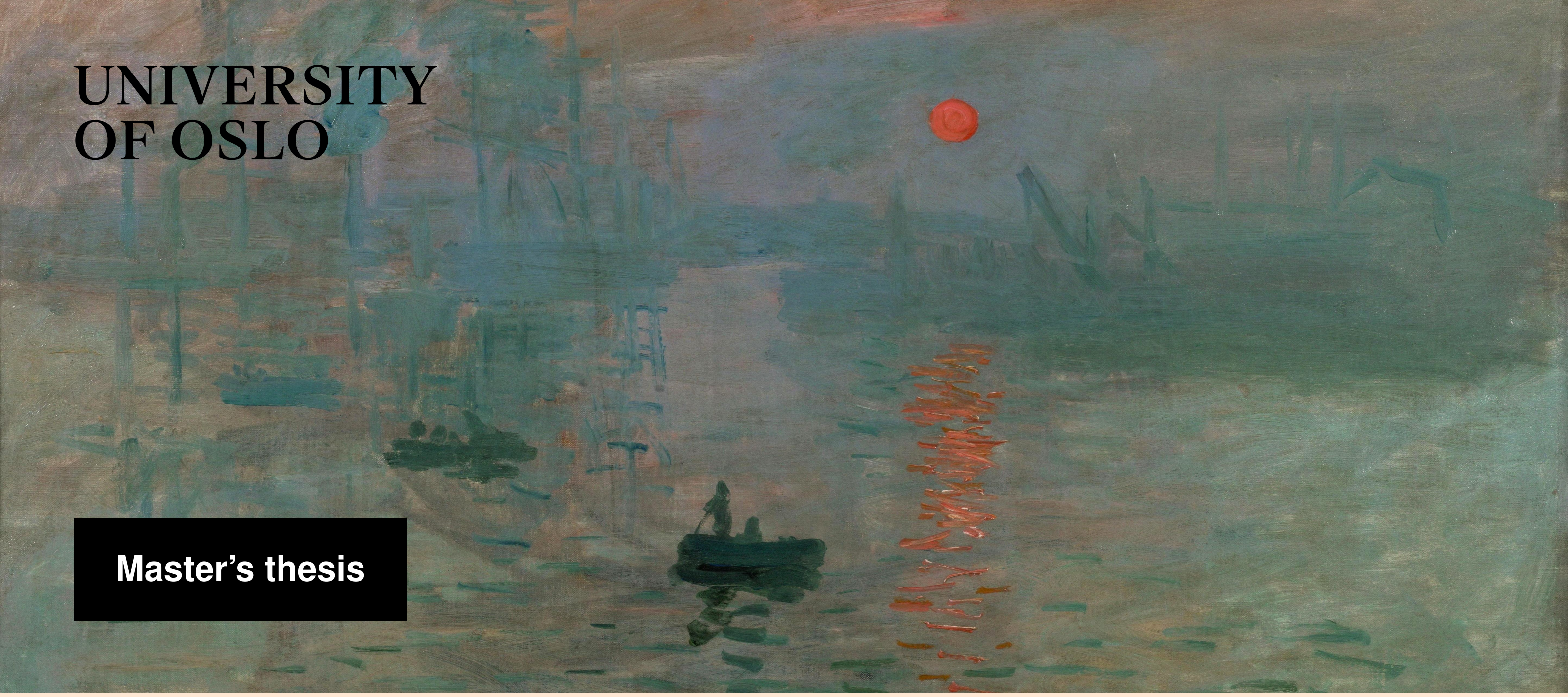

Iterated Thom Spectra

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Abstract

We use the monoidal Grothendieck construction and the theory of lifts of lax $\mathcal{O}$-monoidal maps presented in [ACB19] to propose an alternative construction of the relative Thom spectrum defined by J. Beardsley in [Bea17]. The original method relies on the Kan complex to be reduced; our method replaces this condition with an additional monoidality requirement on the fibration. This allows us to apply our construction to systems of invertible modules where the source is a non-connected Kan complex.

The cover image is Impression, Sunrise (1872) by Claude Monet, collection of Musée Marmottan Monet, Paris

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Chapter 1

Introduction

The Thom spectrum has been proven to be a useful machinery in stable homotopy theory since its introduction by J.M. Boardman [Boa69]. In recent years, mathematicians such as M. Ando, A. Blumberg, D. Gepner, M.J. Hopkins, C. Rezk, and others have published a series of papers where they have developed a modern approach to the Thom functor that extended the classic treatment by L.G. Lewis, J.P. May, and M. Steinberger [LMS86], see for example [And+14b], [And+14a] and [ABG18]. In particular, in the last two aforementioned papers, the authors extended the functor to the framework of quasi-categories originally by A. Joyal in [Joy02] and by J. Lurie in [Lur09]. Their work suggests that ∞-categories constitute a natural setting for the Thom functor.

Let X be an associative monoidal Kan complex and R be a symmetric ring spectrum, a system of invertible R-modules is a map $\xi : X \to \mathrm{Pic}(R)$; where $\mathrm{Pic}(R) \subseteq \mathrm{LMod}_R(\mathrm{Sp})$ is the core of the full subcategory of invertible left R-modules. In [And+14a] M. Ando et al. defined the ∞-categorical Thom spectrum $\mathrm{Th}_R(\xi)$ of the system ξ as the colimit of the composition of ξ with the inclusion $\mathrm{Pic}(R) \hookrightarrow \mathrm{LMod}_R(\mathrm{Sp})$.

Suppose that, in addition to the system of invertible R-modules $\xi : X \to \mathrm{Pic}(R)$, we are given an essentially surjective left fibration $\pi : X \to B$ where B is another associative monoidal Kan complex. To each element $b \in B$ we can associate the system of invertible R-modules $\xi_b : \pi^{-1}(b) \hookrightarrow X \to \mathrm{Pic}(R)$, and consequentially the left R-module $\mathrm{Th}_R(\xi_b)$. We can make this association functorial by defining it as the left Kan extension of ξ composed with $\mathrm{Pic}(R) \hookrightarrow \mathrm{LMod}_R(\mathrm{Sp})$ along π

$$\begin{tikzcd} X \arrow[r, "\xi"] \arrow[dr, "\pi"'] & \mathrm{Pic}(R) \arrow[r, hook] \arrow[d, phantom, "\Downarrow"] & \mathrm{LMod}_R(\mathrm{Sp}) \\ & B. \arrow[ur, dashed, "\mathrm{Th}_R(\xi)^B"'] & \end{tikzcd}$$

Utilizing the theory of operadic left Kan extension or the monoidal Grothendieck construction, it is possible to define the map $\mathrm{Th}_R(\xi)^B$ as a lax monoidal map and use its monoidality to equip the spectra $\mathrm{Th}_R(\xi_b)$ with a left $\mathrm{Th}_R(\xi_1)$-module structure; effectively producing a lax monoidal functor

$$\mathrm{Th}_R(\xi)^B : B \to \mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}(\mathrm{Sp}).$$

Now that we have defined a map from a Kan complex to the category of left $\mathrm{Th}_R(\xi_1)$-modules it is natural to ask if the map factors through $\mathrm{Pic}(\mathrm{Th}_R(\xi_1))$, that is to say, if we can consider the map $\mathrm{Th}_R(\xi)^B$ as a system of invertible $\mathrm{Th}_R(\xi_1)$-modules with base the Kan complex B. The answer is in general no since the spectra $\mathrm{Th}_R(\xi_b)$ are usually not invertible left $\mathrm{Th}_R(\xi_1)$-modules. However, if you assume that X and B are grouplike Kan complexes and the map $\mathrm{Th}_R(\xi)^B$ is (strong) monoidal, we can use the fact that B is grouplike to produce the following equivalence for each $b \in B$

$$\mathrm{Th}_R(\xi_b) \otimes_{\mathrm{Th}_R(\xi_1)} \mathrm{Th}_R(\xi_{\bar{b}}) \simeq \mathrm{Th}_R(\xi_{b \cdot \bar{b}}) \simeq \mathrm{Th}_R(\xi_1),$$

where $\bar{b}$ is the homotopy inverse of the element b. The equivalence ensures that R-module $\mathrm{Th}_R(\xi_b)$ admits a structure as an invertible $\mathrm{Th}_R(\xi_1)$-module.

J. Beardsley in [Bea17], utilizing the theory of operadic left Kan extensions, proved that if π is an $\mathbb{E}_n$-monoidal map of reduced grouplike Kan complexes and ξ is $\mathbb{E}_n$-monoidal, then the map $\mathrm{Th}_R(\xi)^B$ is $\mathbb{E}_{n-1}$-monoidal and it defines an $\mathbb{E}_{n-1}$-monoidal system of invertible $\mathrm{Th}_R(\xi_1)$-modules

$$\mathrm{Th}_R(\xi)^B : B^{\otimes} \to \mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes}.$$

After defining this new system, the author applied once again the Thom functor to obtain a left $\mathrm{Th}_R(\xi_1)$-module which he called relative Thom spectrum of ξ along π, and proved that it recovers the original Thom spectrum $\mathrm{Th}_R(\xi)$ as an $\mathbb{E}_{n-1}$-algebra.

