exact sequence

$$Im(\partial_{k-1}) \rightarrowtail \bullet_k \twoheadrightarrow Im(\partial_b).$$

And so by additivity $K(\bullet_k) = K(Im(\partial_{k-1})) + K(Im(\partial_k))$, which implies that the map $F : \prod_{a+1}^{b} K(\mathcal{C}) \to \prod_{a}^{b} K(\mathcal{C})$ induced by the functor which sends $(B^{a+1}, B^{a+2}, ..., B^b)$ to $(B^{a+1}, B^{a+1} \oplus B^{a+2}, B^{a+2} \oplus B^{a+3}, ..., B^{b-1} \oplus B^b, B^b)$ fits as the bottom horizontal map in our diagram. Now we need to understand the cofiber of this map.
Recall that we are working the additive category of spectra (proposition 3.2.9 in [7]), so products and coproducts coincide. In particular, we can view our map as a map $F : \bigvee_{a+1}^{b} K(\mathcal{C}) \to \prod_{a}^{b} K(\mathcal{C})$.

Written this way it is clear that we can represent F by the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Now because in time we will take the colimit as a tends to negative ∞ and b tends to positive ∞ we will assume that $a < 0 < b$ which will allow us to identify $cofib(F)$ in a way that is compatible with the colimit. We also introduce the notation F_a^b to avoid confusion when we will notice compatibility with the colimit. The sequence $\bigvee_{a+1}^b K(\mathcal{C}) \xrightarrow{F_a^b} \prod_a^b K(\mathcal{C}) \to cofib(F_a^b)$ is also a fiber sequence because we are working in a stable category. So there is a long exact sequence of homotopy groups, notice that $\pi_i(F_a^b)$ can also be represented by the matrix representing F_a^b (though the interpretation is slightly different). We now show that as a map of abelian groups $\pi_i(F_a^b) : \oplus_{a+1}^b K_i(\mathcal{C}) \to \prod_a^b K_i(\mathcal{C})$ is injective. Consider the matrix representing F_a^b as a $\mathbb{Z}$ matrix, call this map of free abelian groups Γ_a^b. Then notice we have $\Gamma_a^b \otimes K_i(\mathcal{C}) = K_i(F_a^b)$, which motivates studying Γ_a^b. It is a simple exercise in linear algebra to obtain a short exact sequence

$$0 \to \bigoplus_{a+1}^{b} \mathbb{Z} \xrightarrow{\Gamma_a^b} \prod_a^b \mathbb{Z} \xrightarrow{\chi} \mathbb{Z} \to 0.$$

Where $\chi : \prod_a^b \mathbb{Z} \to \mathbb{Z}$ sends $(x_a, ..., x_b)$ to $\sum_{i=a}^b (-1)^i x_i$ is the Euler characteristic map. Recall that in the category of abelian groups finite coproducts and finite products agree, we use both notations as pedagogical aid in following the key ideas. This sequence has to be split by projectivity of $\mathbb{Z}$, and tensoring preserves the exactness of split exact sequences. In particular $\pi_i(F_a^b) = \Gamma_a^b \otimes K(\mathcal{C})$ is injective.

This implies that the homotopy long exact sequence of the cofiber sequence $\bigvee_{a+1}^b K(\mathcal{C}) \xrightarrow{F_a^b} \prod_a^b K(\mathcal{C}) \to cofib(F_a^b)$ splits into short exact sequences

$$0 \to \bigoplus_{a+1}^{b} K_i(\mathcal{C}) \xrightarrow{\pi_i(F_a^b)} \prod_a^b K(\mathcal{C}) \to \pi_i(cofib(F_a^b)) \to 0.$$

By the same reasoning we used to show the injectivity of $\pi_i(F_a^b)$ we can see that $\pi_i(cofib(F_a^b)) = K_i(\mathcal{C})$ and that the map is given by the tensor of the Euler characteristic. However, this does not show that $K(\mathcal{C}) \simeq cofib(F_a^b)$. We need a map of spectra to realize the isomorphisms of homotopy groups.
For this, consider the following diagram where ι_0 is inclusion into the 0th factor, with the map being constant equal to the base point on all other factors

$$\begin{tikzcd} & K(\mathcal{C}) \arrow[d, "\iota_0"] \arrow[dr] & \\ \bigvee_{a+1}^b K(\mathcal{C}) \arrow[r, "F_a^b"] & \prod_a^b K(\mathcal{C}) \arrow[r] & cofib(F_a^b) \end{tikzcd}.$$

By applying the functor π_i we see that the map $K(\mathcal{C}) \to cofib(F_a^b)$ is the identity on homotopy groups, thus is a homotopy equivalence. Viewing $\prod_a^b K(\mathcal{C})$ as a coproduct, we can view the map into the cofiber as a matrix, which because we know its effect on homotopy groups, we can deduce to be the Euler characteristic map. Finally, we notice that the homotopy equivalence is compatible with colimit by construction.

What we have shown, is that we have cofiber sequence

$$\prod_{a+1}^{b} K(\mathcal{C}) \to \prod_a^b K(\mathcal{C}) \to K(\mathcal{C})$$

and as we have already mentioned, we can replace the first map by the inclusion $K((Ch^{[a,b]})^w) \to K(Ch^{[a,b]})$. The weak equivalence we use to do this is compatible with the colimit, thus we have a cofiber sequence

$$K((Ch^b)^w) \to K(Ch^b) \to K(\mathcal{C}).$$

Comparing with what we obtained via Waldhausen localization and using uniqueness up to homotopy equivalence of the cofiber, we get $K(\mathcal{C}) \simeq K(Ch^b, w)$. □

We close of this section with an important corollary of additivity, the statement of the cofinality theorem, which has its importance in K-theory.

Theorem 5.1.5. *(Theorem 2.3 in [6] chapter V) Let $(\mathcal{C}, w)$ be a Waldhausen category with a cylinder functor satisfying the cylinder axiom. Assume we are given a surjective homomorphism from $\pi : K_0(\mathcal{C}) \to G$ some group, let $\mathcal{D}$ be the subcategory of elements such that $\pi([D]) = 0$. Then we have a homotopy fibration $K(\mathcal{D}) \to K(\mathcal{C}) \to G$.*

We will not being use it however, so we state it without proof. For a proof one can consult section 2 of [18].

5.2 Approximation Theorem

The approximation theorem is a very useful result in replacing categories with simpler categories. In the proof of the approximation theorem we will need one little definition about posets and a Waldhausen structure on a certain functor category. We will not make an explicit distinction between a poset and the corresponding category.

Definition 5.2.1. (Beginning of section A.2 in [17]) Let $\mathcal{P}$ be a poset. We call a subposet $\mathcal{S} \subset \mathcal{P}$ saturated if whenever $x < s, s \in \mathcal{S}$ implies that $x \in \mathcal{S}$.

Definition 5.2.2. (Beginning of section A.2 in [17]) Let $\mathcal{C}$ be a Waldhausen category, and $\mathcal{P}$ be a poset. We give the functor category $\mathcal{C}^{\mathcal{P}}$ a Waldhausen structure as follows. The weak equivalences are exactly the pointwise weak equivalences. The cofibrations are trickier to describe.
For any $P \in \mathcal{P}$, let $\mathcal{S}_P$ be the poset of elements strictly smaller than P. For any saturated subposet $\mathcal{S} \subset \mathcal{P}$, let $I(\mathcal{S}, P)$ be the full subcategory of $[1] \times \mathcal{P}$ containing precisely all the objects of the form $[1] \times \mathcal{S}$ and (O, P). Maps of functors $\mathcal{P} \to \mathcal{C}$ can be viewed as functors $\mathcal{P} \times [1] \to \mathcal{C}$, thus by restriction as functors $I(S, P)$. We call a map $X : [1] \times \mathcal{P} \to \mathcal{C}$ a cofibration if for all $P \in \mathcal{P}$ the colimit $\varinjlim_{I(S_P,P)} X$ exists and the canonical map $\varinjlim_{I(S_P,P)} X \to X_1(P)$ is a cofibration.

With this in hand we may state and prove the approximation theorem.

