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Filtration Shifting Maps and Differentials

Connecting Filtration Shifting Maps to Differentials in the Spectral Sequences Associated to their Mapping Cones

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

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Abstract

In this master's thesis we derive a connection between filtration shifts and differentials in a spectral sequence. We assume that the spectral sequence comes from a Cartan–Eilenberg system, and we develop a framework to fit the mapping cones of maps of filtered spectra or chain complexes into a sequence of Cartan–Eilenberg systems. Restricting to three-stage filtrations of the Cartan–Eilenberg systems, we give a complete description of this connection. Specifically, we show that a filtration shift leads to a non-zero differential in the spectral sequence associated to the mapping cone, and vice versa. We also give a slight generalisation of this result for longer filtrations, determining conditions at the level of the Cartan–Eilenberg systems that lets us reduce to the case of three-stage filtrations.

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Table of Contents

Introduction

A common problem solving technique when faced with a difficult problem is to break the problem into smaller pieces. A reflection of this technique in mathematics is how we study objects by filtering them. For instance, thinking of a topological pair as a filtration with the subspace filtering the whole space, long exact sequences let us use what we know of the subspace to understand the whole space. It is tempting to emulate this approach, only with longer filtrations. This runs into the question of how we should combine knowledge of the pieces into that of the whole. When working with topological pairs, we only need to understand the subspace and how it fits within the whole space. If we instead have several pieces, each with a relationship to the others, a combinatorial explosion of data ensues. This is where spectral sequences enter the stage. They are algebraic bookkeeping devices tracking all the data, and they give us a procedure for getting information out. Still, spectral sequences do not turn difficult problems into easy ones. A spectral sequence renders information through better and better approximations. Each approximation improves upon the previous ones through a homology computation. To perform this computation, we need to understand a set of differentials. This is often hard, and there are no generally applicable methods to decide such differentials. In practice, we often have to rely on clever arguments or on exploiting structure inherent in the exact problem we are studying.

Conceived by Leray [Ler46] during the second world war, spectral sequences have become invaluable computational tools in algebraic topology and homological algebra. Serre gave an early demonstration of their power in [Ser51]. Introducing the Serre spectral sequence, he computed the homology and cohomology of Eilenberg–MacLane spaces. These calculations had broad consequences in homotopy theory. However, Serre spectral sequences often have many nontrivial differentials. Without sufficient inherent structure, computing such a spectral sequence can be an intractable problem. To combat such problems, mathematicians have devised many other spectral sequences. The underlying example in this thesis is the Adams spectral sequence for the sphere spectrum. The sphere spectrum is a homotopy commutative ring spectrum, making the stable homotopy groups a graded commutative ring. This multiplicative structure is reflected in the Adams spectral sequence. The presence of such abundant structure has made the Adams spectral sequence a powerful tool to deduce information about the homotopy groups of spheres.

In this master's thesis we seek to develop a method to compute certain differentials without relying on any extra structure. What motivates our approach is the question of how filtration shifts appear from the perspective of a spectral sequence. Given a map $f\colon X \to Y$ of filtered objects, a "filtration shift" happens when an element taken from the filtration of X hits an element of Y with a lift to a higher filtration. The objects we have in mind are filtered chain complexes or spectra. The homology or homotopy groups of these give rise to the Cartan–Eilenberg systems introduced in [CE56]. Consequently, we choose to only consider spectral sequences with underlying Cartan–Eilenberg systems.

The main goal of this thesis is to prove a connection between filtration shifting maps and differentials in the spectral sequences associated to their mapping cones. Given a homotopy cofiber sequence of filtered chain complexes or spectra, we devise a framework to fit the associated homology or homotopy groups into a sequence of Cartan–Eilenberg systems. It is in this sense that we encapsulate data about mapping cones. The direct path towards this goal begins with the precise definition of a filtration shift in Definition 2.3.2. We formulate our notion of an exact sequence of Cartan–Eilenberg systems in Definition 2.1.6. This forms the context of our subsequent results. Restricting our attention to only three filtration indices, we prove Theorem 4.1.7. This is our main result, and it shows that there is a one-to-one correspondence between filtration shifts and differentials in this restricted setting. We give a slight generalisation of this theorem in Section 4.2.

While our method has the benefit of being concrete, the restrictions we require are significant. Even so, practical examples suggest that our results can still be useful.

Outline

We begin the first chapter by introducing triangulated categories. We discuss their core features, and we encounter the first instance of a push-lift argument. Such arguments lie at the heart of the development of our results in Chapter 4. We end the chapter by looking at categories that are both symmetric monoidal and triangulated. The relationship between these structures is intricate. We discuss this, and end on another push-lift type result.

Chapter 2 and 3 develop the theory we need to formulate the results of the final chapter. Chapter 2 begins by introducing our perspective on filtrations. We then give the definition of a Cartan–Eilenberg system and look at a couple of examples. Two notable definitions appear at the end of the first section. We explain what we shall mean by an exact sequence of Cartan–Eilenberg systems, and what it means for such a system to be suspended. These establish the context of our work in Chapter 4. In the next section, we derive a push-lift lemma for grids of exact sequences. Finally, we give a precise definition of filtration shifts. The third chapter concerns itself with spectral sequences. We present our grading conventions and construct the spectral sequences we need. All the spectral sequences in Chapter 4 come from Cartan–Eilenberg systems, and we describe the structure of these in detail.

The bulk of our original work appears in Chapter 4. The introduction of this chapter motivates our approach. Section 4.1 is dedicated to the proof of our central theorem. The final section gives a slight generalisation of this result. This generalisation relies on certain structure morphisms of the Cartan–Eilenberg systems being injective. We finish the section by describing what these restrictions mean for the spectral sequences.

Conventions

We apply suspensions from the left, and we let this guide our sign conventions. Wherever arbitrary categories appear, we assume that they are locally small, and we let $\mathscr{C}(X, Y)$ denote the set of morphisms in $\mathscr{C}$ from X to Y. In diagrams, an arrow

$$X \rightarrowtail Y$$

decorated with a tail indicates that the morphism is a monomorphism. A two-headed arrow

$$X \twoheadrightarrow Y$$

indicates that it is an epimorphism.

1 Triangulated Categories

Triangulated categories were introduced by Verdier in his 1967 thesis under Grothendieck. In that same thesis, Verdier introduced the derived category of an abelian category. Passing from an abelian category to its derived category, we retain additivity, but the category is no longer abelian. The purpose of a triangulation is to capture some of the structure left behind. Most essentially, we will be able to meaningfully recover a notion of exactness.

Whereas Verdier's original focus was the derived category, triangulated categories are now found in many different areas of mathematics. A prime example of a triangulated category in algebraic topology is the stable homotopy category. Incidentally, this example is only an instance of the more general fact that the homotopy category of a stable model category is triangulated.

Dold and Puppe gave axioms similar to those of Verdier some years earlier, but they did not include the octahedral axiom. We present the axioms as given by May in [May01], and follow his exposition through the basic definitions. May gives a slightly smaller number of axioms, most notably deducing the existence of "fill-in" morphisms as a consequence, rather than including it as an axiom.

Let $\mathscr{C}$ be an additive category and suppose that we have an additive self-equivalence $\Sigma\colon \mathscr{C} \to \mathscr{C}$. Let 0 denote the zero object of $\mathscr{C}$. Given an object X of $\mathscr{C}$, we call ΣX the *suspension* of X. A *triangle* on an ordered triple (X, Y, Z) of objects of $\mathscr{C}$ is a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

of morphisms in $\mathscr{C}$, where the last object is the suspension of the first object. We often denote such a triangle by $(X, Y, Z; f, g, h)$, or simply (f, g, h) if the names of the objects are less relevant or can be inferred. A *morphism of triangles* (f, g, h) *and* (f', g', h') is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

in $\mathscr{C}$. A morphism of triangles is an isomorphism if each vertical arrow is an isomorphism.

Definition 1.0.1 ([May01, Definition 2.1])**.** A *triangulation* on an additive category $\mathscr{C}$ is an additive self-equivalence $\Sigma\colon \mathscr{C} \to \mathscr{C}$ together with a class of *distinguished triangles* satisfying the following axioms:

(T1) Let X be any object of $\mathscr{C}$ and $f\colon X \to Y$ a morphism in $\mathscr{C}$, then:

a) The triangle $X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow \Sigma X$ is distinguished.

b) The morphism $f\colon X \to Y$ is part of a distinguished triangle

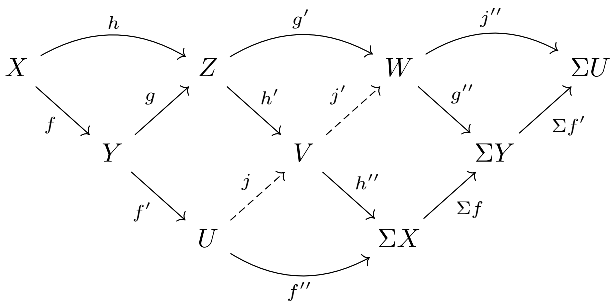
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

c) Any triangle isomorphic to a distinguished triangle is distinguished.

(T2) (Rotation) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is distinguished, then so is

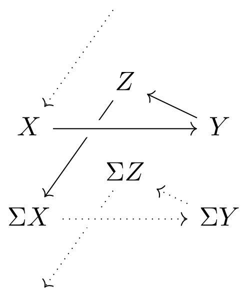
$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y.$$

(T3) (Verdier's Axiom) Consider the diagram



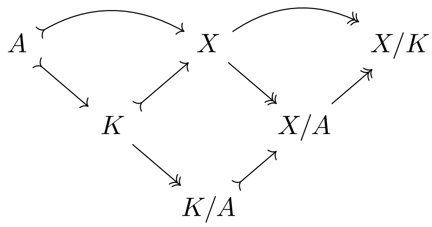
Assume that (f, f', f'') and (g, g', g'') are distinguished, and that $h = g \circ f$ so that the upper left triangle commutes. Given morphisms h' and h'' such that (h, h', h'') is distinguished, then there are morphisms j and j' forming a distinguished triangle (j, j', j'') and making the entire diagram commutative.

A category equipped with a triangulation is a *triangulated category*. To motivate this choice of name, it is instructive to replace $\mathscr{C}$ with a $\mathbb{Z}$-graded category $\mathscr{C}_*$ where a morphism from X to Y of degree n is an element of $\mathscr{C}(X, \Sigma^{-n}Y) = \mathscr{C}(\Sigma^n X, Y)$. Then $\mathscr{C}_*(X, Y) = \bigoplus_n \mathscr{C}(\Sigma^n X, Y)$, and a triangle (f, g, h) along with its rotations form a helix



Projecting the solid arrows into the plane produces a triangle.

Remark 1.0.2. Stating Verdier's axiom as a braid of distinguished triangles is only one of many equivalent descriptions. The axiom as Verdier initially described it is often referred to as the octahedral axiom for the way the objects and morphisms form the skeleton of an octahedra. The axiom can also be disposed of altogether. In [Nee01], Neeman proposes instead a notion of a "good" morphism of triangles, requiring that in a diagram like the one of Lemma 1.0.5, there should exist certain good choices of morphisms filling in for the dashed vertical arrow. The content of the axiom is a close analogue of the Noether isomorphism $(X/A)/(K/A) \cong X/K$ of abelian groups for a sequence of inclusions $A \subset K \subset X$ of subgroups. If we form the cofibers of the inclusions $A \subset X$ and $A \subset K$, then the short exact sequences align in a commutative diagram



We now turn to discuss some implications of the axioms. First among these is that swapping any two signs of a distinguished triangle keeps the triangle distinguished.

Lemma 1.0.3. *If (f, g, h) is a distinguished triangle, then all of $(f, -g, -h)$, $(-f, g, -h)$ and $(-f, -g, h)$ are also distinguished triangles.*

Proof. The diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \| & & \| & & \downarrow{\scriptstyle -\mathrm{id}} & & \| \\ X & \xrightarrow{f} & Y & \xrightarrow{-g} & Z & \xrightarrow{-h} & \Sigma X \end{array}$$

commutes and exhibits an isomorphism of triangles (f, g, h) and $(f, -g, -h)$. When the upper row triangle is distinguished, so is the lower row triangle. The proof of the other cases is similar. $\square$

The next two lemmas were originally axioms of Verdier, but can be derived from the axioms above. The first lemma shows that a distinguished triangle (f, g, h) is distinguished if and only if $(g, h, -\Sigma)$ is distinguished. The second lemma ensures the existence of a *fill-in* between certain distinguished triangles, and is a central feature of a triangulated category.

Lemma 1.0.4 ([May01, Lemma 2.4])**.** *If $(g, h, -\Sigma f)$ is a distinguished triangle, then so is (f, g, h).*

Lemma 1.0.5 ([May01, Lemma 2.2])**.** *If the rows are distinguished triangles and the left square commutes in the following diagram, then there is a morphism $\gamma \colon Z \to Z'$ making the remaining squares commute*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

Note that this lemma only guarantees that such a fill-in exists, not that there is a unique or even a preferred choice of one. In fact, in a commutative diagram with distinguished rows like

$$\begin{array}{ccccccc} X & \longrightarrow & 0 & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0, \end{array}$$

any morphism $Z \to Z'$ may serve as a fill-in making all the squares commute.

Remark 1.0.6. Although triangulated categories are widely used and have proven valuable, the axioms are considered unsatisfactory by some [GM03, Chapter IV]. Among the issues contributing to this view is how forming cofibers is not functorial. Given a morphism $f\colon X \to Y$, we may complete it to distinguished triangles (f, g, h) and (f, g', h'). This leads to a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \| & & \| & & \downarrow{\scriptstyle\gamma} & & \| \\ X & \xrightarrow{f} & Y & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X. \end{array}$$

The fill-in lemma provides a morphism $\gamma\colon Z \to Z'$ making all the squares commute. A triangulated version of the 5-lemma (Proposition 1.0.11) shows that γ is an isomorphism, making $(\mathrm{id}, \mathrm{id}, \gamma)$ an isomorphism of triangles. Thus the distinguished triangle generated by a morphism f is unique up to isomorphism, but the isomorphism is *not* unique due to the different possible choices of γ.

We proceed to discuss how distinguished triangles relates to long exact sequences, first encountering a familiar property.

Lemma 1.0.7 ([BR20, Lemma 4.1.4])**.** *The composition of any two consecutive morphisms in a distinguished triangle is zero.*

Proof. Consider a distinguished triangle $(X, Y, Z; f, g, h)$. To see that the composite hg is zero, we begin with the distinguished triangle

$$X \longrightarrow 0 \longrightarrow \Sigma X \xrightarrow{-\mathrm{id}} \Sigma X$$

obtained from (T1) by rotation. Note that $-\Sigma\mathrm{id} = -\mathrm{id}$ by functoriality. Flipping two signs keeps this triangle distinguished and leads to a diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \| & & \downarrow & & \downarrow{\scriptstyle k} & & \| \\ X & \longrightarrow & 0 & \longrightarrow & \Sigma X & \xrightarrow{\mathrm{id}} & \Sigma X \end{array}$$

with distinguished rows. The left square commutes trivially, so that there is a fill-in $k\colon Z \to \Sigma X$ by Lemma 1.0.5 making all the squares commute. From the right square it follows that $k = h$, and from the middle square we conclude that $hg = 0$. More generally, this argument shows the composition of the second and third morphism of any distinguished triangle vanishes. In particular, it applies to the rotation $(h, -\Sigma f, -\Sigma g)$ of the initial triangle. As Σ is faithful, $\Sigma(gf) = 0$ implies $gf = 0$, concluding the proof. $\square$

Now suppose that $(X, Y, Z; f, g, h)$ is a distinguished triangle, and consider the distinguished triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

we get by rotation. Rotating this triangle gives yet another distinguished triangle, and continuing like this results in a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma g} \Sigma Z \xrightarrow{-\Sigma h} \Sigma^2 X \longrightarrow \cdots \tag{1.0.8}$$

consisting of sequences of distinguished triangles. Certain functors into abelian categories take such sequences into exact sequences. We give these functors names in the case when the target category is the category Ab of abelian groups and homomorphisms.

Definition 1.0.9 ([Ver96, Définition 1.1.5])**.** Let $\mathscr{C}$ be a triangulated category. An additive functor $H\colon \mathscr{C} \to \mathrm{Ab}$ is *homological* if it is half-exact in the sense that for every distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

the sequence

$$H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z)$$

of abelian groups is exact. An additive contravariant functor $H\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Ab}$ is *cohomological* if it is half-exact in the same sense, with the obvious change in variance.

If $H\colon \mathscr{C} \to \mathrm{Ab}$ is a homological functor and (f, g, h) a distinguished triangle, then we get a long exact sequence

$$H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z) \xrightarrow{H(h)} H(\Sigma X) \xrightarrow{-H(\Sigma f)} H(\Sigma Y) \longrightarrow \cdots$$

by applying H to the sequence (1.0.8).

Next, we introduce two particularly interesting instances of homological and cohomological functors, namely those we get from the bifunctor

$$\mathscr{C}(-,-)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Ab}$$

by locking in one of the arguments. First, we define some notation. Given a morphism $f\colon X \to Y$ in $\mathscr{C}$ and an object W of $\mathscr{C}$, we write

$$f_* := \mathscr{C}(W, f)\colon \mathscr{C}(W, X) \longrightarrow \mathscr{C}(W, Y)$$

for the homomorphism of abelian groups sending $g\colon W \to X$ to the composite $fg\colon W \to Y$. Dually, we write

$$f^* := \mathscr{C}(f, W)\colon \mathscr{C}(Y, W) \longrightarrow \mathscr{C}(X, W)$$

for the homomorphism sending $g\colon Y \to W$ to the composite $gf\colon X \to W$.

Proposition 1.0.10 ([BR20, Proposition 4.1.5])**.** *Let $\mathscr{C}$ be a triangulated category. The functor $\mathscr{C}(W, -)\colon \mathscr{C} \to \mathrm{Ab}$ is homological for any object W of $\mathscr{C}$.*

Proof. Consider a distinguished triangle (f, g, h). It follows from Lemma 1.0.7 that $\operatorname{im} f_* \subset \ker g_*$, so to prove that the sequence

$$\mathscr{C}(W, X) \xrightarrow{f_*} \mathscr{C}(W, Y) \xrightarrow{g_*} \mathscr{C}(W, Z)$$

is exact it remains to show the opposite inclusion. By (T1) and the rotation axiom, any morphism $j\colon W \to Y$ satisfying $gj = 0$ fits in a commutative diagram with distinguished rows as follows:

$$\begin{array}{ccccccc} W & \longrightarrow & 0 & \longrightarrow & \Sigma W & \xrightarrow{-\mathrm{id}} & \Sigma W \\ {\scriptstyle j}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma j} \\ Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y. \end{array}$$

The fill-in lemma provides a morphism $\Sigma i\colon \Sigma W \to \Sigma X$ satisfying $\Sigma j = \Sigma(fi)$. That Σ is an equivalence implies that $j = fi$, hence $j \in \operatorname{im} f_*$, completing the proof. □

The proof that $\mathscr{C}(-, W)\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Ab}$ is cohomological is similar. A simple consequence of this proposition is the following triangulated version of the usual 5-lemma.

Proposition 1.0.11 (Triangulated 5-lemma)**.** *Consider a morphism of distinguished triangles*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X'. \end{array}$$

If both α and β are isomorphisms, then γ is also an isomorphism.

Proof. Applying $\mathscr{C}(Z', -)$ to the distinguished triangles gives a commutative diagram

$$\begin{array}{ccccccccc} \mathscr{C}(Z', X) & \longrightarrow & \mathscr{C}(Z', Y) & \longrightarrow & \mathscr{C}(Z', Z) & \longrightarrow & \mathscr{C}(Z', \Sigma X) & \longrightarrow & \mathscr{C}(Z', \Sigma Y) \\ {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \Sigma\alpha_*} & & \downarrow{\scriptstyle \Sigma\beta_*} \\ \mathscr{C}(Z', X') & \longrightarrow & \mathscr{C}(Z', Y') & \longrightarrow & \mathscr{C}(Z', Z') & \longrightarrow & \mathscr{C}(Z', \Sigma X') & \longrightarrow & \mathscr{C}(Z', \Sigma Y'), \end{array}$$

where the rows are exact sequences of abelian groups. If α and β are isomorphisms, then α_*, β_*, $\Sigma\alpha_*$ and $\Sigma\beta_*$ are all isomorphisms. It follows from the usual 5-lemma that γ_* is an isomorphism. In particular, there is a morphism $\gamma^{-1}\colon Z' \to Z$ such that $\gamma\gamma^{-1} = \mathrm{id}_{Z'}$. Repeating the argument with the functor $\mathscr{C}(-, Z')$ produces the corresponding left inverse, hence γ is an isomorphism. □

Finally, we reproduce the 3×3 lemma, attributed to Verdier in [DBB83].

Proposition 1.0.12 ([May01, Lemma 2.6])**.** *Assume that $jf = f'i$ and that the top two rows and left columns are distinguished triangles in the diagram*

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \Sigma i} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \\
{\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle \Sigma i'} \\
X'' & \xrightarrow{f''} & Y'' & \xrightarrow{g''} & Z'' & \xrightarrow{h''} & \Sigma X'' \\
{\scriptstyle i''}\downarrow & & \downarrow{\scriptstyle j''} & & \downarrow{\scriptstyle k''} & & \downarrow{\scriptstyle -\Sigma i''} \\
\Sigma X & \xrightarrow[\Sigma f]{} & \Sigma Y & \xrightarrow[\Sigma g]{} & \Sigma Z & \xrightarrow[-\Sigma h]{} & \Sigma^2 X.
\end{array}
$$

Then there exists an object Z'' and morphisms f'', g'', h'', k, k' and k'' making it a diagram of distinguished rows and columns. Moreover, the diagram commutes, apart for the bottom right square which commutes up to the sign -1.

1.1 Push-Lift and Fill-In Morphisms

Now that we have seen the definition of a triangulated category and the basic properties such a structure entails, we look at a slightly deeper consequence of the fill-in lemma. This will be the first of many times that we encounter diagrams where certain elements push and lift repeatedly to eventually form a cycle. This lies at the heart of the arguments in Chapter 4, but whereas those only rely on exactness, this section explores how such cycles arise from choosing fill-ins.

Let $\mathscr{C}$ be a triangulated category and consider distinguished triangles

$$A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} \Sigma A$$

and

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

If we apply the homological functor $\mathscr{C}(A, -)$ to the second distinguished triangle, we get an exact sequence of abelian groups

$$\cdots \longrightarrow \mathscr{C}(A, X) \xrightarrow{f_*} \mathscr{C}(A, Y) \xrightarrow{g_*} \mathscr{C}(A, Z) \xrightarrow{h_*} \mathscr{C}(A, \Sigma X) \longrightarrow \cdots$$

by Proposition 1.0.10. Similarly, we may apply the cohomological functor $\mathscr{C}(-, X)$ to the first distinguished triangle to get an exact sequence

$$\cdots \longrightarrow \mathscr{C}(X, \Sigma A) \xrightarrow{k^*} \mathscr{C}(X, C) \xrightarrow{j^*} \mathscr{C}(X, B) \xrightarrow{i^*} \mathscr{C}(X, A) \longrightarrow \cdots .$$

Repeating this for the remaining objects of each triangle, the resulting sequences align in a commutative diagram with exact rows and columns extending in each direction:

$$\begin{array}{ccccccc}
\mathscr{C}(\Sigma A, X) & \xrightarrow{f_*} & \mathscr{C}(\Sigma A, Y) & \xrightarrow{g_*} & \mathscr{C}(\Sigma A, Z) & \xrightarrow{h_*} & \mathscr{C}(\Sigma A, \Sigma X) \\
\downarrow{\scriptstyle k^*} & & \downarrow{\scriptstyle k^*} & & \downarrow{\scriptstyle k^*} & & \downarrow{\scriptstyle k^*} \\
\mathscr{C}(C, X) & \xrightarrow{f_*} & \mathscr{C}(C, Y) & \xrightarrow{g_*} & \mathscr{C}(C, Z) & \xrightarrow{h_*} & \mathscr{C}(C, \Sigma X) \\
\downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\
\mathscr{C}(B, X) & \xrightarrow{f_*} & \mathscr{C}(B, Y) & \xrightarrow{g_*} & \mathscr{C}(B, Z) & \xrightarrow{h_*} & \mathscr{C}(B, \Sigma X) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
\mathscr{C}(A, X) & \xrightarrow{f_*} & \mathscr{C}(A, Y) & \xrightarrow{g_*} & \mathscr{C}(A, Z) & \xrightarrow{h_*} & \mathscr{C}(A, \Sigma X).
\end{array}$$

Now suppose $\beta\colon B \to X$ and $\gamma\colon C \to Y$ are morphisms of $\mathscr{C}$ making the diagram

$$\begin{array}{ccccccc}
B & \xrightarrow{j} & C & \xrightarrow{k} & \Sigma A & \xrightarrow{-\Sigma i} & \Sigma B \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \vdots & & \downarrow{\scriptstyle \Sigma\beta} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X
\end{array} \tag{1.1.1}$$

commutative, where the upper row is the distinguished triangle we get by rotating the triangle (i, j, k) once. Any such diagram admits a fill-in by Lemma 1.0.5, and any morphism $\alpha\colon \Sigma A \to Z$ filling in the dashed arrow sits in the group $\mathscr{C}(\Sigma A, Z)$ while satisfying relations $k^*(\alpha) = g_*(\gamma)$ and $h_*(\alpha) = (-\Sigma i)^*(\Sigma\beta)$. These relations imply that the image of α in $\mathscr{C}(C, Z)$ lifts over g_* to γ in $\mathscr{C}(C, Y)$. The left square of (1.1.1) commuting gives $f\beta = \gamma j$, so if we push γ down to $\mathscr{C}(B, Y)$, then the image lifts over f_* to $\beta \in \mathscr{C}(B, X)$. Pushing β down along i^* we have $\beta i \in \mathscr{C}(A, X)$. The right-hand square of (1.1.1) commuting is equivalent to $-\Sigma\beta\Sigma i = h\alpha$. This in turn implies that $\beta i = -\Sigma^{-1}h\Sigma^{-1}\alpha$, so if we push α right along h_*, we find that the images $i^*(\beta)$ in $\mathscr{C}(A, X)$ and $h_*(\alpha) \in \mathscr{C}(\Sigma A, \Sigma X)$ coincide up to a sign under the isomorphism $E\colon \mathscr{C}(A, X) \to \mathscr{C}(\Sigma A, \Sigma X)$ induced by the suspension. This gives an inkling that pushing and lifting six times in this way reduces to the negative of the identity whenever the morphisms come from a fill-in diagram. We give precise meaning to this inkling in the

$$\begin{array}{llll}
 & & \alpha \longmapsto h \circ \alpha & \\
 & & \downarrow & \\
 & \gamma \longmapsto g \circ \alpha = \alpha \circ k & & \\
 & \downarrow & & \\
\beta \longmapsto \gamma \circ j = f \circ \beta & & & \\
\downarrow & & & \\
\beta \circ i = -\Sigma^{-1} h \circ \Sigma^{-1}\alpha & & &
\end{array}$$

Figure 1.1: The push-lift cycle coming from the fill-in in (1.1.1).

slightly more general setting where the first triangle is suspended. After $3n$ rotations, we are left with a distinguished triangle

$$\Sigma^n A \xrightarrow{(-1)^n \Sigma^n i} \Sigma^n B \xrightarrow{(-1)^n \Sigma^n j} \Sigma^n C \xrightarrow{(-1)^n \Sigma^n k} \Sigma^{1+n} A.$$