While the grouplike condition is not too restrictive, the reduced condition prevents us from applying the construction to some interesting cases such as, for example, the symmetric spherical fibration $J_{\mathrm{gp}} : \mathbb{Z} \times \mathrm{BU} \to \mathrm{Pic}(\mathbb{S})$ given by the group-completion of the so-called J map [Hop18].

In this thesis, we provide an alternative construction of the relative Thom spectrum, which we will call the iterated Thom spectrum. This construction relies on the monoidal Grothendieck construction instead of the theory of operadic left Kan extensions and allows us to generalize Beardsley's results to non-connected grouplike Kan complexes; provided that we are willing to assume additional monoidality on the fibration π. The main advantage of our construction is that we can consider for the fibration π the projection on the path components of X. This provides plenty of examples where it is possible to apply the iterated Thom spectrum construction, and in particular, will allow us to produce a symmetric monoidal system of invertible MU-modules $\mathbb{Z} \to \mathrm{Pic}(\mathrm{MU})$ from the symmetric monoidal spherical fibration $J_{\mathrm{gp}} : \mathbb{Z} \times \mathrm{BU} \to \mathrm{Pic}(\mathbb{S})$ along the projection on the path components of $\mathbb{Z} \times \mathrm{BU}$. From our main theorem, Theorem 5.4.4, will follow that these two systems produce the same $\mathbb{E}_\infty$-structure of the periodic complex cobordism spectrum MUP.

1.1 Outline

The thesis is constituted of four further chapters. In the next three chapters, we will present the reader with some basic definitions and results of the theory of ∞-operads and the ∞-categorical version of the Thom functor; no claim of originality is made on the content of these preliminary chapters. In chapter five we will use the results presented in the previous part to develop the construction of the iterated Thom spectrum.

Let us explain in more detail what is the content of the specific chapters.

In chapter two we will give an introduction to the theory of ∞-operads developed by J. Lurie in Higher Algebra, [Lur17]; presenting some of the fundamental definitions and constructions involving ∞-operads with the intention of providing, when possible, an intuition by presenting first the analogous constructions and definitions on the 1-categorical level. This chapter will cover Section 2.1 and the first half of Section 2.2 of [Lur17].

In chapter three, we will see how common algebraic notions can be generalized to the context of ∞-operads. In particular, we will give the definitions of the ∞-categories of associative left modules and bimodules; with a focus on the characterization of the relative tensor product via the bar construction. In the rest of the chapter, we will generalize these notions to a general ∞-operad $\mathcal{O}^\otimes$ and then introduce the ∞-categorical version of the little cubes operads.

Chapter four will conclude the exposition of the preliminary material. We will present the ∞-categorical construction of the Thom spectrum functor, starting from the additive case defined by M. Ando et al. in [And+14a] and then following with the monoidal case defined in [ABG18]. We will then present the work of O. Antolín-Camarena and T. Barthel where the authors developed a theory of lifts of lax $\mathcal{O}$-monoidal maps; which they then used to define a microcosmic version of the monoidal Thom functor [ACB19].

The last chapter is dedicated to the construction of the iterated Thom spectrum. In the first two sections of this chapter, we will see how it is possible to use the theory of monoidal principal G-bundles to factor certain lax monoidal pre-sheaves through (strong) monoidal maps; this is the content of Proposition 5.3.3 which, as we will see, will play a crucial role in the construction of the iterated Thom spectrum. In Section 5.3, we will prove the main theorem of the thesis, Theorem 5.4.4, which states that starting from the iterated Thom spectrum it is possible to partially recover the original Thom spectrum.

1.2 For the reader who is not familiar with ∞-categories

With the idea that a reader who is not familiar with the ∞-categorical framework developed by J.Lurie in [Lur09] might still be interested in understanding the main ideas behind the construction of the relative Thom spectrum; we will provide some informal intuitions on how ∞-categories work. These are not to be considered formal statements.

- **∞-categories are simplicial sets that satisfy certain properties.** We can use the simplicial structure to define the ∞-categorical analogue of familiar 1-categorical notions. Let $\mathcal{C}$ be an ∞-category, then:

  - The zero-cells, or vertices, play the role of the objects of the ∞-category.

  - The one-cells are to be considered morphisms of $\mathcal{C}$. For example, the degenerate one-cell of a vertex $v \in \mathcal{C}$ is the identity morphism of v.

  - A two-cell σ of $\mathcal{C}$ represent a "homotopy" that portrays the morphism $d_1(\sigma)$ as the composition of the morphisms $d_2(\sigma)$ and $d_0(\sigma)$. For example, the degenerate two-cells of a morphism $\alpha : v \to w$ express the unitality of the identity under composition

$$\begin{tikzcd} & w & \\ v \arrow[ur, "\alpha"] \arrow[rr, "\alpha"] & & w \arrow[ul, "id"'] \end{tikzcd} \qquad \begin{tikzcd} & w & \\ v \arrow[ur, "\alpha"] \arrow[rr, "id"] & & v. \arrow[ul, "\alpha"'] \end{tikzcd}$$

  - The three-cells indicate that the "homotopies" represented by the faces commute, and so on for higher dimensional cells.

  The condition that we required on the simplicial sets ensures that it is always possible to compose two subsequent morphisms, i.e., there always exists a two-cell that represents the commutativity between two subsequent morphisms and a third morphism that plays the role of their composition; even if the composition is in general not unique.

  By identifying homotopic morphisms of $\mathcal{C}$ we obtain a unique composition of classes of morphisms that equips the set of homotopic objects with the structure of a 1-category. This 1-category is called the homotopic category of $\mathcal{C}$ and it is usually denoted by $h\mathcal{C}$.

- **For a diagram to be commutative is a structure, not a property.** When we want to prove the commutativity of a diagram we need to specify cells of the ∞-category that make the diagram commute.

- **Functor of ∞-categories are just morphisms between their underlying simplicial sets.** Moreover, there is a natural way to define the simplicial set of functors between two ∞-categories; which can be proven to be an ∞-category itself [Lur17, Prop. 1.2.7.3].

- **We can model topological spaces with ∞-categories.** Starting from a topological space X we can consider its singular set $\mathrm{Sing}(X)$. This simplicial set belongs to a particular class of ∞-categories called Kan complexes. Loosely speaking, a Kan complex is an ∞-category where every morphism is invertible. It can be proven that the Kan complexes constitute a model for the weak homotopic category of topological spaces [Gep20, Remark 1.4.2].
- **Most of the familiar 1-categorical constructions have their analogue in the ∞-categorical framework.** Notions like adjunctions, left and right Kan extensions, and the Grothendieck construction have been generalized to ∞-categories.

If, somehow, we sparked the interest of the reader the author believes that [Gep20] by D. Gepner and [Gro20] by M. Groth constitute a good introduction to the theory of ∞-categories.

1.3 Notation

We have chosen to use a notation that is as close as possible to the one used by J. Lurie in Higher Algebra; with the idea that an interested reader will be facilitated to integrate the preliminary chapters of this thesis with some of the more in-depth results presented in Higher Algebra. In this document, we will use $\mathcal{S}$ to refer to the ∞-category of Kan complexes and Sp to refer to the stable ∞-category of spectra. Moreover, we will usually denote ∞-categories with calligraphic capital letters and Kan complexes with capital letters. Suppose that $p : \mathcal{C} \to \mathcal{D}$ is a map of ∞-categories; if the functor p is clear from context we will denote by $\mathcal{C}_X$ the fiber of p over the object X of $\mathcal{D}$ and by p_X the composition $\mathcal{C}_X \hookrightarrow \mathcal{C} \xrightarrow{p} \mathcal{D}$.

1.4 Acknowledgements

I would like to begin by expressing my profound gratitude to my supervisor, John Rognes. Throughout these two years, John has supported and motivated me far beyond what any student could hope for. Thank you for your patience and your help.

To my family, none of this would have been possible without your support. For this, I will be forever grateful. It is impossible to put into words how happy I am knowing that regardless of the distance, I can always share beautiful moments like this with you. Vi voglio bene

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Chapter 2

∞-operads

In this chapter, we will give a brief introduction to the theory of ∞-operads following Higher Algebra by J. Lurie, [Lur17], presenting some of the basic definitions and results. In order to maintain the content as accessible as possible we will often state weaker versions of the results present in Higher Algebra. This will allow us to introduce less technical machinery and make the connection with the analogous 1-categorical concepts more explicit.

Before introducing the definition of ∞-operads let us focus first on how to generalize the notion of a symmetric monoidal 1-category to the ∞-categorical framework. This example is crucial for the understanding of the definition of ∞-operads and will motivate most of the constructions that we will present in Chapter 2 and Chapter 3.

We recall that a symmetric monoidal 1-category $\mathcal{C}$ is a 1-category equipped with:

(1) A unit object $1 \in \mathcal{C}$.