Theorem 5.2.3. *(Theorem 2.4 in [6] chapter V)(Theorem 10 in [17]) Suppose $F : \mathcal{C} \to \mathcal{D}$ is an exact functor of saturated Waldhausen categories which satisfies the following requirements.*

(i) A morphism in $\mathcal{C}$ is a weak equivalence if and only if its image is a weak equivalence.

(ii) Every morphism in $\mathcal{C}$ can be factored as a cofibration followed by a weak equivalence (this is in particular true if $\mathcal{C}$ has a cylinder functor satisfying the cylinder axiom).

(iii) F satisfies the approximate lifting property which states that for every map $\beta : F(C) \to D$ there is a cofibration $\alpha : C \to C'$ and a weak equivalence $\beta' : F(C') \to D$ such that $\beta' \circ F(\alpha) = \beta$.

Then the induced map on K-theory spaces $F : K(\mathcal{C}) \to K(\mathcal{D})$ is a homotopy equivalence.

Proof. We will first show that if $F : \mathcal{C} \to \mathcal{D}$ satisfies the assumption of the theorem, then so does $S_n F : S_n\mathcal{C} \to S_n\mathcal{D}$. We will then show that whenever $F : \mathcal{C} \to \mathcal{D}$ satisfies the assumptions of the theorem, then F is a homotopy equivalence. Which by using the realization lemma (A.0.5) will show the desired claim that $S.F : S.\mathcal{C} \to S.\mathcal{D}$ is a homotopy equivalence. The fact that $S_n F$ satisfies (i) is obvious as weak equivalences are defined pointwise. For the second point, notice it is equivalent to proving the third point in the case $F = Id_{\mathcal{C}}$. We may thus move on to proving that S_nF satisfies the approximate lifting property, for this we follow [14] lemma 1.6.6. Suppose we have a map

$$\begin{tikzcd} F(A_0) \arrow[r, tail] \arrow[d] & F(A_1) \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & F(A_n) \arrow[d] \\ B_0 \arrow[r, tail] & B_1 \arrow[r, tail] & \cdots \arrow[r, tail] & B_n \end{tikzcd},$$

which we wish to factor by finding a cofibration

$$\begin{tikzcd} A_0 \arrow[r, tail] \arrow[d, tail] & A_1 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & A_n \arrow[d, tail] \\ A'_0 \arrow[r, tail] & A'_1 \arrow[r, tail] & \cdots \arrow[r, tail] & A'_n \end{tikzcd}$$

and a weak equivalence

$$\begin{tikzcd} F(A'_0) \arrow[r, tail] \arrow[d] & F(A'_1) \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & F(A'_n) \arrow[d] \\ B_0 \arrow[r, tail] & B_1 \arrow[r, tail] & \cdots \arrow[r, tail] & B_n \end{tikzcd}.$$

We may proceed by induction on n. Notice that the base case is dealt with by assumption as $S_1F = F$. Suppose we have the above data except for the degree n term, how can we add the degree n term? We can form the pushout

$$\begin{tikzcd} A_{n-1} \arrow[r, tail] \arrow[d] & A'_{n-1} \arrow[d] \\ A_n \arrow[r, tail] & A_{n+1} \cup_{A_{n-1}} A'_{n-1} \end{tikzcd}.$$

Applying F to this diagram, we see that maps into B_n allows us to define a map $F(A_n \cup_{A_{n-1}} A'_{n-1}) \to B_n$. We can apply the approximation property of F on this map to obtain a factorization

$$F(A_n \cup_{A_{n-1}} A'_{n-1}) \rightarrowtail F(A'_n) \xrightarrow{\simeq} B_n.$$

We take the map $F(A'_{n-1}) \rightarrowtail F(A'_n)$ to be the obvious composition $F(A'_{n-1}) \rightarrowtail F(A_n \cup_{A_{n-1}} A'_{n-1}) \rightarrowtail F(A'_n)$. From here it is clear to see that the factorization constructed this way has all the desired properties.

We now move on to showing that whenever $F : \mathcal{C} \to \mathcal{D}$ satisfies the assumptions of the theorem, then $wF : w\mathcal{C} \to w\mathcal{D}$ is a homotopy equivalence. We will do this by using Quillen's theorem A (A.0.2). We want to show, that for each $B \in \mathcal{D}$, the category F/B is contractible. We notice this category is nonempty as the map $0 \to B$ factorizes by assumption as $0 \to F(A) \xrightarrow{\simeq} B$. We will use the following lemma to show the desired contractibility.

Lemma 5.2.4. *(Lemma 14 in [17]) Let $\mathcal{C}$ be a connected category. Suppose that for every finite poset $\mathcal{P}$ and every functor $\Gamma : \mathcal{P} \to \mathcal{C}$ is null homotopic. Then $\mathcal{C}$ is contractible.*

Proof. We will show that it is weakly contractible, which is enough as BC is a CW-complex. Fix some $c \in C$ which will serve as a base point. We want to show $\pi_n(C, c) = 0$, for this consider a pointed map $\alpha : S^n \to BC$. Via simplicial approximation we can consider this map at the simplicial level, i.e. we can assume it is the geometric realization of a map $Sd^kS^n \to N_*\mathcal{C}$ for sufficiently large k. To clear up notation, $S_n = \Delta^n/\partial\Delta^n$ is the simplicial n-sphere and Sd^k is barycentric subdivision. We can, by subdividing again if necessary, assume $k \geq 2$. Now using lemma 5.6 of [38], we see that $Sd^kS^n \cong N_*\mathcal{P}$, where $\mathcal{P}$ is some poset. This shows that the map $\alpha : S^n \to BC$ is nullhomotopic by assumption. To show it is nullhomotopic via a base point preserving homotopy, it suffices to apply proposition 4A.2 in [3]. □

In order to apply this lemma we first need to show F/B is connected. To see this, consider two objects $A_i \xrightarrow{\simeq} B, i \in [2]$ and factor the obvious map $F(A_1 \oplus A_2) \to B$ as $F(A_1 \oplus A_2) \hookrightarrow F(A_{12}^+) \xrightarrow{\simeq} B$. Then by considering the following diagram

$$\begin{tikzcd}
& F(A_i) \arrow[dl] \arrow[dr, "\simeq"] & \\
F(A_1 \oplus A_2) \arrow[dr, hook] & & B \\
& F(A_{12}^+) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

and the fact that $\mathcal{D}$ is saturated, we get weak equivalences $F(A_i) \to F(A_{12}^+)$, which is enough to give a path between A_1 and A_2 in $B(F/B)$. That being said, we fix a finite poset and a functor $\Gamma : \mathcal{P} \to F/B$. This consists of a $\mathcal{P}$-shaped diagram, call it X, in $w\mathcal{C}$, together with a map in $w\mathcal{D}^{\mathcal{P}}$ between $F(X)$ and the constant functor with image B which we denote by cB. We will construct a null homotopy with the use of two lemmas which we now prove.

Lemma 5.2.5. *(Lemma 11 in [17]) Let $\mathcal{C}$ be a Waldhausen category. Suppose we have a map $X : [1] \times \mathcal{P} \to \mathcal{C}$. Let $P \in \mathcal{P}$ and $\mathcal{S}' \subset \mathcal{S} \subset \mathcal{S}_P$ be saturated subposets. Suppose for all $Q < P$ we have that $\varinjlim_{I(\mathcal{S}_Q,Q)} X$ exists and further that the canonical map $\varinjlim_{I(\mathcal{S}_Q,Q)} X \to X_1(Q)$ is a cofibration. Then $\varinjlim_{I(\mathcal{S},P)} X$ exists and the canonical map $\varinjlim_{I(\mathcal{S}',P)} X \to \varinjlim_{I(\mathcal{S},P)} X$ is a cofibration. In particular if X_1 is cofibrant, then $\varinjlim_{\mathcal{P}} X_1$ exists.*