Applying the various homological and cohomological functors as before now gives the following diagram:

$$\begin{array}{ccccccc}
\mathscr{C}(\Sigma^{1+n}A,X) & \xrightarrow{f_*} & \mathscr{C}(\Sigma^{1+n}A,Y) & \xrightarrow{g_*} & \mathscr{C}(\Sigma^{1+n}A,Z) & \xrightarrow{h_*} & \mathscr{C}(\Sigma^{1+n}A,\Sigma X) \\
\big\downarrow{\scriptstyle ((-1)^n\Sigma^n k)^*} & & \big\downarrow{\scriptstyle ((-1)^n\Sigma^n k)^*} & & \big\downarrow{\scriptstyle ((-1)^n\Sigma^n k)^*} & & \big\downarrow{\scriptstyle ((-1)^n\Sigma^n k)^*} \\
\mathscr{C}(\Sigma^{n}C,X) & \xrightarrow{f_*} & \mathscr{C}(\Sigma^{n}C,Y) & \xrightarrow{g_*} & \mathscr{C}(\Sigma^{n}C,Z) & \xrightarrow{h_*} & \mathscr{C}(\Sigma^{n}C,\Sigma X) \\
\big\downarrow{\scriptstyle ((-1)^n\Sigma^n j)^*} & & \big\downarrow{\scriptstyle ((-1)^n\Sigma^n j)^*} & & \big\downarrow{\scriptstyle ((-1)^n\Sigma^n j)^*} & & \big\downarrow{\scriptstyle ((-1)^n\Sigma^n j)^*} \\
\mathscr{C}(\Sigma^{n}B,X) & \xrightarrow{f_*} & \mathscr{C}(\Sigma^{n}B,Y) & \xrightarrow{g_*} & \mathscr{C}(\Sigma^{n}B,Z) & \xrightarrow{h_*} & \mathscr{C}(\Sigma^{n}B,\Sigma X) \\
\big\downarrow{\scriptstyle ((-1)^n\Sigma^n i)^*} & & \big\downarrow{\scriptstyle ((-1)^n\Sigma^n i)^*} & & \big\downarrow{\scriptstyle ((-1)^n\Sigma^n i)^*} & & \big\downarrow{\scriptstyle ((-1)^n\Sigma^n i)^*} \\
\mathscr{C}(\Sigma^{n}A,X) & \xrightarrow{f_*} & \mathscr{C}(\Sigma^{n}A,Y) & \xrightarrow{g_*} & \mathscr{C}(\Sigma^{n}A,Z) & \xrightarrow{h_*} & \mathscr{C}(\Sigma^{n}A,\Sigma X).
\end{array}$$

Proposition 1.1.2. *Let (i,j,k) and (f,g,h) be the distinguished triangles above. Assume $b \in \mathscr{C}(\Sigma^n B, Y)$ maps to zero in both $\mathscr{C}(\Sigma^n A, Y)$ and $\mathscr{C}(\Sigma^n B, Z)$. Then*

$$((-1)^n\Sigma^n i)^* f_*^{-1}(b) = -h_*\big(((-1)^n\Sigma^n k)^*\big)^{-1} g_*\big(((-1)^n\Sigma^n j)^*\big)^{-1}(b),$$

where the indeterminacy of either expression is the image of

$$(-1)^{n+1}(\Sigma^{-1}h)_*(\Sigma^n i)^* = (-1)^{n+1}(\Sigma^n i)^*(\Sigma^{-1}h)_*.$$

Proof. If b maps to zero in both $\mathscr{C}(\Sigma^n A, Y)$ and $\mathscr{C}(\Sigma^n B, Z)$, then exactness gives morphisms $\gamma\colon \Sigma^n C \to Y$ and $\beta\colon \Sigma^n B \to X$ making the left square of the following diagram commute.

$$\begin{array}{ccccccc}
\Sigma^n B & \xrightarrow{(-1)^n\Sigma^n j} & \Sigma^n C & \xrightarrow{(-1)^n\Sigma^n k} & \Sigma^{1+n} A & \xrightarrow{(-1)^{1+n}\Sigma^{1+n} i} & \Sigma^{1+n} B \\
{\scriptstyle \beta}\big\downarrow & & \big\downarrow{\scriptstyle \gamma} & & \vdots & & \big\downarrow{\scriptstyle \Sigma\beta} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X.
\end{array}$$

This diagram admits a fill-in, and if $\alpha\colon \Sigma^{n+1}A \to Z$ is such a fill-in, then

$$h \circ \alpha = \Sigma\beta \circ (-1)^{1+n}\Sigma^{1+n} i \quad \text{and} \quad g \circ \gamma = \alpha \circ (-1)^n \Sigma^n k.$$

The second equality ensures that the images of α and γ meet in $\mathscr{C}(\Sigma^n C, Z)$. The first equality implies that $\Sigma^{-1}h\Sigma^{-1}\alpha = (-1)^{1+n}\beta\Sigma^n i$. Pushing β down we have $(-1)^n\beta\Sigma^n i \in \mathscr{C}(\Sigma^n A, X)$, and pushing α right we have $h\alpha \in \mathscr{C}(\Sigma^{1+n}A, \Sigma X)$. In particular, we see that the isomorphism

$$E\colon \mathscr{C}(\Sigma^n A, X) \longrightarrow \mathscr{C}(\Sigma^{1+n}A, \Sigma X)$$

takes $(-1)^n\beta \circ \Sigma^n i$ to $-h \circ \alpha$. The indeterminacy due to the lift γ vanishes in $\mathscr{C}(\Sigma^{n+1}A, \Sigma X)$, while the indeterminacy due to β is the image

$$\operatorname{im}\big((-\Sigma^{-1}h)_*\colon \mathscr{C}(\Sigma^n B, \Sigma^{-1}Z) \longrightarrow \mathscr{C}(\Sigma^n B, X)\big),$$

This set is reflected in $\mathscr{C}(\Sigma^n A, X)$ as the image of

$$(-1)^{1+1}(\Sigma^{-1}h)_*(\Sigma^n i)^* = (-1)^{1+n}(\Sigma^n i)^*(\Sigma^{-1}h)_*$$

Finally, the indeterminacy involved in choosing the fill-in is

$$\operatorname{im}\big((-1)^{1+n}(\Sigma^{1+n}i)^*\colon \mathscr{C}(\Sigma^{1+n}B, Z) \longrightarrow \mathscr{C}(\Sigma^{1+n}A, Z)),$$

and this appears in $\mathscr{C}(\Sigma^{n+1}A, \Sigma X)$ as the image of

$$(-1)^{1+n}h_*(\Sigma^{1+n}i)^* = (-1)^{1+n}(\Sigma^{1+n}i)^* h_*. \qquad \square$$

1.2 Symmetric Monoidal Structure

Both the derived category and the stable homotopy category carry a product structure, represented by the tensor product and smash product, respectively. In the language of category theory, they are instances of monoidal categories. Such a category comes with a bifunctor $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$ behaving in a way we expect a product would. If this product is also commutative, then we say that the monoidal category is symmetric. It is these categories that we will look at in this section, and both the derived category and the stable homotopy category are in fact symmetric monoidal. Moreover, as we know, they are also triangulated. When both of these structures are present, we would like to know how they interact. This relationship is intricate, and there are several attempts at elucidating it in the literature. Hovey, Palmieri and Strickland proposes some compatibility axioms in [HPS97, A.2], and May proposes more axioms in [May01, §4]. We will keep from going too deep into all the intricacies, and shall lean on the slightly more compact axioms of [HPS97].

We begin this section by describing the symmetric monoidal structure in more detail, and make sense of what it means for such a category to be "closed". Next, we give a brief introduction to categorical duality theory in the context of closed symmetric monoidal categories. After the setup, we discuss the compatibility axioms restraining the relationship between the triangulated and closed symmetric monoidal structure. We finish off by discussing another push-lift result, this time for diagrams arising as products of distinguished triangles.

A *monoidal category* is a category $\mathscr{C}$ equipped with a functor

$$\wedge\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C},$$

called the *monoidal product*, and a *unit object* S of $\mathscr{C}$. We require that the monoidal product is associative and unital in the sense that there are specified natural isomorphisms

$$X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z \qquad S \wedge X \cong X \cong X \wedge S$$

for all objects X, Y and Z of $\mathscr{C}$. Each of these natural transformations are subject to certain coherence conditions, as explained in [Lan71, Chapter XI]. A *symmetric monoidal category* is a monoidal category $(\mathscr{C}, \wedge, S)$ together with a twist isomorphism

$$\tau\colon X \wedge Y \xrightarrow{\cong} Y \wedge X,$$

natural in both X and Y, making the monoidal product coherently commutative. A symmetric monoidal category is *closed* if the functor $L\colon \mathscr{C} \to \mathscr{C}$ given by $L\colon X \mapsto X \wedge Y$ has a right adjoint $R\colon Z \mapsto F(Y, Z)$ for all objects Y of $\mathscr{C}$. In this case, we refer to $F(X, Y)$ as the *internal hom*, and there are isomorphisms

$$\theta = \theta_{X,Z}\colon \mathscr{C}(X \wedge Y, Z) \cong \mathscr{C}(X, F(Y, Z)),$$

natural in all three variables. We let $\eta\colon X \to F(Y, X \wedge Y)$ denote the unit of this adjunction, and $\epsilon\colon F(Y, Z) \wedge Y \to Y$ the counit. Note that the counit has the appearance of an evaluation map, and we think of it as such. We recall the following fact about adjunctions.

Proposition 1.2.1 ([Lan71, Theorem IV.1.1])**.** *Let* $R\colon \mathscr{D} \leftrightarrows \mathscr{C} : L$ *be a pair of adjoint functors, with* L *left adjoint and* R *right adjoint. The right adjoint of any morphism* $f\colon LX \to Z$ *is the composite*

$$\tilde{f}\colon X \xrightarrow{\eta} RLX \xrightarrow{Rf} RZ.$$

Conversely, the left adjoint of any morphism $\tilde{f}\colon X \to RZ$ *is the composite*

$$f\colon LX \xrightarrow{L\tilde{f}} LRZ \xrightarrow{\epsilon} Z.$$

Remark 1.2.2. Our choice of notation is inspired by the stable homotopy category of spectra, where $\wedge$ would denote the smash product, S the sphere spectrum and $F(X, Y)$ the function spectrum of spectra X and Y. In the derived category of an abelian category, we would write $\otimes$ for the monoidal unit and Hom for the internal hom.

Following [LMS86, §III.1], we construct some general nonsense morphisms central to categorical duality theory. Before we begin, we define what we mean by duality.

Definition 1.2.3. The *dual* of an object X of a closed symmetric monoidal category $(\mathscr{C}, \wedge, S, F; \tau)$ is the object $DX := F(X, S)$.

For the rest of this section we fix a closed symmetric monoidal category $(\mathscr{C}, \wedge, S, F; \tau)$. Applying the adjunction twice to the morphism

$$\epsilon\colon F(X \wedge Y, Z) \wedge X \wedge Y \to Z$$

gives an isomorphism

$$F(X \wedge Y, Z) \xrightarrow{\cong} F(X, F(Y, Z)).$$

With $X \wedge X'$ as the fixed object, the right adjoint of the composite

$$L(F(X, Y), F(X', Y')) \xrightarrow{\mathrm{id}\wedge\tau\wedge\mathrm{id}} F(X, Y) \wedge X \wedge F(X', Y') \wedge X' \xrightarrow{\epsilon\wedge\epsilon} Y \wedge Y'$$

defines a pairing

$$\wedge\colon F(X, Y) \wedge F(X', Y') \longrightarrow F(X \wedge X', Y \wedge Y').$$

With $\eta\colon Z \to F(S, Z \wedge S)$, this pairing specialises to a natural morphism

$$\nu\colon F(X, Y) \wedge Z \xrightarrow{\mathrm{id}\wedge\eta} F(X, Y) \wedge F(S, Z) \xrightarrow{\wedge} F(X, Y \wedge Z),$$

recalling that $F(S, Z \wedge S) \cong F(S, Z)$. Finally, the right adjoint of the composite

$$X \wedge F(X, S) \xrightarrow{\tau} F(X, S) \wedge X \xrightarrow{\epsilon} S$$

defines a natural morphism

$$\rho\colon X \longrightarrow F(F(X, S), S).$$

Expressing this using the notation of duals, this is a morphism $\rho\colon X \to DDX$. Moreover, taking $Y = S$ in ν, we get a morphism $\nu\colon DX \wedge X \to F(X, X)$.

With all of these morphisms in hand, we are ready to classify certain objects as "dualisable". This notion also appears in the literature as "strongly dualisable" or "finite", as in [LMS86, Definition III.1.1].

Definition 1.2.4. An object X of a $\mathscr{C}$ is *dualisable* if the canonical morphism

$$\nu\colon DX \wedge X \longrightarrow F(X,X)$$

is an isomorphism in $\mathscr{C}$.

Dualisable objects enjoy several nice properties, and we summarise the ones we need in the next proposition.

Proposition 1.2.5 ([LMS86, Proposition III.1.3])**.** *If X is a dualisable object of $\mathscr{C}$, then $\rho\colon X \to DDX$ is an isomorphism. If either X or Z are dualisable and Y is any object of $\mathscr{C}$, then*

$$\nu\colon F(X,Y) \wedge Z \longrightarrow F(X, Y \wedge Z)$$

is an isomorphism. When $Y = S$, this isomorphism specialises to an isomorphism $\nu\colon DX \wedge Z \to F(X,Z)$.

Assume now that $\mathscr{C}$ also carries a triangulation. Looking to the derived category and the stable homotopy category, it is clear that the suspension and the monoidal product interact in rich ways. We reproduce the axioms of [HPS97, A.2], which attempts to describe this interaction. May proposes more axioms in [May01, §4], and demonstrates how a subset of his axioms imply these ones.

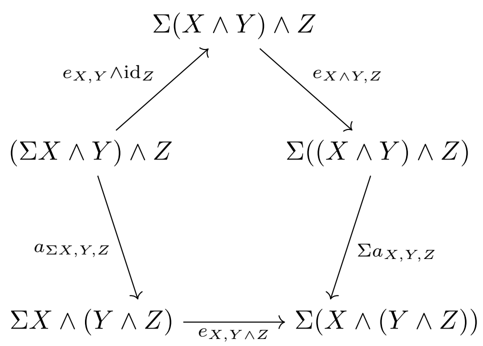
We follow the convention that suspensions happen on the left, so that the suspension of an object X should be the object $\Sigma X = S^1 \wedge X$. Here we shall mean $S^1 := \Sigma S$. More generally, we define $S^n := \Sigma^n S$.

Definition 1.2.6 ([HPS97, Definition A.2.1])**.** Let $(\mathscr{C}, \wedge, S, F)$ be a closed symmetric monoidal category with a triangulation. We say that the triangulation is *compatible* with the closed symmetric monoidal structure if:

i) There is a natural isomorphism

$$e_{X,Y}\colon \Sigma X \wedge Y \xrightarrow{\cong} \Sigma(X \wedge Y)$$

for each object X and Y of $\mathscr{C}$. We require that e is compatible with the associativity and unit morphisms in the sense that $\Sigma r_X \circ e_{X,S} = r_{\Sigma X}$, and that the following diagram commutes



ii) For each distinguished triangle $(X, Y, Z; f, g, h)$ and object W of $\mathscr{C}$, the triangles

$$X \wedge W \xrightarrow{f \wedge \mathrm{id}} Y \wedge W \xrightarrow{g \wedge \mathrm{id}} Z \wedge W \xrightarrow{h \wedge \mathrm{id}} \Sigma(X \wedge W),$$

and

$$\Sigma^{-1}F(X,W) \xrightarrow{-F(h,\mathrm{id})} F(Z,W) \xrightarrow{F(g,\mathrm{id})} F(Y,W) \xrightarrow{F(f,\mathrm{id})} F(X,W)$$

are distinguished. Here we identify $\Sigma X \wedge W$ and $\Sigma(X \wedge W)$ through $e_{X,W}$ in the first triangle, and we identify $F(\Sigma X, W)$ and $\Sigma^{-1}F(X,W)$ through the adjoint of e in the second triangle.

iii) The following diagram is commutative for each integer m and n:

$$\begin{array}{ccc} S^m \wedge S^n & \xrightarrow{\cong} & S^{m+n} \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle(-1)^{mn}} \\ S^n \wedge S^m & \xrightarrow{\cong} & S^{n+m}. \end{array}$$

Specifically, the monoidal product interacts with suspensions in a graded-commutative manner.

The natural isomorphism $e_{S,X}$ gives a natural isomorphism

$$\sigma\colon S^1 \wedge X = \Sigma S \wedge X \xrightarrow{e_{S,X}} \Sigma(S \wedge X) = \Sigma X,$$

and this again determines a natural isomorphism $\Sigma X \wedge Y \cong \Sigma(X \wedge Y)$. As these identifications essentially reduce to associativity of the monoidal product, we often suppress them. When the right-hand side term of the product is the one suspended, we use the twist to construct a natural isomorphism $e^{\tau}_{X,Y}\colon X \wedge \Sigma Y \to \Sigma(X \wedge Y)$ making the diagram

$$\begin{array}{ccc} X \wedge \Sigma Y & \xrightarrow{e^{\tau}_{X,Y}} & \Sigma(X \wedge Y) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\Sigma\tau} \\ \Sigma Y \wedge X & \xrightarrow{\cong} & \Sigma(Y \wedge X). \end{array}$$

commutative. Given a distinguished triangle $(X, Y, Z; f, g, h)$ and an object W of $\mathscr{C}$, then the top row of the following diagram is distinguished

$$\begin{array}{ccccccc} X \wedge W & \xrightarrow{f\wedge\mathrm{id}} & Y \wedge W & \xrightarrow{g\wedge\mathrm{id}} & Z \wedge W & \xrightarrow{h\wedge\mathrm{id}} & \Sigma(X \wedge W) \\ {\scriptstyle\tau}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\tau} & & {\scriptstyle\cong}\downarrow{\scriptstyle\tau} & & {\scriptstyle\cong}\downarrow{\scriptstyle\Sigma\tau} \\ W \wedge X & \xrightarrow{\mathrm{id}\wedge f} & W \wedge Y & \xrightarrow{\mathrm{id}\wedge g} & W \wedge Z & \xrightarrow{e^{\tau}(\mathrm{id}\wedge h)} & \Sigma(W \wedge X). \end{array}$$

The vertical arrows exhibit an isomorphism of triangles, hence the bottom row is also a distinguished triangle.

Our final construction for now demonstrates how we extend the homological functor $\mathscr{C}(S, -)$ to a functor $\pi_*\colon \mathscr{C} \to \mathrm{grAb}$ taking values in the category of graded abelian groups. Given a non-negative integer n and an object X of $\mathscr{C}$, we define

$$\pi_n(X) := \mathscr{C}(S, \Sigma^{-n}X) = \mathscr{C}(\Sigma^n S, X) = \mathscr{C}(S^n, X).$$

As we assemble these groups we get a graded abelian group $\pi_*(X)$.

With all the theory in place, we return to the setting of the previous section. Consider distinguished triangles

$$A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} \Sigma A$$

and

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

of $\mathscr{C}$. Now that $\mathscr{C}$ also carries a product, we take the opportunity to use it. In particular, taking the product of each object of the first triangle with the second triangle in turn, and vice versa, we get a commutative diagram

$$\begin{array}{ccccccc}
A \wedge X & \xrightarrow{\mathrm{id}\wedge f} & A \wedge Y & \xrightarrow{\mathrm{id}\wedge g} & A \wedge Z & \xrightarrow{\mathrm{id}\wedge h} & A \wedge \Sigma X \\
{\scriptstyle i\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle i\wedge\mathrm{id}} & & \downarrow{\scriptstyle i\wedge\mathrm{id}} & & \downarrow{\scriptstyle i\wedge\mathrm{id}} \\
B \wedge X & \xrightarrow{\mathrm{id}\wedge f} & B \wedge Y & \xrightarrow{\mathrm{id}\wedge g} & B \wedge Z & \xrightarrow{\mathrm{id}\wedge h} & B \wedge \Sigma X \\
{\scriptstyle j\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle j\wedge\mathrm{id}} & & \downarrow{\scriptstyle j\wedge\mathrm{id}} & & \downarrow{\scriptstyle j\wedge\mathrm{id}} \\
C \wedge X & \xrightarrow{\mathrm{id}\wedge f} & C \wedge Y & \xrightarrow{\mathrm{id}\wedge g} & C \wedge Z & \xrightarrow{\mathrm{id}\wedge h} & C \wedge \Sigma X \\
{\scriptstyle k\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle k\wedge\mathrm{id}} & & \downarrow{\scriptstyle k\wedge\mathrm{id}} & & \downarrow{\scriptstyle k\wedge\mathrm{id}} \\
\Sigma A \wedge X & \xrightarrow{\mathrm{id}\wedge f} & \Sigma A \wedge Y & \xrightarrow{\mathrm{id}\wedge g} & \Sigma A \wedge Z & \xrightarrow{\mathrm{id}\wedge h} & \Sigma A \wedge \Sigma X.
\end{array}$$

Neither the rows nor the columns of this diagram are distinguished triangles. We remedy this through the identifications above. Explicitly, we replace $\Sigma A \wedge X$ with $\Sigma(A \wedge X)$, $\Sigma A \wedge Y$ with $\Sigma(A \wedge Y)$ and so on without further ado, so that the columns all become distinguished triangles. To make the rows distinguished, we use the isomorphisms e^τ. After we make these replacements, we get the following diagram of distinguished rows and columns, where the squares commute apart from the one in the bottom right, which commutes up to the sign -1.

$$\begin{array}{ccccccc}
A \wedge X & \xrightarrow{\mathrm{id}\wedge f} & A \wedge Y & \xrightarrow{\mathrm{id}\wedge g} & A \wedge Z & \xrightarrow{e^\tau(\mathrm{id}\wedge h)} & \Sigma(A \wedge X) \\
{\scriptstyle i\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle i\wedge\mathrm{id}} & & \downarrow{\scriptstyle i\wedge\mathrm{id}} & & \downarrow{\scriptstyle \Sigma(i\wedge\mathrm{id})} \\
B \wedge X & \xrightarrow{\mathrm{id}\wedge f} & B \wedge Y & \xrightarrow{\mathrm{id}\wedge g} & B \wedge Z & \xrightarrow{e^\tau(\mathrm{id}\wedge h)} & \Sigma(B \wedge X) \\
{\scriptstyle j\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle j\wedge\mathrm{id}} & & \downarrow{\scriptstyle j\wedge\mathrm{id}} & & \downarrow{\scriptstyle \Sigma(j\wedge\mathrm{id})} \\
C \wedge X & \xrightarrow{\mathrm{id}\wedge f} & C \wedge Y & \xrightarrow{\mathrm{id}\wedge g} & C \wedge Z & \xrightarrow{e^\tau(\mathrm{id}\wedge h)} & \Sigma(C \wedge X) \\
{\scriptstyle k\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle k\wedge\mathrm{id}} & & \downarrow{\scriptstyle k\wedge\mathrm{id}} & \ominus & \downarrow{\scriptstyle -\Sigma(k\wedge\mathrm{id})} \\
\Sigma(A \wedge X) & \xrightarrow{\Sigma(\mathrm{id}\wedge f)} & \Sigma(A \wedge Y) & \xrightarrow{\Sigma(\mathrm{id}\wedge g)} & \Sigma(A \wedge Z) & \xrightarrow{-\Sigma(e^\tau(\mathrm{id}\wedge h))} & \Sigma^2(A \wedge X)
\end{array}$$

Note that we add signs to the morphisms in the lower right to ensure that the triangles are distinguished. That the bottom right square anti-commutes is due to the two different suspensions involved. We differentiate them by adding a decorative dot above one of them in the following diagram.

$$\begin{array}{ccccc}
C \wedge Z & \xrightarrow{\mathrm{id}\wedge h} & C \wedge \dot{\Sigma} X & \xrightarrow{e^\tau_{X,C}} & \dot{\Sigma}(C \wedge X) \\
{\scriptstyle k\wedge\mathrm{id}}\downarrow & & {\scriptstyle k\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \Sigma(k\wedge\mathrm{id})} \\
\Sigma(A \wedge Z) & \xrightarrow[\Sigma(\mathrm{id}\wedge h)]{} & \Sigma(A \wedge \dot{\Sigma} X) & \xrightarrow[e^\tau_{X,\Sigma A}]{} & \dot{\Sigma}\Sigma(A \wedge X)
\end{array}$$

This diagram commutes by naturality. We focus on the lower horizontal morphism in the right square. As we shift one suspension over the other, we twist the product $S^1 \wedge S^1$, introducing a sign. In particular, we see that

$$e^{\tau}_{X,\Sigma A} \colon \Sigma(A \wedge \dot{\Sigma} X) \xrightarrow{\cong} \dot{\Sigma}\Sigma(A \wedge X)$$

is the negative of

$$\Sigma e^{\tau}_{X,A} \colon \Sigma(A \wedge \dot{\Sigma} X) \xrightarrow{\cong} \Sigma\dot{\Sigma}(A \wedge X).$$

The diagram of distinguished triangles above appears in [May01], and Andrews and Miller derive a push-lift result in the associated homotopy groups for this diagram in [AM17, Lemma 9.3.2]. We take a slightly different approach, proving a similar result under the assumption that a certain fill-in morphism exists.

We begin by dualising the first distinguished triangle (i, j, k). This leads to a distinguished triangle

$$\Sigma^{-1} DA \xrightarrow{-Dk} DC \xrightarrow{Dj} DB \xrightarrow{Di} DA.$$

If we rotate this triangle $3n + 1$ times, then we get a distinguished triangle

$$\Sigma^n DC \xrightarrow{(-1)^n \Sigma^n Dj} \Sigma^n DB \xrightarrow{(-1)^n \Sigma^n Di} \Sigma^n DA \xrightarrow{(-1)^n \Sigma^{1+n} Dk} \Sigma^{1+n} DC.$$

It is this distinguished triangle that will serve as our jumping-off point. Applying the various homological and cohomological functors as we did in the last section, this triangle combines with the triangle (f, g, h) to construct a commutative diagram consisting of exact columns and rows. This time around, the diagram is the one in Figure 1.2. Suppose that there are morphisms $\alpha \colon \Sigma^n DC \to X$ and

$$\begin{array}{ccccccc}
\mathscr{C}(\Sigma^{1+n} DC, X) & \to & \mathscr{C}(\Sigma^{1+n} DC, Y) & \to & \mathscr{C}(\Sigma^{1+n} DC, Z) & \to & \mathscr{C}(\Sigma^{1+n} DC, \Sigma X) \\
\big\downarrow ((-1)^n \Sigma^{1+n} Dk)^* & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathscr{C}(\Sigma^n DA, X) & \longrightarrow & \mathscr{C}(\Sigma^n DA, Y) & \longrightarrow & \mathscr{C}(\Sigma^n DA, Z) & \longrightarrow & \mathscr{C}(\Sigma^n DA, \Sigma X) \\
\big\downarrow ((-1)^n \Sigma^n Di)^* & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathscr{C}(\Sigma^n DB, X) & \longrightarrow & \mathscr{C}(\Sigma^n DB, Y) & \longrightarrow & \mathscr{C}(\Sigma^n DB, Z) & \longrightarrow & \mathscr{C}(\Sigma^n DB, \Sigma X) \\
\big\downarrow ((-1)^n \Sigma^n Dj)^* & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathscr{C}(\Sigma^n DC, X) & \xrightarrow{f_*} & \mathscr{C}(\Sigma^n DC, Y) & \xrightarrow{g_*} & \mathscr{C}(\Sigma^n DC, Z) & \xrightarrow{h_*} & \mathscr{C}(\Sigma^n DC, \Sigma X)
\end{array}$$

Figure 1.2: Horizontal and vertical exact sequences associated to the distinguished triangles. The unlabeled arrows are the ones of the corresponding row or column.

$\beta \colon \Sigma^n DB \to Y$ such that $f \circ \alpha = \beta \circ (-1)^n \Sigma^n Dj$. This corresponds to elements of $\mathscr{C}(\Sigma^n DC, X)$ and $\mathscr{C}(\Sigma^n DB, Y)$ with common image in $\mathscr{C}(\Sigma^n DC, Y)$, all of this happening in the lower left corner of the Figure 1.2. Moreover, these data fit in a commutative diagram

$$\begin{array}{ccccccc}
\Sigma^n DC & \xrightarrow{(-1)^n\Sigma^n Dj} & \Sigma^n DB & \xrightarrow{(-1)^n\Sigma^n Di} & \Sigma^n DA & \xrightarrow{(-1)^n\Sigma^{1+n} Dk} & \Sigma^{1+n} DC \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X.
\end{array}$$

Here both the rows are distinguished, so there is a fill-in $\gamma\colon \Sigma^n DA \to Z$ making all the squares commute. This implies that there is a push-lift cycle, tracing a path alternatingly touching the diagonal and the subdiagonal of the diagram, taking α from the bottom left corner to $\Sigma\alpha$ in the top right. If we assume that the objects of the triangle $(A, B, C; i, j, k)$ are dualisable, then the next lemma shows how we might translate between the diagram in Figure 1.2, and the diagram we get as the products of the two triangles.