(2) A product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

(3) An associative natural equivalence $\alpha : (a \otimes b) \otimes c \simeq a \otimes (b \otimes c)$ such that the following diagram commmute

$$\begin{tikzcd}
a \otimes (b \otimes (c \otimes d)) \arrow[r, "\alpha"] \arrow[d, "id\otimes\alpha"] & (a \otimes b) \otimes (c \otimes d) \arrow[r, "\alpha"] & ((a \otimes b) \otimes c) \otimes d \arrow[d, "\alpha\otimes id"] \\
a \otimes ((b \otimes c) \otimes d) \arrow[rr, "\alpha"] & & (a \otimes (b \otimes c)) \otimes d
\end{tikzcd} \quad (\star)$$

for each $a, b, c, d \in \mathcal{C}$.

(4) Two unit natural equivalences $\rho : a \otimes 1 \simeq a$ and $\lambda : 1 \otimes b \simeq b$ such that the following diagrams commute

$$
\begin{tikzcd}
a \otimes (1 \otimes b) \arrow[rr, "\alpha"] \arrow[dr, "id\otimes\lambda"'] & & (a \otimes 1) \otimes b \arrow[dl, "\rho\otimes id"] \\
& a \otimes b &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \otimes 1 \arrow[dl, "\lambda"'] \arrow[dr, "\rho"] & \\
1 \arrow[rr, "id"] & & 1
\end{tikzcd}
$$

for each $a, b \in \mathcal{C}$.

Diagram ($\star$) is just the first of an infinite hierarchy of commutative diagrams that encode the associativity of the product of more and more elements of $\mathcal{C}$. In the 1-categorical case it is easy to prove that the commutativity of these diagrams follows from the commutativity of ($\star$) [ML98, Section XI.1].

One can try to generalize the previous definition to ∞-categories by defining a naive symmetric monoidal ∞-category as an ∞-category equipped with analogous structures. However, providing the two-cells that make the analogue of diagram ($\star$) commute is no longer sufficient to ensure the commutativity of the higher hierarchy diagrams; for each diagram involving the associativity of multiple objects of $\mathcal{C}$ one has to provide cells that make the diagram commute. Therefore, giving an example of a symmetric monoidal ∞-category would require specifying an infinite hierarchy of cells of the ∞-category $\mathcal{C}$.

In order to avoid this we will repackage the definition of symmetric monoidal 1-category by giving an equivalent definition that will generalize easily to ∞-categories.

Construction 2.0.1. [Lur17, Construction 2.0.0.1] Let $\mathcal{C}$ be a symmetric monoidal 1-category. We define $\mathcal{C}^{\otimes}$ to be the 1-category where:

(1) The objects are finite sequences of objects of $\mathcal{C}$, possibly empty.

(2) A morphism f between two objects $[C_1, \dots, C_n]$, $[C'_1, \dots, C'_m]$ of $\mathcal{C}^{\otimes}$ consists of:

- a subset $S \subseteq \{1, \dots, n\}$;
- a map of finite sets $\alpha : S \to \{1, \dots, m\}$;
- and a collection of morphisms $\{f_j : \bigotimes_{\alpha(i)=j} C_i \to C'_j\}_{1 \le j \le m}$ of $\mathcal{C}$.

(3) The composition of two morphisms $f = (S, \alpha, \{f_j\}) : [C_1, \dots, C_n] \to [C'_1, \dots, C'_m]$ and $g = (T, \beta, \{g_k\}) : [C'_1, \dots, C'_m] \to [C''_1, \dots, C''_l]$ of $\mathcal{C}^{\otimes}$ is given by:

- the subset $U = \alpha^{-1}T \subseteq \{1, \dots, n\}$ of S;
- the composition $\beta \circ \alpha : U \to \{1, \dots, l\}$;
- and the collection of maps

$$\bigotimes_{(\beta\circ\alpha)(i)=k} C_i \simeq \bigotimes_{\beta(j)=k} \bigotimes_{\alpha(i)=j} C_i \xrightarrow{\otimes f_j} \bigotimes_{\beta(j)=k} C'_j \xrightarrow{g_k} C''_k$$

for $1 \leq k \leq l$.

The 1-category $\mathcal{C}^{\otimes}$ admits a natural functor to another 1-category that in some ways plays the role of the "prototype" symmetric monoidal 1-category; we will see that this functor encodes the symmetric monoidal structure of $\mathcal{C}$.

Definition 2.0.2. [Lur17, Notation 2.0.0.2] Let I be a finite set, we denote by I_* the set $I \coprod \{*\}$. For each $n \geq 0$ we denote by $\langle n \rangle^{\circ}$ the finite set $\{1, \ldots, n\}$ and by $\langle n \rangle$ the pointed finite set $\langle n \rangle^{\circ}_{*} = \{*, 1, \ldots, n\}$.

We define $\mathcal{F}\mathrm{in}_*$ to be the 1-category where:

(1) The objects are the sets $\langle n \rangle$ where $n \geq 0$.

(2) A morphism between two objects $\langle m \rangle$ and $\langle n \rangle$ of $\mathcal{F}\mathrm{in}_*$ is a function $\alpha : \langle m \rangle \to \langle n \rangle$ with $\alpha(*) = *$.

(3) And the composition of two morphisms is defined in the obvious way.

Let $n \geq 0$. For each $1 \leq i \leq n$ we denote by $\rho^i : \langle n \rangle \to \langle 1 \rangle$ the morphism of $\mathcal{F}\mathrm{in}_*$ given by the formula

$$\rho^i(j) = \begin{cases} 1 & \text{if} \quad i = j \\ * & \text{otherwise.} \end{cases}$$

We observe that $\mathcal{F}\mathrm{in}_*$ is the 1-category obtained by applying Construction 2.0.1 to the simplest symmetric monoidal 1-category, i.e., the final 1-category $\{*\}$ equipped with the trivial product.

The 1-category $\mathcal{C}^{\otimes}$ defined in Construction 2.0.1 admits a forgetful functor $p : \mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$, which satisfies two special properties:

(1) Is a (op-fibration) coCartesian fibration; therefore each morphism $\alpha : \langle m \rangle \to \langle n \rangle$ of $\mathcal{F}\mathrm{in}_*$ defines a unique, up to isomorphism, functor $\alpha_! : \mathcal{C}^{\otimes}_{\langle m \rangle} \to \mathcal{C}^{\otimes}_{\langle n \rangle}$ between the fibers.

(2) The functors $\rho^i_!$ define an isomorphism between the fiber $\mathcal{C}^{\otimes}_{\langle n \rangle}$ of p over the object $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$ and the n-fold Cartesian product of $\mathcal{C}$.

These two properties ensure that it is possible to recover the symmetric monoidal 1-category $\mathcal{C}$ from the forgetful functor p up to isomorphism, [Lur17, Remark 2.0.0.6]. The discussion above motivates the correct definition of symmetric monoidal ∞-category.

Definition 2.0.3. A symmetric monoidal ∞-category is a coCartesian functor $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ between ∞-categories, such that: for each $n \geq 0$ the functors $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ induce an equivalence $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n$.

CoCartesian morphisms and coCartesian fibrations play a central role in the theory of

∞-operad. In Proposition A.1.1 we give a characterization of coCartesian morphisms. For a more in-depth exposition we refer the reader to Section 2.4.1 of [Lur09].

2.1 Definitions

Let us start by recalling the 1-categorical definition of a colored operad.

Definition 2.1.1. [Lur17, Def. 2.1.1.1] A colored operad $\mathcal{O}$ is a 1-category equipped with the following structures:

(1) A collection of objects $\{X, Y, Z, \dots\}$ which we will refer to as the colors of the operad $\mathcal{O}$.

(2) For every finite set I, every I-indexed collection of objects $\{X_i\}_{i\in I}$, and every object $Y \in \mathcal{O}$, there is a set $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$ which will refer to as the set of morphisms, or operations, from $\{X_i\}_{i\in I}$ to Y.

(3) For every map of finite sets $I \to J$ having fibers $\{I_j\}_{j\in J}$, every I-indexed collection of objects $\{X_i\}_{i\in I}$, every J-indexed collection of objects $\{Y_j\}_{j\in J}$, and every object $Z \in \mathcal{O}$, there is map

$$\prod_{j\in J} \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I_j}, Y_j) \times \mathrm{Mul}_{\mathcal{O}}(\{Y_j\}_{j\in J}, Z) \to \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Z),$$

that we will call the composition map.

(4) A collection of morphisms $\{id_X \in \mathrm{Mul}_{\mathcal{O}}(\{X\}, X)\}$ which are left and right units for the composition on $\mathcal{O}$, that is to say: for every I-indexed collection $\{X_i\}_{i\in I}$ and every object $Y \in \mathcal{O}$, the compositions

$$\begin{aligned}\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y) \simeq & \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y) \times \{id_Y\} \\ \subseteq & \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y) \times \mathrm{Mul}_{\mathcal{O}}(\{Y\}, Y) \\ \to & \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y) \simeq & \prod_{i\in I}\{id_{X_i}\} \times \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y) \\ \subseteq & \prod_{i\in I}\{\mathrm{Mul}_{\mathcal{O}}(\{X_i\}, X_i)\} \times \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y) \\ \to & \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)\end{aligned}$$

coincide with the identity map from $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$ to itself.