Proof. We proceed by induction on the cardinality of $\mathcal{S}$, which we henceforth call n. First observe that for $n = 0$ the colimit must be $X_0(P)$ and the map is a cofibration as it must be the identity map (indeed in the case $\mathcal{S}' = \mathcal{S} = \emptyset$). Now assume the result holds for all cardinality up to and including n, we want to show it holds in the case $n + 1$.
If $\mathcal{S}' = \mathcal{S}$, then once we show the colimit exists, the resulting map is obviously a cofibration, and we can choose $Q \in \mathcal{S}$ maximal. Otherwise, $Q \in \mathcal{S}$ maximal with the additional property that $Q \in \mathcal{S}$. The idea is to use the following diagram, the left term exists by induction hypothesis, the middle term by assumption and the arrow on the right is a cofibration also by induction

$$\varinjlim_{I(\mathcal{S}\setminus\{Q\},P)} X \longleftarrow \varinjlim_{I(\mathcal{S}_Q,Q)} X \hookrightarrow X_1(Q) \; .$$

As pushouts along cofibrations exist in Waldhausen categories, the above diagram admits a pushout. This pushout realizes the desired colimit as it has the universal property of $\varinjlim_{I(\mathcal{S},P)} X$, which proves this colimit exists. As pushouts of cofibrations are cofibrations, we have that the map $\varinjlim_{I(\mathcal{S}\setminus\{Q\},P)} X \to \varinjlim_{I(\mathcal{S},P)} X$ is a cofibration. By induction hypothesis the map $\varinjlim_{I(\mathcal{S}',P)} X \to \varinjlim_{I(\mathcal{S}\setminus\{Q\},P)} X$ is a cofibration as well. By composition $\varinjlim_{I(\mathcal{S}',P)} X \to \varinjlim_{I(\mathcal{S},P)} X$ is a cofibration as well.
We may now prove the remark in the specific case that X_1 is cofibrant in $\mathcal{C}^{\mathcal{P}}$. In this case, denote by $\mathcal{P}'$ the poset obtained from $\mathcal{P}$ by appending a single maximal element $+$. Let $X_0(+) = 0$, restating the definition of X_1 being cofibrant, the map $0 \to X_1$ is a cofibration. By definition of cofibrations in $\mathcal{C}^{\mathcal{P}}$ the condition required to apply the first part of the statement is satisfied. This implies the existence of $\varinjlim_{I(\mathcal{P}',+)}$, but this is in fact the colimit $\varinjlim_{\mathcal{P}} X_1$ by comparison of universal properties. □

Lemma 5.2.6. *(Lemma 13 in [17]) Let $\mathcal{C}$ be a Waldhausen category such that every morphism can be factored as a cofibration followed by a weak equivalence. Then the same holds true for $\mathcal{C}^{\mathcal{P}}$, whenever $\mathcal{P}$ is a finite poset.*

Proof. We induct on the cardinality of $\mathcal{P}$ which we henceforth call n. The case $n = 1$ is just the assumption in $\mathcal{C}$. Now assume we have the result for all posets of cardinality n, we prove the result holds true for $n+1$. Fix a morphism $X \to Y$ we wish to factor. Choose a maximal element $P \in \mathcal{P}$, by induction hypothesis we have a factorization

$$X|_{\mathcal{P}\setminus\{P\}} \hookrightarrow Z|_{\mathcal{P}\setminus\{P\}} \xrightarrow{\simeq} Y|_{\mathcal{P}\setminus\{P\}}.$$

By the previous lemma, because $X \hookrightarrow Z$ is a cofibration, the colimit $\varinjlim_{I(\mathcal{S}_P,P)}(X|_{\mathcal{P}\setminus\{P\}} \hookrightarrow Z|_{\mathcal{P}\setminus\{P\}})$ exists. Now we apply the factorization assumption in $\mathcal{C}$ to the map obtained by the universal property $\varinjlim_{I(\mathcal{S}_P,P)}(X|_{\mathcal{P}\setminus\{P\}} \hookrightarrow Z|_{\mathcal{P}\setminus\{P\}}) \to Y(P)$. Thus, we obtain $\varinjlim_{I(\mathcal{S}_P,P)}(X|_{\mathcal{P}\setminus\{P\}} \hookrightarrow Z|_{\mathcal{P}\setminus\{P\}}) \hookrightarrow \tilde{P} \xrightarrow{\simeq} Y(P)$. Set $Z(P) = \tilde{P}$, it is clear how to extend the map $Z|_{\mathcal{P}\setminus\{P\}} \to Y|_{\mathcal{P}\setminus\{P\}}$ to a map $Z \to Y$, and because weak equivalence are computed pointwise it is also clear that this is a weak equivalence. Now we need to extend $X|_{\mathcal{P}\setminus\{P\}} \hookrightarrow Z|_{\mathcal{P}\setminus\{P\}}$ to a map $X \hookrightarrow Z$. For this apply the previous lemma with $\mathcal{S} = \mathcal{S}_P, \mathcal{S}' = \emptyset, P = P$. This shows that the map $X(P) \to \varinjlim_{I(\mathcal{S}_P,P)}(X|_{\mathcal{P}\setminus\{P\}} \hookrightarrow Z|_{\mathcal{P}\setminus\{P\}})$ is a cofibration, thus we can post compose with $\varinjlim_{I(\mathcal{S}_P,P)}(X|_{\mathcal{P}\setminus\{P\}} \hookrightarrow Z|_{\mathcal{P}\setminus\{P\}}) \to Z(P)$, thus we have a cofibration $X(P) \to Z(P)$ which we can use to define $X \to Z$. The fact that this map is a cofibration is clear by construction. □

With these lemmas in hand, use the first of the two to factor the unique map from the constant $\mathcal{P}$-shaped diagram $c0$ to X as $0 \hookrightarrow Y \xrightarrow{\simeq} X$. Now Y is cofibrant, so by the second lemma the colimit $\varinjlim_{\mathcal{P}} Y$ exists. The functor F commutes with this colimit as it preserves pushouts along cofibration and this colimit can be constructed via successive pushouts. And so the composite $F(Y) \to F(X) \to cB$ defines a map $\varinjlim_{\mathcal{P}} F(Y) \to B$, which we can write as $F(\varinjlim_{\mathcal{P}} Y) \to B$. Written this way, we can use property (ii) which holds by assumption to factor this map as $F(\varinjlim_{\mathcal{P}} Y) \hookrightarrow F(Z) \xrightarrow{\simeq} B$. We consider the constant diagram cZ. Then, using that F preserves weak equivalences, we have a diagram

$$\begin{tikzcd}
 & F(X) \arrow[dr, "\simeq"] & \\
F(Y) \arrow[ur, "\simeq"] \arrow[dr] & & F(cB) \\
 & F(cZ) \arrow[ur, "\simeq"] &
\end{tikzcd} .$$

Now use that weak equivalence satisfy two out of three, are saturated in $\mathcal{D}$ (thus in $\mathcal{D}^{\mathcal{P}}$) as well) and that F reflects weak equivalences to see that $Y \to cZ$ is a weak-equivalence. This shows that we have natural transformations $X \xleftarrow{\simeq} Y \xrightarrow{\simeq} cZ$. This is the desired nullhomotopy. □

For ease convenience we include without proof another version of the approximation theorem which is very useful for dealing with subcategories of categories of chain complexes. We will in particular need this in section §6.

Proposition 5.2.7. *(Theorem 1.9.8 in [16]) Let $\mathcal{A}, \mathcal{B}$ be two complicial biWaldhausen categories which are closed under the canonical homotopy pushouts and homotopy pullbacks and $F : \mathcal{A} \to \mathcal{B}$ a complicial exact functor. Suppose that F induces an equivalence of the derived categories, then it induces a homotopy equivalence in K-theory.*

We refer the reader to [16] for the details of all the relevant vocabulary. For our purposes it is important to know that complicial Waldhausen categories are in particular subcategories of a category of chain complexes $Ch(\mathcal{M})$ with $\mathcal{M}$ abelian and that an exact complicial functor is simply an exact functor which is computed by pointwise application. All the categories we will encounter will obviously satisfy the necessary requirements (see example 1.2.12 to 1.2.15 in [16]). So when using the above theorem, what one needs to verify is that the induced functor on the derived categories is an equivalence.
Again for ease of reference, we include the following theorem which will give us sufficient condition for an exact complicial functor of complicial biWaldhausen categories closed under homotopy pullbacks and pushouts to induce an equivalence on the derived categories.