Lemma 1.2.7. *Let A and X be objects of $\mathscr{C}$, and assume that A is dualisable. For each integer $n \geq 0$, there is a preferred isomorphism*

$$\pi_n(A \wedge X) \xrightarrow{\cong} \mathscr{C}(\Sigma^n DA, X),$$

natural in A and X.

Proof. If A is dualisable, then $\rho\colon A \to DDA$ and $\nu\colon DAA \wedge X \to F(DA, X)$ are both isomorphisms, hence the composite

$$A \wedge X \xrightarrow{\rho\wedge\mathrm{id}} DDA \wedge X \xrightarrow{\nu} F(DA, X).$$

is an isomorphism. The closed structure provides an isomorphism

$$\mathscr{C}(S^n, F(DA, X)) \cong \mathscr{C}(S^n \wedge DA, X),$$

where we get $S^n \wedge DA \cong \Sigma^n DA$ by repeatedly applying $\sigma\colon S^1 \wedge DA \to \Sigma DA$. Recalling that $\pi_n(A \wedge X) = \mathscr{C}(S^n, X \wedge A)$ by definition, this combines to the desired isomorphism

$$\pi_n(A \wedge X) \xrightarrow{\nu_*(\rho\wedge\mathrm{id})_*} \mathscr{C}(S^n, F(DA, X)) \xrightarrow{\cong} \mathscr{C}(\Sigma^n DA, X). \qquad \square$$

Taking the cue of this lemma, we get an isomorphic version of Figure 1.2, where we identify $\Sigma^{1+n}DC$ with $\Sigma^n D\Sigma^{-1}C$ before we apply the isomorphism.

$$\begin{array}{ccccccc}
\pi_n(\Sigma^{-1}C \wedge X) & \longrightarrow & \pi_n(\Sigma^{-1}C \wedge Y) & \longrightarrow & \pi_n(\Sigma^{-1}C \wedge Z) & \longrightarrow & \pi_n(\Sigma^{-1}C \wedge \Sigma X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_n(A \wedge X) & \longrightarrow & \pi_n(A \wedge Y) & \longrightarrow & \pi_n(A \wedge Z) & \longrightarrow & \pi_n(A \wedge \Sigma X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_n(B \wedge X) & \longrightarrow & \pi_n(B \wedge Y) & \longrightarrow & \pi_n(B \wedge Z) & \longrightarrow & \pi_n(B \wedge \Sigma X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_n(C \wedge X) & \longrightarrow & \pi_n(C \wedge Y) & \longrightarrow & \pi_n(C \wedge Z) & \longrightarrow & \pi_n(C \wedge \Sigma X)
\end{array}$$

We still have to determine the homomorphisms. We walk through the identifications. Take $x \in \pi_n(A \wedge X)$, and note that the composite

$$\tilde{x}'\colon S^n \xrightarrow{x} A \wedge X \xrightarrow{\rho\wedge\mathrm{id}} DDA \wedge X \xrightarrow{\nu} F(DA, X)$$

is left adjoint to $x'\colon S^n \wedge DA \to X$. This describes chasing x from the bottom left corner of the following diagram up to the top of the left column, where the diagram commutes by naturality.

$$\begin{array}{ccc}
\mathscr{C}(\Sigma^n DA, X) & \xrightarrow{(-1)^n(\Sigma^n Di)^*} & \mathscr{C}(\Sigma^n DB, X) \\
\| & & \| \\
\mathscr{C}(S^n \wedge DA, X) & \xrightarrow{(-1)^n(\mathrm{id}\wedge Di)^*} & \mathscr{C}(S^n \wedge DB, X) \\
\theta\downarrow\cong & & \cong\downarrow\theta \\
\mathscr{C}(S^n, F(DA, X)) & \xrightarrow{(-1)^n(F(Di,\mathrm{id}))_*} & \mathscr{C}(S^n, F(DB, X)) \\
\nu_*\uparrow\cong & & \cong\uparrow\nu_* \\
\mathscr{C}(S^n, DDA \wedge X) & \xrightarrow{(-1)^n(DDi\wedge\mathrm{id})_*} & \mathscr{C}(S^n, DDB \wedge X) \\
(\rho\wedge\mathrm{id})_*\uparrow\cong & & \cong\uparrow(\rho\wedge\mathrm{id})_* \\
\mathscr{C}(S^n, A \wedge X) & \xrightarrow{(-1)^n(i\wedge\mathrm{id})_*} & \mathscr{C}(S^n, B \wedge X) \\
\| & & \| \\
\pi_n(A \wedge X) & \xrightarrow{(-1)^n(i\wedge\mathrm{id})_*} & \pi_n(B \wedge X)
\end{array}$$

Mapping x' across and moving the result back down along the right column, we deduce that $(-1)(\Sigma^n Di)^*\colon \mathscr{C}(\Sigma^n DA, X) \to \mathscr{C}(\Sigma^n DB, X)$ corresponds to $(-1)^n(i \wedge \mathrm{id})_*\colon \pi_n(A \wedge X) \to \pi_n(B \wedge X)$ under the preferred identifications of domains and codomains. Repeating these identifications for the remaining homomorphisms, we obtain the commutative diagram below, where the vertical homomorphisms all carry the sign $(-1)^n$:

$$\begin{array}{ccccccc}
\pi_n(\Sigma^{-1}C \wedge X) & \to & \pi_n(\Sigma^{-1}C \wedge Y) & \to & \pi_n(\Sigma^{-1}C \wedge Z) & \to & \pi_n(\Sigma^{-1}C \wedge \Sigma X) \\
{\scriptstyle(\Sigma^{-1}k\wedge\mathrm{id})_*}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_n(A \wedge X) & \longrightarrow & \pi_n(A \wedge Y) & \longrightarrow & \pi_n(A \wedge Z) & \longrightarrow & \pi_n(A \wedge \Sigma X) \\
{\scriptstyle(i\wedge\mathrm{id})_*}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_n(B \wedge X) & \longrightarrow & \pi_n(B \wedge Y) & \longrightarrow & \pi_n(B \wedge Z) & \longrightarrow & \pi_n(B \wedge \Sigma X) \\
{\scriptstyle(j\wedge\mathrm{id})_*}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_n(C \wedge X) & \xrightarrow{(\mathrm{id}\wedge f)_*} & \pi_n(C \wedge Y) & \xrightarrow{(\mathrm{id}\wedge g)_*} & \pi_n(C \wedge Z) & \xrightarrow{(\mathrm{id}\wedge h)_*} & \pi_n(C \wedge \Sigma X)
\end{array}$$

Now let $a \in \pi_n(C \wedge X)$ be the element corresponding to $\alpha \in \mathscr{C}(\Sigma^n DC, X)$ under the induced identification of groups, and let $a' \in \pi_n(\Sigma^{-1}C \wedge \Sigma X)$ be the element corresponding to $\Sigma\alpha \in \mathscr{C}(\Sigma^{1+n}DC, \Sigma X)$. It is clear that there is a push-lift cycle connecting these elements. However, it is not clear what the exact relationship between them is. This depends on the way we identify $\pi_n(C \wedge X) \cong \pi_{1+n}(\Sigma(C \wedge X))$. Time not permitting a thorough exploration of this, we give a handwavy summary of the above in the next proposition.

Proposition 1.2.8. *Let $(A, B, C; i, j, k)$ and $(X, Y, Z; f, g, h)$ be distinguished triangles and suppose that the objects A, B and C are all dualisable. Assume given a commutative diagram*

$$\begin{array}{ccccccc}
\Sigma^n DC & \xrightarrow{(-1)^n \Sigma^n Dj} & \Sigma^n DB & \xrightarrow{(-1)^n \Sigma^n Di} & \Sigma^n DA & \xrightarrow{(-1)^n \Sigma^{1+n} Dk} & \Sigma^{1+n} DC \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\
X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\quad g \quad} & Z & \xrightarrow{\quad h \quad} & \Sigma X,
\end{array}$$

and let $a \in \pi_n(C \wedge X)$ be the element corresponding to α under the identification $\mathscr{C}(\Sigma^n DC, X) \cong \pi_n(C \wedge X)$, and $a' \in \pi_n(\Sigma^{-1} C \wedge \Sigma X)$ the element corresponding to $\Sigma\alpha$. Then there is a push-lift cycle connecting a to a'.

2 Cartan–Eilenberg Systems

In [CE56, §XV.7], Cartan and Eilenberg gave axioms for what has since become known as Cartan–Eilenberg systems. Their choice of axioms are geared towards providing a spectral sequence with certain good properties. We shall discuss this spectral sequence in Section 3.3, but rather than get ahead of ourselves, we spend this chapter introducing Cartan–Eilenberg systems and set up the theoretical framework we will be working within. We start off by giving the definition of a Cartan–Eilenberg system in the form most useful to us and derive some simple consequences of this definition. Then, we introduce exact sequences of Cartan–Eilenberg systems. This definition naturally extends the notion of exactness already present in each system, and is central to the work we do later. In the second section of this chapter, we derive a push-lift lemma for grids of exact sequences. We finish the chapter by formulating a notion of a filtration shift. It is through filtration shifts that we tie together exact sequences of Cartan–Eilenberg systems and spectral sequences in Chapter 4.

Before we go on, we detour briefly into filtrations and filtered objects. These serve as a nice source of examples and will play an important role when we discuss the convergence of spectral sequences. Working with collections of filtered objects, bifiltered objects and spectral sequences, there are usually enough indices around to cause confusion. As such, we do most of our work in the category of graded abelian groups, which lets us hide at least one of the indices. The objects of this category are sequences $X = (X_t)_t$ of abelian groups, where we call $t \in \mathbb{Z}$ the *internal degree* for how it usually stays internal and out of sight. The morphisms are sequences of homomorphisms of abelian groups.

Definition 2.0.1. A (descending) *filtration* $(F^sX)_s$ of a graded abelian group X is a sequence

$$\cdots \subset F^{s+1}X \subset F^sX \subset F^{s-1}X \subset \cdots \subset X$$

of inclusions, where the *filtration degree* s runs through the integers. A graded abelian group X is *filtered* if it comes equipped with a filtration.

A familiar and rudimentary example of a filtration is that of a pair (X, A) of topological spaces, with A a subspace of X. Given such a pair, there are various long exact sequences that might let us use our understanding of A to learn about X. Letting the filtrations grow longer builds on this idea. We split an object into increasingly granular pieces, learn about the pieces, and then attempt to assemble what we learn to knowledge about that object.

To formalise this idea, let X be a graded abelian group with filtration $(F^sX)_s$. For each filtration degree s there is a short exact sequence

$$0 \longrightarrow F^{s+1}X \longrightarrow F^sX \longrightarrow \frac{F^sX}{F^{s+1}X} \longrightarrow 0 \tag{2.0.2}$$

including $F^{s+1}X$ into F^sX, followed by the canonical projection onto the *filtration quotient* $F^sX/F^{s+1}X$. This sequence expresses the graded group F^sX as an extension of the filtration quotient by the subsequent group $F^{s+1}X$.

Definition 2.0.3. Given a filtered graded abelian group X, we define the *associated graded* of X as the graded abelian group $\mathrm{Gr}^s X := F^s X / F^{s+1} X$.

With a sufficiently good filtration, the associated graded inductively determine X through the extensions (2.0.2). We return to this in more detail in the context of spectral sequences. For now, we content ourselves with hinting at what we might mean by a sufficiently good filtration.

Definition 2.0.4. The filtration $(F^s X)_s$ of a graded abelian group X is *exhaustive* if the canonical morphism $\operatorname{colim}_s F^s X \to X$ is an isomorphism. It is *bounded* if $F^s X = 0$ for s sufficiently large and $F^s X = X$ for s sufficiently small.

Finally, a word on how homomorphisms of graded abelian groups interact with filtrations. If X and Y are graded abelian groups filtered by $(F^s X)_s$ and $(F^s Y)_s$, respectively, then a morphism $f\colon X \to Y$ is *filtration-preserving* if $f(F^s X) \subset F^s(Y)$ for each integer s.

All of the above generalises to any abelian category. For example, in the category of chain complexes of abelian groups, a filtration of a chain complex X_* is a decreasing sequence $(F^s X_*)_s$ of subcomplexes of X_*.

2.1 Cartan–Eilenberg Systems

Cartan and Eilenberg introduced their axioms in the context of cohomology and cohomological spectral sequences. Our definition is essentially the one given by Douady in [Dou59], which is geared towards applications in the Adams spectral sequence. Consequently, our indexing scheme is homotopical rather than cohomological.

Let an *extended integer* be an element of the set $\mathbb{Z}$ of integers with symbols $-\infty$ and ∞ added in. We equip the extended integers with the usual ordering of integers, but extended to have $-\infty$ minimal and ∞ maximal.

Definition 2.1.1. A (homotopical) *Cartan–Eilenberg system* (π_*, η, ∂) consists of graded abelian groups $\pi_*(i,j)$ for each pair (i,j) of extended integers with $i \leq j$, structure morphisms

$$\eta\colon \pi_*(i,j) \longrightarrow \pi_*(i',j')$$

preserving the degree for all pairs (i,j) and (i',j') of extended integers with $i \leq j$ and $i' \leq j'$ satisfying $i' \leq i$ and $j' \leq j$, and connecting morphisms

$$\partial\colon \pi_*(i,j) \longrightarrow \pi_{*-1}(j,k)$$

reducing the degree by 1 for all extended integers $i \leq j \leq k$. These must satisfy the following:

i) (Functoriality) The morphism $\eta\colon \pi_*(i,j) \to \pi_*(i,j)$ is the identity, and the composition

$$\eta \circ \eta\colon \pi_*(i,j) \longrightarrow \pi_*(i',j') \longrightarrow \pi_*(i'',j'')$$

equals $\eta\colon \pi_*(i,j) \to \pi_*(i'',j'')$ for all extended integers $i \leq j$, $i' \leq j'$ and $i'' \leq j''$ with $i'' \leq i' \leq i$ and $j'' \leq j' \leq j$.

ii) (Naturality) For all pairs of triples $i \le j \le k$ and $i' \le j' \le k'$ of extended integers with $i' \le i$, $j' \le j$ and $k' \le k$, the diagram

$$\begin{array}{ccc} \pi_*(i,j) & \xrightarrow{\partial} & \pi_{*-1}(j,k) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ \pi_*(i',j') & \xrightarrow{\partial} & \pi_{*-1}(j',k') \end{array}$$

is commutative.

iii) (Exactness) The sequence

$$\cdots \longrightarrow \pi_*(j,k) \xrightarrow{\eta} \pi_*(i,k) \xrightarrow{\eta} \pi_*(i,j) \xrightarrow{\partial} \pi_{*-1}(j,k) \longrightarrow \cdots$$

is exact for all extended integers $i \le j \le k$.

iv) (Colimit) The canonical homomorphism

$$\operatorname*{colim}_i \pi_*(i,j) \xrightarrow{\cong} \pi_*(-\infty, j)$$

is an isomorphism for each extended integer j.

To simplify notation later on, we will often use the shorthand $\pi_*(i)$ to mean the group $\pi_*(i,\infty)$.

Before we give some examples of Cartan–Eilenberg systems, we discuss some easy implications of the axioms. First among these is that $\pi_*(i,i) = 0$ for any extended integer i. This follows from the exactness axiom providing an exact sequence

$$\pi_{*+1}(i,i+1) \xrightarrow{\text{id}} \pi_{*+1}(i,i+1) \xrightarrow{\partial} \pi_*(i,i) \xrightarrow{\eta} \pi_*(i,i+1) \xrightarrow{\text{id}} \pi_*(i,i+1),$$

where the first and last morphisms are identities by functoriality. Exactness implies that $\partial = 0$, so that $\operatorname{im}(\eta) = \ker(\eta) = 0$. This is only possible if $\pi_*(i,i)$ is trivial in each degree.

Lemma 2.1.2. *Given a Cartan–Eilenberg system (π_*, η, ∂) and extended integers $i' \le i \le j \le k \le k'$, the connecting morphism $\partial\colon \pi_*(i,j) \to \pi_{*-1}(j,k)$ factors as both*

$$\pi_*(i,j) \xrightarrow{\partial'} \pi_{*-1}(j,k') \xrightarrow{\eta'} \pi_{*-1}(j,k)$$

and

$$\pi_*(i,j) \xrightarrow{\eta''} \pi_*(i',j) \xrightarrow{\partial''} \pi_{*-1}(j,k).$$

Proof. Two applications of naturality gives a commutative diagram

$$\begin{array}{ccc} & & \pi_{*-1}(j,k') \\ & \overset{\partial'}{\nearrow} & \downarrow{\scriptstyle\eta'} \\ \pi_*(i,j) & \xrightarrow{\partial} & \pi_{*-1}(j,k) \\ {\scriptstyle\eta''}\downarrow & \underset{\partial''}{\nearrow} & \\ \pi_*(i',j), & & \end{array}$$

whence the conclusion follows. ∎

We now construct some Cartan–Eilenberg systems.

Example 2.1.3. Given a chain complex X_*, a subcomplex A_* of X_*, and an extended integer i, there is a Cartan–Eilenberg system (π_*, η, ∂) with

$$\pi_*(i) = H_*(X), \quad \pi_*(i+1) = H_*(A), \quad \text{and} \quad \pi_*(i, i+1) = H_*(X/A).$$

All the other groups are trivial. The structure morphisms η are the maps induced on homology by the inclusion of A_* into X_* and the projection of X_* onto X_*/A_*. The connecting morphism ∂ is the connecting homomorphism from the long exact sequence in homology associated to the pair (X_*, A_*).

Example 2.1.4. Let $(F^s X_*)_s$ be an exhaustive filtration of a chain complex X_* with $F^{-\infty} X_* = X_*$ and $F^{\infty} X_* = 0$. Given two pairs (i, j) and (i', j') of extended integers with $i \leq j$, $i' \leq j'$, $i' \leq i$ and $j' \leq j$, there are horizontal short exact sequences aligning in a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^j X_* & \longrightarrow & F^i X_* & \longrightarrow & F^i X_*/F^j X_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & F^{j'} X_* & \longrightarrow & F^{i'} X_* & \longrightarrow & F^{i'} X_*/F^{j'} X_* & \longrightarrow & 0. \end{array}$$

Commutativity ensures that there is a well-defined chain map

$$h \colon F^i X_*/F^j X_* \to F^{i'} X_*/F^{j'} X_*$$

connecting the filtration quotients. Adding in a third extended integer k with $i, j \leq k$, there is a short exact sequence

$$0 \longrightarrow F^j X_*/F^k X_* \xrightarrow{h'} F^i X_*/F^k X_* \xrightarrow{h''} F^i X_*/F^j X_* \longrightarrow 0$$

of chain complexes. Passing to homology, h induces a homomorphism

$$\eta := H_*(h) \colon H_*(F^i X_*/F^j X_*) \longrightarrow H_*(F^{i'} X_*/F^{j'} X_*),$$

and the long exact sequence associated to the short exact sequence of chain complexes hands us a connecting homomorphism

$$\partial \colon H_*(F^i X_*/F^j X_*) \longrightarrow H_{*-1}(F^j X_*/F^k X_*).$$

In particular, defining $\pi_*(i, j) := H_*(F^i X_*/F^j X_*)$ gives a Cartan–Eilenberg system (π_*, η, ∂).

Definition 2.1.5. A *morphism* $f \colon (\pi_*, \eta, \partial) \to (\pi'_*, \eta', \partial')$ *of Cartan–Eilenberg systems* is a collection of degree-preserving homomorphisms

$$f_{i,j} \colon \pi_*(i, j) \longrightarrow \pi'_*(i, j)$$

commuting with both the structure and connecting morphisms for each pair of extended integers (i, j) with $i \leq j$. Explicitly, we require that the two diagrams

$$\begin{array}{ccc} \pi_*(i, j) & \xrightarrow{f_{i,j}} & \pi'_*(i, j) \\ \eta \downarrow & & \downarrow \eta' \\ \pi_*(i', j') & \xrightarrow{f_{i',j'}} & \pi'_*(i', j'), \end{array} \qquad \begin{array}{ccc} \pi_*(i, j) & \xrightarrow{f_{i,j}} & \pi'_*(i, j) \\ \partial \downarrow & & \downarrow \partial' \\ \pi_*(j, k) & \xrightarrow{f_{j,k}} & \pi'_*(j', k'). \end{array}$$

commute for all pairs of triples $i \leq j \leq k$ and $i' \leq j' \leq k'$ of extended integers with $i' \leq i$, $j' \leq j$ and $k' \leq k$.

Now suppose that we have a composable pair of morphisms of Cartan–Eilenberg systems. As both morphisms are collections of homomorphisms of graded abelian groups, it makes sense to ask whether a composite of such homomorphisms is exact at their common graded group. The next definition pinpoints the case when exactness occurs across all the homomorphisms.

Definition 2.1.6. A sequence

$$(\pi'_*, \eta', \partial') \xrightarrow{f} (\pi_*, \eta, \partial) \xrightarrow{g} (\pi''_*, \eta'', \partial'')$$

of Cartan–Eilenberg systems is *exact at* (π_*, η, ∂) if the sequence

$$\pi'_*(i,j) \xrightarrow{f} \pi_*(i,j) \xrightarrow{g} \pi''_*(i,j)$$

of graded abelian groups is exact at $\pi_*(i,j)$ for all pairs of extended integers (i,j) with $i \leq j$. A *long exact sequence of Cartan–Eilenberg systems* is a sequence of Cartan–Eilenberg systems exact at each position.

Example 2.1.7. Consider a morphism $f\colon X_* \to Y_*$ of filtered chain complexes and form the mapping cone Cf_* as in Definition A.1.3. We equip the mapping cone with a filtration $(F^sCf)_s$ making the inclusion $i\colon Y_* \to Cf_*$ filtration-preserving and each sequence

$$F^sY_* \xrightarrow{i} F^sCf_* \xrightarrow{q} F^s\Sigma X_* \cong F^sX_{*-1}$$

short exact, where q is the chain map projection $Cf_* \to \Sigma X_*$. In each filtration degree, the restriction $f^s\colon F^sX_* \to F^sY_*$ of f to the subcomplex F^sX_* has mapping cone Cf^s_* as a subcomplex of Cf_*. Moreover, the componentwise inclusions

$$Cf^{s+1}_n \cong F^{s+1}Y_n \oplus F^{s+1}X_{n-1} \subseteq F^sY_n \oplus F^sX_{n-1} \cong Cf^s_n$$

ensures that $Cf^{s+1}_* \subseteq Cf^s_*$ for each s. Defining $F^sCf := Cf^s$ thus gives the desired filtration of Cf_*. Following Example 2.1.4, we associate Cartan–Eilenberg systems

$$\begin{aligned}
\pi'_*(i,j) &:= H_*(F^iY/F^jY, \eta', \partial'),\\
\pi_*(i,j) &:= H_*(F^iCf/F^jCf, \eta, \partial),\\
\pi''_*(i,j) &:= H_*(F^i\Sigma X/\Sigma X^j, \eta'', \partial'')
\end{aligned}$$

to Y_*, Cf_* and ΣX_*, respectively. Then the sequence

$$(\pi'_*, \eta', \partial') \xrightarrow{f_*} (\pi_*, \eta, \partial) \xrightarrow{i_*} (\pi''_{*-1}, \eta'', \partial'')$$

is an exact sequence of Cartan–Eilenberg systems, as

$$H_*(F^iY/F^jY) \xrightarrow{i_*} H_*(F^iCf/F^jCf) \xrightarrow{q_*} H_*(F^iX/F^jX)$$

is exact at $H_*(F^iY/F^jY)$ for each $i \leq j$.

The final definition for now lets us suspend a Cartan–Eilenberg system. Essentially, this reduces to shifting the components of the graded groups, but we get a sign on the connecting morphism. The sign is necessary to ensure that the suspended Cartan–Eilenberg system of a filtered chain complex or spectrum is the Cartan–Eilenberg system associated to the suspension of that object.

Definition 2.1.8. The *suspension* of a Cartan–Eilenberg system (π_*, η, ∂) is the Cartan–Eilenberg system $(\Sigma\pi_*, \eta, \partial)$ consisting of graded abelian groups

$$\Sigma\pi_*(i,j) = \pi_{*-1}(i,j)$$

for each pair (i,j) of extended integers with $i \le j$. The structure morphisms

$$\eta\colon \Sigma\pi_*(i,j) \to \Sigma\pi_*(i',j')$$

are the shifts $\eta\colon \pi_{*-1}(i,j) \to \pi_{*-1}(i',j')$ of the structure morphisms of (π_*, η, ∂), while the connecting morphisms

$$\partial\colon \Sigma\pi_*(i,j) \to \Sigma\pi_{*-1}(j,k)$$

are given by the connecting morphisms $-\partial\colon \pi_{*-1}(i,j) \to \pi_{*-2}(j,k)$ of (π_*, η, ∂), but with a sign.

Remark 2.1.9. To elaborate on the sign convention, let (π_*, η, ∂) be the Cartan–Eilenberg system associated to a filtered chain complex X_*. In Example 2.1.7, we associated a Cartan–Eilenberg system to the suspension ΣX_* of X_*, and this is the system we want to call $\Sigma\pi_*$. This forces a sign on the connecting morphism, as we discuss at the end of Appendix A.1.

Suspending a Cartan–Eilenberg system (π_*, η, ∂) a number n times leads to the system $\Sigma^n\pi_*$ with groups $\Sigma^n\pi_*(i,j) = \pi_{*-n}(i,j)$, where the structure and connecting morphisms are shifted correspondingly, and the connecting morphisms carry the sign $(-1)^n$.

2.2 A Push-Lift Lemma

To begin this section, we take a step back and look at grids of exact sequences of abelian groups. Consider a commutative diagram

$$\begin{array}{ccccccc}
A^{s+2} & \xrightarrow{i} & A^{s+1} & \xrightarrow{i} & A^{s} & \xrightarrow{i} & A^{s-1} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\
B^{s+2} & \xrightarrow{j} & B^{s+1} & \xrightarrow{j} & B^{s} & \xrightarrow{j} & B^{s-1} \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\
C^{s+2} & \xrightarrow{k} & C^{s+1} & \xrightarrow{k} & C^{s} & \xrightarrow{k} & C^{s-1} \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\
D^{s+2} & \xrightarrow{\ell} & D^{s+1} & \xrightarrow{\ell} & D^{s} & \xrightarrow{\ell} & D^{s-1}.
\end{array}$$

Assume that each row forms a horizontal exact sequence, and that each column forms a vertical exact sequence. We shall see that commutativity and exactness is enough to identify a certain subgroup moving through isomorphic guises along a path the shape of a staircase through the diagram. To get a feel for this subgroup, let c be an element of C^{s+1} mapping to zero across the diagonal $C^{s+1} \to D^s$. If we push c into C^s along $k\colon C^{s+1} \to C^s$, then commutativity tells us that $k(c)$ is in the kernel of $h\colon C^s \to D^s$. By exactness of the vertical sequence involving C^s, there is an element $b \in B^s$ hitting $k(c)$ under $g\colon B^s \to C^s$. Now, nothing suggests that b is the only such element. Specifically, we have to control for the choices we have when picking b. If we assume that c was only determined up to images of sums of elements of C^{s+2} and B^{s+1}, then it turns out that $k(c)$ is determined up to images coming across the diagonal $B^{s+1} \to C^s$. The next lemma concerns itself with proving this.

Lemma 2.2.1 (Push-lift)**.** *The homomorphism $k\colon C^{s+1} \to C^s$ induces an isomorphism*

$$\frac{\ker(C^{s+1} \to D^s)}{\operatorname{im}(C^{s+2} \oplus B^{s+1} \to C^{s+1})} \xrightarrow{\cong} \frac{\operatorname{im}(C^{s+1} \to C^s) \cap \operatorname{im}(B^s \to C^s)}{\operatorname{im}(B^{s+1} \to C^s)}$$

Proof. We prove that the induced homomorphism is both surjective and injective. Let $c \in C^s$ satisfy $k(c') = c = g(b)$ for $c' \in C^{s+1}$ and $b \in B^s$. Then

$$hk(c') = h(c) = hg(b) = 0$$

as hg vanishes by exactness, leaving $c' \in \ker(C^{s+1} \to D^s)$. This shows that

$$\bar{k}\colon \ker(C^{s+1} \to D^s) \twoheadrightarrow \frac{\operatorname{im}(C^{s+1} \to C^s) \cap \operatorname{im}(B^s \to C^s)}{\operatorname{im}(B^{s+1} \to C^s)}$$

is surjective. To see that $\bar{k}$ has kernel $\operatorname{im}(C^{s+2} \oplus B^{s+1} \to C^{s+1})$, note that $k(c') = kg(b')$ in C^s for $b' \in B^{s+1}$ if and only if,

$$c' - g(b') \in \ker(k\colon C^{s+1} \to C^s) = \operatorname{im}(k\colon C^{s+2} \to C^{s+1}).$$

Thus $c' - g(b') = k(c'')$ for $c'' \in C^{s+2}$, as desired. □

Note that the proof of this lemma only relies on exactness. As such, the result is still valid if we only require that the squares of the diagram commute up to a sign.