(5) The composition is associative in the following sense: for every sequence of maps $I \to J \to K$ of finite sets, every collection of objects $\{W_i\}_{i\in I}$, $\{X_j\}_{j\in J}$, $\{Y_k\}_{k\in K}$,

and every object $Z \in \mathcal{O}$, the diagram

$$\begin{tikzcd}
& \prod_{j\in J} \mathrm{Mul}_{\mathcal{O}}(\{W_i\}_{i\in I_j}, X_j) \times \prod_{k\in K} \mathrm{Mul}_{\mathcal{O}}(\{X_j\}_{j\in J_k}, Y_k) \times \mathrm{Mul}_{\mathcal{O}}(\{Y_k\}_{k\in K}, Z) \arrow[dl] \arrow[ddr] & \\
\prod_{k\in K} \mathrm{Mul}_{\mathcal{O}}(\{W_i\}_{i\in I_k}, Y_k) \times \mathrm{Mul}_{\mathcal{O}}(\{Y_k\}_{k\in K}, Z) \arrow[ddr] & & \\
& & \prod_{j\in J} \mathrm{Mul}_{\mathcal{O}}(\{W_i\}_{i\in I_j}, X_j) \times \prod_{k\in K} \mathrm{Mul}_{\mathcal{O}}(\{X_j\}_{j\in J}, Z) \arrow[dl] \\
& \mathrm{Mul}_{\mathcal{O}}(\{W_i\}_{i\in I}, Z) &
\end{tikzcd}$$

commutes.

Remark 2.1.2. It is possible to recover from Definition 2.1.1 the notion of operads as originally defined by J.M. Boardman, R. M. Vogt, and J.P. May in [BV68] and [May72]; by considering a topological colored operad $\mathcal{O}$ with a single color, meaning that the sets of morphisms are equipped with topological structures and the compositions are continuous maps. In this section, we will see that the notion of ∞-operads generalizes to ∞-categories the notion of colored operad rather than the one of single-colored operad.

Similar to the case of symmetric monoidal categories, if we try to generalize Definition 2.1.1 directly to the context of ∞-categories the structures associated with a colored operad will require presenting an infinite hierarchy of coherent diagrams. We can instead repackage the structures that define a colored operad $\mathcal{O}$ in a 1-category equipped with a forgetful functor. The properties of this 1-category will motivate our definition of ∞-operads.

Construction 2.1.3. [Lur17, Construction 2.1.1.7] Let $\mathcal{O}$ be a colored operad. We define the 1-category $\mathcal{O}^{\otimes}$ as follows:

(1) The objects of $\mathcal{O}^{\otimes}$ are finite sequences of colors $X_1, \ldots, X_n \in \mathcal{O}$, possibility empty.

(2) Given two sequences of objects

$$X_1, \ldots, X_n \in \mathcal{O}, \quad Y_1, \ldots, Y_m \in \mathcal{O},$$

a morphism from $\{X_i\}_{1\leq i\leq n}$ to $\{Y_j\}_{1\leq j\leq m}$ is given by:

- a morphism $\alpha : \langle m\rangle \to \langle n\rangle$ of $\mathcal{F}\mathrm{in}_*$;
- together with a collection of morphisms

$$\{\phi_j \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{\alpha(i)=j}, Y_j)\}_{1\leq j\leq m}$$

of $\mathcal{O}$.

(3) Composition of morphisms in $\mathcal{O}^{\otimes}$ is given by the composition law of $\mathcal{F}\mathrm{in}_*$ and the composition of the colored operad $\mathcal{O}$.

We observe that, by construction, the 1-category $\mathcal{O}^{\otimes}$ comes equipped with a forgetful functor $\pi : \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$. It is possible to prove, [Lur17, Construction 2.1.17], that starting from the 1-category $\mathcal{O}^{\otimes}$ and the functor π we can recover the colored operad $\mathcal{O}$ up to a canonical equivalence. In particular:

- for each $\langle n\rangle \in \mathcal{F}\mathrm{in}_*$, the fiber $\mathcal{O}^{\otimes}_{\langle n\rangle} := \pi^{-1}(\langle n\rangle)$ is canonically equivalent to the 1-category $\mathcal{O}^n$;
- for every finite set I with $|I| = n$, every I-indexed collection $\{X_i\}_{i\in I}$, and every object $Y \in \mathcal{O}$ the set $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$ can be identified with the set of morphisms f of $\mathcal{O}^{\otimes}$ between $X_1, \ldots, X_n$ to Y such that $\pi(f)^{-1}(*) = *$;
- the composition of $\mathcal{O}$ can be recovered from the composition law of $\mathcal{O}^{\otimes}$.

This observation suggests that it is possible to give an equivalent definition of colored operads by considering an ordinary 1-category $\mathcal{O}^{\otimes}$ equipped with a forgetful functor $\pi : \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$; provided that we require that the forgetful functor satisfies certain properties that will allow us to reconstruct from π a unique colored operad. This approach will give us a characterization of colored operads that can easily be generalized to ∞-categories.

Definition 2.1.4. [Lur17, Def. 2.1.1.10] An ∞-operad is a functor $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ between ∞-categories which satisfies the following conditions:

(1) For every object $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$ and every inert morphism $f : \langle m\rangle \to \langle n\rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, that is a morphism of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ such that for each $i \in \langle n\rangle$ the inverse image $f^{-1}(i)$ has exactly one element, there exists a p-coCartesian morphism $\bar{f} : C \to C'$ in $\mathcal{O}^{\otimes}$ covering f. In particular, the morphism f induces a unique, up to equivalence, functor $f_! : \mathcal{O}^{\otimes}_{\langle m\rangle} \to \mathcal{O}^{\otimes}_{\langle n\rangle}$.

(2) Let $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$ and $C' \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ be two objects of $\mathcal{O}^{\otimes}$, let $f : \langle m\rangle \to \langle n\rangle$ ba a morphism of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and $\mathrm{Map}^{f}_{\mathcal{O}^{\otimes}}(C, C')$ be the union of the connected components of $\mathrm{Map}_{\mathcal{O}^{\otimes}}(C, C')$ that lie over f. For every choice of p-coCartesian morphisms $\bar{\rho}_i : C' \to C'_i$ of $\mathcal{O}^{\otimes}$ lying over the inert morphisms $\rho^i : \langle n\rangle \to \langle 1\rangle$ for $1 \leq i \leq n$, the induced map
$$\mathrm{Map}^{f}_{\mathcal{O}^{\otimes}}(C, C') \to \prod_{1\leq i\leq n} \mathrm{Map}^{\rho^i\circ f}_{\mathcal{O}^{\otimes}}(C, C'_i)$$
is a homotopy equivalence.

(3) For each $n \geq 0$, the functors $\{\rho^i_! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}\}$ define an equivalence of ∞-categories $\mathcal{O}^{\otimes}_{\langle n\rangle} \to (\mathcal{O}^{\otimes}_{\langle 1\rangle})^n$. We will refer to $\mathcal{O} := \mathcal{O}^{\otimes}_{\langle 1\rangle}$ as the underlying ∞-category of $\mathcal{O}^{\otimes}$.

It is common to use ∞-operad to refer to the ∞-category $\mathcal{O}^{\otimes}$ rather than the functor

$p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Every time that we will refer to an ∞-category as an ∞-operad we will always assume that the ∞-category is equipped with a functor that satisfies the conditions of Definition 2.1.4.

Example 2.1.5. Let $\mathcal{O}$ be a colored operad. We have seen that $\mathcal{O}$ defines a unique forgetful functor $\mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ where $\mathcal{O}^{\otimes}$ is the 1-category defined in Construction 2.1.3; then, we can use the nerve of the forgetful functor $\mathrm{N}(\mathcal{O}^{\otimes}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ to define an ∞-operad. This observation allows us to realize our first examples of ∞-operads:

- The commutative ∞-operad $\mathrm{Comm}^{\otimes}$ which corresponds to the nerve of the identity functor $id : \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.
- The trivial ∞-operad $\mathrm{Triv}^{\otimes}$ which corresponds to the nerve of the inclusion of the subcategory $\mathbf{Triv}^{\otimes} \subseteq \mathcal{F}\mathrm{in}_*$ spanned by the inert morphisms of $\mathcal{F}\mathrm{in}_*$.