Proposition 5.2.8. *(1.9.7 in [16]) Let $\mathcal{A}, \mathcal{B}$ be two complicial biWaldhausen categories which are closed under the canonical homotopy pushouts and homotopy pullbacks and $F : \mathcal{A} \to \mathcal{B}$ a complicial exact functor. Then F induces an equivalence of categories if*

(i) A morphism in $\mathcal{A}$ is a weak equivalence if and only if its image is an equivalence in $\mathcal{B}$.

(ii) For any $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ such that there is a weak equivalence $b : F(A) \to B$ in $\mathcal{B}$.

(iii) For any map $b : FA' \to FA''$ in $\mathcal{B}$, there is a weak equivalence $a' : A \to A'$ and a map $a'' : A \to A''$ in $\mathcal{A}$ such that $b \circ F(a')$ is chain homotopic to $F(a'')$ in $\mathcal{B}$.

(iv) For any map $a'; A' \to A''$ in $\mathcal{A}$ such that $F(a')$ is nullhomotopic in $\mathcal{B}$ there exists a weak equivalence $a : A \to A'$ in $\mathcal{A}$ such that $a' \circ a$ is nullhomotopic in $\mathcal{A}$.

5.3 Resolution theorem

In this section we present one of the two main tools to replace a category by a subcategory whose K-theory is (hopefully) simpler to understand. We do not follow the proof given in section 3 of chapter V of the K-book [6], but instead follow Staffeldt's proof in section 3 of [18] which places emphasis on the $S.$ construction.

Theorem 5.3.1. *(Theorem 3.1 in [18]) Suppose $\mathcal{A}$ is a full exact subcategory of an exact category $\mathcal{B}$ such that a sequence of three objects in $\mathcal{A}$ which is exact in $\mathcal{B}$ is exact in $\mathcal{A}$, $\mathcal{A}$ is closed under extension and cokernels in $\mathcal{B}$. Assume further that every object B in $\mathcal{B}$ has a resolution, i.e. an exact sequence, $0 \to B \to A \to A' \to 0$ with $A, A' \in \mathcal{A}$.*
Then the inclusion $\mathcal{A} \to \mathcal{B}$ induces a homotopy equivalence $K(\mathcal{A}) \to K(\mathcal{B})$ on K-theory spaces.

Proof. Let $\mathcal{A} \subset \mathcal{B}$ be as in the statement of the theorem. We can apply proposition 4.4.1 to the inclusion $\iota : \mathcal{A} \to \mathcal{B}$ and obtain this way a homotopy fibration

$$iS.\mathcal{B} \to iS.S.\mathcal{A} \to iS.S.\iota.$$

It is obvious that this reduces our problem to showing that $iS.S.\iota$ is contractible. Because the class of weak equivalences is the isomorphisms we can apply proposition 4.2.3 to study $s.S.\iota$ and then use the realization lemma to reduce further to showing the contractibility of $s.S_n\iota$. It turns out this allows for quite an efficient proof, as we can show that whenever an inclusion $\iota : \mathcal{A} \to \mathcal{B}$ satisfies the hypothesis of the theorem, then $s.\iota$ is contractible and that then $S_n\iota : S_n\mathcal{A} \to S_n\mathcal{B}$ also satisfies the hypothesis of the theorem. Proving these two claims will obviously be enough to prove the desired claim.
Let's first show that $s.\iota$ is contractible. For convenience, we recall that $s.\iota = Ob(S.\iota)$ and that $S.\iota$ is the simplicial category whose n-simplices is the category of pairs

$$(A_1 \rightarrowtail \cdots \rightarrowtail A_n, B_0 \rightarrowtail \cdots \rightarrowtail B_n)$$

with the additional requirement that $B_i/B_0 \cong A_i$. It is not hard to notice that this is isomorphic to the simplicial set given by taking the nerve of the category $\mathcal{C}$ whose objects are the same as $\mathcal{B}$, but the morphisms are the maps which are admissible monomorphisms in $\mathcal{B}$ with quotient lying in $\mathcal{A}$. Verifying this is just a matter of comparing the two definition, so we do not detail it any further. Denote by $m\mathcal{A}$ the subcategory of $\mathcal{A}$ with all objects and maps being exactly the admissible monics. The inclusion $\iota : \mathcal{A} \to \mathcal{B}$ obviously restricts and corestricts to a map, which we denote $\gamma : m\mathcal{A} \to \mathcal{C}$. Now notice that $m\mathcal{A}$ is obviously contractible because it admits 0 as an initial object, thus if γ is a homotopy equivalence, we are done. So naturally our strategy will be to apply Quillen's theorem A A.0.2, to do this let $B \in \mathcal{C}$ and consider the category B/γ.
We will show that the identity functor of this category is homotopic to a constant functor. For this fix a resolution $0 \to B \to A_0 \to A_0'' \to 0$ of B which can be done by assumption. Consider an object

$B \rightarrowtail A$ of B/γ and consider the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & B \arrow[r, tail] \arrow[d, tail] & A \arrow[r, two heads] \arrow[d] & A/B \arrow[r] \arrow[d, "Id"] & 0 \\
 & A_0 \arrow[r] \arrow[d, two heads] & A_0 \oplus_B A \arrow[r] \arrow[d] & A/B & \\
 & A_0'' \arrow[r, "Id"] \arrow[d] & A_0'' & & \\
 & 0 & & &
\end{tikzcd}.$$

We can easily notice that all the rows and columns are exact sequences because $\mathcal{B}$ embeds in an abelian category where we can check exactness. When proving results about diagrams in an abelian category we are allowed to diagram chase, thus the claim follows easily. What this exactness allows is to define functors given by the above diagram $c_{A_0} : (B \rightarrowtail A) \mapsto (B \rightarrowtail A_0)$ and $A_0 \oplus_B - : (B \rightarrowtail A) \mapsto (B \rightarrowtail A_0 \oplus_B A)$ and to use the above diagram to define the natural transformations

$$Id \to A_0 \oplus_B - \leftarrow c_{A_0}.$$

This shows the desired contractability allowing the application of Quillen's theorem A to γ which in turn provides the desired contractability.
Thus, all that remains to show is that the inclusion $\iota : \mathcal{A} \to \mathcal{B}$ satisfying the hypothesis of the theorem implies that also $S_n\iota : S_n\mathcal{A} \to S_n\mathcal{B}$ satisfies the hypothesis. All the closure properties are not hard to verify, and we leave them to the reader. The only point requiring attention is the fact that objects in S_nB admit resolutions via objects in $S_n\mathcal{A}$.
We do this by inductively constructing partial resolutions such as

$$\begin{tikzcd}
B_1 \arrow[r, tail] \arrow[d, tail] & B_2 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & B_i \arrow[r, tail] \arrow[d, tail] & B_{i+1} \arrow[r, tail] & \cdots \arrow[r, tail] & B_n \\
A_1 \arrow[r, tail] \arrow[d, two heads] & A_2 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_i \arrow[d, two heads] & & & \\
A_1'' \arrow[r, tail] & A_2'' \arrow[r, tail] & \cdots \arrow[r, tail] & A_i'' & & &
\end{tikzcd}.$$

When we say that a diagram as above is partial resolution, we mean it is pointwise a resolution in $\mathcal{B}$, and we require further that the natural maps $A_j \oplus_{B'_j} B_{j+1} \to A'_{j+1}$ are all cofibrations. The case $i = 1$ can be dealt with by assumption that objects in $\mathcal{B}$ can be resolved via objects in $\mathcal{A}$. So to proceed with the induction, assume we have a diagram as above, we wish to add the $(i+1)$th column. Notice $A_i \oplus_{B_i} B_{i+1}$ is an object in $\mathcal{B}$ and thus can be resolved, i.e. we have an admissible exact sequence $A_i \oplus_{B_i} B_{i+1} \rightarrowtail A_{i+1} \twoheadrightarrow C_{i+1}$. We have a natural map $B_{i+1} \rightarrowtail A_i \oplus_{B_i} B_{i+1} \rightarrowtail A_{i+1}$, we set A''_{i+1} to be the cokernel of this map. Thus, we can extend the above diagram to

$$\begin{tikzcd}
B_1 \arrow[r, tail] \arrow[d, tail] & B_2 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & B_i \arrow[r, tail] \arrow[d, tail] & B_{i+1} \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & B_n \\
A_1 \arrow[r, tail] \arrow[d, two heads] & A_2 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_i \arrow[r, tail] \arrow[d, two heads] & A_{i+1} \arrow[d, two heads] & & \\
A_1'' \arrow[r, tail] & A_2'' \arrow[r, tail] & \cdots \arrow[r, tail] & A_i'' \arrow[r] & A_{i+1}'' & &
\end{tikzcd}.$$