The diagram permitting, these isomorphisms chain together to produce as long a sequence of isomorphic subquotients as we would like. This is not very interesting on its own. In practice, however, we often encounter long exact sequences that are periodic. A typical example is the 3-periodic long exact sequence in homology associated to a pair of topological spaces. In such cases, we can continue the sequence of isomorphisms until we meet the group we started in. This gets us an automorphism of subquotients, and a natural thing to wonder is what we can say about this automorphism.

Example 2.2.2. Let $Y_* = (Y_*, \partial)$ be a chain complex of abelian groups with subcomplexes X_* and B_*, and define $A_* := X_* \cap B_*$. In each degree n, we form the pushout $B_n \leftarrow A_n \to X_n$ of the inclusions $f'\colon A_n \to B_n$ and $i'\colon A_n \to X_n$. This is the subgroup

$$\begin{aligned} P_n &:= (B_n \oplus X_n)/\{(f'(a), -i'(a)) : a \in A_n\} \\ &= (B_n \oplus X_n)/\{(a, -a) : a \in A_n = B_n \cap X_n\} \\ &\cong B_n + X_n \end{aligned}$$

of Y_n, where the isomorphism identifying P_n and $B_n + X_n$ is the one induced by $(b, x) \mapsto b + x$. Assembling these subgroups to a subcomplex P_* of Y_* with differential $\partial^B + \partial^X$, the quotient complex Y_*/P_* fits into the following diagram of short exact sequences of chain complexes

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_* & \xrightarrow{i'} & X_* & \xrightarrow{i} & X_*/A_* & \longrightarrow & 0 \\
 & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & B_* & \xrightarrow{j'} & Y_* & \xrightarrow{j} & Y_*/B_* & \longrightarrow & 0 \\
 & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g''} & & \\
0 & \longrightarrow & B_*/A_* & \xrightarrow{k'} & Y_*/X_* & \xrightarrow{k} & Y_*/P_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Passing to homology, we combine the associated long exact sequences into a diagram where the unmarked squares commute, and the squares marked $\ominus$ anti-commute.

$$
\begin{array}{ccccccc}
H_{n+2}(Y/P) & \xrightarrow{\partial} & H_{n+1}(B/A) & \xrightarrow{k'_*} & H_{n+1}(Y/X) & \xrightarrow{k_*} & H_{n+1}(Y/P) \\
{\scriptstyle \partial}\downarrow & \ominus & {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
H_{n+1}(X/A) & \xrightarrow{\partial} & H_n(A) & \xrightarrow{i'_*} & H_n(X) & \xrightarrow{i_*} & H_n(X/A) \\
{\scriptstyle f''_*}\downarrow & & \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f''_*} \\
H_{n+1}(Y/B) & \xrightarrow{\partial} & H_n(B) & \xrightarrow{j'_*} & H_n(Y) & \xrightarrow{j_*} & H_n(Y/B) \\
{\scriptstyle g''_*}\downarrow & & \downarrow{\scriptstyle g'_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g''_*} \\
H_{n+1}(Y/P) & \xrightarrow{\partial} & H_n(B/A) & \xrightarrow{k'_*} & H_n(Y/X) & \xrightarrow{k_*} & H_n(Y/P) \\
{\scriptstyle \partial}\downarrow & \ominus & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
H_n(X/A) & \xrightarrow{\partial} & H_{n-1}(A) & \xrightarrow{i'_*} & H_{n-1}(X) & \xrightarrow{i_*} & H_{n-1}(X/A)
\end{array}
$$

Choose cycles b of B_n and x of X_n and suppose that $x = b + \partial y$ in Y_n. This determines a homology class in $H_n(Y)$ in the image of f_n with a lift to $H_n(B)$. Pushing and lifting these cycles in either direction until we get to Y_n/P_n, we have the following maps:

$$
\begin{array}{ccccccc}
 & & & & & & y + P_* \\
 & & & & & & \downarrow \\
 & & & & x & \longmapsto & x + A_* \\
 & & & & \downarrow & & \\
 & & b & \longmapsto & x = \partial y + b & & \\
 & & \downarrow & & & & \\
-y + P_* & \longmapsto & b + A_* & & & &
\end{array}
$$

Specifically, we push b down to a cycle $b + A_n$ in B_n/A_n. By assumption, this cycle is the image of $-y + P_{n+1}$ under the connecting homomorphism

$\partial\colon H_{n+1}(Y/P) \to H_n(B/A)$. Going the other way, the image $x + A_n$ of x in X_n/A_n lifts to a cycle $y + P_{n+1}$ in Y_{n+1}/P_{n+1}. What this shows, is that a sequence of six push-lift isomorphisms sends $[y]$ to $[-y]$ in $H_{n+1}(Y/P)$. As such we might suspect that the automorphism of subquotients reduces to $-\mathrm{id}$ in this case. However, the resulting sign depends on what cycle of isomorphisms we follow. If we begin instead with a cycle $a \in A_n$ satisfying $a = \partial y$, then a includes to a cycle in both X_n and B_n. These cycles again are images of $y + X_*$ and $y + B_*$, hence the automorphism takes $[y]$ to $[y]$ in $H_{n+1}(Y/P)$.

We now specialise this discussion to the case where we have a 3-periodic long exact sequence of Cartan–Eilenberg systems

$$\cdots \longrightarrow (\pi'_*, \eta', \partial') \xrightarrow{f} (\pi_*, \eta, \partial) \xrightarrow{g} (\pi''_*, \eta'', \partial'') \xrightarrow{h} (\Sigma\pi'_*, \eta', \partial') \longrightarrow \cdots .$$

Fix extended integers i, j and k satisfying $i \le j \le k$. Laying the long exact sequences internal to each system out horizontally, and aligning them with the sequences from the exact sequence of Cartan–Eilenberg systems vertically, we get a diagram

$$\begin{array}{ccccccc}
\Sigma^{-1}\pi''_{*+1}(i,j) & \xrightarrow{\partial''} & \Sigma^{-1}\pi''_*(j,k) & \xrightarrow{\eta''} & \Sigma^{-1}\pi''_*(i,k) & \xrightarrow{\eta''} & \Sigma^{-1}\pi''_*(i,j) \\
\downarrow{\scriptstyle \Sigma^{-1}h} & & \downarrow{\scriptstyle \Sigma^{-1}h} & & \downarrow{\scriptstyle \Sigma^{-1}h} & & \downarrow{\scriptstyle \Sigma^{-1}h} \\
\pi'_{*+1}(i,j) & \xrightarrow{\partial'} & \pi'_*(j,k) & \xrightarrow{\eta'} & \pi'_*(i,k) & \xrightarrow{\eta'} & \pi'_*(i,j) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\
\pi_*(i,j) & \xrightarrow{\partial} & \pi_*(j,k) & \xrightarrow{\eta} & \pi_*(i,k) & \xrightarrow{\eta} & \pi_*(i,j) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\
\pi''_*(i,j) & \xrightarrow{\partial''} & \pi''_*(j,k) & \xrightarrow{\eta''} & \pi''_*(i,k) & \xrightarrow{\eta''} & \pi''_*(i,j) \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\
\Sigma\pi'_*(i,j) & \xrightarrow{\partial'} & \Sigma\pi'_*(j,k) & \xrightarrow{\eta'} & \Sigma\pi'_*(i,k) & \xrightarrow{\eta'} & \Sigma\pi'_*(i,j).
\end{array}$$

This diagram extends in each direction, commutes, and both the columns and the rows are exact sequences. If we identify the suspended systems with their underlying systems, this diagram slots into the upper left of the diagram below, with an extra column added on the right.

$$\begin{array}{ccccccccc}
\pi''_{*+2}(i,j) & \overset{-\partial''}{\dashrightarrow} & \pi''_{*+1}(j,k) & \overset{\eta''}{\dashrightarrow} & \pi''_{*+1}(i,k) & \overset{\eta''}{\dashrightarrow} & \pi''_{*+1}(i,j) & \overset{-\partial''}{\dashrightarrow} & \pi''_*(j,k) \\
\vdots{\scriptstyle h} & \ominus & \vdots{\scriptstyle h} & & \vdots{\scriptstyle h} & & \downarrow{\scriptstyle h} & \ominus & \vdots{\scriptstyle h} \\
\pi'_{*+1}(i,j) & \overset{\partial'}{\dashrightarrow} & \pi'_*(j,k) & \overset{\eta'}{\dashrightarrow} & \pi'_*(i,k) & \xrightarrow{\eta'} & \pi'_*(i,j) & \overset{\partial'}{\dashrightarrow} & \pi'_{*-1}(j,k) \\
\vdots{\scriptstyle f} & & \vdots{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \vdots{\scriptstyle f} & & \vdots{\scriptstyle f} \\
\pi_{*+1}(i,j) & \overset{\partial}{\dashrightarrow} & \pi_*(j,k) & \xrightarrow{\eta} & \pi_*(i,k) & \overset{\eta}{\dashrightarrow} & \pi_*(i,j) & \overset{\partial}{\dashrightarrow} & \pi_{*-1}(j,k) \\
\vdots{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle g} & & \vdots{\scriptstyle g} & & \vdots{\scriptstyle g} \\
\pi''_{*+1}(i,j) & \xrightarrow{\partial''} & \pi''_*(j,k) & \overset{\eta''}{\dashrightarrow} & \pi''_*(i,k) & \overset{\eta''}{\dashrightarrow} & \pi''_*(i,j) & \overset{\partial''}{\dashrightarrow} & \pi''_{*-1}(j,k) \\
\vdots{\scriptstyle h} & \ominus & \vdots{\scriptstyle h} & & \vdots{\scriptstyle h} & & \vdots{\scriptstyle h} & \ominus & \vdots{\scriptstyle h} \\
\pi'_*(i,j) & \overset{-\partial'}{\dashrightarrow} & \pi'_{*-1}(j,k) & \overset{\eta'}{\dashrightarrow} & \pi'_{*-1}(i,k) & \overset{\eta'}{\dashrightarrow} & \pi'_{*-1}(i,j) & \overset{-\partial'}{\dashrightarrow} & \pi'_{*-2}(j,k)
\end{array}$$

The rows and columns of this diagram are still exact sequences. However, it is now only commutative up to sign, with the squares marked $\ominus$ anti-commuting. Still, we get a chain of push-lift isomorphisms. Starting in the group $\pi''_{*+1}(i,j)$ in the left column, we get the following sequence of isomorphic subquotients along the path traced by the solid arrows in the diagram:

$$\begin{aligned}
&\frac{\ker(\pi''_{*+1}(i,j) \to \pi'_{*-1}(j,k))}{\operatorname{im}(\pi''_{*+1}(i,k) \oplus \pi_{*+1}(i,j) \to \pi''_{*+1}(i,j))} \\
&\quad \xrightarrow{\cong} \frac{\operatorname{im}(\pi''_{*+1}(i,j) \to \pi''_{*}(j,k)) \cap \operatorname{im}(\pi_{*}(j,k) \to \pi''_{*}(j,k))}{\operatorname{im}(\pi_{*+1}(i,j) \to \pi''_{*}(j,k))} \\
&\quad \xleftarrow{\cong} \frac{\ker(\pi_{*}(j,k) \to \pi''_{*}(i,k))}{\operatorname{im}(\pi_{*+1}(i,j) \oplus \pi'_{*}(j,k) \to \pi_{*}(j,k))} \\
&\quad \xrightarrow{\cong} \frac{\operatorname{im}(\pi_{*}(j,k) \to \pi_{*}(i,k)) \cap \operatorname{im}(\pi'_{*}(i,k) \to \pi_{*}(i,k))}{\operatorname{im}(\pi'_{*}(j,k) \to \pi_{*}(i,k))} \\
&\quad \xleftarrow{\cong} \frac{\ker(\pi'_{*}(i,k) \to \pi_{*}(i,j))}{\operatorname{im}(\pi'_{*}(j,k) \oplus \pi''_{*+1}(i,k) \to \pi'_{*}(i,k))} \\
&\quad \xrightarrow{\cong} \frac{\operatorname{im}(\pi'_{*}(i,k) \to \pi'_{*}(i,j)) \cap \operatorname{im}(\pi''_{*+1}(i,j) \to \pi'_{*}(i,j))}{\operatorname{im}(\pi''_{*+1}(i,k) \to \pi'_{*}(i,j))} \\
&\quad \xleftarrow{\cong} \frac{\ker(\pi''_{*+1}(i,j) \to \pi'_{*-1}(j,k))}{\operatorname{im}(\pi''_{*}(i,k) \oplus \pi_{*+1}(i,j) \to \pi''_{*+1}(i,j))}
\end{aligned}$$

In particular, this sequence gives us an automorphism of

$$\frac{\ker(\Sigma\pi'_{*+1}(i,j) \to \Sigma\pi_{*}(j,k))}{\operatorname{im}(\Sigma\pi'_{*+1}(i,k) \oplus \pi''_{*+1}(i,j) \to \Sigma\pi'_{*+1}(i,j))}. \tag{2.2.3}$$

Definition 2.2.4. We say that a long exact sequence

$$\cdots \longrightarrow (\pi'_*, \eta', \partial') \xrightarrow{f} (\pi_*, \eta, \partial) \xrightarrow{g} (\pi''_*, \eta'', \partial'') \xrightarrow{h} (\Sigma\pi'_*, \eta', \partial') \longrightarrow \cdots$$

of Cartan–Eilenberg systems is *good* if the automorphism (2.2.3) of subquotients, arising as the composite of a sequence of six push-lift isomorphisms, is equal to the negative of the identity.

2.3 Filtration Shifts

Suppose that X and Y are filtered abelian groups and that $f\colon X \to Y$ is a filtration-preserving homomorphism connecting them. If we choose an element x from somewhere in the filtration of X, then its image under f lands in the same stage of the filtration of Y. However, $f(x)$ might also lift to a later stage in the filtration of Y. When this happens, we think of x as shifting up in filtration as it is mapped into Y. Whenever x itself lifts higher in the filtration of X, such a shift happens trivially. Also, if, say $f(x)$ lifts over five filtration degrees, then it also lifts over four and three and so on. As we spend this section formalising this intuitive notion of a filtration shift, we will take care to rule out these trivial cases. That is, our goal is to describe only the furthest shifts: the ones from the first occurrence of x in the filtration of X, to the last occurrence of $f(x)$ in the filtration of Y. Before we move on to the details, we visit a simple example.

Example 2.3.1. Consider the abelian group $\mathbb{Z}$ of integers under addition, and equip it with an exhaustive filtration $F^s\mathbb{Z} := 2^s\mathbb{Z}$ for $s \geq 0$, with each group consisting of multiples of integers by increasingly higher powers of 2. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be the filtration-preserving homomorphism taking each integer n to $2n$. Then f fits in a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \rightarrowtail & 2^{s+1}\mathbb{Z} & \rightarrowtail & 2^s\mathbb{Z} & \rightarrowtail & \cdots & \rightarrowtail & 2\mathbb{Z} & \rightarrowtail & \mathbb{Z} \\ & & \downarrow f & & \downarrow f & & & & \downarrow f & & \downarrow f \\ \cdots & \rightarrowtail & 2^{s+1}\mathbb{Z} & \rightarrowtail & 2^s\mathbb{Z} & \rightarrowtail & \cdots & \rightarrowtail & 2\mathbb{Z} & \rightarrowtail & \mathbb{Z}. \end{array}$$

Now take any $s \geq 0$ and $n \in 2^s\mathbb{Z}$ from the top row. By the way we constructed the filtrations, the image of n under f always shifts up a degree. In this case, we would like to say that f is a strictly filtration shifting map.

Looking ahead, we return to and discuss filtration shifts in the context of Cartan–Eilenberg systems. Consequently, we choose to formalise them in this context. Let $f\colon (\pi'_*, \eta', \partial') \to (\pi_*, \eta, \partial)$ be a morphism of Cartan–Eilenberg systems and fix an extended integer i. To reduce the number of visible indices as we introduce a bifiltration relating these two systems, we shall write $X_* := \pi'_*(i) = \pi'_*(i, \infty)$ and $Y_* := \pi_*(i) = \pi_*(i, \infty)$. We begin by defining filtrations $(F^sX_*)_s$ and $(F^sY_*)_s$ of X_* and Y_*, respectively, each consisting of increasingly longer images into X_* or Y_*. For each integer $s \geq i$, let F^sX_* be given by

$$F^sX_* := \operatorname{im}(\eta'\colon \pi'_*(s) \longrightarrow \pi'_*(i)),$$

and F^sY_* by

$$F^sY_* := \operatorname{im}(\eta\colon \pi_*(s) \longrightarrow \pi_*(i)).$$

When $s < i$, we set $F^sX_* := \pi'_*(i)$ and $F^sY_* := \pi_*(i)$. These are both exhaustive filtrations. When i is finite, this is immediate as the image of the identity homomorphism on either X_* or Y_* is part of the filtration. When $i = -\infty$, we rely on the isomorphism $\operatorname{colim}_s \pi'_*(s) \cong \pi'_*(-\infty, \infty)$ required by the axioms of a Cartan–Eilenberg system. As each element of the sequential colimit is the image along a structure morphism of some $x \in \pi'_*(s)$ where s is finite, we have

$$\operatorname*{colim}_s F^sX_* = \operatorname*{colim}_s \operatorname{im}(\eta'\colon \pi'_*(s) \longrightarrow \pi'_*(-\infty, \infty)) \cong \pi'_*(-\infty, \infty) = X_*.$$

The same argument applies to F^sY_*, only with fewer primes.

Next, we define a bifiltration of Y_* taking into account how f interacts with these filtrations. We are mapping between filtrations, so we naturally get two indices to think about: the filtration degree we are mapping out of and the one we are landing in. Before we give the definition, note that f respects these filtrations in the sense that $f(F^sX_*) \subset F^sY_*$. This follows from the commutative diagram

$$\begin{array}{ccc} \pi'_*(s) & \xrightarrow{\eta'} & \pi'_*(i) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \pi_*(s) & \xrightarrow{\eta} & \pi_*(i) \end{array}$$

$$
\begin{array}{ccc}
F_f^{s-1,s+a+1} & F_f^{s,s+a+1} & F_f^{s+1,s+a+1} \\
F_f^{s-1,s+a} & F_f^{s,s+a} & F_f^{s+1,s+a} \\
F_f^{s-1,s+a-1} & F_f^{s,s+a-1} & F_f^{s+1,s+a-1}
\end{array}
$$

Figure 2.1: The bifiltration $F_f^{s,s+a}Y_*$ of Y_* depicted in the $(s, s+a)$-plane.

implied by f being a morphism of Cartan–Eilenberg systems. For each pair of integers s and a with $a \geq 0$, let

$$F_f^{s,s+a}Y_* := f(F^sX_*) \cap F^{s+a}Y_*$$

be the subgroup of Y_* in filtration degree $(s, s+a)$ of a bifiltration $(F_f^{s,s+a}Y_*)_{s,a}$ of Y_*. When $s \leq i$, the group in bifiltration $(s, s+a)$ is simply the image subgroup $\mathrm{im}(f)$ in Y_*. When $s + a \leq i$, it is the image subgroup of the composite $f \circ (\eta' \colon \pi'_*(s) \to \pi'_*(i))$. Finally, when $s, s+a \geq i$, we have

$$F_f^{s,s+a}Y_* = f(\mathrm{im}(\eta' \colon \pi'_*(s) \to \pi'_*(i))) \cap \mathrm{im}(\eta \colon \pi_*(s+a) \to \pi_*(i)).$$

We proceed to inspect the various filtration quotients. As we fix the second index and let the first run through the integers, we get a sequence

$$\cdots \subseteq \frac{F_f^{s+1,s+a}Y_*}{F_f^{s+1,s+a+1}Y_*} \subseteq \frac{F_f^{s,s+a}Y_*}{F_f^{s,s+a+1}Y_*} \subseteq \frac{F_f^{s-1,s+a}Y_*}{F_f^{s-1,s+a+1}Y_*} \subseteq \cdots$$

of subquotients of $F^{s+a}Y_*/F^{s+a+1}Y_*$, where each group in the sequence are quotients of vertically adjacent groups of Figure 2.1. Letting the second index vary, we get a filtration of the associated graded

$$\mathrm{Gr}^sY_* = \frac{F^sY_*}{F^{s+1}Y_*}$$

of Y_*. The filtration quotients of this filtration are the groups

$$\frac{F_f^{s,s+a}Y_*}{F_f^{s,s+a+1}Y_* + F^{s+1,s+a}Y_*} = \frac{f(F^sX_*) \cap F^{s+a}Y_*}{f(F^sX_*) \cap F^{s+a+1}Y_* + F^{s+1}X_* \cap F^{s+a}Y_*}.$$

These admit the interpretation we are looking for. An element y of such a quotient is the image $f(x)$ from filtration degree s, and lifts a steps in the filtration of Y_*, where neither x nor y lifts any further.

Definition 2.3.2. We say that an element of $F_f^{s,s+a}Y_*$ is a *filtration shift* from filtration degree s of F^sX_* to filtration degree $s+a$ of F^sY_*.

3 Spectral Sequences

Spectral sequences are neat pieces of algebraic machinery. Conceived by Leray during the second world war and reworked into their modern form by Kozsul some years later, they have since become an invaluable tool in algebraic topology and homological algebra. At their core, spectral sequences look like generalisations of exact sequences, and they are often used for the same purpose. Where spectral sequences and exact sequences differ is in the amount of data they handle. Exact sequences usually busy themselves managing pairs or triples of objects, while spectral sequences happily accepts objects broken up into infinitely fine pieces. This freedom to deconstruct objects comes at a cost, however, and to keep the machine running we often need our objects to be richly structured, or to reach for clever tricks. To develop such a trick is precisely the goal of the next chapter.

This chapter gives an introduction to spectral sequences and our preferred way of constructing them using exact couples. We briefly discuss convergence to ensure that the spectral sequences we use actually compute what we want them to compute. To finish the chapter, we introduce the spectral sequences arising from Cartan–Eilenberg systems, which are the ones we are most interested in.

Our spectral sequences will be sequences of bigraded objects, so before we give the definition we need to establish some terminology. A *bigraded abelian group* $X = X^{*,*}$ is a sequence $X^{*,*} = (X^{s,t})_{s,t}$ of abelian groups indexed over pairs of integers (s,t). We refer to such pairs of integers as *bidegrees*. A *morphism* $f\colon X \to Y$ of bigraded abelian groups is a sequence of group homomorphisms

$$f^{s,t}\colon X^{s,t} \longrightarrow Y^{s,t}$$

for all $s,t \in \mathbb{Z}$. More generally, a morphism $f\colon X \to Y$ of *bidegree* (u,v) is a sequence of group homomorphisms

$$f^{s,t}\colon A^{s,t} \longrightarrow B^{s+u,t+v}$$

for all $s,t \in \mathbb{Z}$. Composing f with a morphism $g\colon Y \to Z$ of bidegree (u',v') gives a morphism $gf\colon X \to Z$ of bidegree $(u+u', v+v')$. We say that a morphism of bidegree $(0,0)$ is *degree-preserving*.

Definition 3.0.1. A *spectral sequence* is a sequence $(E_r)_{r\geq 1}$ of bigraded abelian groups $E_r = E_r^{*,*}$, along with the following data:

i) For every integer $r \geq 1$, a morphism $d_r\colon E_r \to E_r$ of bidegree $(r, r-1)$ satisfying $d_r \circ d_r = 0$.

ii) For every integer $r \geq 1$, an isomorphism $E_{r+1} \cong H(E_r, d_r)$, identifying E_{r+1} with the homology of E_r with respect to d_r, i.e.

$$E_{r+1}^{s,t} \cong \ker(d_r^{s,t}) / \operatorname{im}(d_r^{s-r,t-r+1}).$$

We call E_r the *E_r-term* of the spectral sequence, and d_r the *d_r-differential.*

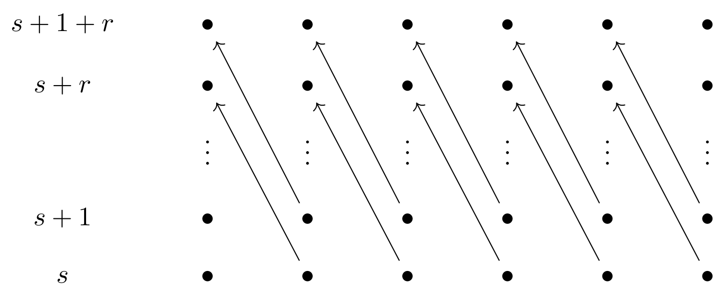


Figure 3.1: The E_r-term of the spectral sequence in the $(t-s,s)$-plane

This definition assigns roles to three indices represented by the letters r, s and t. The index r determines the term of the spectral sequence and the destination of the differentials. The indices s and t give the bidegree of the objects E_r, and we call s the *filtration degree* and t the *internal degree*. The difference $t-s$ is the *total degree*. Given a class x of the E_r-term with total degree $t-s$, the d_r differential maps x to a class of total degree $t-s-1$, reducing the total degree by 1.

The terms of a spectral sequence are alternatively called pages. This illustrates how we might view a spectral sequence as a book, with each page of the book a term of the spectral sequence. Turning the page reflects how we move from one term onto the next by taking homology. The analogy extends to how we often depict spectral sequences; drawing the terms one at a time into planar diagrams as in Figure 3.1. If we let the total degree $t-s$ increase along the horizontal axis towards the right, and the filtration degree s along the vertical axis and upwards, then a d_r differential takes one step to the left and r steps up.

Remark 3.0.2. There are many other ways to index spectral sequences. We use Adams type grading with a view towards applications in the Adams spectral sequence. Other common grading schemes include homological and cohomological spectral sequences. In a homological spectral sequence, we write the E^r terms using homological indexing, and the differentials $d^r_s\colon E^r_s \to E^r_{s-r}$ have bidegree $(-r, r-1)$.

We can identify each subsequent term of a spectral sequence with a quotient of subgroups of the E_1-term. Let us write Z_2 for the kernel of the d_1-differential and B_2 for the image. As we identify the E_2-term with the homology of the E_1-term, the d_2-differential becomes a homomorphism $Z_2/B_2 \to Z_2/B_2$. Proceeding in this way, we identify E_r with the quotient Z_r/B_r, and find that the d_r-differential $Z_r/B_r \to Z_r/B_r$ has kernel Z_{r+1}/B_r and image B_{r+1}/B_r. Continuing inductively gives the following lemma.

Lemma 3.0.3 ([HR19, Lemma 2.2]). *For any spectral $(E_r, d_r)_{r\geq 1}$ and all integers s and $r \geq 1$, there are inclusions*

$$0 = B^s_1 \subset \cdots \subset B^s_r \subset \cdots Z^s_r \subset \cdots \subset Z^s_1 = E^s_1,$$

with $E^s_r \cong Z^s_r/B^s_r$.

Before we move on to actually constructing spectral sequences, we make sense of what it means to map between them.

Definition 3.0.4. A *morphism* $f\colon (E_r, d_r)_{r\geq 1} \to ('E_r, 'd_r)_{r\geq 1}$ *of spectral sequences* is a sequence of morphisms $f_r\colon E_r \to 'E_r$ of bigraded abelian groups, compatible with the differentials and the isomorphisms $E_{r+1} \cong H(E_r, d_r)$ and $'E_{r+1} \cong H('E_r, 'd_r)$. Explicitly, we require that the diagram

$$\begin{array}{ccc} E_r^{*,*} & \xrightarrow{f_r} & 'E_r^{*,*} \\ {\scriptstyle d_r}\downarrow & & \downarrow{\scriptstyle 'd_r} \\ E_r^{*,*} & \xrightarrow{f_r} & 'E_r^{*,*} \end{array}$$

commutes, and that f_{r+1} equals the induced homomorphism $H(f_r)$ for each $r \geq 1$, making the following diagram commutative:

$$\begin{array}{ccc} H_{*,*}(E_r, d_r) & \xrightarrow{H(f_r)} & H_{*,*}('E_r, 'd_r) \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ E_{r+1}^{*,*} & \xrightarrow{f_{r+1}} & 'E_{r+1}^{*,*}. \end{array}$$

There are various ways to construct spectral sequences. We reach for them when confronted with understanding filtered objects, and the exact couples we introduce next provides a nice stepping stone from filtrations to spectral sequences.