In Definition 2.1.4 we said that a morphism $f : \langle m \rangle \to \langle n \rangle$ of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is inert if it induces an injective map $\langle m \rangle^{\circ} \to \langle n \rangle^{\circ}$ of finite sets. By definition, if $\mathcal{O}^{\otimes}$ is an ∞-operad an inert morphism f induces a functor

$$\begin{aligned} f_! :\ \mathcal{O}^m \simeq \mathcal{O}^{\otimes}_{\langle m \rangle} &\longrightarrow\ \mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^n. \\ (X_1, \ldots, X_m) &\longmapsto (X_{f^{-1}(1)}, \ldots, X_{f^{-1}(n)}) \end{aligned}$$

The functor $f_!$ can be interpreted as a projection map composed with the natural action of the m-th symmetric group on $\mathcal{O}^m$ determined by the morphism f. Therefore, we expect that the inert morphisms will not provide any useful information on the monoidal structure of the ∞-category $\mathcal{O}^{\otimes}$. We will now define another class of morphisms of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ that, on the contrary, will play a crucial role in studying the ∞-category $\mathcal{O}^{\otimes}$.

Definition 2.1.6. We say that a morphism $g : \langle m \rangle \to \langle n \rangle$ of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is active if $g^{-1}(*) = *$. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. We say that a morphism γ of $\mathcal{O}^{\otimes}$ is inert if its image $p(\gamma)$ is an inert morphism of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and γ is p-coCartesian; and that γ is active if its image $p(\gamma)$ is an active morphism of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

In many practical cases, when one needs to prove a property involving the morphisms of $\mathcal{O}^{\otimes}$ it is usually easy to prove that the property is satisfied for the inert morphisms of $\mathcal{O}^{\otimes}$ and that is stable under compositions. Then, one can use the following proposition to reduce the proof to the case of active morphisms of $\mathcal{O}^{\otimes}$.

Proposition 2.1.7. [Lur17, Prop. 2.1.2.4] *Let $\mathcal{O}^{\otimes}$ be an ∞-operad. The collection of active and inert morphisms determines a factorization system, [Lur09, Def. 5.2.8.8], of the ∞-category $\mathcal{O}^{\otimes}$.*

Definition 2.1.8. [Lur17, Def. 2.1.2.7] Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and $p' : \mathcal{O}'^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be two ∞-operads. A map of ∞-operads, or operadic map, is a map of simplicial sets

$F : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ satisfying the following conditions:

(1) The diagram of simplicial sets

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{O}'^{\otimes} \arrow[dl, "p'"] \\ & \mathrm{N}(\mathcal{F}\mathrm{in}_*) & \end{tikzcd}$$

commutes.

(2) The functor F carries inert morphisms of $\mathcal{O}^{\otimes}$ to inert morphisms of $\mathcal{O}'^{\otimes}$. That is to say, the functor F preserves coCartesian morphisms if their image in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is inert.

We denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ spanned by maps of ∞-operads. The notation is justified by the fact that in some cases it is possible to interpret maps of ∞-operads from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$ as $\mathcal{O}$-algebras of $\mathcal{O}'^{\otimes}$. We will discuss this in more detail in the next section after defining $\mathcal{O}$-monoidal ∞-categories.

2.2 $\mathcal{O}$-monoidal categories and their algebras

In this section, we will present a particular class of operadic maps, the coCartesian fibrations of ∞-operads, that we will then use to define an $\mathcal{O}$-monoidal generalization of the notion of symmetric monoidal ∞-categories. We will then follow by introducing the $\mathcal{O}$-monoidal analogue of algebras and monoidal functors.

Proposition 2.2.1. [Lur17, Prop. 2.1.2.12] *Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration. Then, the following conditions are equivalent:*

(1) The composite map $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad.

(2) For every object $T \simeq T_1 \oplus \cdots \oplus T_n \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, the inert morphisms $T \to T_i$ induce an equivalence of ∞-categories $\mathcal{C}^{\otimes}_T \to \prod_{1 \leq i \leq n} \mathcal{C}^{\otimes}_{T_i}$.

Definition 2.2.2. [Lur17, Def. 2.1.2.13] A map $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads if it satisfies the equivalent conditions of Proposition 2.2.1. In this case, we say that p exhibits $\mathcal{C}^{\otimes}$ as an $\mathcal{O}$-monoidal category, or that p is an $\mathcal{O}$-monoidal structure for $\mathcal{C}^{\otimes}$. We will denote the pullback of simplicial sets $\mathcal{C} := \mathcal{C}^{\otimes} \underset{\mathcal{O}^{\otimes}}{\times} \mathcal{O}$ as the underlying category of the $\mathcal{O}$-monoidal category of $\mathcal{C}^{\otimes}$.

Informally, we can think of a coCartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ as a generalization of the forgetful functor defined in Definition 2.0.3, where the structure on the ∞-category $\mathcal{C}^{\otimes}$ is no longer encoded by the "prototype" symmetric monoidal ∞-category $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, but it is instead encoded by the "prototype" $\mathcal{O}$-monoidal category, which is the ∞-operad $\mathcal{O}^{\otimes}$ itself.

We observe that, since the map that defines an $\mathcal{O}$-monoidal category $\mathcal{C}^{\otimes}$ as an ∞-operad factors through the ∞-operad structure of $\mathcal{O}^{\otimes}$; the two notions of underlying category of $\mathcal{C}^{\otimes}$, i.e., the underlying category $\mathcal{C}^{\otimes}_{\langle 1\rangle}$ of $\mathcal{C}^{\otimes}$ considered as an ∞-operad, and the underlying category $\mathcal{C}^{\otimes} \underset{\mathcal{O}^{\otimes}}{\times} \mathcal{O}$ of $\mathcal{C}^{\otimes}$ considered as an $\mathcal{O}$-monoidal category, coincide.

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{O} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \{\langle 1\rangle\} \arrow[d, hook] \\
\mathcal{C}^{\otimes} \arrow[r, "p"] & \mathcal{O}^{\otimes} \arrow[r] & \mathrm{N}(\mathcal{F}\mathrm{in}_*).
\end{tikzcd}$$

By definition, if $\mathcal{O}^{\otimes}$ is an ∞-operad the inert morphisms of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ define functors between the fibers. Similarly, if $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is an $\mathcal{O}$-monoidal category the morphisms of $\mathcal{O}^{\otimes}$ define functors between the fibers of p; we will call these functors $\mathcal{O}$-operations. In contrast to the case of inert morphisms of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, the $\mathcal{O}$-operations defined by active morphisms usually provide useful information on the structure of the $\mathcal{O}$-monoidal category $\mathcal{C}^{\otimes}$. Let us formally define the $\mathcal{O}$-operations, and then see which information they provide in the case where $\mathcal{C}^{\otimes}$ is a Comm-monoidal category.

Definition 2.2.3. [Lur17, Remark 2.1.2.16] Let $\mathcal{O}^{\otimes}$ be an ∞-operad, $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads, and let $f : X \to Y$ be a morphism of $\mathcal{O}^{\otimes}$, where $X \in \mathcal{O}^{\otimes}_{\langle m\rangle}$ and $Y \in \mathcal{O}^{\otimes}_{\langle n\rangle}$. The coCartesian fibration p defines, up to equivalence, a functor

$$\otimes_f : \mathcal{C}^{\otimes}_X \to \mathcal{C}^{\otimes}_Y,$$

we will refer to this functor as the $\mathcal{O}$-operation defined by f. Let $\bar{X} \in \mathcal{C}^{\otimes}_X$ be an object of $\mathcal{C}^{\otimes}$, the image of $\bar{X}$ by the functor $\otimes_f$ is the target of the unique, up to equivalence, p-coCartesian morphisms of $\mathcal{C}^{\otimes}$ with source $\bar{X}$ covering f.

Let us see how we can recover the definition of symmetric monoidal ∞-category that we presented in the beginning of the chapter as a special case of Definition 2.2.2.

Example 2.2.4. A symmetric monoidal ∞-category is a coCartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathrm{Comm}^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Let $\beta : \langle 2\rangle \to \langle 1\rangle$ be the unique active morphism of $\mathrm{Comm}^{\otimes}$, we refer to the Comm-operation defined by β

$$\otimes_\beta : \mathcal{C}^2 \simeq \mathcal{C}^{\otimes}_{\langle 2\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{C}$$

as the tensor product of $\mathcal{C}$. Let $\eta : \langle 0\rangle \to \langle 1\rangle$ be the unique active morphism of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, we denote the Comm-operation defined by η

$$\otimes_\eta : \Delta^0 \simeq \mathcal{C}^{\otimes}_{\langle 0\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{C}$$

as the unit of $\mathcal{C}$. Using the universal property of coCartesian morphisms, it is possible to prove that the tensor product and the unit satisfy the usual axioms of a symmetric monoidal category up to homotopy [Lur17, Remark 2.1.2.20]. In particular, these operations endow the homotopy category $h\mathcal{C}$ with a symmetric monoidal structure.

Example 2.2.5. If $\mathcal{C}$ is an ∞-category that admits all finite products, then it is possible to define an essentially unique symmetric monoidal ∞-category $\mathcal{C}^{\times} \to \mathrm{Comm}^{\otimes}$ called Cartesian symmetric monoidal structure; which, with the process described above, will equip $\mathcal{C}$ with the Cartesian product $(C, C') \to C \times C'$. The ∞-category $\mathcal{C}^{\times}$ is formally defined in [Lur17, Section 2.4.1].