In order to complete the induction step we need to verify that $A_i \oplus_{B_i} B_{i+1} \to A_{i+1}$ and $A''_i \to A''_{i+1}$ are cofibrations and that A''_{i+1} is an object in $\mathcal{A}$. The first assertion is completely obvious by construction of A_{i+1}, as the first term in the resolution of $A_i \oplus_{B_i} B_{i+1}$. The fact that the second map is monic follows from a diagram chase showing that A''_{i+1} is the pushout of $A''_i \leftarrow A_i \rightarrowtail A_{i+1}$, and that pushouts preserve cofibrations. Now to show that A''_{i+1} is in fact an object in $\mathcal{A}$ we consider the colimit of the following diagram in two different ways

$$\begin{tikzcd} 0 & A_i \arrow[l] \arrow[r, "Id"] & A_i \\ 0 \arrow[u] \arrow[d] & B_i \arrow[l] \arrow[r] \arrow[u, tail] \arrow[d, tail] & A_i \arrow[u, "Id"'] \arrow[d, tail] \\ 0 & B_{i+1} \arrow[l] \arrow[r, tail] & A_{i+1} \end{tikzcd} .$$

We first take horizontal pushouts, and take the pushout of the resulting vertical diagram. This yields the pushout of the diagram $0 \leftarrow A''_i \rightarrowtail A''_{i+1}$, i.e. the cokernel of the map $A''_i \rightarrowtail A''_{i+1}$. Then we take first the vertical pushouts, resulting in $0 \leftarrow A_i \oplus_{B_i} B_{i+1} \to A_{i+1}$. We take the pushout of this diagram, which gives C_{i+1}, which by construction is an object of $\mathcal{A}$. Commutativity of colimits shows then that A''_{i+1} fits into a short exact sequence $A''_i \to A''_{i+1} \to C_{i+1}$ with both outer terms in $\mathcal{A}$. Now recall that $\mathcal{A}$ is closed under extensions in $\mathcal{B}$, which implies the desired claim that A''_{i+1}. This in turn concludes the proof of the resolution theorem. □

There are ways to generalize this theorem, such as allowing for resolutions on the left or allowing longer resolutions. One possible source to find proofs of these alternative statements is in §3 of chapter V of [6]. For ease of reference, we cite here the dual of our theorem

Proposition 5.3.2. *(Proposition 3.1.1 in chapter V of [6]) Let $\mathcal{B}$ be an exact category and $\mathcal{A}$ be a full exact subcategory closed under kernels and extensions such that every object $B \in \mathcal{B}$ admits a resolution $0 \to A \to A' \to B \to 0$ with $A, A' \in \mathcal{A}$. Then the inclusion $\mathcal{A} \to \mathcal{B}$ induces an equivalence in K-theory.*

5.4 Devissage theorem

The devissage theorem which we prove in this section is based, like many results in K-theory, on the intuition that K-theory splits exact sequences. Just like the theorems of the two previous sections it serves to replace a category by a hopefully simpler category with the same K-theory.

Theorem 5.4.1. *(Theorem 4.1 in [18]) Let $\mathcal{A} \subset \mathcal{B}$ be an inclusion of abelian categories such that $\mathcal{A}$ is closed under direct sum, subobject and quotient in $\mathcal{B}$. Suppose every object in $\mathcal{B}$ has a finite filtration by monics such that all the filtration quotients are in $\mathcal{A}$. Then the inclusion $\mathcal{A} \to \mathcal{B}$ induces a homotopy equivalence $K(\mathcal{A}) \to K(\mathcal{B})$ of K-theory spaces.*

Proof. The proof we present is in spirit very similar to our proof of the resolution theorem. We can use proposition 4.4.1 to obtain, after one delooping, a fiber sequence

$$iS.\mathcal{A} \to iS.\mathcal{B} \to iS.S.(\mathcal{A} \subset \mathcal{B}).$$

This reduces the problem to showing that $iS.S.(\mathcal{A} \subset \mathcal{B})$ is contractible. We use proposition 4.2.3 to reduce to showing that $s.S.(\mathcal{A} \subset \mathcal{B})$ is contractible and then use the realization lemma to perform our final reduction to showing that each $s.S_n(\mathcal{A} \subset \mathcal{B})$ is contractible. Now we use the same strategy as in the previous section and show that if $\mathcal{A} \subset \mathcal{B}$ satisfies the hypothesis of the theorem, then so does $S_n\mathcal{A} \subset S_n\mathcal{B}$ and whenever $\iota : A \to B$ satisfies the hypothesis of the theorem, then we have that $s.\iota$ is contractible.

We prove first that $\iota : S_n\mathcal{A} \subset S_n\mathcal{B}$ also satisfies the hypothesis of the theorem. The only part we will make explicit is the fibration condition, all the others being sufficiently clear. Take an element in $S_n\mathcal{B}$, as usual we suppress the choices of subquotient, and so we denote such an element by

$$B_{1,p} \rightarrowtail \cdots \rightarrowtail B_{n.p}.$$

There is a slight abuse of notation in using the symbol p whose value we have not defined yet. Filter the object $B_{n,p}$ in the way given by assumption to obtain a sequence $B_{n,1} \rightarrowtail \cdots \rightarrowtail B_{n,p}$ (this is what determines p). Taking successive pullbacks, we can obtain a diagram

$$\begin{tikzcd}
B_{1,1} \arrow[r, tail] \arrow[d, tail] & B_{2,1} \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & B_{n,1} \arrow[d, tail] \\
B_{1,2} \arrow[r, tail] \arrow[d, tail] & B_{2,2} \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & B_{n,2} \arrow[d, tail] \\
\vdots \arrow[d, tail] & \vdots \arrow[d, tail] & & \vdots \arrow[d, tail] \\
B_{1,p} \arrow[r, tail] & B_{2,p} \arrow[r, tail] & \cdots \arrow[r, tail] & B_{n,p}
\end{tikzcd} .$$

One can extend this diagram to include choices of quotients and maps between them, thus this defines a sequence of maps $B_{\bullet,1} \to \cdots \to B_{\bullet,n}$ such that each map is pointwise monic. We can observe that this is in fact a sequence of maps which are admissible monic in $S_n\mathcal{B}$. Indeed, we can show that each of the natural maps $B_{i,j+1} \oplus_{B_{i,j}} B_{i+1,j} \to B_{i+1,j+1}$ is monic. We are working in an abelian category, so we can use our intuition from the category of R-modules by Freyd-Mitchel embedding (theorem 1.6.1 in [4]). We know that by construction $B_{i,j} = ker(B_{i,j+1} \oplus B_{i+1,j} \to B_{i+1,j+1})$. The map we wish to show is injective is the one obtained from $B_{i,j+1} \oplus B_{i+1,j} \to B_{i+1,j+1}$ by quotienting the domain by the kernel, thus must have trivial kernel, which by embedding into a category of R-modules implies the desired monicity. The fact that the filtration quotients are in $S_n\mathcal{A}$ comes from the fact that these are computed pointwise, that $B_{n,i+1}/B_{n,i} \in Ob(\mathcal{A})$ and that $\mathcal{A}$ is closed under subobject, direct sum and quotient. We do not spell out the details.
Now we may assume we are given an inclusion of abelian categories $\iota : \mathcal{A} \to \mathcal{B}$ satisfying the assumptions of the theorem. We want to show that $s.\iota$ is contractible, which will conclude the proof. To show this it would be useful to find a category whose nerve is our simplicial set, so that we may use Quillen's theorem A (A.0.2) to replace our simplicial set by an easier one. This is accomplished by proposition A.0.7. So we want to show that the category $Simp(s.\iota)$ is contractible. The vertices of a given simplex are totally ordered, thus there is a natural last vertex, which one can see in our case, by definition of $s.\iota$, defines a functor $L : Simp(s.\iota) \to m\mathcal{B}$ with $m\mathcal{B}$ the category of monic maps in $\mathcal{B}$. This category is contractible as 0 is an initial object of it. To be clear the category $Simp(s.\iota)$ has as objects pairs $(q, B \in s_q\iota)$ and maps $\alpha : (q,B) \to (r,B')$ are defined by maps $\alpha : [q] \to [r]$ with the added condition that $\alpha^*(B') = B$. The functor L maps (q,B) to B_q and a map $\alpha : (q,B) \to (r,B')$ to $B_q = B'_{\alpha(q)} \rightarrowtail B'_r$.
We will show that this functor is a homotopy equivalence by using Quillen's theorem A, thus reducing us further to showing that $L/\bar{B}$ is contractible. It is not hard to see that this category is equivalent to the category of simplicies of the simplicial set $N_{\bar{B}}$ whose q simplicies are the $q+1$ simplices of $N.(m\mathcal{B})$ subject to the two conditions the final vertex must be $\bar{B}$ and that if $B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_q \rightarrowtail \bar{B}$ is a q-simplex of $N_{\bar{B}}$ then $B_i/B_0 \in Ob(\mathcal{A})$. By proposition A.0.7 we have now replaced our problem by showing that $N_{\bar{B}}$ is contractible. We do this by providing a chain of homotopies from the identity map on $N_{\bar{B}}$ to a constant map on $N_{\bar{B}}$. For this we filter $\bar{B}$