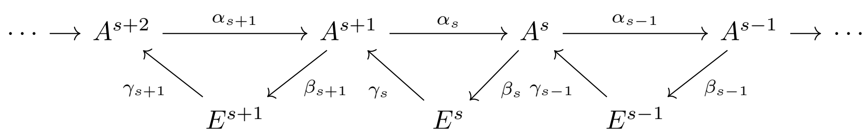
3.1 Exact Couples

Exact couples will be our tool of choice to construct spectral sequences. We follow Massey [Mas52, §1.4], but define them in their unrolled form as in Boardman's paper [Boa99, §0].

Definition 3.1.1. Let $(A^s)_s$ and $(E^s)_s$ be sequences of graded abelian groups, and suppose $\alpha_s\colon A^s \to A^{s+1}$, $\beta_s\colon A^s \to E^s$ and $\gamma_s\colon E^s \to A^{s+1}$ are graded morphisms of graded abelian groups. An *unrolled exact couple*

$$(A, E) = (A^s, E^s; \alpha_s, \beta_s, \gamma_s)_s$$

is a diagram



where each triangle forms a long exact sequence

$$\cdots \longrightarrow A^{s+1} \xrightarrow{\alpha_s} A^s \xrightarrow{\beta_s} E^s \xrightarrow{\gamma_s} A^{s+1} \longrightarrow \cdots .$$

The internal degrees of the morphisms α_s, β_s and γ_s are -1, 0 and 0, respectively, so that α_s has bidegree $(-1, -1)$, β_s has bidegree $(0, 0)$ and γ_s has bidegree $(1, 0)$.

Given exact couples (A, E) and $('A, 'E)$, a morphism $f\colon (A, E) \to ('A, 'E)$ between them is a collection of degree-preserving homomorphisms $f^s\colon A^s \to 'A^s$ and $f^s\colon E^s \to 'E^s$ for each integer s making the following diagrams commutative

$$\begin{array}{ccccccc} A^{s+1} & \xrightarrow{\alpha_s} & A^s & \xrightarrow{\beta_s} & E^s & \xrightarrow{\gamma_s} & A^{s+1} \\ \downarrow{\scriptstyle f^{s+1}} & & \downarrow{\scriptstyle f^s} & & \downarrow{\scriptstyle f^s} & & \downarrow{\scriptstyle f^{s+1}} \\ 'A^{s+1} & \xrightarrow{\alpha_s} & 'A^s & \xrightarrow{\beta_s} & 'E^s & \xrightarrow{\gamma_s} & 'A^{s+1}. \end{array}$$

Theorem 3.1.2 ([McC01, Theorem 2.8])**.** *Let $(A^s, E^s; \alpha_s, \beta_s, \gamma_s)_s$ be an exact couple. There is a spectral sequence $(E_r, d_r)_{r\geq 1}$ with $E_1^s := E^s$ and $d_1^s := \beta_{s+1}\gamma_s\colon E_1^s \to E_1^{s+1}$ for all integers s. If α_s and β_s have total degree 0 and γ_s has total degree -1, then*

$$d_r^{s,t}\colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$$

has bidegree $(r, r-1)$, where $E_r^{s,t} = (E_r^s)_{t-s}$ is a subquotient of $E_1^{s,t} = (E^s)_{t-s}$.

We sketch the construction of this spectral sequence. Given an exact couple $(A, E; \alpha, \beta, \gamma)$ and an integer $r \geq 1$, we define the *rth cycle group* as

$$Z_r^s := \gamma_s^{-1}\,\mathrm{im}(\alpha^{r-1}\colon A^{s+r} \to A^{s+1}),$$

and the *rth boundary group* as

$$B_r^s = \beta_s \ker(\alpha^{r-1}\colon A^s \to A^{s-r+1}).$$

Both the cycle groups and boundary groups are graded subgroups of E^s for each s, with $E^{s,t} = (E^s)_{t-s}$ containing the groups $Z_r^{s,t}$ and $B_r^{s,t}$ in bidegree (s, t). As r increases, we get inclusions $\ker(\gamma_s) \subset Z_r^s \subset Z_{r+1}^s$ and $B_{r+1}^s \subset B_r^s \subset \mathrm{im}(\beta_s)$ for each $s \in \mathbb{Z}$, and we define

$$E_r^s := Z_r^s / B_r^s.$$

with $E_r^{s,t} := Z_r^{s,t}/B_r^{s,t}$, making $E_r = E_r^{*,*}$ the E_r-term of the spectral sequence. This settles the objects, and the group E^s of the exact couple appears as on the E_1-term as E_1^s, as desired. It remains to define an appropriate differential. If $x \in Z_r^s$ and we write $[x] \in E_r^s$ for its equivalence class modulo B_r^s, then there is a morphism

$$\begin{aligned} d_r^s\colon E_r^s &\longrightarrow E_r^{s+r} \\ [x] &\longmapsto [\beta_{s+r}(y)], \end{aligned}$$

where $y \in A^{s+r}$ is a lift of $\gamma_s(x)$ over $\alpha^{r-1}\colon A^{s+r} \to A^{s+1}$. Assuming that this is well-defined, then this reduces to $d_1^s = \beta_{s+1}\gamma_s$ when $r = 1$ in line with the theorem. The next two lemmas provide the final missing pieces.

Lemma 3.1.3. *The d_r-differential is well-defined.*

Proof. An element x of Z_r^s has image $\gamma_s(x)$ lifting over $\alpha^{r-1}\colon A^{s+r} \to A^{s+1}$ to $y \in A^{s+r}$ by definition. As Z_r^{s+r} contains the image of β_{s+r}, it follows that $\beta_{s+r}(y)$ defines a class $[\beta_{s+r}(y)]$ in E_r^{s+r}.

Now suppose $y' \in A^{s+r}$ is another element satisfying $\alpha^{r-1}(y') = \gamma_s(x)$. This places the difference $y - y'$ in $\ker(\alpha^{r-1}\colon A^{s+r} \to A^{s+1})$, hence $\beta_{s+r}(y')$ differs from $\beta_{s+r}(y)$ by an element in B_r^{s+r}. In particular, both $\beta_{s+r}(y)$ and $\beta_{s+r}(y')$ represent the same class in E_r^{s+r}. Finally, an element $x' \in Z_r^s$ representing the same class as x in E_r^s differs from x by an element $x - x' \in B_r^s$. It follows from the inclusion $B_r^s \subset \ker(\gamma_s)$ that the image of x and x' coincide under γ_s, so that y remains a valid choice of lift. □

Lemma 3.1.4. $\ker(d_r^s) = Z_{r+1}^s / B_r^s$ *and* $\operatorname{im}(d_r^{s-r}) = B_{r+1}^s / B_r^s$.

Proof. To establish the first equality, take $[x] \in \ker(d_r^s)$ represented by $x \in Z_r^s$ with $\gamma_s(x) = \alpha^{r-1}(y)$ in A^{s+1} for a choice of $y \in A^{s+r}$. That $d_r^s([x]) = 0$ implies that $\beta_{s+r}(y) \in B_r^{s+r}$, giving an element $y' \in \ker(\alpha^{r-1}) \subset A_{s+r}$ satisfying $\beta_{s+r}(y) = \beta_{s+r}(y')$. Then $y - y'$ defines an element of $\ker(\beta_{s+r}) = \operatorname{im}(\alpha_{s+r+1})$ with image $\gamma_s(x)$ in A^{s+1}, so that $x \in Z_{r+1}^s$. This proves the inclusion $\ker(d_r^s) \subset Z_{r+1}^s / B_r^s$. Conversely, suppose $x \in Z_{r+1}^s$, such that $\gamma_s(x) = \alpha^r(z)$ for $z \in A^{s+r+1}$. Then $\alpha_{s+r}(z)$ defines an element of A^{s+r} with image $\gamma_s(x)$ in A^{s+1} and trivial image under β_{s+r}, hence $x \in d_r^s([x]) = [0]$.

Next, consider $x \in Z_r^{s-r}$ with $\gamma_{s-r}(x) = \alpha^{r-1}(y)$ in A^{s-r+1} for some $y \in A^s$, so that $[\beta_s(y)] \in \operatorname{im}(d_r^{s-r})$. Then $\alpha^{r-1}(y) \in \ker(A^{s-r+1} \to A^{s+r})$ by exactness, so that $\beta_s(y) \in B_s^{r+1}$. To prove the opposite inclusion, choose $y \in A^s$ with $\alpha^r(y) = 0$ in A^{s-r} such that $\beta_s(y) \in B_{r+1}^s$. Then the image of y in A^{s-r+1} defines an element of $\ker(\alpha_{s-r}) = \operatorname{im}(\gamma_{s-r})$, and there exists $x \in Z_r^{s-r}$ with $d_r^{s-r}([x]) = [\beta_s(y)]$, as desired. □

So where does this leave us? We have constructed a sequence of bigraded objects $(E_r)_s$ and well-defined morphisms $d_r\colon E_r \to E_r$ of the right bidegree $(r, r-1)$. Moreover, we can conclude that $d_r \circ d_r = 0$ as the inclusion of B_{r+1}^s into Z_{r+1}^s implies that $(\operatorname{im}(d_r))_s \subset (\ker(d_r))_s$ by the last lemma. Finally, the Noether isomorphism

$$Z_{r+1}^s / B_{r+1}^s \xrightarrow{\cong} \frac{Z_{r+1}^s / B_r^s}{B_{r+1}^s / B_r^s} = \frac{\ker(d_r^s)}{\operatorname{im}(d_r^{s-r})}$$

shows that projecting from Z_{r+1}^s onto $\ker(d_r^s)$ induces an isomorphism $E_{r+1} \cong (H(E_r, d_r))_s$. Thus we conclude that we have indeed constructed a spectral sequence.

3.2 Convergence

We need a notion of convergence to ensure that a spectral sequence is actually computing what we want it to compute. A well-behaved spectral sequence renders information about its target through increasingly better approximations. The first issue we have to deal with is whether this target is the right one, and this is tightly wound up in how we filter the objects we pass to the spectral sequence. The second issue is whether the spectral sequence terminates. This often happens in practice, as there is typically something finite about the data we feed the spectral sequence, and the differentials will simply all vanish after a certain point. In this case all later terms of the spectral sequence are equal, and we say that the spectral sequence *collapses*. Other times, there are no such restrictions on our data, and we need a more granular description of convergence.

Recall that for any spectral sequence $(E_r, d_r)_{r\geq 1}$ there are inclusions

$$0 = B_1^s \subset \cdots \subset B_r^s \subset \cdots Z_r^s \subset \cdots \subset Z_1^s = E_1^s,$$

and the E_r-term is isomorphic to the quotient Z_r/B_r. When the spectral sequence is the spectral sequence associated to an exact couple, Lemma 3.1.4 tells us that the subgroups Z_r^s and B_r^s of E_1^s are precisely the ones we defined in the previous section.

Definition 3.2.1 ([Boa99, §5])**.** Let $(E_r, d_r)_{r\geq 1}$ be a spectral sequence. Define the group of *infinite cycles* as the limit

$$Z_\infty^s := \lim_{r\geq 1} Z_r^s = \bigcap_{r\geq 1} Z_r^s,$$

and the group of *infinite boundaries* as the colimit

$$B_\infty^s := \operatorname*{colim}_{r\geq 1} B_r^s = \bigcup_{r\geq 1} B_r^s.$$

From this definition we deduce inclusions $B_\infty^s \subset Z_\infty^s$, $B_\infty^s \subset B_r^s$ and $Z_\infty^s \subset Z_r^s$ for all integers s and $r \geq 1$. We define the E_∞*-term* of the spectral sequence as the bigraded group $E_\infty = (E_\infty^s)_s = E_\infty^{*,*}$ with

$$E_\infty^s := Z_\infty^s / B_\infty^s$$

for each $s \in \mathbb{Z}$. Now the point of convergence is to relate this E_∞-term to the filtration $(F^s X)_s$ of some group X. Before we can do this, we need another piece of language. To this end, view the integers $\mathbb{Z}$ along with the usual order relation $\geq$ as a category $(\mathbb{Z}, \geq)$. Explicitly, this is a category with the integers as objects and hom-sets $\mathbb{Z}(m, n)$ that contain a single morphism if $m \geq n$, and that are empty otherwise. Let $\mathrm{Ab}^{\mathbb{Z}}$ denote the category of functors $\mathbb{Z} \to \mathrm{Ab}$. Recall that the categorical limit $\lim\colon \mathrm{Ab}^{\mathbb{Z}} \to \mathrm{Ab}$ defines a functor right adjoint to the constant diagram functor. As such, lim is left exact and has a right derived functor. We denote this functor by Rlim and construct it as the cokernel of a certain morphism, following [Wei94, §3.5]. Given a sequence $(A^s)_s$ of graded abelian groups

$$\cdots \longrightarrow A^{s+1} \xrightarrow{\alpha_s} A^s \xrightarrow{\alpha_{s-1}} A^{s-1} \longrightarrow \cdots,$$

we define a homomorphism

$$(\mathrm{id} - \alpha)\colon \prod_s A^s \longrightarrow \prod_s A^s.$$

Here α is the homomorphism making the next diagram commute, where the vertical arrows are the canonical projections onto the corresponding factor of the direct product.

$$\begin{array}{ccc} \prod_s A^s & \xrightarrow{\alpha} & \prod_s A^s \\ \downarrow & & \downarrow \\ A^{s+1} & \xrightarrow{\alpha_s} & A^s \end{array}$$

The kernel of this homomorphism is the limit $\lim_s A^s$, and we define Rlim as the cokernel $\operatorname{Rlim}_s A^s := \operatorname{coker}(\mathrm{id} - \alpha)$. This results in an exact sequence

$$0 \to \lim_s A^s \longrightarrow \prod_s A^s \xrightarrow{\mathrm{id}-\alpha} \prod_s A^s \longrightarrow \operatorname{Rlim}_s A^s \to 0.$$

Definition 3.2.2. A filtration $(F^s X)_s$ of a graded abelian group X is *Hausdorff* if $\lim_s F^s X = 0$. It is *complete* if $\operatorname{Rlim}_s F^s X = 0$.

Definition 3.2.3 ([Boa99, Definition 5.2])**.** A spectral sequence $(E_r, d_r)_{r\geq 1}$ *converges* to a filtration $(F^s X)_s$ of a graded abelian group X if there are isomorphisms

$$\operatorname{Gr}^s X = F^s X / F^{s+1} X \cong E^s_\infty$$

for each integer s, and the filtration is exhaustive and Hausdorff. If the filtration is also complete, then the spectral sequence *converges strongly.*

Even if a spectral sequence $(E_r, d_r)_r$ converges strongly to a filtered object X and we know the E_∞-term, we might still not be able to determine X. This is because the spectral sequence only determines the associated graded, and we would need to uniquely solve all the extension problems

$$0 \to F^{s+1} X \longrightarrow F^s X \longrightarrow \operatorname{Gr}^s X \to 0$$

to have certain information about X itself.

Remark 3.2.4. Technically, we are abusing language when we say that a spectral sequence converges (strongly) to a certain object. Convergence is really additional information and not a property of the spectral sequence. However, as this is a natural thing to say, we shall continue doing so.

We now return to the spectral sequences arising from exact couples. If $(A, E; \alpha, \beta, \gamma)$ is an exact couple, then we say that the sequence

$$\cdots \longrightarrow A^{s+2} \xrightarrow{\alpha_{s+1}} A^{s+1} \xrightarrow{\alpha_s} A^s \xrightarrow{\alpha_{s-1}} A^{s-1} \longrightarrow \cdots$$

is *bounded above* if $A^k = 0$ for k sufficiently large. We say that it is *degreewise bounded above* if for each total degree n there is an integer $k(n)$ such that $(A^s)_n = 0$ for all $s > k(n)$. A natural target for the associated spectral sequence is the sequential colimit $A^{-\infty} := \operatorname{colim}_s A^s$. If we let $\iota_s \colon A^s \to A^{-\infty}$ denote the structure morphisms into the colimit, then

$$F^s A^{-\infty} := \operatorname{im}(\iota_s \colon A^s \to A^{\infty})$$

defines an exhaustive filtration of A^∞. To see this, note that any $y \in A^\infty$ is the image $\iota_s(x)$ of some $x \in A^s$ for some integer s. Then $y \in F^s A^{-\infty}$, and $\bigcup_s F^s A^{-\infty} = A^{-\infty}$.

Theorem 3.2.5 ([Boa99, Theorem 8.10])**.** *Let (A, E) be an exact couple with associated spectral sequence $(E_r, d_r)_{r\geq 1}$ and assume that the sequence*

$$\cdots \longrightarrow A^{s+2} \xrightarrow{\alpha_{s+1}} A^{s+1} \xrightarrow{\alpha_s} A^s \xrightarrow{\alpha_{s-1}} A^{s-1} \longrightarrow \cdots$$

is degreewise bounded above. Then the spectral sequence converges strongly to the filtration $F^s A^{-\infty} = \operatorname{im}(\iota_s \colon A^s \to A^{-\infty})$.

3.3 Spectral Sequences and Cartan–Eilenberg Systems

At this point we have seen all we need to see of general spectral sequences, and we return to the Cartan–Eilenberg systems introduced in the previous chapter. As we hinted at then, a Cartan–Eilenberg system gives rise to a spectral sequence much in the same way as an exact couple. In fact, when we construct the spectral sequence associated to a Cartan–Eilenberg system we go through a certain exact couple. Several reasons justify constructing spectral sequences in this way. A Cartan–Eilenberg system carries more structure than an exact couple, and this extra structure lets us express the terms and differentials of the associated spectral sequences more concretely. Also, although we will not get into this here, we can pair Cartan–Eilenberg systems in a way that induces a pairing of spectral sequences. This way, we may endow our spectral sequences with multiplicative structure, which can be very valuable in calculations.

The content of this section follows the unpublished notes of Rognes [Rog21].

Definition 3.3.1. To each Cartan–Eilenberg system (π_*, η, ∂) we associate an exact couple $(A^s, E^s)_s$ with $(A^s)_* = \pi_*(s, \infty)$ and $(E^s)_* = \pi_*(s, s+1)$. The morphisms α_s and β_s are given by η, and γ_s is given by ∂.

$$\cdots \longrightarrow \pi_*(s+1, \infty) \xrightarrow{\eta} \pi_*(s, \infty) \longrightarrow \cdots$$
$$\pi_*(s, \infty) \xrightarrow{\eta} \pi_*(s, s+1), \qquad \pi_*(s, s+1) \xrightarrow{\partial} \pi_*(s+1, \infty)$$

The spectral sequence associated to a Cartan–Eilenberg system is the spectral sequence associated to the exact couple $(A^s, E^s; \eta, \eta, \partial)_s$.

The next proposition reveals how the extra structure present in a Cartan–Eilenberg system lets us give more concrete descriptions of the internals of the spectral sequence.

Proposition 3.3.2. *The spectral sequence associated to a Cartan–Eilenberg system (π_*, η, ∂) satisfies the following:*

i) The rth cycle group is given by

$$\begin{aligned} Z_r^s &= \partial^{-1} \operatorname{im}(\eta\colon \pi_{*-1}(s+r, \infty) \to \pi_{*-1}(s+1, \infty)) \\ &= \ker(\partial\colon \pi_*(s, s+1) \to \pi_{*-1}(s+1, s+r)) \\ &= \operatorname{im}(\eta\colon \pi_*(s, s+r) \to \pi_*(s, s+1)). \end{aligned}$$

ii) The rth boundary group is given by

$$\begin{aligned} B_r^s &= \eta \ker(\eta\colon \pi_*(s, \infty) \to \pi_*(s-r+1, \infty)) \\ &= \operatorname{im}(\partial\colon \pi_{*+1}(s-r+1, s) \to \pi_*(s, s+1)) \\ &= \ker(\eta\colon \pi_*(s, s+1) \to \pi_*(s-r+1, s+1)). \end{aligned}$$

In particular, η induces an isomorphism

$$E_r^s \xrightarrow{\cong} \operatorname{im}(\eta\colon \pi_*(s, s+r) \to \pi_*(s-r+1, s+1))$$

The d_r-differential is given by $d_r^s\colon E_r^s \to E_r^{s+r}$ sending $[x] \in E_r^s$ represented by $x \in \pi_*(s, s+1)$ to $\partial(z) \in \pi_{*-1}(s+r, s+r+1)$, where $z \in \pi_*(s, s+r)$ is a lift of x over η.

Proof. We first prove the identities i) and ii), relying on Lemma 2.1.2 to factor the connecting morphisms.

i) Exactness gives the first equality in the following computation:

$$\begin{aligned}\partial^{-1}\operatorname{im}(\eta\colon \pi_{*-1}(s+r,\infty) \to \pi_{*-1}(s+1,\infty))\\ = \partial^{-1}\ker(\eta\colon \pi_{*-1}(s+1,\infty) \to \pi_{*-1}(s+1,s+r))\\ = \ker(\partial\colon \pi_*(s,s+1) \to \pi_*(s+1,s+r)).\end{aligned}$$

Factoring $\partial\colon \pi_*(s,s+1) \to \pi_{*-1}(s+1,s+r)$ as the composite

$$\pi_*(s,s+1) \xrightarrow{\partial} \pi_{*-1}(s+1) \xrightarrow{\eta} \pi_{*-1}(s+1,s+r).$$

gives the second equality.

ii) Similarly:

$$\begin{aligned}\eta\ker(\eta\colon \pi_*(s,\infty) \to \pi_*(s-r+1,\infty)\\ = \eta\operatorname{im}(\partial\colon \pi_{*+1}(s-r+1,s) \to \pi_*(s,\infty))\\ = \operatorname{im}(\partial\colon \pi_{*+1}(s-r+1,s) \to \pi_*(s,s+1)).\end{aligned}$$

The first equality is a consequence of exactness, while the second follows from factorisation.

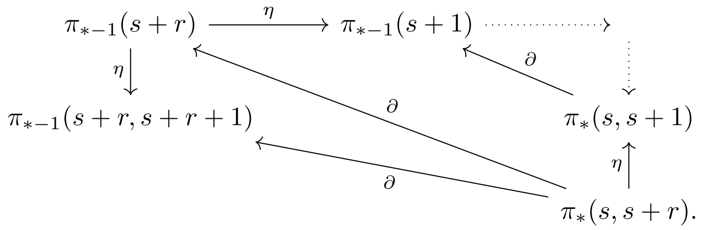
To establish the isomorphism of E_r-terms, note that $\eta\colon \pi_*(s,s+r) \to \pi_*(s-r+1,s+1)$ factors as

$$\pi_*(s,s+r) \xrightarrow{\eta'} \pi_*(s,s+1) \xrightarrow{\eta''} \pi_*(s-r+1,s+1)$$

by functoriality. Using the first part of the proof, we identify $\operatorname{im}\eta'$ as Z_r^s and $\ker\eta''$ as B_r^s. We know that $B_r^s \subset Z_r^s$, hence the canonical isomorphism $\pi_*(s,s+1)/\ker(\eta'') \to \operatorname{im}(\eta'')$ induced by η'' restricts to an isomorphism

$$E_r^s = Z_r^s/B_s^r = \operatorname{im}(\eta')/\ker(\eta'') \xrightarrow{\cong} \operatorname{im}(\eta).$$

Finally, suppose $x \in \pi_*(s,s+1)$ is contained in Z_r^s. Then there is $z \in \pi_*(s,s+r)$ such that $\eta(z) = x$, and naturality of the solid square in the diagram below gives an element $y := \partial z \in \pi_{*-1}(s+r)$ with $\eta(y) = \partial x$ in $\pi_{*-1}(s+1)$. Naturality of the solid triangle implies that $\eta(y) = \partial z$ in $\pi_{*-1}(s+r,s+r+1)$, and it follows that $d_r^s([x]) = [\eta(y)] = [\partial(z)]$.



This concludes the proof. □

In similar fashion, we can say a bit more about the Z_∞-term and B_∞-term. If (π_*, η, ∂) is a Cartan–Eilenberg system, then the colimit

$$\pi_* = \pi_*(-\infty, \infty) \cong \operatorname*{colim}_s \pi_*(s, \infty)$$

is exhaustively filtered by

$$F^s\pi_* := \operatorname{im}(\eta\colon \pi_*(s, \infty) \to \pi_*(-\infty, \infty)).$$

This forms a natural target for the associated spectral sequence.

Proposition 3.3.3. *Let (π_*, η, ∂) be a Cartan–Eilenberg system with associated spectral sequence $(E_r, d_r)_{r\geq 1}$, and equip the target $\pi_* := \pi_*(-\infty, \infty)$ with its canonical filtration. The infinite boundaries of this spectral sequence are the groups*

$$\begin{aligned} B^s_\infty &= \operatorname{im}(\partial\colon \pi_{*+1}(-\infty, s) \to \pi_*(s, s+1)) \\ &= \ker(\eta\colon \pi_*(s, s+1) \to \pi_*(-\infty, s+1)). \end{aligned}$$

If the sequence

$$\cdots \longrightarrow \pi_*(s+1) \xrightarrow{\eta} \pi_*(s) \xrightarrow{\eta} \pi_*(s-1) \longrightarrow \cdots \tag{3.3.4}$$

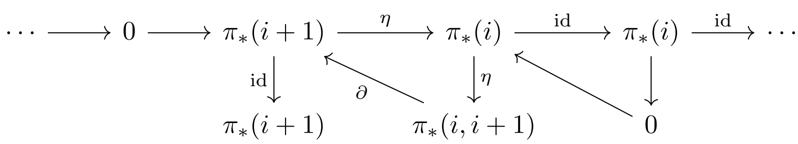
is bounded above, then the infinite cycles are the groups

$$\begin{aligned} Z^s_\infty &= \ker(\partial\colon \pi_*(s, s+1) \to \pi_{*-1}(s+1, \infty)) \\ &= \operatorname{im}(\eta\colon \pi_*(s, \infty) \to \pi_*(s, s+1)), \end{aligned}$$

and the filtration $(F^s\pi_*)_s$ is bounded above. In particular, this ensures that $E_r^{*,*}$ converges strongly to π_*.

With all of the theory and construction work out of the way, we take a look at a very simple example. Starting with a Cartan–Eilenberg system, we shall forget all but two of the groups in the top row of the associated exact couple and study the behaviour of the spectral sequences in some detail. This example appears again in the next chapter to illustrate how we connect filtration shifts and differentials.

Example 3.3.5. Let (π_*, η, ∂) be a Cartan–Eilenberg system and fix an integer i. Extracting just the three groups $\pi_*(i)$, $\pi_*(i+1)$ and $\pi_*(i, i+1)$ from this system, we get an exact couple as follows:



Here the groups $A^{s,t} = (A^s)_t$ of the top row are given by $\pi_{t-s}(i)$ for $s \geq i$, $\pi_{t-s}(i+1)$ for $s = i+1$, and are trivial otherwise. In the bottom row, the groups $E^{s,t} = (E^s)_t$ are all trivial unless $s = i$ or $s = i+1$, in which case $E^{s,t} = \pi_{t-s}(i, i+1)$ and $\pi_{t-s}(i+1)$, respectively. We proceed to describe the associated spectral sequence $(E_r, d_r)_{r\geq 1}$ in detail. At the E_1-term, only the two groups $E^{i,t} = \pi_{t-i}(i, i+1)$ and $E^{i+1,t} = \pi_{t-i-1}(i+1)$ may be non-zero.