Now that we have defined the notion of $\mathcal{O}$-monoidal categories, we can introduce the concept of $\mathcal{O}$-algebras. In order to motivate our definition, we will first look at the 1-categorical case.

Remark 2.2.6. In the 1-categorical setting, a commutative algebra of a symmetric monoidal 1-category $(\mathcal{C}, \otimes, 1)$ is an object $A \in \mathcal{C}$ equipped with a unit

$$\eta : 1 \to A,$$

and an associative, unital, and commutative algebraic structure

$$\mu : A \otimes A \to A.$$

We now consider the 1-category $\mathcal{C}^{\otimes}$ as in Construction 2.0.1 and see how we can repackage the structure of the algebra A in terms of $\mathcal{C}^{\otimes}$. The algebra object defines:

- for each $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$ an object $A_{\langle n \rangle} := (A, \dots, A) \in \mathcal{C}^{\otimes}_{\langle n \rangle}$,
- for each morphism $f : \langle m \rangle \to \langle n \rangle$ a unique morphism $\bar{f} : A_{\langle m \rangle} \to A_{\langle n \rangle}$ covering f given by composing projections, the unit map, and the algebraic structure of A accordingly.

In particular, we can associate to the algebra object A of $\mathcal{C}$ a section of the forgetful functor π

$$\begin{tikzcd} \mathcal{F}\mathrm{in}_* \arrow[r, "A(\cdot)"] \arrow[dr, equal] & \mathcal{C}^{\otimes} \arrow[d, "\pi"] \\ & \mathcal{F}\mathrm{in}_*. \end{tikzcd}$$

The section $A(\cdot)$ maps the object $\langle m \rangle$ of $\mathcal{F}\mathrm{in}_*$ to the object $A_{\langle m \rangle}$ of $\mathcal{C}^{\otimes}$, and maps the morphism $f : \langle m \rangle \to \langle n \rangle$ to the morphisms $\bar{f} : A_{\langle m \rangle} \to A_{\langle n \rangle}$. Here the associativity, unitality, and commutativity of the algebraic structure ensure that the section is a well-defined functor.

Definition 2.2.7. [Lur17, Def. 2.1.3.1] Let $\mathcal{O}^{\otimes}$ be an ∞-operad, $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal ∞-category, and let $\alpha : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be an ∞-operad over $\mathcal{O}^{\otimes}$. We define the ∞-category of $\mathcal{O}'$-algebras of $\mathcal{C}^{\otimes}$ to be the full subcategory $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$ spanned by operadic maps. We observe that an object F of the ∞-category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ corresponds to an operadic map over the ∞-operad $\mathcal{O}^{\otimes}$

$$\begin{tikzcd} \mathcal{O}'^{\otimes} \arrow[r, "F"] \arrow[dr, "\alpha"'] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\ & \mathcal{O}^{\otimes}. \end{tikzcd}$$

In the special case where $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes}$ and the map $\alpha : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ is the identity of $\mathcal{O}^{\otimes}$ we denote the ∞-category of $\mathcal{O}$-algebra objects of $\mathcal{C}^{\otimes}$ as $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$.

Remark 2.2.8. Let $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) = \mathrm{Comm}^{\otimes}$ be an ∞-operad and let $p : \mathcal{C}^{\otimes} \to \mathrm{Comm}^{\otimes}$ be a symmetric monoidal ∞-category. We can define an $\mathcal{O}$-monoidal category $p' : \mathcal{C}'^{\otimes} \to \mathcal{O}^{\otimes}$ by considering the pullback of the symmetric monoidal structure p of $\mathcal{C}^{\otimes}$ along the map q

$$\begin{tikzcd} \mathcal{C}'^{\otimes} \arrow[r] \arrow[d, "p'"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\ \mathcal{O}^{\otimes} \arrow[r, "q"] & \mathrm{Comm}^{\otimes}. \end{tikzcd}$$

Limits of ∞-operads exist and are computed on the underlying ∞-categories. This, combined with the fact that coCartesian fibrations are stable under pullbacks [Lur09, Prop. 2.4.2.3], ensures that the map $p' : \mathcal{C}'^{\otimes} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads, i.e., $\mathcal{C}'^{\otimes}$ is an $\mathcal{O}$-monoidal category. By construction, the ∞-category $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}')$ of $\mathcal{O}$-algebras of $\mathcal{C}'^{\otimes}$ is equivalent to the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of operadic maps from $\mathcal{O}^{\otimes}$ to $\mathcal{C}^{\otimes}$. We will often use this equivalence implicitly by talking about $\mathcal{O}$-algebras of a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$.

In many practical cases, such as the little cubes ∞-operads that we will define in Section 3.3, the ∞-operad $\mathcal{O}^{\otimes}$ enjoys enough properties that will allow us to start from an $\mathcal{O}$-algebra $A : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ of $\mathcal{C}^{\otimes}$ and recover the classical notion of an object of $\mathcal{C}$ equipped with an algebraic structure and a unit. In particular, let $\mathcal{O}^{\otimes}$ be an ∞-operad that:

- is single-colored, meaning that there exists a full and faithful map $\Delta^0 \to \mathcal{O}^{\otimes}_{\langle 1 \rangle}$. Since $\mathcal{O}^{\otimes}$ is an ∞-operad every fiber $\mathcal{O}^{\otimes}_{\langle n \rangle}$ is equivalent to the discrete ∞-category with a single object
$$\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^{\times n} \simeq (\Delta^0)^{\times n} \simeq \Delta^0.$$
Abusing the notation, we will usually denote the unique object of the fiber $\mathcal{O}^{\otimes}_{\langle n \rangle}$ by $\langle n \rangle$; except for $n = 0$ where the standard notation for the object is $\emptyset \in \mathcal{O}^{\otimes}_{\langle 0 \rangle}$.
- Has a distinguished active morphism $\beta : \langle 2 \rangle \to \langle 1 \rangle$ covering the unique active

morphism $\langle 2\rangle \to \langle 1\rangle$ of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

- Is unital, meaning that the unique object $\emptyset \in \mathcal{O}^{\otimes}_{\langle 0\rangle}$ is both an initial and a final object of $\mathcal{O}^{\otimes}$.

Under these conditions, if $\mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal category and $A : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ is an $\mathcal{O}$-algebra of $\mathcal{C}^{\otimes}$, then:

- the $\mathcal{O}$-operation induced by the distinguished active morphism β defines a product functor
$$\otimes_{\beta} : \mathcal{C}^2 \simeq \mathcal{C}^{\otimes}_{\langle 2\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{C}$$
on the underlying category of $\mathcal{C}^{\otimes}$. Let ν be the unique morphism of $\mathcal{O}^{\otimes}$ from $\emptyset$ to $\langle 1\rangle$, the target of the unique p-coCartesian morphism $\bar{\nu}$ of $\mathcal{C}^{\otimes}$ covering ν defines an object $1_{\mathcal{C}}$ of $\mathcal{C}$ that plays the role of the unit of the product.

- The algebra A defines a unique object of the underlying category $\mathcal{C}$, naming the image of the unique point $\langle 1\rangle$ of $\mathcal{O}$ by the functor A. Since the functor A preserves inert morphisms and $\mathcal{C}^{\otimes}$ is $\mathcal{O}$-monoidal then
$$A(\langle n\rangle) \simeq \prod_{1\le i\le n} \rho^i_! A(\langle n\rangle) \simeq \prod_{1\le i\le n} A(\langle 1\rangle).$$
We will usually use the same notation to denote both the object $A(\langle 1\rangle) \in \mathcal{C}$ and the operadic map $A : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$.

- The image of the unique active morphism $\nu : \emptyset \to \langle 1\rangle$ by the functor $A : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ defines the unit morphism $A(\nu) : 1_{\mathcal{C}} \to A$ of the algebra. To construct the algebraic structure of A we start by considering the image of the distinguished active morphism β by the operadic map A, that is the morphism $A(\beta) : (A, A) \to A$ of $\mathcal{C}^{\otimes}$. Since $\mathcal{C}^{\otimes}$ is $\mathcal{O}$-monoidal we know that there exists a p-coCartesian $\beta_!$ morphism covering β and with source (A, A). The universal property of coCartesian morphisms allows, starting from the diagram of solid arrows, to fill the commutative diagram of $\mathcal{C}^{\otimes}$ with the dashed arrow

$$\begin{tikzcd} \left[\begin{array}{ccc} \langle 2\rangle & \xrightarrow{\beta} & \langle 1\rangle \end{array}\right] \arrow[rr, mapsto, "A(\cdot)"] \arrow[dr, mapsto, "id"'] & & \left[\begin{array}{ccc} (A,A) & \xrightarrow{A(\beta)} & A \\ & \searrow^{\beta_!} & \uparrow{\scriptstyle \mu} \\ & & A \otimes A \end{array}\right] \arrow[dl, mapsto, "p"] \\ & \left[\begin{array}{ccc} \langle 2\rangle & \xrightarrow{\beta} & \langle 1\rangle \\ & \searrow^{\beta} & \uparrow{\scriptstyle id} \\ & & \langle 1\rangle \end{array}\right]. & \end{tikzcd}$$

The dashed arrow defined above is the algebraic structure of A.