$$0 = C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_{n-1} \rightarrowtail C_n = \bar{B}$$

, and we may by assumption assume that $B_{i+1}/B_i \in Ob(\mathcal{A})$. For each C_i we define $F_i : N_{\bar{B}} \to N_{\bar{B}}$ which maps

$$B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_q \rightarrowtail \bar{B}$$

to

$$B_0 + C_i \rightarrowtail B_1 + C_i \rightarrowtail \cdots \rightarrowtail B_q + C_i \rightarrowtail \bar{B},$$

where $B_j + C_i = B_j \oplus C_i / Ker(B_j \oplus C_i \to \bar{B})$. To verify that this map has $N_{\bar{B}}$ as a codomain as we claim, we need to verify that $B_j + C_i / B_0 + C_i \in Ob(\mathcal{A})$. For this we can observe the following diagram, where $K_{j,i}$ denotes $Ker(B_j \oplus C_i \to \bar{B})$:

$$
\begin{tikzcd}
K_{0,i} \arrow[r, tail] \arrow[d, tail] & K_{j,i} \arrow[r, two heads] \arrow[d, tail] & K_{j,i}/K_{0,i} \arrow[d, "\alpha"] & \\
B_0 \oplus C_i \arrow[r, tail] \arrow[dd, two heads] & B_j \oplus C_i \arrow[r, two heads] \arrow[dd, two heads] & B_j \oplus C_i/B_0 \oplus C_i \arrow[r, "\cong"] \arrow[d, two heads] \arrow[dd, dashed, bend left, "\mathbf{!1}"] & B_j/B_0 \in \mathcal{A} \\
& & coker(\alpha) \arrow[d, dashed, bend left, "\mathbf{!2}"] & \\
B_0 + C_i \arrow[r] & B_j + C_i \arrow[r, two heads] \arrow[ur, dashed, bend left, "\mathbf{!1'}"] & B_j + C_i/B_0 + C_i \arrow[u, dashed, bend left, "\mathbf{!2'}"] &
\end{tikzcd}
.
$$

The maps marked by bold numbers are obtained by appropriate universal properties, the maps **1, 1'** are obtained before the other two. Extending this reasoning slightly with the usual trick of including the identity shows that **2** and **2'** are mutually inverse isomorphisms. This shows that $B_j + C_i/B_0 + C_i$ is quotient of an object in $\mathcal{A}$ which by quotient closure shows that the F_i are in fact self-maps as claimed.

Now notice that F_0 is the identity whereas F_n sends every simplex to the degenerate simplex of the same dimension corresponding to the 0 simplex $\bar{B}$. So we are done if we construct homotopies $h_i : N_{\bar{B}} \times \Delta[1] \to N_{\bar{B}}$ from F_{i-1} to F_i. To define h_i we define t_α which for an order preserving map $\alpha : [q] \to [1]$ returns the largest element of $[q]$ which is mapped to 0. Define h_i to send a pair

$$(B_0 \rightarrowtail \cdots B_{t_\alpha} \rightarrowtail B_{t_\alpha+1} \rightarrowtail \cdots \rightarrowtail B_q, \alpha : [q] \to [1])$$

to

$$B_0 + C_{i-1} \rightarrowtail \cdots B_{t_\alpha} + C_{i-1} \rightarrowtail B_{t_\alpha+1} + C_i \rightarrowtail \cdots \rightarrowtail B_q + C_i.$$

To show that this map is well-defined one needs to verify that $B_j + C_i/B_0 + C_{i-1}$ is an object of $\mathcal{A}$ for all $j > t_\alpha$ (for all the other quotients we require to be in $\mathcal{A}$, this has already been shown above). This follows from a reasoning wholly analogous to the one above showing that $B_j + C_i/B_0 + C_i \in Ob(\mathcal{A})$. Verifying that the h_i indeed form homotopies is routine verification, we thus do not make it explicit. This concludes the proof of the devissage theorem. $\square$

6 Concluding with a fundamental homotopy fibration

We end this bachelor project with the proof of the following theorem.

Theorem 6.0.1. *Denote by $K(R)$ the K-theory space of the category of finitely generated projective R-modules with weak equivalences being the isomorphisms and the cofibration being monics. Then we have a homotopy fibration*

$$\bigvee_p K(\mathbb{F}_p) \to K(\mathbb{Z}) \to K(\mathbb{Q}).$$

The wedge is taken over all primes.

Proof. Recall that we write the category of finitely generated projective R-modules by $P(R)$. The reason one could expect this to be true is with the heuristic that the category $P(\mathbb{Q})$ is very similar to the category $P(\mathbb{Z})$ where weak equivalences are rational equivalences, i.e. maps that become isomorphisms after tensoring with $\mathbb{Q}$. Denoting the class rational equivalence by $w_{\mathbb{Q}}$, we have a natural exact functor given by the identity on the underlying categories $(P(\mathbb{Z}), i) \to (P(\mathbb{Z}), w_{\mathbb{Q}})$ respectively. If categories of modules had a cylinder functor, we would be able to use Waldhausen localization (theorem 5.1.1) to study this map better, but in general this is not the case. We can make this the case by replacing these categories by the appropriate categories of bounded chain complexes, with weak equivalences given by requiring the induced map on homology to be a weak equivalence in $(P(\mathbb{Z}), i)$ or $(P(\mathbb{Z}), w_{\mathbb{Q}})$. Which can be done by using the Gillet-Waldhausen theorem (theorem 5.1.4).
Denote the category of bounded projective finitely generated $\mathbb{Z}$ chain complexes by $C(P(\mathbb{Z}))$, the class of weak equivalences by we and the class of rational weak equivalences by $we_{\mathbb{Q}}$. So we apply Waldhuasen localization on the family of weak equivalences $we \subset we_{\mathbb{Q}}$ and get the homotopy fibration

$$K((C(P(\mathbb{Z})))^{we_{\mathbb{Q}}}, we) \to K(C(P(\mathbb{Z})), we) \to K(C(P(\mathbb{Z})), we_{\mathbb{Q}}).$$