As such, $d_1^{i,t} = \partial\colon \pi_{t-i}(i,i+1) \to \pi_{t-i-1}(i+1)$ is the only differential that can hope to be non-zero. This makes computing the E_2-term quite simple, and we have

$$E_2^{s,t} = \begin{cases} \ker(\partial\colon \pi_{t-i}(i,i+1) \to \pi_{t-i-1}(i+1)) & \text{for } s = i \\ \operatorname{coker}(\partial\colon \pi_{t-i}(i,i+1) \to \pi_{t-i-1}(i+1)) & \text{for } s = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

At the E_2-term there is no room for further differentials, and the spectral sequence collapses. The canonical filtration of the target group $\pi_*(i)$ reduces to

$$0 \subset \cdots \subset 0 \subset \operatorname{im}(\eta\colon \pi_*(i+1) \to \pi_*(i)) \subset \pi_*(i) \subset \cdots \subset \pi_*(i),$$

which is clearly bounded above. It follows that the spectral sequence is convergent, and there are isomorphisms $E_\infty^{s,t} \cong F^s\pi_{t-s}(s)/F^{s+1}\pi_{t-s}(s)$ for each s. When $s = i$, this isomorphism is

$$\ker(\partial\colon \pi_{t-i}(i,i+1) \to \pi_{t-i-1}(i+1)) \cong \frac{\pi_{t-i}(i)}{\operatorname{im}(\eta\colon \pi_{t-i}(i+1) \to \pi_{t-i}(i))},$$

and when $s = i+1$, it is

$$\operatorname{coker}(\partial\colon \pi_{t-i}(i,i+1) \to \pi_{t-i-1}(i+1)) \cong \operatorname{im}(\eta\colon \pi_{t-i}(i+1) \to \pi_{t-i}(i)).$$

Both of these follow from the exactness axiom.

4 Filtration Shifts and Differentials

A convergent spectral sequence approaches its target one term at a time, and each term arises as the homology of the previous one. As such, it is vital that we can determine the differentials on the current term of the spectral sequence. There is no set way of doing this. Among the spectral sequences that mathematicians know how to compute, many carry extra structure like an algebra or a ring structure. This opens avenues for giving ad hoc arguments deciding a single or a series of connected differentials. In this chapter, we present an alternative method that does not rely on any such extra structure. We assume that the spectral sequence in question comes from a Cartan–Eilenberg system, and that this system sits in an exact sequence. Given this, we derive a connection between filtration shifts and differentials in the associated spectral sequences. Specifically, we describe situations in which a filtration shift leads to a non-zero differential, and vice versa. Although this approach seems promising in general, the results of this chapter only apply to restricted filtrations.

For the length of this chapter, we will be working with an exact sequence

$$(\pi'_*, \eta', \partial') \xrightarrow{f} (\pi_*, \eta, \partial) \xrightarrow{g} (\pi''_*, \eta'', \partial'') \xrightarrow{h} (\Sigma\pi'_*, \eta', \partial') \tag{4.0.1}$$

of Cartan–Eilenberg systems. Before we get too deep into the details, we attempt to motivate the approach we take towards deriving this connection between filtration shifts and differentials. To this end, let i be an integer and extract the two adjacent groups in position (i, ∞) and $(i+1, \infty)$ from each system, along with the connected group in position $(i, i+1)$. To lessen the notational burden, we relabel these graded groups as follows:

$$\begin{aligned}
A_* &:= \pi'_*(i+1), & X_* &:= \pi'_*(i), & (X,A)_* &:= \pi'_*(i,i+1),\\
B_* &:= \pi_*(i+1), & Y_* &:= \pi_*(i), & (Y,B)_* &:= \pi_*(i,i+1),\\
C_* &:= \pi''_*(i+1), & Z_* &:= \pi''_*(i), & (Z,C)_* &:= \pi''_*(i,i+1).
\end{aligned}$$

Recall that we write $\pi_*(i)$ to mean the group $\pi_*(i,\infty)$. To each of the systems we associate spectral sequences $(E_r(X), d_r)_{r\geq 1}$, $(E_r(Y), d_r)_{r\geq 1}$ and $(E_r(Z), d_r)_{r\geq 1}$. We computed these spectral sequences in Example 3.3.5. They all collapse at the second term and converge to the very short filtrations containing only the image from $i+1$ into i. In Section 2.3, we defined filtration shifts under f from i to $i+1$ as elements of the quotient

$$\frac{F_f^{i,i+1}Y_*}{F_f^{i,i+2}Y_* + F_f^{i+1,i+1}Y_*} = \frac{\operatorname{im}(f\colon X_* \to Y_*) \cap \operatorname{im}(\eta\colon B_* \to Y_*)}{\operatorname{im}(\eta f = f\eta'\colon A_* \to Y_*)}.$$

The right-hand side simplification is due to the vanishing of $F_f^{i,i+2}Y_*$, and A_* factoring through B_*. If we lay out the exact sequences internal to each system horizontally, and the exact sequences coming from (4.0.1) vertically, there is a large commutative diagram with exact columns and rows containing a piece like the one below.

$$
\begin{array}{ccccccccc}
(X,A)_{*+1} & \xrightarrow{\partial'} & A_* & \xrightarrow{\eta'} & X_* & \xrightarrow{\eta'} & (X,A)_* & \xrightarrow{\partial'} & A_{*-1} \\
\downarrow f & & \downarrow f & & \downarrow f & & \downarrow f & & \downarrow f \\
(Y,B)_{*+1} & \xrightarrow{\partial} & B_* & \xrightarrow{\eta} & Y_* & \xrightarrow{\eta} & (Y,B)_* & \xrightarrow{\partial} & B_{*-1} \\
\downarrow g & & \downarrow g & & \downarrow g & & \downarrow g & & \downarrow g \\
(Z,C)_{*+1} & \xrightarrow{\partial''} & C_* & \xrightarrow{\eta''} & Z_* & \xrightarrow{\eta''} & (Z,C)_* & \xrightarrow{\partial''} & C_{*-1} \\
\downarrow h & & \downarrow h & & \downarrow h & & \downarrow h & & \downarrow h \\
(X,A)_* & \xrightarrow{\partial'} & A_{*-1} & \xrightarrow{\eta'} & X_{*-1} & \xrightarrow{\eta'} & (X,A)_{*-1} & \xrightarrow{\partial'} & A_{*-2}
\end{array}
$$

Looking back to the quotient representing filtration shifts, it speaks of elements of Y_* in the image of both X_* and B_*. This is precisely the setting of our push-lift lemma (Lemma 2.2.1). Following through with two applications of this lemma, lifting from Y_* to B_* and pushing down to C_*, we derive an isomorphism of subquotients

$$
\begin{aligned}
&\frac{\operatorname{im}(f\colon X_* \to Y_*) \cap \operatorname{im}(\eta\colon B_* \to Y_*)}{\operatorname{im}(\eta f\colon A_* \to Y_*)} \\
&\qquad \cong \frac{\operatorname{im}(g\colon B_* \to C_*) \cap \operatorname{im}(\partial''\colon (Z,C)_{*+1} \to C_*)}{\operatorname{im}(\partial'' g\colon (Y,B)_{*+1} \to C_*)).}
\end{aligned} \tag{4.0.2}
$$

From Example 3.3.5, we know that the d_1-differential of $E_1(Z)$ is just the connecting homomorphism $\partial''\colon (Z,C)_{*+1} \to C_*$. Also, the E_1-terms in filtration degree $i+1$ are simply the groups A_*, B_* and C_*. Thus the component of the morphism $E_1(g)\colon E_1(Y) \to E_1(Z)$ mapping from $E_1^{i+1}(Y)$ to $E_1^{i+1}(Z)$ is $g\colon B_* \to C_*$ itself. That is, we have $\operatorname{im}(g\colon B_* \to C_*) = \operatorname{im}(E_1^{i+1}(g))$. This translates two of the three expressions in the target subquotient of (4.0.2) to information about the spectral sequence. The final piece of the puzzle is to recognise that the sequence

$$E_1(Y) \xrightarrow{g} E_1(Z) \xrightarrow{h} E_1(\Sigma X)$$

is exact, as the E_1-terms reduce to the corresponding groups of the Cartan–Eilenberg systems. It follows that

$$\ker(h\colon (Z,C)_{*+1} \to (X,A)_*) = \operatorname{im}(g\colon (Y,B)_{*+1} \to (Z,C)_{*+1}),$$

and this group maps onto $\operatorname{im}(\partial'' g : (Y,B)_{*+1} \to C_*)$ under $\partial''\colon (Z,C)_{*+1} \to C_*$. Putting this all together, we get an isomorphism

$$\frac{F_f^{i,i+1} Y_*}{F_f^{i+1,i+1} Y_*} \cong \frac{\operatorname{im}({}''d_1^i) \cap \operatorname{im}(E_1^{i+1}(g))}{{}''d_1^i(\ker {}''E_1^i(h)) \cap \operatorname{im}(E_1^{i+1}(g))}$$

connecting filtration shifts and differentials.

4.1 Three-Stage Filtrations

The goal of this section is to establish the promised connection between filtration shifts and differentials. When we restrict attention to filtrations with only three stages, we can give a complete description of this connection. This is the content of Theorem 4.1.7. The proof of this result follows the path outlined in the introduction to this chapter. We begin by describing a homomorphism with domain reminiscent of filtration shifts and codomain close to where we expect our differentials to land. The question is how this can be made an isomorphism and what that implies for the spectral sequences. Before we get going, we introduce the notation

$$\operatorname{im}_\eta^{i+j} \pi_*(i) := \operatorname{im}(\eta\colon \pi_*(i+j) \to \pi_*(i))$$

to describe the image in $\pi_*(i)$ of the structure morphism leaving position $(i+j, \infty)$, and

$$\operatorname{im}_\partial^{i,j} \pi_*(j) := \operatorname{im}(\partial\colon \pi_{*+1}(i,j) \to \pi_*(j))$$

the to describe the image in $\pi_*(j)$ of the connecting morphism leaving position (i,j).

Proposition 4.1.1. *Let*

$$(\pi'_*, \eta', \partial') \xrightarrow{f} (\pi_*, \eta, \partial) \xrightarrow{g} (\pi''_*, \eta'', \partial'')$$

be an exact sequence of Cartan–Eilenberg systems, and suppose i and j are integers with $j \geq 1$. The composite $g\eta = \eta'' g\colon \pi_*(i+j) \to \pi''_*(i+1)$ induces a homomorphism

$$\phi\colon \operatorname{im}(f_i) \cap \operatorname{im}^{i+j}\pi_*(i) \longrightarrow \frac{\pi''_*(i+1)}{\operatorname{im}(\partial'' g = g\partial\colon \pi_{*+1}(i,i+j) \to \pi''_*(i+1))}.$$

with $\ker\phi \subset f_i(\operatorname{im}_{\eta'}^{i+1}\pi'_*(i)) \cap \operatorname{im}_\eta^{i+j}\pi_*(i)$. Explicitly, ϕ is given by $\phi(y) = \eta'' g(\tilde{y})$ for a lift $\tilde{y}$ of $y \in \pi_*(i)$ over $\eta\colon \pi_*(i+j) \to \pi_*(i)$.

Proof. Choose $y \in \pi_*(i)$ satisfying $y = f(x)$ for $x \in \pi'_*(i)$. Any two choices of lifts of y over $\eta\colon \pi_*(i+j) \to \pi_*(i)$ differ by ∂ applied to an element of $\pi_{*+1}(i, i+j)$ by exactness of the sequence

$$\pi_{*+1}(i, i+j) \xrightarrow{\partial} \pi_*(i+j) \xrightarrow{\eta} \pi_*(i).$$

In particular, both lifts have the same image in

$$\frac{\pi''_*(i+1)}{\operatorname{im}(\pi_{*+1}(i, i+j) \to \pi''_*(i+1))}$$

under the homomorphism induced by $g\eta = \eta'' g$, hence ϕ is well-defined.

To determine the kernel, consider $y \in \ker\phi$ with a lift $\tilde{y} \in \pi_*(i+j)$. We may assume that $g\eta(\tilde{y})$ is zero in $\pi''_*(i+1)$. If not, there is an element of $\pi_{*+1}(i, i+j)$ with image $y' \in \pi_*(i+j)$ such that $g\eta(\tilde{y}) = g\eta(y')$. Then $\tilde{y} - y'$ is a lift of y mapping to zero in $\pi''_*(i+1)$, and we may replace $\tilde{y}$ with this lift without affecting the value of $\phi(y)$. Now, $g\eta(\tilde{y}) = 0$ places $\eta(\tilde{y})$ in $\ker(g) \subset \pi_*(i+1)$ by commutativity, hence $\eta(\tilde{y})$ lifts over f to an element $\tilde{x} \in \pi'_*(i+1)$ by exactness. It follows that $\eta f(\tilde{x})) = y = \eta(\tilde{y})$, as desired. □

With this homomorphism in place, we specialise our discussion to the case where we focus on a triple of filtration indices. Fixing integers i and $j \geq 1$, we consider the groups of the diagram

$$\begin{array}{ccccc}
\pi'_*(i+j) & \xrightarrow{\eta'} & \pi'_*(i+1) & \xrightarrow{\eta'} & \pi'_*(i) \\
\downarrow f & & \downarrow f & & \downarrow f \\
\pi_*(i+j) & \xrightarrow{\eta} & \pi_*(i+1) & \xrightarrow{\eta} & \pi_*(i) \\
\downarrow g & & \downarrow g & & \downarrow g \\
\pi''_*(i+j) & \xrightarrow{\eta''} & \pi''_*(i+1) & \xrightarrow{\eta''} & \pi''_*(i) \\
\downarrow h & & \downarrow h & & \downarrow h \\
\Sigma\pi'_*(i+j) & \xrightarrow{\eta'} & \Sigma\pi'_*(i+1) & \xrightarrow{\eta'} & \Sigma\pi'_*(i).
\end{array}$$

Let $(F^s\pi'_*(i))_s$, $(F^s\pi_*(i))_s$, $(F^s\pi''_*(i))_s$ and $(F^s\Sigma\pi'_*(i))_s$ denote the exhaustive filtrations of each row of the diagram, with the right-hand group receiving images from the left. Explicitly, $(F^s\pi_*(i))_s$ is the filtration

$$\begin{array}{ccccccccc}
 & & F^{i+j}\pi_*(i) & & F^{i+1}\pi_*(i) & & F^{i}\pi_*(i) & & \\
 & & \| & & \| & & \| & & \\
\cdots \subset 0 & \subset & \mathrm{im}_\eta^{i+j}\pi_*(i) & \subset & \mathrm{im}_\eta^{i+1}\pi_*(i) & \subset & \pi_*(i) & \subset & \cdots \subset \pi_*(i)
\end{array}$$

with $\pi_*(i)$ in filtration degrees less than or equal to i. As before, we take the opportunity to relabel these groups, writing $(X,K)_*$ for the group $\pi'_*(i,i+1)$, $(X,A)_*$ for the group $\pi_*(i,i+j)$, and so on:

$$\begin{aligned}
A_* &:= \pi'_*(i+j), & K_* &:= \pi'_*(i+1), & X_* &:= \pi'_*(i), \\
B_* &:= \pi_*(i+j), & L_* &:= \pi_*(i+1), & Y_* &:= \pi_*(i), \\
C_* &:= \pi''_*(i+j), & M_* &:= \pi''_*(i+1), & Z_* &:= \pi''_*(i).
\end{aligned}$$

The domain of the homomorphism ϕ from Proposition 4.1.1 is clearly reminiscent of filtration shifts under f. With our choice of integers i and j, the domain describes elements of Y_* in the image of $f\colon X_* \to Y_*$ that lift to B_*. Our first goal is to restrict ϕ to an isomorphism to a subquotient of the codomain, with the domain of ϕ expressing the actual filtration shifts

$$\frac{F_f^{i,i+j}Y_*}{F_f^{i,i+j+1}Y_* + F_f^{i+1,i+j}Y_*} = \frac{\mathrm{im}(f\colon X_* \to Y_*) \cap \mathrm{im}(\eta\colon B_* \to Y_*)}{\mathrm{im}(\eta f\colon K_* \to Y_*) \cap \mathrm{im}(\eta\colon B_* \to Y_*)}.$$

Starting with a sequence of push-lift isomorphisms beginning in the subgroup $\mathrm{im}(f\colon X_* \to Y_*) \cap \mathrm{im}(\eta\colon B_* \to Y_*)$ of Y_*, we shall derive the following sequence of isomorphic subquotients:

$$\begin{aligned}
&\frac{\operatorname{im}(f\colon X_*\to Y_*)\cap\operatorname{im}(\eta\colon B_*\to Y_*)}{\operatorname{im}(\eta f\colon K_*\to Y_*)\cap\operatorname{im}(\eta\colon B_*\to Y_*)}\\
&\qquad\xleftarrow{\cong}\frac{(\eta\colon B_*\to Y_*)^{-1}\operatorname{im}(f\colon X_*\to Y_*)}{(\eta\colon B_*\to Y_*)^{-1}\operatorname{im}(\eta f\colon K_*\to Y_*)}\\
&\qquad\xrightarrow{\cong}\frac{\operatorname{im}(g\colon B_*\to C_*)\cap\operatorname{im}(\partial\colon (Z,C)_{*+1}\to C_*)}{(g\colon B_*\to C_*)\big((\eta\colon B_*\to Y_*)^{-1}\operatorname{im}(\eta f\colon K_*\to Y_*)\big)}\\
&\qquad\xleftarrow{\cong}\frac{\operatorname{im}(g\eta\colon B_*\to M_*)\cap\operatorname{im}(\partial\colon (Z,M)_{*+1}\to M_*)}{(g\eta\colon B_*\to M_*)\big((\eta\colon B_*\to Y_*)^{-1}\operatorname{im}(\eta f\colon K_*\to Y_*)\big)}.
\end{aligned}\tag{4.1.2}$$

The first isomorphism is simply a Noether isomorphism

$$\begin{aligned}
&\frac{(\eta\colon B_*\to Y_*)^{-1}\operatorname{im}(f\colon X_*\to Y_*)/\ker(\eta\colon B_*\to Y_*)}{(\eta\colon B_*\to Y_*)^{-1}\operatorname{im}(\eta f\colon K_*\to Y_*)/\ker(\eta\colon B_*\to Y_*)}\\
&\qquad\cong\frac{\operatorname{im}(f\colon X_*\to Y_*)\cap\operatorname{im}(\eta\colon B_*\to Y_*)}{\operatorname{im}(\eta f\colon K_*\to Y_*)\cap\operatorname{im}(\eta\colon B_*\to Y_*)}.
\end{aligned}$$

We spend the next two lemmas justifying the second and third isomorphisms, the proofs of which reduce to simple diagram chases.

Lemma 4.1.3. *The homomorphism $g\colon B_*\to C_*$ induces an isomorphism*

$$\frac{(B_*\to Y_*)^{-1}\operatorname{im}(X_*\to Y_*)}{(B_*\to Y_*)^{-1}\operatorname{im}(K_*\to Y_*)}\xrightarrow{\cong}\frac{\operatorname{im}(B_*\to C_*)\cap\operatorname{im}((Z,C)_{*+1}\to C_*)}{(B_*\to C_*)\big((B_*\to Y_*)^{-1}\operatorname{im}(K_*\to Y_*)\big)}.$$

Proof. We begin by proving that g induces a surjection

$$\bar g\colon (B_*\to Y_*)^{-1}\operatorname{im}(X_*\to Y_*)\longrightarrow\frac{\operatorname{im}(B_*\to C_*)\cap\operatorname{im}((Z,C)_{*+1}\to C_*)}{(B_*\to C_*)\big((B_*\to Y_*)^{-1}\operatorname{im}(K_*\to Y_*)\big)}.$$

Given any $b\in B_*$ in the domain of $\bar g$, the image of $g(b)$ in Z_* lifts through Y_* to X_*, and must be zero by exactness of the sequence $X_*\to Y_*\to Z_*$. In particular, we have $\bar g(b)\in\ker(\eta''\colon C_*\to Z_*)=\operatorname{im}((Z,C)_{*+1}\to C_*)$. To see that $\bar g$ is surjective, let c denote an element of $\operatorname{im}(g\colon B_*\to C_*)$ mapping to zero in Z_*. If $b\in B_*$ is an element with $(g\colon B_*\to C_*)(b)=c$, then the image of b in Y_* lifts to X_* by exactness. This shows that b resides in $(\eta\colon B_*\to Y_*)^{-1}\operatorname{im}(f\colon X_*\to Y_*)$, proving that $\bar g$ is surjective.

To prove injectivity, we prove the equality

$$\ker\bar g=(\eta\colon B_*\to Y_*)^{-1}\operatorname{im}(\eta f\colon K_*\to Y_*)$$

To establish the forward inclusion, suppose $b\in B_*$ is an element with

$$(\eta\colon B_*\to Y_*)(b)=(f\colon X_*\to Y_*)(x)$$

in Y_* for some $x\in X_*$, and assume that $\bar g(b)=0$. Then there exists $b'\in B_*$ with $(g\colon B_*\to C_*)(b)=(g\colon B_*\to C_*)(b')$, where the image of b' in Y_* meets the set $\operatorname{im}(\eta f\colon K_*\to Y_*)$. Take $k'\in K_*$ satisfying

$$(\eta f\colon K_*\to X_*)(k')=(\eta\colon B_*\to Y_*)(b'),$$

and define $b^* = b - b'$. That b and b' have common image under $g\colon B_* \to C_*$ implies that $(\eta\colon B_* \to Y_*)(b^*) \in \ker(g\colon L_* \to M_*) = \operatorname{im}(f\colon K_* \to L_*)$, so that $(\eta\colon B_* \to Y_*)(b^*)$ lifts to an element $k^* \in K_*$. It follows that

$$\begin{aligned}(\eta f\colon K_* \to X_*)(k^* + k') &= (\eta\colon B_* \to Y_*)(b^*) + (\eta f\colon K_* \to Y_*)(k') \\ &= (\eta\colon B_* \to Y_*)(b) - (\eta\colon B_* \to Y_*)(b') + (\eta f\colon K_* \to Y_*)(k') \\ &= (\eta\colon B_* \to Y_*)(b).\end{aligned}$$

As $(f\colon X_* \to Y_*)(x) = (\eta\colon B_* \to Y_*)(b)$, this shows that the image of b in Y_* has the desired lift $k := k^* + k'$ to K_*. The other inclusion follows immediately from observing that an element $b \in B_*$ with $(\eta\colon B_* \to Y_*)(b) = (\eta f\colon K_* \to Y_*)(k)$ in Y_* has $\bar{g}(b)$ contained in $(g\colon B_* \to C_*)\big((\eta\colon B_* \to Y_*)^{-1}\operatorname{im}(\eta f\colon K_* \to Y_*)\big)$. □

Lemma 4.1.4. *The homomorphism $\eta''\colon C_* \to M_*$ induces an isomorphism*

$$\begin{aligned}&\frac{\operatorname{im}(B_* \to C_*) \cap \operatorname{im}((Z,C)_{*+1} \to C_*)}{(B_* \to C_*)\big((B_* \to Y_*)^{-1}\operatorname{im}(K_* \to Y_*)\big)} \\ &\qquad \xrightarrow{\cong} \frac{\operatorname{im}(B_* \to M_*) \cap \operatorname{im}((Z,M)_{*+1} \to M_*)}{(B_* \to M_*)\big((B_* \to Y_*)^{-1}\operatorname{im}(K_* \to Y_*)\big)}.\end{aligned}$$

Proof. Observe that we have equalities $\operatorname{im}((Z,C)_{*+1} \to C_*) = \ker(C_* \to Z_*)$ and $\operatorname{im}((Z,M)_{*+1} \to M_*) = \ker(M_* \to Z_*)$ by exactness. As $g\eta\colon B_* \to M_*$ factors through C_*, any element of $\ker(M_* \to Z_*)$ in the image from B_* is also the image of an element of $\ker(C_* \to Z_*)$. Consequently, η'' induces a surjection

$$\operatorname{im}(B_* \to C_*) \cap \ker(C_* \to Z_*) \twoheadrightarrow \operatorname{im}(B_* \to M_*) \cap \ker(M_* \to Z_*).$$

In terms of images, we see that η'' induces a surjective homomorphism $\bar{\eta}''$ from

$$\operatorname{im}(g\colon B_* \to C_*) \cap \operatorname{im}(\partial''\colon (Z,C)_{*+1} \to C_*)$$

onto

$$\frac{\operatorname{im}(\eta'' g\colon B_* \to M_*) \cap \operatorname{im}(\partial''\colon (Z,M)_{*+1} \to M_*)}{(\eta'' g\colon B_* \to M_*)\big((\eta\colon B_* \to Y_*)^{-1}\operatorname{im}(\eta f\colon K_* \to Y_*)\big)}.$$

We claim that $\bar{\eta}''$ has kernel

$$\ker \bar{\eta}'' = (g\colon B_* \to C_*)\big((\eta\colon B_* \to Y_*)^{-1}\operatorname{im}(\eta f\colon K_* \to Y_*)\big).$$

To prove the forward inclusion, suppose $\bar{\eta}''(c) = 0$ for some element $c \in C_*$ with $(g\colon B_* \to C_*)(b) = c = (\partial''\colon (Z,M)_{*+1} \to M_*)(\bar{z})$ for $b \in B_*$ and $\bar{z} \in (Z,M)_{*+1}$. Then the image of c in M_* satisfies

$$(\eta'' g\colon B_* \to M_*)(b') = (\eta''\colon C_* \to M_*)(c)$$

for some $b' \in B_*$, where $(\eta\colon B_* \to Y_*)(b') = (\eta f\colon K_* \to Y_*)(k')$ in Y_* for some $k' \in K_*$. Now both b and b' have common image in M_*, hence the image of their difference $b^* = b - b'$ in L_* lifts to an element k^* of K_* by exactness. Defining $k := k^* + k'$, we have

$$\begin{aligned}(\eta f\colon K_* \to Y_*)(k) &= (\eta\colon B_* \to Y_*)(b^*) + (\eta f\colon K_* \to Y_*)(k') \\ &= (\eta\colon B_* \to Y_*)(b) - (\eta\colon B_* \to Y_*)(b') + (\eta f\colon K_* \to Y_*)(k') \\ &= (\eta\colon B_* \to Y_*)(b).\end{aligned}$$

This shows that $b \in (\eta\colon B_* \to Y_*)^{-1}(\mathrm{im}(\eta f\colon K_* \to X_*))$, so that

$$c \in (g\colon B_* \to C_*)\big((\eta\colon B_* \to Y_*)^{-1}\,\mathrm{im}(\eta f\colon K_* \to Y_*)\big).$$

To prove the opposite inclusion, suppose $c \in C_*$ satisfies $c = (g\colon B_* \to C_*)(b)$ for some $b \in B_*$, where $(\eta\colon B_* \to Y_*)(b) = (\eta f\colon K_* \to Y_*)(k)$. Then the image $(\eta''\colon C_* \to M_*)(c)$ of C_* in M_* defines an element of

$$(g\eta\colon B_* \to M_*)\big((\eta\colon B_* \to Y_*)^{-1}\,\mathrm{im}(\eta f\colon K_* \to Y_*)\big),$$

as desired. □

With these diagram chases out of the way, we are ready to demonstrate how the homomorphism ϕ of Proposition 4.1.1 restricts to an isomorphism.