In the special case of $\mathcal{O}^{\otimes} = \mathrm{Comm}^{\otimes}$ this procedure will define an algebra object of the symmetric monoidal homotopy category $h\mathcal{C}$.

Remark 2.2.9. Let $\mathcal{O}^{\otimes}$ be an ∞-operad and $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, $q : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be two $\mathcal{O}$-monoidal categories. The $\mathcal{O}$-monoidal category $\mathcal{C}^{\otimes}$ is, in particular, an ∞-operad over $\mathcal{O}^{\otimes}$ and we can consider the ∞-category $\mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$ of $\mathcal{C}$-algebras of $\mathcal{D}^{\otimes}$. In this special case, is more useful to think of this ∞-category as the ∞-category of lax monoidal functors between $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ rather than interpreting it as the ∞-category of algebra objects of $\mathcal{D}^{\otimes}$. Let us explain why this is the case; suppose that the ∞-operad is single-colored and admits a morphism β covering the unique active morphism $\langle 2\rangle \to \langle 1\rangle$ of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and let F be an object of $\mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$. By definition, the object F corresponds to an operadic map $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over $\mathcal{O}^{\otimes}$. We will now prove that for each $C, C' \in \mathcal{C}$ the functor F defines a natural morphism

$$\gamma : F(C) \underset{\mathcal{D}}{\otimes} F(C') \to F(C \underset{\mathcal{C}}{\otimes} C')$$

of $\mathcal{D}$, where we are considering the products of $\mathcal{C}$ and $\mathcal{D}$ defined by β as described above. Let g be the p-coCartesian morphism of $\mathcal{C}^{\otimes}$ covering β with source (C, C') and h be the q-coCartesian morphism of $\mathcal{D}^{\otimes}$ covering β with source $(F(C), F(C'))$. We define the morphism γ as the dashed arrow obtained by applying the universal property of the q-coCartesian morphism h to the following diagram of solid arrows

$$\begin{tikzcd}
\left[\begin{tikzcd} (C,C') \arrow[r, "g"] & C \underset{\mathcal{C}}{\otimes} C' \end{tikzcd}\right] \arrow[rr, mapsto, "F"] \arrow[dr, "p"'] & & \left[\begin{tikzcd} (F(C),F(C')) \arrow[r, "F(g)"] \arrow[dr, "h"'] & F(C \underset{\mathcal{C}}{\otimes} C') \\ & F(C) \underset{\mathcal{D}}{\otimes} F(C') \arrow[u, dashed, "\gamma"'] \end{tikzcd}\right] \arrow[dl, "q"] \\
& \left[\begin{tikzcd} \langle 2\rangle \arrow[r, "\beta"] \arrow[dr, "\beta"'] & \langle 1\rangle \\ & \langle 1\rangle \arrow[u, "id"'] \end{tikzcd}\right]. &
\end{tikzcd} \tag{$\star$}$$

This is why, in this particular case, we will refer to the objects of $\mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$ as lax $\mathcal{O}$-monoidal maps from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$.

Remark 2.2.10. We observe that if $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a lax $\mathcal{O}$-monoidal map, for each ∞-operad $\mathcal{O}'^{\otimes}$ over $\mathcal{O}^{\otimes}$ the postcomposition with F induces a functor between the ∞-

categories of $\mathcal{O}'$-algebras

$$(F \circ -) : \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}).$$

We will refer to this functor as the functor induced by F on the $\mathcal{O}'$-algebras.

Definition 2.2.11. [Lur17, Def. 2.1.3.7] Let $\mathcal{O}^{\otimes}$ be an ∞-operad and $\mathcal{C}^{\otimes}$, $\mathcal{D}^{\otimes}$ be two $\mathcal{O}$-monoidal categories. We say that an ∞-operad map $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over $\mathcal{O}^{\otimes}$ is an $\mathcal{O}$-monoidal functor if it preserves coCartesian morphisms, i.e., it sends p-coCartesian morphisms of $\mathcal{C}^{\otimes}$ to q-coCartesian morphisms of $\mathcal{D}^{\otimes}$. We will denote by $\mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by $\mathcal{O}$-monoidal functors. Some authors refer to $\mathcal{O}$-monoidal functors as strong $\mathcal{O}$-monoidal functors and use $\mathcal{O}$-monoidal functors to refer to what we defined as lax $\mathcal{O}$-monoidal maps.

Let $\mathcal{O}^{\otimes}$ be a single-colored ∞-operad such that there exists a morphism β of $\mathcal{O}^{\otimes}$ covering the unqiue active morphism $\langle 2 \rangle \to \langle 1 \rangle$ and suppose that we have an $\mathcal{O}$-monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ of $\mathcal{O}$-monoidal categories. Applying the same procedure described above, for each pair of objects $C, C' \in \mathcal{C}$, we can define a morphism

$$\gamma : F(C) \underset{\mathcal{D}}{\otimes} F(C') \to F(C \underset{\mathcal{C}}{\otimes} C')$$

of $\mathcal{D}$. In this case, however, both the morphisms $F(g)$ and h of diagram $(\star)$ are q-coCartesian, and from the dual version of [Lur09, Prop. 2.4.1.7] follows that the morphism γ is q-coCartesian too. Now γ is a q-coCartesian morphism covering an equivalence of $\mathcal{O}^{\otimes}$; so we can apply [Lur09, Prop. 2.4.1.5] to prove that γ is an equivalence of $\mathcal{D}^{\otimes}$.

The difference between lax $\mathcal{O}$-monoidal maps and $\mathcal{O}$-monoidal maps will play a central role in Chapter 5. For example, the focus of Section 5.2 and Section 5.3 will be to prove that certain lax $\mathcal{O}$-monoidal maps are actually $\mathcal{O}$-monoidal. Let us introduce a useful proposition regarding the composition of $\mathcal{O}$-monoidal maps that we will use multiple times during the construction of the iterated Thom spectra. As a consequence of this result we will be able to prove that different notions of equivalence of $\mathcal{O}$-monoidal Kan complexes are equivalent.

Proposition 2.2.12. *Let $\mathcal{A}^{\otimes}$, $\mathcal{B}^{\otimes}$ and $\mathcal{C}^{\otimes}$ be three $\mathcal{O}$-monoidal categories. Suppose that we have the following commutative diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{A}^{\otimes} \arrow[r, "F"] \arrow[dr, "p"'] & \mathcal{B}^{\otimes} \arrow[r, "G"] \arrow[d, "p'"] & \mathcal{C}^{\otimes} \arrow[dl, "p''"] \\
& \mathcal{O}^{\otimes}, &
\end{tikzcd}$$

where F is $\mathcal{O}$-monoidal and essentially surjective. Then, G is $\mathcal{O}$-monoidal if and only if $G \circ F$ is $\mathcal{O}$-monoidal.

Proof. Suppose that the map G is $\mathcal{O}$-monoidal, since a composition of $\mathcal{O}$-monoidal maps is $\mathcal{O}$-monoidal then $G \circ F$ is $\mathcal{O}$-monoidal.

Suppose now that $G \circ F$ is $\mathcal{O}$-monoidal. Let $f : B \to B'$ be a p'-coCartesian morphism of $\mathcal{B}^{\otimes}$ covering the morphism $\alpha : X \to X'$ of $\mathcal{O}^{\otimes}$. We want to prove that $G(f)$ is p''-coCartesian. Let us consider first the case where B is in the image of the functor F and let $A \in \mathcal{A}^{\otimes}$ be such that $F(A) = B$.

Since $\mathcal{A}^{\otimes}$ is $\mathcal{O}$-monoidal there exists a p-coCartesian morphism $h : A \to A''$ covering α. F is an $\mathcal{O}$-monoidal map and h is p-coCartesian, then $F(h)$ is p'-coCartesian and its universal property defines the morphism ℓ covering the identity up to a contractible space of choices. Since both f and $F(h)$ are p'-coCartesian morphisms with the same source covering α, the morphism ℓ must be an equivalence.

$$\begin{tikzcd}
\begin{bmatrix} \begin{tikzcd} A \arrow[dr, "h"'] & \\ & A'' \end{tikzcd} \end{bmatrix} \arrow[r, mapsto, "F"] \arrow[dr, mapsto, "p"'] & \begin{bmatrix} \begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "F(h)"'] & B' \\ & B'' \arrow[u, "\ell", "\simeq"'] \end{tikzcd} \end{bmatrix} \arrow[r, mapsto, "G"] \arrow[d, mapsto, "p'"] & \begin{bmatrix} \begin{tikzcd} C \arrow[r, "G(f)"] \arrow[dr, "(G\circ F)(h)"'] & C' \\ & C'' \arrow[u, "G(\ell)", "\simeq"'] \end{tikzcd} \end{bmatrix} \arrow[dl, mapsto, "p''"] \\
& \begin{bmatrix} \begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\alpha"'] & X' \\ & X' \arrow[u, "id"'] \end{tikzcd} \end{bmatrix} . &
\end{tikzcd}$$

The image of ℓ by the functor G is again an equivalence, therefore a p''-coCartesian morphism [Lur09, Prop. 2.4.1.5]. Since $G \circ F$ is $\mathcal{O}$-monoidal and h is p-coCartesian then $(G \circ F)(h)$ is p''-coCartesian covering α. Applying the dual version of [Lur09, Prop. 2.4.1.7] we obtain that $G(f)$ is p''-coCartesian.