We leave it to the reader to recall the basics of homological algebra needed to verify that we can indeed apply the Waldhausen localization theorem.
Now, by the K-theory equivalence given by the Gillet-Waldhausen theorem, the middle term is already as desired. We need to use the theorems we have at our disposal to replace the two outer terms by what we desire.
We first deal with the right term. We will do this by using the approximation theorem (theorem 5.2.3). As we will not need to juggle different weak equivalences, for this part of the proof we denote by $C(P(\mathbb{Z}))$ the category of bounded $\mathbb{Z}$ chain complexes with rational quasi-isomorphisms as weak equivalences and $C(\mathbb{Q})$ the category of bounded $\mathbb{Q}$ chain complexes with quasi-isomorphisms as weak equivalences. We have an exact functor $C(P(\mathbb{Z})) \to C(\mathbb{Q})$ given by tensoring each term in the chain complex by $\mathbb{Q}$. This functor sends weak equivalences to weak equivalences by construction, cofibrations to cofibrations as $\mathbb{Q}$ is an injective $\mathbb{Z}$-module and preserves pushouts (in particular along cofibrations) as these are computed degreewise and $- \otimes \mathbb{Q} : \mathbb{Z}\text{-}mod \to \mathbb{Q}\text{-}mod$ is a left adjoint. So this functor is exact. To show this functor induces an isomorphism in K-theory we need to show this functor satisfies properties $(i), (ii), (iii)$ using the same notation as in the statement of the approximation theorem. Point (i) is again by construction and point (ii) follows from the fact that $C(P(\mathbb{Z}))$ has a cylinder functor. So only the third point requires attention.
Given a bounded chain complexes $C_\bullet$ in $C(P(\mathbb{Z}))$ and $D_\bullet$ in $C(\mathbb{Q})$ consider an arbitrary map $\alpha : C_\bullet \otimes \mathbb{Q} \to D_\bullet$ in $C(\mathbb{Q})$ we need to find a cofibration $\beta : C_\bullet \to C'_\bullet$ in $C(P(\mathbb{Z}))$ and a weak equivalence $\sigma : C'_\bullet \otimes \mathbb{Q} \to D_\bullet$ such that $\sigma \circ \beta = \alpha$. To do this use that $C(\mathbb{Q})$ has a cylinder functor to get a factorization $C_\bullet \otimes \mathbb{Q} \hookrightarrow E_\bullet \xrightarrow{\simeq} D_\bullet$. Focusing around the ith index, this data gives

$$C_i \otimes \mathbb{Q} \xrightarrow{\lambda_i B_i} \mathbb{Q}^{n_i} \xrightarrow{S_i} \mathbb{Q}^{m_i}$$

with $\lambda \in \mathbb{Q}, B_i \in Mat_{n_i \times rk(C_i)}(\mathbb{Z}), S_i \in M_{m_i \times n_i}(\mathbb{Q})$. We can replace the S_i by $\lambda_i S_i$ without changing the fact that this is a quasi-isomorphism and replace $\lambda_i B_i$ by B_i, this does not change the composition. Now for the map out of $C_\bullet \otimes \mathbb{Q}$ to still be a chain map we need to multiply the differentials of $E_\bullet$ by some scalars, this can be done by induction as our chain complexes are bounded. Now notice that all the differentials in $E_\bullet$ can, up to multiplication by scalars, be assumed to be matrices with coefficients in $\mathbb{Z}$. In order to ensure that the map $E_\bullet \to D_\bullet$ is a still a map of chain complexes, we similarly as before need to multiply all the S_i by some scalars and again as our complexes are all bounded this can be done by induction. What all of this moving of scalars means is that the map $C_\bullet \otimes \mathbb{Q} \to E_\bullet$ is actually in the image of $- \otimes \mathbb{Q} : C(P(\mathbb{Z})) \to C(\mathbb{Q})$. We choose some preimage of this map which is free in each degree, call it $\beta : C_\bullet \to C'_\bullet$. It is not hard to verify that restricting to projective modules makes tensoring with $\otimes\mathbb{Q}$ reflect monics, thus this is the desired β and the weak equivalence $C'_\bullet \otimes \mathbb{Q} \cong E_\bullet \xrightarrow{\simeq} D_\bullet$ is the desired weak equivalence. This proves that $K(C(P(\mathbb{Z})), we_{\mathbb{Q}}) \simeq K(C(\mathbb{Q}), we)$ which by the Gillet-Waldhausen theorem is homotopy equivalent to $K(\mathbb{Q})$ as desired.

Now we work towards proving that $K((C(P(\mathbb{Z})))^{we_{\mathbb{Q}}}, we)$ is homotopy equivalent to $\bigvee_p K(\mathbb{F}_p)$. The first step is going to be to replace $K((C(P(\mathbb{Z})))^{we_{\mathbb{Q}}}, we)$ with the category of finitely generated torsion abelian groups, which we denote by $\mathbb{Z}^{w_{\mathbb{Q}}}$. We denote categories of bounded chain complexes by $C(-)$, the argument saying what modules our chain complexes are composed of. If we consider chain complexes with torsion homology we add a $we_{\mathbb{Q}}$ in the exponent. We denote the category of finitely generated $\mathbb{Z}$-modules simply by $\mathbb{Z}$. All of our modules are finitely generated and all of our comlexes are bounded. We as usual denote F_f for the homotopy fiber of $f : X \to Y$.
With this notation in hand, we can present our proof strategy for the first step. This is easiest to understand when accompanied by the following diagram

$$\begin{tikzcd}
K(\mathbb{Z}^{w_{\mathbb{Q}}}, i) \arrow[d] & & \\
K(C(\mathbb{Z}^{w_{\mathbb{Q}}}), we)) \arrow[d] & & \\
K(C(\mathbb{Z})^{we_{\mathbb{Q}}}, we) \arrow[r] & K(C(\mathbb{Z}), we) \arrow[r] & K(C(\mathbb{Z}), we_{\mathbb{Q}}) \\
F_\iota \arrow[r] \arrow[u] & K(\mathbb{Z}, \iota) \arrow[r, "\iota"] \arrow[u] & K(\mathbb{Z}, w_{\mathbb{Q}}) \arrow[u] \\
F_{P\iota} \arrow[r] \arrow[u] \arrow[d] & K(P(\mathbb{Z}), i) \arrow[r, "P\iota"] \arrow[u] \arrow[d] & K(P(\mathbb{Z}), w_{\mathbb{Q}}) \arrow[u] \arrow[d] \\
K(C(P(\mathbb{Z}))^{we_{\mathbb{Q}}}, we) \arrow[r] & K(C(P(\mathbb{Z})), we) \arrow[r] & K(C(P(\mathbb{Z})), we_{\mathbb{Q}}))
\end{tikzcd}.$$

What we aim to show is that the bottom left and top left K-theory spaces are homotopy equivalent. We do this by showing that each of the vertical maps is a homotopy equivalences. Notice that the pairs of horizontal composable morphisms are all homotopy fibrations either by Waldhausen localization (theorem 5.1.1) or by construction. So by the five lemma if we show that the middle and right most vertical maps are homotopy equivalences we get the fact that the left one is a homotopy equivalence for free. For the top and bottom triples of vertical homotopy equivalences this follows from the Gillet Waldhausen theorem. For the middle triple of vertical homotopy equivalences, this follows from the resolution theorem. Indeed, finitely generated $\mathbb{Z}$-modules all admit a length two resolution by projective modules, the category of which is closed under extensions and kernels, thus we can apply proposition 5.3.2. This means only the top lone vertical maps need to be shown to be homotopy equivalences. For the very top map this follows from Gillet-Waldhausen (theorem 5.1.4). So we need to show that the inclusion $C(\mathbb{Z}^{w_{\mathbb{Q}}}) \to C(\mathbb{Z})^{we_{\mathbb{Q}}}$ induces a K-theory equivalence.
To do this, we follow the method explained in theorem 1.5.2 of [39], in particular the proof of lemma 1.5.3. We use proposition 5.2.8 which by proposition 5.2.7 will show the desired homotopy equivalence. We need to verify the 4 requirements of proposition 5.2.8 for the inclusion $C(\mathbb{Z}^{w_{\mathbb{Q}}}) \to C(\mathbb{Z})^{we_{\mathbb{Q}}}$. Because this is a fully faithful inclusion the only point that is not obvious is that for any complex $C_\bullet \in C(\mathbb{Z})^{we_{\mathbb{Q}}}$ we can find a complex $C'_\bullet \in C(\mathbb{Z}^{w_{\mathbb{Q}}})$ and a quasi-isomorphism $C_\bullet \to C'_\bullet$. We construct $C'_\bullet$ by induction. Let $m > 1$ be such that every homology group of $C_\bullet$ is annihilated by m, this integer exists by the