Proposition 4.1.5. *The homomorphism ϕ descends to an isomorphism*

$$\frac{\mathrm{im}(X_* \to Y_*) \cap \mathrm{im}(B_* \to Y_*)}{\mathrm{im}(K_* \to Y_*) \cap \mathrm{im}(B_* \to Y_*)} \xrightarrow{\cong} \frac{\mathrm{im}(B_* \to M_*) \cap \mathrm{im}((Z,M)_{*+1} \to M_*)}{\mathrm{im}(B_* \to M_*) \cap \mathrm{im}((Y,L)_{*+1} \to M_*)}.$$

Proof. Let Ω denote the preimage $\Omega := (\eta\colon B_* \to Y_*)^{-1}\,\mathrm{im}(\eta f\colon K_* \to Y_*)$. We begin by establishing the equality

$$(\eta'' g\colon B_* \to M_*)(\Omega) = \mathrm{im}(\eta'' g\colon B_* \to M_*) \cap \mathrm{im}(\partial'' g\colon (Y,L)_{*+1} \to M_*)$$

To prove the forward inclusion, let $m \in M_*$ satisfy $(\eta'' g\colon B_* \to M_*)(b) = m$ for $b \in B_*$ with $(\eta\colon B_* \to Y_*)(b) = (\eta f\colon K_* \to Y_*)(k)$ in Y_* for $k \in K_*$. Define

$$\ell^* := (\eta\colon B_* \to L_*)(b) - (f\colon K_* \to L_*)(k)$$

in L_*. Note that $(\eta\colon L_* \to Y_*)(\ell^*) = 0$ in Y_*, so there exists $\bar{y} \in (Y,L)_{*+1}$ mapping to ℓ^* under $\partial\colon (Y,L)_{*+1} \to L_*$ by exactness. Furthermore,

$$\begin{aligned}(\partial'' g\colon (Y,L)_{*+1} \to M_*)(\bar{y}) &= (g\colon L_* \to M_*)(\ell^*)\\ &= (g\colon L_* \to M_*)(\eta(b) - f(k))\\ &= (g\colon L_* \to M_*)(\eta(b))\\ &= (\eta'' g\colon B_* \to M_*)(b).\end{aligned}$$

This shows that $m \in \mathrm{im}(\eta'' g\colon B_* \to M_*) \cap \mathrm{im}(\partial'' g\colon (Y,L)_{*+1} \to M_*)$. To prove the opposite inclusion, suppose

$$m = (\eta'' g\colon B_* \to M_*)(b) = (\partial'' g\colon (,/L)_{*+1} \to M_*)(\bar{y})$$

for $b \in B_*$ and $\bar{y} \in (Y,L)_{*+1}$. Let $\ell^* := (\eta\colon B_* \to L_*)(b) - (\partial\colon (Y,L)_{*+1} \to L_*)$, so that ℓ^* goes to zero along $g\colon L_* \to M_*$. Then there exists $k^* \in K_*$ with $(f\colon K_* \to L_*)(k^*) = \ell^*$ by exactness. As the composite $\eta\partial\colon (Y,L)_{*+1} \to Y_*$ vanishes, $(\eta\partial\colon (Y,L)_{*+1} \to Y_*)(\bar{y})$ is zero in Y_*, and

$$(\eta f\colon K_* \to Y_*(k^*) = (\eta\colon B_* \to Y_*)(b).$$

This proves the desired equality. The proposition now follows readily from the sequence of isomorphisms in (4.1.2). □

This completes our first goal: determining the isomorphism induced by ϕ. The next step is to make sense of this in the associated spectral sequences. We return to the full notation and restate the previous proposition in that language.

Proposition 4.1.6. *The homomorphism ϕ descends to an isomorphism*

$$\bar{\phi}\colon \frac{\operatorname{im}(f_i) \cap \operatorname{im}_\eta^{i+j} \pi_*(i)}{\operatorname{im}(f_i \eta') \cap \operatorname{im}_\eta^{i+j} \pi_*(i)} \xrightarrow{\cong} \frac{g_{i+1}(\operatorname{im}_\eta^{i+j} \pi_*(i+1)) \cap \operatorname{im}_{\partial''}^{i,i+1} \pi''_*(i+1)}{g_{i+1}(\operatorname{im}_\eta^{i+j} \pi_*(i+1)) \cap g_{i+1}(\operatorname{im}_\partial^{i,i+1} \pi_*(i+1))}$$

Note that we may pass from the case where we focus on three filtration indices to the case where we include all filtration indices between $i+1$ and $i+j$ with no additional effort. The proofs above are simply independent of the extra information. As we move to the associated spectral sequences, however, we will be more diligent. To this end, we introduce truncated Cartan–Eilenberg systems that we pass through the formalism of exact couples to obtain spectral sequences.

We illustrate the construction of such a truncated Cartan–Eilenberg system in the case of the system (π_*, η, ∂). The essence of the construction is to forget all the groups seen by the exact couple apart from the ones in the range (i, ∞) to $(i+j, \infty)$. Define a Cartan–Eilenberg system $(\bar{\pi}_*, \eta, \partial)$ from (π_*, η, ∂) by replacing the groups in position (i', k) with trivial groups whenever $i' < i+j$ and $i' \leq k$, modifying the structure morphisms and connecting morphisms accordingly. Next, let $\bar{\pi}_*(i', \infty) = \pi_*(i)$ for $i' < i$ and $\bar{\pi}_*(i', k) = 0$ for $i' < i$ and $i' \leq k$, again making the obvious corrections to the morphisms. Repeating this procedure for the other systems gives truncated Cartan–Eilenberg systems $(\bar{\pi}'_*, \eta', \partial')$, $(\bar{\pi}_*, \eta, \partial)$ and $(\bar{\pi}''_*, \eta'', \partial'')$, where the associated unrolled exact couples are the ones appearing in Figure 4.1. The canonical filtrations $(F^s\bar{\pi}'_*)_s$, $(F^s\bar{\pi}_*)_s$ and $(F^s\bar{\pi}''_*)_s$ are exhaustive and bounded above, hence the associated spectral sequences $({}'\bar{E}_r, {}'d_r)_{r\geq 1}$, $(\bar{E}_r, d_r)_{r\geq 1}$ and $({}''\bar{E}_r, {}''d_r)_{r\geq 1}$ all converge to the respective filtered targets $\bar{\pi}'_*(i)$, $\bar{\pi}_*(i)$ and $\bar{\pi}''_*(i)$.

Theorem 4.1.7. *Let*

$$(\pi'_*, \eta', \partial') \xrightarrow{f} (\pi_*, \eta, \partial) \xrightarrow{g} (\pi''_*, \eta'', \partial'') \xrightarrow{h} (\Sigma\pi'_*, \eta', \partial')$$

be an exact sequence of Cartan–Eilenberg systems. Fix integers i and $j \geq 1$, and consider the truncated systems $(\bar{\pi}'_*, \eta', \partial')$, $(\bar{\pi}_*, \eta, \partial)$ and $(\bar{\pi}''_*, \eta'', \partial'')$ along with the associated spectral sequences $({}'\bar{E}_r, {}'d_r)_{r\geq 1}$, $(\bar{E}_r, d_r)_{r\geq 1}$ and $({}''\bar{E}_r, {}''d_r)_{r\geq 1}$. There is an isomorphism

$$\frac{F_f^{i,i+j}\bar{\pi}_*(i)}{F_f^{i+1,i+j}\bar{\pi}_*(i)} \xrightarrow{\cong} \frac{\operatorname{im}({}''d_j^i) \cap \operatorname{im}(\bar{E}_j^{i+j}(g))}{{}''d_j^i(\ker {}''\bar{E}_j^i(h)) \cap \operatorname{im}(\bar{E}_j^{i+j}(g))}$$

relating filtration shifts under f from $\bar{\pi}'_*(i)$ to $\bar{\pi}_*(i+j)$, and d_j-differentials in the spectral sequence associated to $\bar{\pi}''_*$.

Proof. This follows from the isomorphism $\bar{\phi}$ of Proposition 4.1.6, after interpreting the codomain of $\bar{\phi}$ in the context of the spectral sequences associated to the truncated Cartan–Eilenberg systems. Specifically, we will argue for the following two identifications:

i) $\operatorname{im}_{\partial''}^{i,i+1} \bar{\pi}''_*(i+1) \cap g(\operatorname{im}_\eta^{i+j} \bar{\pi}_*(i+1)) \cong \operatorname{im}({}''d_j^i) \cap \operatorname{im}(\bar{E}_j^{i+j}(g))$,

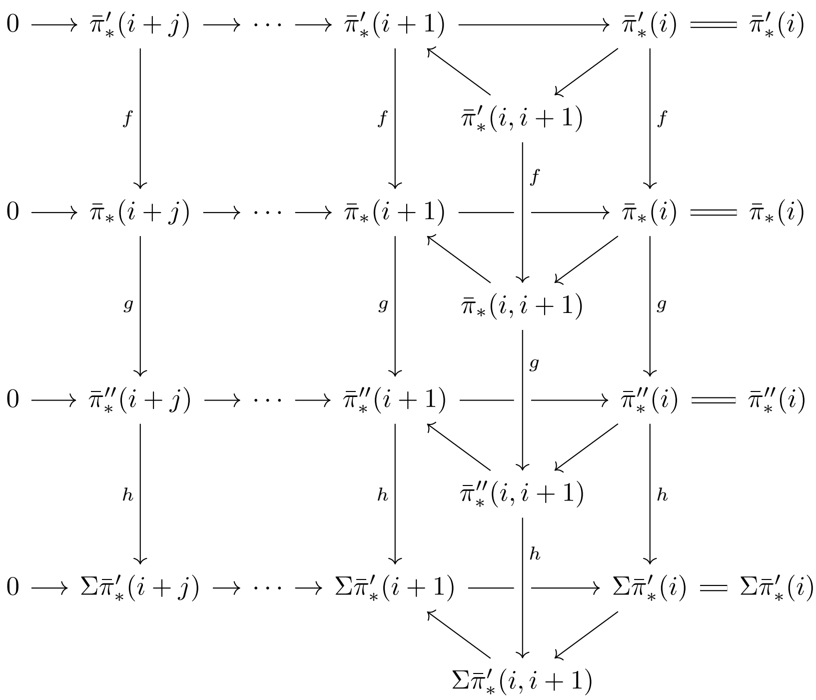


Figure 4.1: Exact couples associated to the sequence of truncated Cartan–Eilenberg systems.

ii) $g(\operatorname{im}^{i,i+1}_{\partial} \bar\pi_*(i+1)) \cap g(\operatorname{im}^{i+j}_{\eta} \bar\pi_*(i+1)) \cong {}''d^i_j(\ker {}''\bar{E}^i_j(h)) \cap \operatorname{im}(\bar{E}^{i+j}_j(g))$.

We shall establish each of these in turn, but begin by describing the image of $\bar{E}^{i+j}_j(g)$ appearing in both identifications. Observe first that the structure of the truncated Cartan–Eilenberg systems lead to isomorphisms

$$\begin{aligned} \bar{E}^{i+j}_j &\cong \operatorname{im}(\eta\colon \bar\pi_*(i+j) \to \bar\pi_*(i+1)) = \operatorname{im}^{i+j}_{\eta} \bar\pi_*(i+1), \\ {}''\bar{E}^{i+j}_j &\cong \operatorname{im}(\eta''\colon \bar\pi''_*(i+j) \to \bar\pi''_*(i+1)) = \operatorname{im}^{i+j}_{\eta''} \bar\pi''_*(i+1) \end{aligned} \tag{4.1.8}$$

by Proposition 3.3.2. The compatibility of g with the structure morphisms ensures that the composite $g\eta = \eta'' g\colon \bar\pi_*(i+j) \to \bar\pi''_*(i+1)$ factors through $\bar\pi''_*(i+j)$, so that $g(\operatorname{im}^{i+j}_{\eta} \bar\pi_*(i+1)) \subset \operatorname{im}^{i+j}_{\eta''} \bar\pi''_*(i+1)$. In particular, the horizontal arrows in the left-hand square of

$$\begin{array}{ccccc} \bar\pi_*(i+j) & \twoheadrightarrow & \bar{E}^{i+j}_j & \rightarrowtail & \bar\pi_*(i+1) \\ \downarrow g & & \downarrow \bar{E}^{i+j}_j(g) & & \downarrow g \\ \bar\pi''_*(i+j) & \twoheadrightarrow & {}''\bar{E}^{i+j}_j & \rightarrowtail & \bar\pi''_*(i+1) \end{array}$$

are surjections, hence $\operatorname{im}(\bar{E}^{i+j}_j(g)) = \operatorname{im}(\bar\pi_*(i+j) \to {}''\bar{E}^{i+j}_j) \cong g(\operatorname{im}^{i+j}_{\eta} \bar\pi_*(i+1))$.

To obtain the first identification i), note that

$$\begin{aligned}\operatorname{im} {}''d^i_j = \frac{{}''\bar{B}^{i+j}_{j+1}}{{}''\bar{B}^{i+j}_j} &= \frac{\operatorname{im}(\partial''\colon \bar{\pi}''_{*+1}(i,i+j) \to \bar{\pi}''_*(i+j))}{\ker(\eta''\colon \bar{\pi}''_*(i+j) \to \bar{\pi}''_*(i+1))} \\ &\cong \operatorname{im}(\eta''\partial''\colon \bar{\pi}''_{*+1}(i,i+j) \to \bar{\pi}''_*(i+j) \to \bar{\pi}''_*(i+1)) \\ &= \operatorname{im}^{i,i+1}_{\partial''} \bar{\pi}''_*(i+1) \cap \operatorname{im}^{i+j}_{\eta''} \bar{\pi}''_*(i+1).\end{aligned}$$

Here the last equality is a consequence of the commutative diagram

$$\begin{array}{ccccc} \bar{\pi}''_*(i,i+j) & \xrightarrow{\eta''} & \bar{\pi}''_*(i,i+1) & \xrightarrow{\partial''} & \bar{\pi}''_{*-1}(i+1,i+j) \\ \downarrow{\scriptstyle \partial''} & & \downarrow{\scriptstyle \partial''} & & \| \\ \bar{\pi}''_{*-1}(i+j) & \xrightarrow{\eta''} & \bar{\pi}''_{*-1}(i+1) & \xrightarrow{\eta''} & \bar{\pi}''_{*-1}(i+1,i+j) \end{array}$$

with exact rows. That is, any $z \in \pi''_{*-1}(i+1)$ in $\operatorname{im}^{i,i+1}_{\partial''} \bar{\pi}''_*(i+1) \cap \operatorname{im}^{i+j}_{\eta''} \bar{\pi}''_*(i+1)$ maps to zero along η'' in the lower right-hand corner, so that an element $\bar{z} \in \pi''_*(i,i+1)$ with image $\partial''(\bar{z}) = z$ lifts to $\pi''_*(i,i+j)$. Combining this with the image of $\bar{E}^{i+j}_j(g)$ above gives the correct intersection:

$$\begin{aligned}&\operatorname{im}({}''d^i_j) \cap \operatorname{im}(\bar{E}^{i+j}_j(g)) \\ &\quad\cong \operatorname{im}^{i,i+1}_{\partial''} \bar{\pi}''_*(i+1) \cap \operatorname{im}^{i+j}_{\eta''} \bar{\pi}''_*(i+1) \cap g(\operatorname{im}^{i+j}_{\eta} \bar{\pi}_*(i+1)) \\ &\quad= \operatorname{im}^{i,i+1}_{\partial''} \bar{\pi}''_*(i+1) \cap g(\operatorname{im}^{i+j}_{\eta} \bar{\pi}_*(i+1)).\end{aligned}$$

To prove ii), we first observe how the identifications

$$\begin{aligned}{}'\bar{E}^i_j &\cong \operatorname{im}(\eta\colon \bar{\pi}_*(i,i+j) \to \bar{\pi}_*(i,i+1)) \\ {}''\bar{E}^i_j &\cong \operatorname{im}(\eta''\colon \bar{\pi}''_*(i,i+j) \to \bar{\pi}''_*(i,i+1))\end{aligned}$$

lead to a commutative diagram

$$\begin{array}{ccccc} \bar{\pi}''_*(i,i+j) & \twoheadrightarrow & {}''\bar{E}^i_j & \rightarrowtail & \bar{\pi}''_*(i,i+1) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle {}''\bar{E}^i_j(h)} & & \downarrow{\scriptstyle h} \\ \Sigma\bar{\pi}'_*(i,i+j) & \twoheadrightarrow & \Sigma\, {}'\bar{E}^i_j & \rightarrowtail & \Sigma\bar{\pi}'_*(i,i+1), \end{array}$$

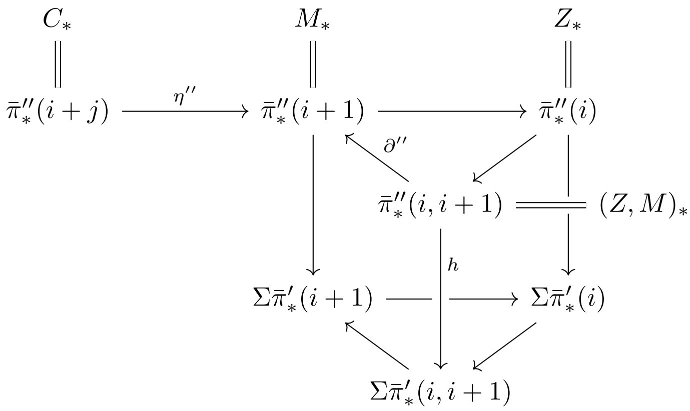
where the horizontal morphisms in the right-hand square are injections. It follows that

$$\ker({}''\bar{E}^i_j(h)) \cong \operatorname{im}(\eta''\colon \bar{\pi}''_*(i,i+j) \to \bar{\pi}''_*(i,i+1)) \cap g(\bar{\pi}_*(i,i+1)),$$

where the exactness of the sequence of Cartan–Eilenberg systems gives the equality $\ker(h\colon \bar{\pi}''_*(i,i+1) \to \Sigma\bar{\pi}'_*(i,i+1)) = g(\bar{\pi}_*(i,i+1))$. Via the isomorphisms (4.1.8), the image of this kernel under the ${}''d^i_j$-differential is

$$\partial(\partial^{-1}(\operatorname{im}^{i+j}_{\eta''} \bar{\pi}''_*(i+1)) \cap \ker(h\colon \bar{\pi}''_*(i,i+1) \to \Sigma\bar{\pi}'_*(i,i+1))), \qquad (4.1.9)$$

where ∂ denotes the connecting morphism $\partial''\colon \bar{\pi}''_{*+1}(i,i+1) \to \bar{\pi}''_*(i+1)$. This follows directly from the way we constructed differentials in spectral sequences arising from exact couples. The diagram below depicts the region of the linked exact couples contributing to the construction of the differential, where the arrows carrying labels are the ones taking parting in forming the set (4.1.9). To simplify notation for the final part of the proof, this diagram also shows how we relabel some of the groups.



We claim that (4.1.9) is equal to the set

$$g(\operatorname{im}_{\partial}^{i,i+1} \bar{\pi}_*(i+1)) \cap \operatorname{im}_{\eta''}^{i+j} \bar{\pi}''_*(i+1).$$

To see this, choose $[m] \in d^i_j(\ker E^i_j(h))$ represented by an element $m \in M_*$ with $m = \partial''\bar{z}$ for some $\bar{z} \in \ker(h)$, where m lifts over η'' to an element $c \in C_*$. Exactness gives an element $\bar{y} \in \bar{\pi}_*(i,i+1)$ with $g(\bar{y}) = \bar{z}$, hence $\eta'' g(\bar{y}) = m$. This shows that

$$m \in g(\operatorname{im}_{\partial}^{i,i+1} \bar{\pi}_*(i+1)) \cap \operatorname{im}_{\eta''}^{i+j} \bar{\pi}''_*(i+1).$$

Conversely, suppose $\eta''(c) = m = \partial'' g(\bar{y})$ in M_* for some $c \in C_*$ and $\bar{y} \in \bar{\pi}_{*+1}(i,i+1)$. Then $\bar{z} := g(\bar{y})$ defines an element of M_* contained in $\ker(h)$ by exactness, and $\partial''(\bar{z}) = m$. Pulling m back along ∂'' shows that m is contained in $d^i_j(\ker E^i_j(h))$. This establishes the identity ii):

$$\begin{aligned}
&{}''d^i_j(\ker({}''\bar{E}^i_j(h))) \cap \operatorname{im}(\bar{E}^{i+j}_j(g)) \\
&\quad = g(\operatorname{im}_{\partial}^{i,i+1} \bar{\pi}_*(i+1)) \cap \operatorname{im}_{\eta''}^{i+j} \bar{\pi}''_*(i+1) \cap g(\operatorname{im}_{\eta}^{i+j} \bar{\pi}_*(i+1)) \\
&\quad = g(\operatorname{im}_{\partial}^{i,i+1} \bar{\pi}_*(i+1)) \cap g(\operatorname{im}_{\eta}^{i+j} \bar{\pi}_*(i+1)),
\end{aligned}$$

completing the proof. □

4.2 Adding More Stages to the Filtrations

There are several natural ways to extend the results of the previous section. In this section, we keep the three filtration indices from before, but place no restriction on what happens in higher filtration degrees. Our goal is to derive conditions ensuring that this situation reduces to that of Theorem 4.1.7.

We return again to the sequence (4.0.1) and fix integers i and $j \geq 1$ as before. Let us write $Y_* := \pi_*(i)$ and equip Y_* with its usual filtration F^sY_* of

η-images. This time around, this filtration has $F^{i'}Y_* = Y_*$ for $i' \leq i$, but we impose no restrictions for $i' \geq i+j$. Instead, we let $\bar{Y}_* := \pi_*(i, i+j+1)$ and exhaustively filter this group by

$$F^{i+s}\bar{Y}_* = \operatorname{im}(\eta\colon \pi_*(i+s, i+j+1) \to \pi_*(i, i+j+1)). \tag{4.2.1}$$

We seek to determine conditions on the level of the Cartan–Eilenberg systems making filtration shifts for the first filtration correspond to that of the second. Specifically, we determine conditions under which $\eta\colon \pi_*(i) \to \pi_*(i, i+j+1)$ induces an isomorphism

$$\frac{F_f^{i,i+j}Y_*}{F_f^{i,i+j+1}Y_* + F_f^{i+1,i+j}Y_*} \xrightarrow{\cong} \frac{F_f^{i,i+j}\bar{Y}_*}{F_f^{i,i+j+1}\bar{Y}_* + F_f^{i+1,i+j}\bar{Y}_*} = \frac{F_f^{i,i+j}\bar{Y}_*}{F_f^{i+1,i+j}\bar{Y}_*}.$$

These conditions take the form of injectivity requirements, leading to the vanishing of certain differentials in the associated spectral sequences.

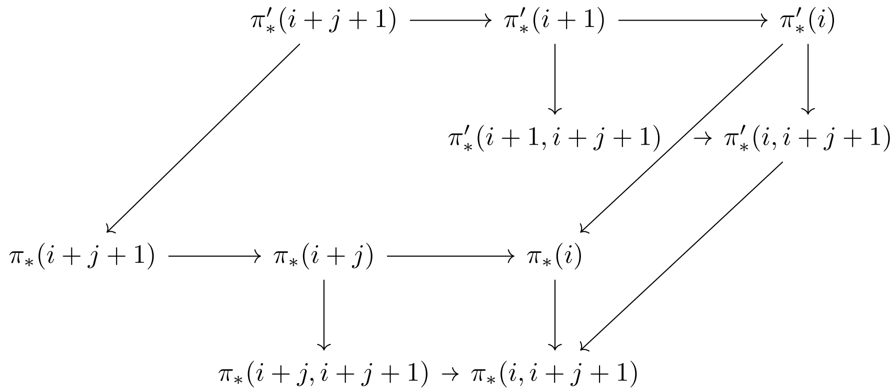


Figure 4.2: The relevant groups of the Cartan–Eilenberg systems.

If y is an element of $F_f^{i,i+j}Y_* = \operatorname{im}(f_i) \cap \operatorname{im}_\eta^{i+j} \pi_*(i)$, then its image $\eta(y)$ in $\pi_*(i, i+j+1)$ defines an element of $\operatorname{im}(f_{i,i+j+1}) \cap \operatorname{im}_\eta^{i+j,i+j+1} \pi_*(i, i+j+1)$, which is just $F_f^{i,i+j}\bar{Y}_*$. In particular, $\eta\colon \pi_*(i) \to \pi_*(i, i+j+1)$ induces a well-defined homomorphism

$$\psi\colon\ \operatorname{im}(f_i) \cap \operatorname{im}_\eta^{i+j} \pi_*(i) \longrightarrow \operatorname{im}(f_{i,i+j+1}) \cap \operatorname{im}_\eta^{i+j,i+j+1} \pi_*(i, i+j+1)$$

from a subgroup of $\pi_*(i)$ to a subgroup of $\pi_*(i, i+j+1)$. The next lemmas do the legwork in deriving an isomorphism from this homomorphism. Again, these boil down to diagram chases, where the groups involved appear as in Figure 4.2.

Lemma 4.2.2. *If $\eta\colon \pi'_*(i+j+1) \to \pi'_*(i)$ and $\eta\colon \pi_*(i+j+1) \to \pi_*(i+j)$ are injective, then ψ is surjective.*

Proof. If $\eta\colon \pi_*(i+j+1) \to \pi_*(i+j)$ is injective, then exactness of the sequence

$$\pi_*(i+j) \longrightarrow \pi_*(i+j, i+j+1) \xrightarrow{\partial} \pi_{*-1}(i+j+1) \xrightarrow{\eta} \pi_{*-1}(i+j)$$

implies that $\partial = 0$, hence $\eta\colon \pi_*(i+j) \to \pi_*(i+j, i+j+1)$ is surjective. The same argument shows that $\eta\colon \pi'_*(i) \to \pi'_*(i, i+j+1)$ is surjective whenever $\eta\colon \pi'_*(i+j+1) \to \pi'_*(i)$ is injective. Now consider $\bar{y} \in \pi_*(i, i+j+1)$ satisfying

$$\eta(\bar{y}') = \bar{y} = f(\bar{x})$$

for $\bar{y}' \in \pi_*(i+j, i+j+1)$ and $\bar{x} \in \pi'_*(i, i+j+1)$. Choose lifts $y' \in \pi_*(i+j)$ and $x \in \pi'_*(i)$ of $\bar{y}'$ and $\bar{x}$, respectively. The difference $f(x) - \eta(y')$ in $\pi_*(i)$ defines an element of $\ker(\eta\colon \pi_*(i) \to \pi_*(i, i+j+1))$, and lifts to an element $y'' \in \pi_*(i+j+1)$ by exactness. Define $y^* := \eta(y'') - y'$ in $\pi_*(i+j)$. Then

$$y := \eta(y^*) = f(x)$$

is an element of $\operatorname{im}(f_i) \cap \operatorname{im}_\eta^{i+j} \pi_*(i)$ with $\psi(y) = \bar{y}$ in $\pi_*(i, i+j+1)$, proving that ψ is surjective. □

Lemma 4.2.3. *If $\eta\colon \pi'_*(i+j+1) \to \pi'_*(i+1)$ and $\eta\colon \pi_*(i+j+1) \to \pi_*(i+j)$ are injective, then ψ induces a surjective homomorphism*

$$\begin{aligned} \bar{\psi}\colon\ & \operatorname{im}(f_i) \cap \operatorname{im}_\eta^{i+j} \pi_*(i) \\ & \longrightarrow \frac{\operatorname{im}(f_{i,i+j+1}) \cap \operatorname{im}_\eta^{i+j,i+j+1} \pi_*(i, i+j+1)}{f_{i,i+j+1}(\operatorname{im}_{\eta'}^{i+1,i+j+1} \pi'_*(i, i+j+1)) \cap \operatorname{im}_\eta^{i+j,i+j+1} \pi_*(i, i+j+1)} \end{aligned}$$

with $\ker \psi = f_i(\operatorname{im}_{\eta'}^{i+1} \pi'_*(i)) \cap \operatorname{im}_\eta^{i+j} \pi_*(i) + \operatorname{im}(f_i) \cap \operatorname{im}_\eta^{i+j+1} \pi_*(i)$.

Proof. Surjectivity is clear. To determine the kernel, we prove both inclusions. To establish the forward inclusion, suppose $\bar{\psi}(y) = 0$ for an element y in $\operatorname{im}(f_i) \cap \operatorname{im}_\eta^{i+j} \pi_*(i)$, so that

$$\eta f(\bar{x}') = \bar{\psi}(y) = \eta(\bar{y}')$$

in $\pi_*(i, i+j+1)$ for $\bar{x}' \in \pi'_*(i+1, i+j+1)$ and $\bar{y}' \in \pi_*(i+j, i+j+1)$. Let $x \in \pi'_*(i)$ and $y' \in \pi_*(i+j)$ be elements satisfying $f(x) = y = \eta(y')$ in $\pi_*(i)$. Assuming that $\eta\colon \pi'_*(i+j+1) \to \pi'_*(i+1)$ is injective, it follows from exactness that $\eta\colon \pi'_*(i+1) \to \pi'_*(i+1, i+j+1)$ is surjective. Consequently, choose a lift $x' \in \pi'_*(i+1)$ of $\bar{x}'$, and define elements

$$\pi'_*(i) \ni x^* := x - \eta(x') \quad \text{and} \quad \tilde{y} := \eta f(x') - \eta(y') \in \pi_*(i).$$

Also define $y^* := f(x^*)$ in $\pi_*(i)$. Note that both y^* and $\tilde{y}$ are differences of elements whose images agree in $\pi_*(i, i+j+1)$, hence both y^* and $\tilde{y}$ sit in $\ker(\eta\colon \pi_*(i) \to \pi_*(i, i+j+1))$. As such, exactness gets us lifts z^* and $\tilde{z}$ in $\pi_*(i+j+1)$ of y^* and $\tilde{y}$, respectively. With

$$\pi_*(i+j) \ni z'' := \eta(z') + y' \quad \text{and} \quad y'' := \eta(z'') \in \pi_*(i),$$

it follows that

$$\begin{aligned} y'' + y^* &= (\eta\colon \pi_*(i+j) \to \pi_*(i))(z'') + (f\colon \pi'_*(i) \to \pi_*(i))(x^*) \\ &= \big((\eta\colon \pi_*(i+j+1) \to \pi_*(i)(\tilde{z}) + (\eta\colon \pi_*(i+j) \to \pi_*(i))(y')\big) \\ &\quad + \big((f\colon \pi'_*(i) \to \pi_*(i))(x))(x) - (\eta f\colon \pi'_*(i+1) \to \pi_*(i))(x')\big) \\ &= \tilde{y} + (\eta\colon \pi_*(i+j) \to \pi_*(i))(y') + y - (\eta f\colon \pi'_*(i+1) \to \pi_*(i))(x') \\ &= y. \end{aligned}$$

As we defined them, y'' sits in $f(\mathrm{im}_{\eta'}^{i+1} \pi'_*(i)) \cap \mathrm{im}_\eta^{i+j} \pi_*(i)$ and y^* sits in $\mathrm{im}(f_i) \cap \mathrm{im}_\eta^{i+j+1} \pi_*(i)$, hence the proposed kernel contains y.

To prove the opposite inclusion, choose an element $y \in \pi_*(i)$ in the proposed kernel. We may assume that $y = y_1 + y_2 = y_1$, where and $y_1 = \eta f(x') = \eta(y')$ in $\pi_*(i)$ for $x' \in \pi'_*(i+1)$ and $y' \in \pi_*(i+j)$. To see that we can take $y_2 = 0$, note that exactness of the sequence

$$\pi_*(s+k+1) \xrightarrow{\eta} \pi_*(s) \xrightarrow{\eta} \pi_*(s, s+k+1),$$

places any element of $\mathrm{im}(f_i) \cap \mathrm{im}_\eta^{i+j+1} \pi_*(i)$ in $\ker \psi$. Define elements $\bar{x}' := \eta(x')$ in $\pi'_*(i+1, i+j+1)$ and $\bar{y}' := \eta(y')$ in $\pi_*(i+j, i+j+1)$. Then

$$f(\bar{x}') = \eta(\bar{y}') = \bar{\psi}(y),$$

shows that $\bar{\psi}(y) = 0$. □

Combining the two lemmas we have the following result.

Proposition 4.2.4. *If $\eta\colon \pi'_*(i+j+1) \to \pi'_*(i+1)$ and $\eta\colon \pi_*(i+j+1) \to \pi_*(i+j)$ are injective, then $\eta\colon \pi_*(i) \to \pi_*(i, i+j+1)$ induces an isomorphism*

$$\frac{F_f^{i,i+j} Y_*}{F_f^{i,i+j+1} Y_* + F_f^{i+1,i+j} Y_*} \xrightarrow{\cong} \frac{F_f^{i,i+j} \bar{Y}_*}{F_f^{i+1,i+j} \bar{Y}_*}.$$

Proof. Under the given conditions we deduce an isomorphism

$$\frac{\mathrm{im}(f_i) \cap \mathrm{im}_\eta^{i+j} \pi_*(i)}{f_i(\mathrm{im}_{\eta'}^{i+1} \pi'_*(i)) \cap \mathrm{im}_\eta^{i+j} \pi_*(i) + \mathrm{im}(f_i) \cap \mathrm{im}_\eta^{i+j+1} \pi_*(i)}$$
$$\xrightarrow{\cong} \frac{\mathrm{im}(f_{i,i+j+1}) \cap \mathrm{im}_\eta^{i+j,i+j+1} \pi_*(i, i+j+1)}{f_{i,i+j+1}(\mathrm{im}_{\eta'}^{i+1,i+j+1} \pi'_*(i, i+j+1)) \cap \mathrm{im}_\eta^{i+j,i+j+1} \pi_*(i, i+j+1)}.$$

This is precisely the one we want. □

At this point, we are almost ready to state our conclusion. This conclusion is an immediate consequence of Theorem 4.1.7, but we need to be precise about what spectral sequences we have in mind. Let us write

$$\bar{X}_* := \pi'_*(i, i+j+1) \quad \text{and} \quad \bar{Z}_* := \pi''_*(i, i+j+1),$$

and equip both of these groups with exhaustive filtrations (4.2.1). Consider the respectively truncated systems, and let $(\bar{E}_r(\bar{X}), d_r(\bar{X}))_{r\geq 1}$, $(\bar{E}_r(\bar{Y}), d_r(\bar{Y}))_{r\geq 1}$ and $(\bar{E}_r(\bar{Z}), d_r(\bar{Z}))_{r\geq 1}$ be the associated spectral sequences.

Proposition 4.2.5. *Let*

$$(\pi'_*, \eta', \partial') \xrightarrow{f} (\pi_*, \eta, \partial) \xrightarrow{g} (\pi''_*, \eta'', \partial'') \xrightarrow{h} (\Sigma\pi'_*, \eta', \partial')$$

be an exact sequence of Cartan–Eilenberg systems. Fix integers i and $j \geq 1$, and consider the spectral sequences $(\bar{E}_r(\bar{X}), d_r(\bar{X}))_{r\geq 1}$, $(\bar{E}_r(\bar{Y}), d_r(\bar{Y}))_{r\geq 1}$ and $(\bar{E}_r(\bar{Z}), d_r(\bar{Z}))_{r\geq 1}$ associated to the truncated systems $\bar{X}_*$, $\bar{Y}_*$ and $\bar{Z}_*$. There is an isomorphism

$$\frac{F_f^{i,i+j} Y_*}{F_f^{i,i+j+1} Y_* + F_f^{i+1,i+j} Y_*} \xrightarrow{\cong} \frac{\mathrm{im}(d_j^i(\bar{Z})) \cap \mathrm{im}(\bar{E}_j^{i+j}(g))}{d_j^i(\bar{Z})(\ker{}''\bar{E}_k^i(h)) \cap \mathrm{im}(\bar{E}_j^{i+j}(g))}$$

relating filtration shifts under f from $\pi'_*(i)$ to $\pi_*(i+j)$, and d_j-differentials in the spectral sequence associated to $\bar{Z}_*$.

To finish this section we describe how the injectivity conditions on the morphisms of a Cartan–Eilenberg system affects the differentials of the associated spectral sequence. The way we describe the boundary groups of a spectral sequence using kernels of morphisms along the top row of an exact couple suggests that we can expect differentials to vanish. We restrict our attention to convergent spectral sequences.

Proposition 4.2.6. *Let (π_*, η, ∂) be a Cartan–Eilenberg system and consider the associated spectral sequence $(E_r, d_r)_{r\geq 1}$. Fix integers i and $j \geq 1$. If the spectral sequence is convergent, then*

$$\eta\colon \pi_n(i+j) \longrightarrow \pi_n(i)$$

is injective if and only if the differentials

$$d_r\colon E_r^{a,a+n+1} \longrightarrow E_r^{a+r,a+r+n}$$

are all zero for $i \leq a < i+j$ and $a+r \geq i+j$.

To illustrate this result, assume that $\eta\colon \pi_*(i+j) \to \pi_*(i)$ is the injective morphism of the proposition. Then any differential originating in the range of filtration degrees from i to $i+j+1$, whose target is either in filtration degree $i+j$ or crosses over it, has to be zero. Moreover, the vanishing of these differentials is sufficient to ensure that η is injective. Figure 4.3 depicts the vanishing differentials as solid arrows.

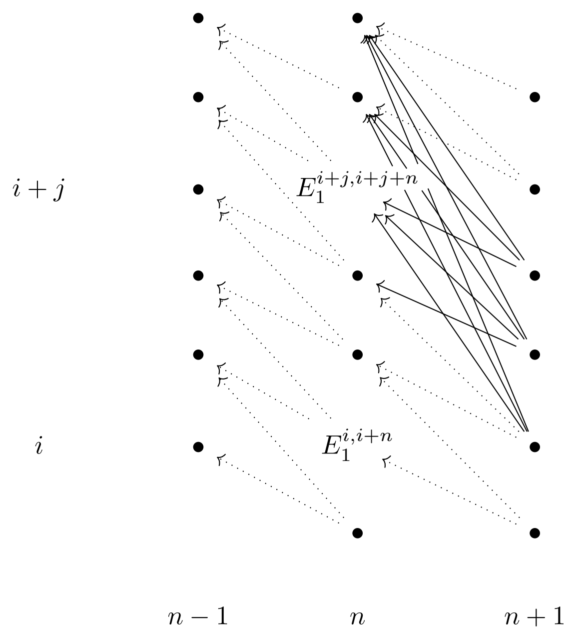


Figure 4.3: The vanishing differentials of Proposition 4.2.6

To prove Proposition 4.2.6 we make use of truncated spectral sequences converging towards filtrations ending at $\pi_*(i+j)$ and $\pi_*(i)$, respectively. Our preliminary goal is to demonstrate that injectivity of $\eta\colon \pi_*(i+j) \to \pi_*(i)$ is equivalent to a claim about the infinite boundaries of these spectral sequences. First, define an exhaustive filtration of $\pi_*(i+j)$ with

$$F^s\pi_n(i+j) := \operatorname{im}(\eta\colon \pi_n(s) \to \pi_n(i+j))$$

for $s \geq i+j$, and an exhaustive filtration of $\pi_*(i)$ with

$$F^s\pi_n(i) := \operatorname{im}(\eta\colon \pi_n(s) \to \pi_n(i)).$$

for $s \geq i$. Naturally, these filtrations are tightly connected, and their differences reveal information about $\eta\colon \pi_*(i+j) \to \pi_*(i)$. Let $(E_r(\tau_{*\geq i+j}), d_r)_{r\geq 1}$ be the spectral sequence associated to the filtration ending at $\pi_*(i+j)$, and $(E_r(\tau_{*\geq i}), d_r)_{r\geq 1}$ the spectral sequence associated to the filtration ending at $\pi_*(i)$.

Lemma 4.2.7. *If $F^s\pi_n(i+j)$ is Hausdorff, then $\eta\colon \pi_n(i+j) \to \pi_n(i)$ is injective if and only if*

$$\frac{F^s\pi_n(i+j)}{F^{s+1}\pi_n(i+j)} \longrightarrow \frac{F^s\pi_n(i)}{F^{s+1}\pi_n(i)}$$

is injective for each $s \geq i+j$.

Proof. Observe first that $\eta\colon \pi_n(i+j) \to \pi_n(i)$ is injective if and only if

$$F^{i+j}\pi_n(i+j) \longrightarrow F^{i+j}\pi_n(i)$$

is injective, as the displayed map is just the inclusion of $\pi_n(i+j)$ into its image in $\pi_n(i)$. Also, when $s \geq i+j$, then $F^s\pi_n(i+j) \to F^s\pi_n(i)$ is surjective as $\eta\colon \pi_n(s) \to \pi_n(i)$ factors through $\pi_n(i+j)$. Suppose inductively that α^s is injective in the map of short exact sequences below, starting from $s = i+j$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^{s+1}\pi_n(i+j) & \longrightarrow & F^s\pi_n(i+j) & \longrightarrow & \dfrac{F^s\pi_n(i+j)}{F^{s+1}\pi_n(i+j)} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \alpha^{s+1}} & & \Big\downarrow{\scriptstyle \alpha^{s}} & & \Big\downarrow{\scriptstyle \bar{\alpha}^{s}} & & \\
0 & \longrightarrow & F^{s+1}\pi_n(i) & \longrightarrow & F^s\pi_n(i) & \longrightarrow & \dfrac{F^s\pi_n(i)}{F^{s+1}\pi_n(i)} & \longrightarrow & 0.
\end{array}$$

By a special case of the snake lemma we have an exact sequence

$$\ker(\alpha^s) \longrightarrow \ker(\bar{\alpha}^s) \longrightarrow \operatorname{coker}(\alpha^{s+1}).$$

As α^{s+1} is surjective, exactness shows that $\bar{\alpha}^s$ is injective. Thus we conclude that $\eta\colon \pi_n(i+j) \to \pi_n(i)$ is injective if and only if

$$\frac{F^s\pi_n(i+j)}{F^{s+1}\pi_n(i+j)} \longrightarrow \frac{F^s\pi_n(i)}{F^{s+1}\pi_n(i)}$$

for each finite $s \geq i+j$. □

Lemma 4.2.8. *Assume that the spectral sequences* $(E_r(\tau_{*\geq i+j}), d_r)_{r\geq 1}$ *and* $(E_r(\tau_{*\geq i}), d_r)_{r\geq 1}$ *are both convergent. Then* $\eta\colon \pi_n(i+j) \to \pi_n(i)$ *is injective if and only if*

$$E_\infty^{s,s+n}(\tau_{*\geq i+j}) \longrightarrow E_\infty^{s,s+n}(\tau_{*\geq i})$$

is injective for each $s \geq i+j$.

Proof. Convergence means that $F^s\pi_{i+j}$ is Hausdorff and that the inclusions

$$\mathrm{Gr}^s\pi_n(i+j) = \frac{F^s\pi_n(i+j)}{F^{s+1}\pi_n(i+j)} \longrightarrow E_\infty^{s,s+n}(\tau_{*\geq i+j})$$

and

$$\mathrm{Gr}^s\pi_n(i) = \frac{F^s\pi_n(i)}{F^{s+1}\pi_n(i)} \longrightarrow E_\infty^{s,s+n}(\tau_{*\geq i})$$

are isomorphisms for each s. Given $s \geq i+j$, Lemma 4.2.7 tells us that $\eta\colon \pi_n(i+j) \to \pi_n(i)$ is an isomorphism if and only if the left-hand side vertical arrow is injective in the commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}^s\pi_n(i+j) & \xrightarrow{\cong} & E_\infty^{s,s+n}(\tau_{*\geq i+j}) \\ \downarrow & & \downarrow \\ \mathrm{Gr}^s\pi_n(i) & \xrightarrow{\cong} & E_\infty^{s,s+n}(\tau_{*\geq i}). \end{array}$$

That is, if and only if $E_\infty^{s,s+n}(\tau_{*\geq i+j}) \to E_\infty^{s,s+n}(\tau_{*\geq i})$ is injective. □

Finally, observe that $Z_r^{s,s+n}(\tau_{*\geq i+j})$ and $Z_r^{s,s+n}(\tau_{*\geq i})$ are isomorphic for each $r \geq 1$ when $s \geq i+j$, as both groups are isomorphic to

$$\mathrm{im}(\eta\colon \pi_{s+n}(s, s+r) \to \pi_{s+n}(s, s+1))$$

by Proposition 3.3.2. As this holds for all finite r, the induced map of limits $Z_\infty^{s,s+n}(\tau_{*\geq i+j}) \to Z_\infty^{s,s+n}(\tau_{*\geq i})$ is an isomorphism. Applying the snake lemma to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_\infty^{s,s+n}(\tau_{*\geq i+j}) & \longrightarrow & Z_\infty^{s,s+n}(\tau_{*\geq i+j}) & \longrightarrow & E_\infty^{s,s+n}(\tau_{*\geq i+j}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\cong & & \downarrow & & \\ 0 & \longrightarrow & B_\infty^{s,s+n}(\tau_{*\geq i}) & \longrightarrow & Z_\infty^{s,s+n}(\tau_{*\geq i}) & \longrightarrow & E_\infty^{s,s+n}(\tau_{*\geq i}) & \longrightarrow & 0, \end{array}$$

we conclude that $B_\infty^{s,s+n}(\tau_{*\geq i+j}) \to B_\infty^{s,s+n}(\tau_{*\geq i})$ is an isomorphism if and only if $\eta\colon \pi_*(i+j) \to \pi_*(i)$ is injective.

Proof of Proposition 4.2.6. Fix an integer $s \geq i+j$. In the spectral sequence $(E_r(\tau_{*\geq i+j}), d_r)$ converging to $F^s\pi_*(i+j)$, the last d_r-differential landing in filtration s is the differential

$$d_{s-i-j}^{i+j,n+i+j+1}\colon E_{s-i-j}^{i+j,n+i+j+1}(\tau_{*\geq i+j}) \longrightarrow E_{s-i-j}^{s,s+n}(\tau_{*\geq i+j}),$$

where $r = s-i-j$. Thus we conclude that

$$B_{s+1-i-j}^{s,s+n}(\tau_{*\geq i+j}) = B_\infty^{s,s+n}(\tau_{*\geq i+j}).$$

Reasoning similarly for the spectral sequence $(E_r(\tau_{r*\geq i}), d_r)$ converging to $F^s\pi_*(i)$, we find that $B^{s,s+n}_{s+1-i}(\tau_{*\geq i}) = B^{s,s+n}_\infty(\tau_{*\geq i})$. Now $B^{s,s+n}_{s+1-i-j}(\tau_{*\geq i+j})$ and $B^{s,s+n}_{s+1-i-j}(\tau_{*\geq i})$ are isomorphic subgroups as both are isomorphic to the group

$$\ker(\eta\colon \pi_{s+n}(s, s+1) \to \pi_{s+n}(i+j, s+1)).$$

Summarizing the above, we have the following groups and relations

$$\begin{array}{ccccc} B^{s,s+n}_{s+1-i-j}(\tau_* \geq i+j) & = & & & B^{s,s+n}_\infty(\tau_{*\geq i+j}) \\ \downarrow \cong & & & & \\ B^{s,s+n}_{s+1-i-j}(\tau_* \geq i) & \subset & B^{s,s+n}_{s+1-i}(\tau_{*\geq i}) & = & B^{s,s+n}_\infty(\tau_{*\geq i}). \end{array}$$

It follows that $B^{s,s+n}_\infty(\tau_{*\geq i+j}) \to B^{s,s+n}_\infty(\tau_{*\geq i})$ is an isomorphism for each $s \geq i+j$ if and only if the inclusion

$$B^{s,s+n}_{s+1-i-j}(\tau_{*\geq i}) \subset B^{s,s+n}_{s+1-i}(\tau_{*\geq i})$$

is the identity for each $s \geq i+j$. For this to hold, we need the differentials

$$d_r\colon E^{s-r,s-r+n+1}_r(\tau_{*\geq 1}) \longrightarrow E^{s,s+n}_r(\tau_{*\geq i})$$

to be zero for $s+1-i-j \leq r \leq s-i$. Defining $a := s-r$, this is equivalent to $i \leq a \leq i+j+1$ and $a+r \geq i+j$. In the full spectral sequence $(E_r, d_r)_r$, this shows that $\eta\colon \pi_*(i+j) \to \pi_*(i)$ is injective if and only if the differentials

$$d_r\colon E^{a,a+n+1}_r \to E^{a+r,a+r+n}_r$$

vanish for $i \leq a < i+j$ and $a+r \geq i+j$. $\square$

A Appendix

A.1 A Category of Chain Complexes

Definition A.1.1. A *chain complex* $X_* = (X_*, \partial)$ *of abelian groups* is a sequence $(X_n)_{n\in\mathbb{Z}}$ of abelian groups together with group homomorphisms $\partial\colon X_n \to X_{n-1}$ for each integer n satisfying $\partial \circ \partial = 0$. The homomorphisms ∂ are called *differentials*.

There is a category Ch(Ab) of chain complexes of abelian groups. The morphisms $f\colon X_* \to Y_*$ are sequences $f_n\colon X_n \to Y_n$ of group homomorphisms that commute with the differentials, and are called *chain maps*. We let the index n represent the *degree*, and say that the group X_n of a chain complex X_* is the group in degree n. The category of chain complexes is both complete and cocomplete, with limits and colimits formed degreewise. The zero object is the chain complex $0_* = (0_*, 0)$ made up of only trivial groups. Moreover, Ch(Ab) is closed symmetric monoidal. The monoidal product is the *tensor product* $X_* \otimes Y_*$ given in degree n by

$$(X_* \otimes Y_*)_n := \bigoplus_{i+j=n} X_i \otimes_{\mathbb{Z}} Y_j,$$

where $\otimes_{\mathbb{Z}}$ denotes the tensor product of abelian groups. Given elements $x \in X_i$ and $y \in Y_j$, the differential is defined by

$$\partial(x \otimes y) = \partial x \otimes y + (-1)^i x \otimes \partial y,$$

picking up a sign as the differential moves across the element of degree i in line with the usual convention in homological algebra. The monoidal unit is the chain complex $U_* = (U_*, 0)$ with $\mathbb{Z}$ concentrated in degree 0. The symmetry is the twist isomorphism defined as

$$\begin{aligned} \tau\colon (X_* \otimes Y_*) &\longrightarrow (X_* \otimes Y_*), \\ x \otimes y &\longmapsto (-1)^{ij} y \otimes x \end{aligned}$$

for elements $x \in X_i$ and $y \in Y_j$. Finally, the closed structure is given by the hom complex $F(X_*, Y_*)_*$ defined as

$$F(X_*, Y_*)_n := \prod_{i+j=n} \mathrm{Hom}_{\mathbb{Z}}(X_{-i}, Y_j),$$

with differential

$$(\partial f)(x) = \partial f(x) - (-1)^i f(\partial x)$$

for $f \in F(X_*, Y_*)_i$.

We proceed to construct some interesting chain complexes. Given a chain complex $X_* = (X_*, \partial)$, a *subcomplex* $A_* = (A_*, \partial|_A)$ of X_* is a sequence $(A_n)_{n\in\mathbb{Z}}$ of subgroups A_n of X_n, where the differential $\partial|_A$ is the restriction of the differential $\partial\colon X_n \to X_{n-1}$ to the subgroup A_n. When A_* is a subcomplex of X_*, we can assemble the quotient groups X_n/A_n into a chain complex

$$\cdots \longrightarrow X_{n+1}/A_{n+1} \xrightarrow{\partial} X_n/A_n \xrightarrow{\partial} X_{n-1}/A_{n-1} \longrightarrow \cdots$$

denoted by X_*/A_*. Next, we define suspensions and mapping cones. We lift these definitions from their topological counterparts, hence we need notions of circle and interval chain complexes. The interval chain complex (I_*, ∂) is given as

$$0 \longrightarrow R\{e_1\} \xrightarrow{\partial} R\{i_1, i_0\} \longrightarrow 0, \quad \partial e_1 = i_1 - i_0,$$

with i_0 and i_1 in degree 0 and e_1 in degree 1. The circle complex is the chain complex $(S_*, 0)$ given as

$$0 \longrightarrow R\{s_1\} \longrightarrow 0,$$

with s_1 in degree 1.

Definition A.1.2. Given a chain complex $X_* = (X_*, \partial)$, the *cylinder on* X_* is the chain complex $(I \otimes X)_* = I_* \otimes X_*$. Explicitly, the symmetric monoidal structure determines the group in degree n as

$$(I \otimes X)_n = \bigoplus_{k+\ell=n} I_k \otimes_R X_\ell \cong X_n \oplus X_n \oplus X_{n-1}.$$

The differential $\partial\colon (I \otimes X)_{n+1} \to (I \otimes X)_n$ is

$$\partial(x_{n+1}, x'_{n+1}, x_n) = (\partial x_{n+1} - x_n, \partial x'_{n+1} + x_n, -\partial x_n).$$

There are inclusions $i_1, i_0\colon X_* \to I_* \otimes X_*$ inserting X_* into either end of the cylinder, with $i_1(x) = (x, 0, 0)$ and $i_0(x) = (0, x, 0)$.

The *mapping cylinder* Mf_* of a chain map $f\colon X_* \to Y_*$ is the pushout

$$\begin{array}{ccc} X_* & \xrightarrow{f} & Y_* \\ {\scriptstyle i_0}\downarrow & & \downarrow \\ I_* \otimes X_* & \longrightarrow & Mf_*. \end{array}$$

Under the identifications for the cylinder, the group in degree n is

$$Mf_n \cong X_n \oplus Y_n \oplus X_{n-1}$$

The differential $\partial\colon Mf_{n+1} \to Mf_n$ is

$$\partial(x_{n+1}, y_{n+1}, x_n) = (\partial x_{n+1} - x_n, \partial y_{n+1} + f(x_n), -\partial x_n).$$

The inclusion $i\colon Y_* \to Mf_*$ is a chain homotopy equivalence, with inverse $\alpha\colon Mf_* \to Y_*$ the chain map $\alpha(x_{n+1}, y_{n+1}, x_n) = f(x_{n+1}) + y_{n+1}$.

Definition A.1.3. Let $f\colon X_* \to Y_*$ be a chain map. The *mapping cone* of f is the chain complex Cf_* given as the quotient of the mapping cylinder by the image of the inclusion $i_1\colon X_* \to I_* \otimes X_*$. Explicitly, the group in degree n is

$$Cf_n \cong Y_n \oplus X_{n-1},$$

and the differential $\partial\colon Cf_{n+1} \to Cf_n$ is given by $\partial(y, x) = (\partial y + f(x), -\partial x)$. Collecting the canonical inclusions $(Y_n \to Y_n \oplus X_{n-1})_{n\in\mathbb{Z}}$ into the first factor gives an inclusion $i\colon Y_* \to Cf_*$.

The *suspension* of a chain complex (X_*, ∂^X) is the chain complex

$$\Sigma X_* := S_* \otimes X_*$$

Explicitly, the symmetric monoidal structure determines the group in degree n as

$$(\Sigma X_*)_n = \bigoplus_{i+j=n} S_i \otimes_R X_j = R\{s_1\} \otimes_R X_{n-1} \cong X_{n-1},$$

with the differential $\partial\colon \Sigma X_{n+1} \to \Sigma X_n$ given by $-\partial^X\colon Xn \to X_{n-1}$. Assembling the canonical projections $(Y_n \oplus X_{n-1} \to X_{n-1})_{n\in\mathbb{Z}}$ onto the second factor gives a chain map $q\colon Cf_* \to \Sigma X_*$. In particular, there is a short exact sequence of chain complexes

$$0 \longrightarrow Y_* \xrightarrow{i} Cf_* \xrightarrow{q} \Sigma X_* \longrightarrow 0. \tag{A.1.4}$$

The *chain homology functor* $H\colon \mathrm{Ch}(\mathrm{Mod}_R) \to \mathrm{grAb}$ associates to each chain complex $X_* = (X_*, \partial)$ a graded abelian group $H_n(X)$, and to each chain map $f\colon X_* \to Y_*$ a homomorphism $f_* = H_*(f)\colon H_n(X) \to H_n(Y)$. If the homomorphism f_* is an isomorphism, we say that the chain map f is a *quasi-isomorphism.*

Lemma A.1.5. *Consider a chain complex X_* and a subcomplex A_* of X_*. Let $i\colon A_* \to X_*$ denote the inclusion and $j\colon X_* \to X_*/A_*$ the projection onto the quotient complex. There is a long exact sequence*

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \longrightarrow H_n(A/X) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

in homology, where the connecting homomorphism is given by $\partial[y + X_*] = [\partial y]$ for where $y \in Y_*$ with $\partial y \in X_*$.

Letting Σ denote the functor $\mathrm{Ch(Ab)} \to \mathrm{Ch(Ab)}$ sending an object to its suspension, there are natural isomorphisms $E\colon H_n(-) \to H_{n+1}(-) \circ \Sigma$ for each n. In the context of the previous previous lemma, this leads to a diagram

$$\begin{array}{ccccccc}
H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X/A) & \xrightarrow{\partial} & H_{n-1}(A) \\
E\downarrow\cong & & \cong\downarrow E & & \cong\downarrow E & & \cong\downarrow E \\
H_{n+1}(\Sigma A) & \longrightarrow & H_{n+1}(\Sigma X) & \longrightarrow & H_{n+1}(\Sigma X/\Sigma A) & \xrightarrow{\partial} & H_n(\Sigma A),
\end{array}$$

where both rows are part of long exact sequences. Naturality tells us that the two first squares commute. To determine the relation between the connecting homomorphisms in the last square, take a cycle $x + A_n \in X_n/A_n$. The image $\partial^X x$ of this cycle in A_{n-1} corresponds to a cycle $\partial^X x \in \Sigma A_n$ under E. Going the other way, we find that $\partial\colon H_{n+1}(\Sigma X/\Sigma A) \to H_n(\Sigma A)$ sends x to $\partial^{\Sigma X} x = -\partial^X x$. In particular, we conclude that the last square anti-commutes.

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