finiteness assumptions. Now assume we have constructed a chain complex $C_\bullet^{(k)}$ which is such that there is a quasi isomorphism $C_\bullet \to C_\bullet^{(k)}$ and such that $C_i^{(k)}$ is annihilated by some power of m for each $i \geq k$. We will construct a complex $C_\bullet^{(k-1)}$ equipped with a quasi isomorphism $C_\bullet^{(k)} \to C_\bullet^{(k-1)}$ and such that $C_i^{(k-1)}$ is annihilated by some power of m for each $i \geq k-1$. Because our complexes are bounded, this will clearly be sufficient as this process will eventually no longer need to be iterated, and we can start with k large enough so that $C_i = 0$ for all $i \geq k$. For this, let p be an integer such that $m^p C_k^{(k)} = 0$, we construct the following short exact sequence of chain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & m^{2p}C_{k-1}^{(k)} \arrow[r] \arrow[d, hook] & ker(q) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_k^{(k)} \arrow[r] \arrow[d, "Id"] & C_{k-1}^{(k)} \arrow[r] \arrow[d, two heads] & C_{k-2}^{(k)} \arrow[r] \arrow[d, two heads, "q"] & C_{k-3}^{(k)} \arrow[d, "Id"] & \cdots . \\
\cdots \arrow[r] & C_k^{(k)} \arrow[r] & C_{k-1}^{(k)}/m^{2p}C_{k-1}^{(k)} \arrow[r] & C_{k-1}^{(k)}/mC_{k-1}^{(k)} \oplus_{C_{k-1}^{(k)}} C_{k-2}^{(k)} \arrow[r] & C_{k-3}^{(k)} \arrow[r] & \cdots
\end{tikzcd}$$

We let the bottom row be $C_\bullet^{(k-1)}$, clearly if we show that the top row has trivial homology we are done by the long exact sequence in homology. For the top row to have trivial homology, it suffices for the morphism $\partial : mC_{k-1}^{(k)} \to ker(q)$ to be an isomorphism. To show injectivity, it suffices to show that the kernel of $C_{k-1}^{(k)} \to C_{k-2}^{(k)}$ is m^{2p} torsion. This follows from the following two exact sequences

$$0 \to Z_k^{(k)} \to C_k^{(k)} \to B_{k-1}^{(k)} \to 0$$

which shows that $B_k^{(k)}$ is m^p torsion and

$$0 \to B_{k-1}^{(k)} \to Z_{k-1}^{(k)} \to H_{k-1}(C_\bullet^{(k)}) \to 0$$

which shows that $Z_{k-1}^{(k)}$ is m^{2p} torsion. The surjectivity of $\partial : mC_{k-1}^{(k)} \to ker(q)$ follows from the surjectivity of q and the fact that the map q is the inclusion into the first coordinate. This proves the final vertical homotopy equivalence $C(\mathbb{Z}^{w_\mathbb{Q}}) \to C(\mathbb{Z})^{we_\mathbb{Q}}$ of the above diagram.

Now, let's show that $\mathbb{Z}^{w_\mathbb{Q}}$ is in fact homotopy equivalent to $\bigvee K(\mathbb{F}_p)$, which will conclude the proof. By the fundamental theorem on finitely generated abelian groups, this is equivalent to the restricted product (i.e. the colimit over the finite products) of the categories of p-torsion abelian groups for prime p. Denoting this category by pT, because K-theory commutes with filtered colimits and finite products (see section 3.2) and by passing to K-theory spectra, we can replace finite products with finite coproducts. So all we have to show is that $K(P(\mathbb{F}_p)) \simeq K(pT)$, where in both cases the cofibrations are the monics and the weak equivalences are the isomorphism. We have an inclusion of categories $P(\mathbb{F}_p) \to pT$. Using the structure of finitely generated p-torsion groups, we know that all of these are of order p^n for some n, and that such a group always has a subgroup of order p^{n-1}. The quotient by this subgroup is $\mathbb{F}_p$ which is obviously in the image of the inclusion $P(\mathbb{F}_p) \to pT$. With this it is easy to see that all the assumptions needed to apply the Devissage theorem (theorem 5.4.1) are satisfied, thus completing the proof. $\square$

A Admitted results

I have tried to keep this bachelor project as honest as possible, in the sense that any result used, I must have at least studied. I do this to minimize the number of bad surprises in terms of lack of prerequisites a reader might have. Nonetheless, there are a handful of results which we collect here which had to be admitted. There are some results mentioned in the text without proof, when this is done it means that with the prerequisites listed in the introduction I was able to work through the proofs with limited struggle. It is in this sense that the results in this section and the material indicated as prerequisites should be sufficient to understand the in this project.

Theorem A.0.1. *(Proposition 5.13 in [40]) Consider a map of fibrations*

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ F' & \longrightarrow & E' & \longrightarrow & B' \end{array},$$

such that for both fibrations the action of the base space on the homology of the fiber is trivial. Then if two of the three maps are isomorphisms on homology with coefficients in a PID, then so is the third.

Other than this first statement, all the results we admit without proof, yet without claiming them as prerequisites are technical results concerning simplicial sets.

The first two of these results serve to relate category theory and topology. These are essential results in a bachelor project in K-theory, a subject which aims at associating topology inspired invariants to different families of categories.

Theorem A.0.2. *(3.7 Quillen's theorem A in [6] chapter IV) Let $F : C \to D$ be a functor such that F/d (dually d/F) is contractible for each $d \in D$. Then F is a homotopy equivalence.*

Theorem A.0.3. *(3.8 Quillen's theorem B in [6] chapter IV) Let $F : C \to D$ be a functor such that for every morphism $d \in d'$ the induced functor $F/d \to F/d'$ (dually $d'/F \to d/F$) is a homotopy equivalence. Then for each $d \in D$ we have a homotopy fibration*

$$F/d \to C \xrightarrow{F} D.$$

The dual statements follow from the fact that $BC \cong BC^{op}$ for every category C and that $Fun(C^{op}, D^{op}) \cong Fun(C, D)$ for every pair of categories C and D.
Both the above results admit a rewording in the language of (pre-)(co)fibered functors (definition 3.7.3 in [6]) which we give.

Proposition A.0.4. *(Corollary 3.7.4 and 3.8.1 in [6] chapter 4)*

(i) Let $F : C \to D$ be a functor which is either pre-fibered or pre-cofibered and assume furthermore that $F^{-1}(d)$ is contractible for all $d \in D$. Then F is a homotopy equivalence.

(ii) Let $F : C \to D$ be a functor which either fibered (or cofibered) and such all the base change (or cobase change) maps are homotopy equivalences, then we have a homotopy fibration

$$F^{-1}(d) \to C \xrightarrow{F} D.$$

The following result, which we call the "realization lemma", allows us to detect homotopy equivalences of bisimplicial sets.

Proposition A.0.5. *(Proposition 1.7 of [5] chapter 4) Suppose $f : X_{\bullet,\bullet} \to Y_{\bullet,\bullet}$ is a map of bisimplicial set such that each $f_n : X_{n,\bullet} \to Y_{n,\bullet}$ is a homotopy equivalence, then f is a homotopy equivalence.*

Similarly, this result allows us to work degreewise when working with bisimplicial sets, thus reducing to simplicial sets.

Proposition A.0.6. *(Lemma 5.2 in [37]) Let $X_{\bullet,\bullet} \to Y_{\bullet,\bullet} \to Z_{\bullet,\bullet}$ be a sequence of bisimplicial sets such that the composite is constant. Suppose furthermore that for each $n \geq 0$, the sequence $X_{n,\bullet} \to Y_{n,\bullet} \to Z_{n,\bullet}$ is a fibration up to homotopy and that $Z_{n,\bullet}$ is connected. Then the sequence of bisimplicial sets is a fibration up to homotopy.*

The following result allows us to go in the opposite direction of the nerve functor. I.e. it allows us to use concepts from category theory to study simplicial sets.

Proposition A.0.7. *(page 359 of [14]) Given a simplicial set X, we have a category $Simp(X)$, corresponding to the poset of simplicies of X which is such that $N.Simp(X) \simeq X$.*

The proof is the discussion on page 359 of [14], for help with what the gluing lemma is I think lemma 8.8 of [5] is more useful than the source Waldhausen refers to.

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