Let us now consider the case where B is not in the image of F. Since F is essentially surjective, there exists an object $\overline{B} \in \mathcal{B}^{\otimes}$ in the image of F and an equivalence $\gamma : \overline{B} \to B$. The composition $f \circ \gamma$ is p'-coCartesian so from the previous case it follows that $G(f \circ \gamma)$ is p''-coCartesian. Since γ is an equivalence $G(\gamma)$ is p''-coCartesian. We can apply again [Lur09, Prop. 2.4.1.7] to conclude that $G(f)$ is p''-coCartesian. □

Applying Proposition 2.2.12 we can relate different notions of equivalence of $\mathcal{O}$-monoidal

Kan complexes.

Corollary 2.2.13. *Let $p : \mathcal{A}^{\otimes} \to \mathcal{O}^{\otimes}$, $p' : \mathcal{B}^{\otimes} \to \mathcal{O}^{\otimes}$ be two $\mathcal{O}$-monoidal Kan complexes, i.e., $\mathcal{O}$-monoidal categories such that their underlying categories are Kan complexes, and let $F : \mathcal{A}^{\otimes} \to \mathcal{B}^{\otimes}$ be an $\mathcal{O}$-monoidal map. The following statements are equivalent:*

(1) The map F is an equivalence of $\mathcal{O}$-monoidal categories, that is, there exists an $\mathcal{O}$-monoidal map $G : \mathcal{B}^{\otimes} \to \mathcal{A}^{\otimes}$ such that $F \circ G \simeq id$ and $G \circ F \simeq id$.

(2) The map F is an equivalence of ∞-categories, i.e., there exists an inverse functor $G : \mathcal{B}^{\otimes} \to \mathcal{A}^{\otimes}$ but we are not asking for the inverse to be $\mathcal{O}$-monoidal.

(3) For each $Z \in \mathcal{O}$ the map F induces an equivalence on the fibers over Z, i.e., the map $F_Z : \mathcal{A}^{\otimes}_Z \to \mathcal{B}^{\otimes}_Z$ is an equivalence of ∞-categories.

Proof. It is trivial to check that (1) $\implies$ (2). Implications (2) $\Leftrightarrow$ (3) are proven in [Lur17, Remark 2.1.3.8].

For (2) $\implies$ (1), we first observe that since $\mathcal{A}$ and $\mathcal{B}$ are Kan complexes then p and p' are right fibrations [Lur09, Prop. 2.4.2.4]. In view of [Lur09, Lemma. 2.2.3.16] the functor F is an equivalence of $\mathrm{Set}_{\Delta/\mathcal{O}^{\otimes}}$, i.e., there exists a functor $G : \mathcal{B}^{\otimes} \to \mathcal{A}^{\otimes}$ over $\mathcal{O}^{\otimes}$ which is inverse to F and such that the following diagram of simplicial sets commutes

$$\begin{tikzcd} \mathcal{A}^{\otimes} \arrow[r, "F"] \arrow[dr, "p"'] & \mathcal{B}^{\otimes} \arrow[r, "G"] \arrow[d, "p'"] & \mathcal{A}^{\otimes} \arrow[dl, "p"] \\ & \mathcal{O}^{\otimes}. & \end{tikzcd}$$

We can apply Proposition 2.2.12 to F and G to prove that G is $\mathcal{O}$-monoidal. Here F is an equivalence and therefore is essentially surjective, and the composition $G \circ F$ is equivalent to the identity, which is an $\mathcal{O}$-monoidal map. $\square$

2.3 Some constructions of $\mathcal{O}$-monoidal categories

In this section, we will see that under some reasonable conditions familiar 1-categorical constructions can be generalized to $\mathcal{O}$-monoidal categories. In particular, we will see under which conditions full subcategories and ∞-overcategories of an $\mathcal{O}$-monoidal category are again $\mathcal{O}$-monoidal.

2.3.1 Full subcategories

Starting from a full subcategory $\mathcal{D}$ of the underlying category $\mathcal{C}$ of an $\mathcal{O}$-monoidal category $\mathcal{C}^{\otimes}$ which is closed under equivalences; we can construct a canonical full subcategory $\mathcal{D}^{\otimes}$ of the $\mathcal{O}$-monoidal category $\mathcal{C}^{\otimes}$ itself. This is defined in the beginning of Section

2.2.1 of [Lur17].

Definition 2.3.1. Let $\mathcal{C}^{\otimes}$ be an $\mathcal{O}$-monoidal category, and $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory that is stable under equivalences. We denote by $\mathcal{D}^{\otimes}$ the full subcategory of $\mathcal{C}^{\otimes}$ spanned by the objects $D \simeq \mathcal{C}^{\otimes}$ equivalent to objects of the form $(D_1, \ldots, D_n)$ where D_i's are objects of $\mathcal{D}$.

If $\mathcal{C}$ is a symmetric monoidal 1-category a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ does not inherit, in general, the monoidal product of the 1-category $\mathcal{C}$, but it is easy to prove that sufficient conditions for the 1-category $\mathcal{D}$ to inherit the monoidal product are:

- being closed with respect to the product;
- and containing the unit object of $\mathcal{C}$.

The following proposition generalizes this result to $\mathcal{O}$-monoidal categories. Here the conditions of being closed under the monoidal product and containing the unit object are replaced by the condition of being closed under the $\mathcal{O}$-operations of $\mathcal{C}^{\otimes}$.

Proposition 2.3.2. [Lur17, Prop. 2.2.1.1] *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal category. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory that is stable under equivalences, and consider $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ as defined in Definition 2.3.1. Suppose that for every morphism $f : X \to Y$ of $\mathcal{O}^{\otimes}$ the ∞-category $\mathcal{D}^{\otimes}$ is closed under the $\mathcal{O}$-operation $\otimes_f : \mathcal{C}^{\otimes}_X \to \mathcal{C}^{\otimes}_Y$. Then:*

(1) The composition $\mathcal{D}^{\otimes} \hookrightarrow \mathcal{C}^{\otimes} \xrightarrow{p} \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads.

(2) The inclusion $\mathcal{D}^{\otimes} \hookrightarrow \mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal functor.

In the special case where $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, it can be proven that the condition of Proposition 2.3.2 is equivalent to the conditions that we had in the 1-categorical case [Lur17, Remark 2.2.1.2]. A full subcategory $\mathcal{D} \subseteq \mathcal{C}$ that is stable under equivalences is closed under Comm-operations if and only if $\mathcal{D}$ contains the unit object of $\mathcal{C}$ and is closed under the product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Another way to construct $\mathcal{O}$-monoidal subcategories starting from an $\mathcal{O}$-monoidal category $\mathcal{C}^{\otimes}$ is by considering localization functors.

Definition 2.3.3. [Lur17, Def. 2.2.1.6] Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal category. Suppose that we have a family of localization functors $\{L_X : \mathcal{C}_X \to \mathcal{C}_X\}_{X \in \mathcal{O}}$. Then, we say that the family of localizations is compatible with the $\mathcal{O}$-monoidal structure of $\mathcal{C}^{\otimes}$ if for each morphism $f : (X_1, \ldots, X_n) \to Y$, where $X_i, Y \in \mathcal{O}$, and every family $\{g_i\}_{1 \leq i \leq n}$ of L_{X_i}-equivalences, then the morphism $\otimes_f(\{g_i\}_{1 \leq i \leq n})$ is a L_Y-equivalence. By L_{X_i}-equivalences we mean morphisms of $\mathcal{C}^{\otimes}_{X_i}$ such that their image by the functor L_{X_i} is an equivalence of $\mathcal{C}^{\otimes}_Y$.

Proposition 2.3.4. [Lur17, Prop. 2.2.1.9] *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal category, and let $\{L_X : \mathcal{C}_X \to \mathcal{C}_X\}_{X \in \mathcal{O}}$ be a family of compatible localization functors. We denote by $\mathcal{D}$ the full subcategory spanned by the objects of $\mathcal{C}$ which are in the image of a functor L_X for some $X \in \mathcal{O}$, and consider the full subcategory $\mathcal{D}^{\otimes}$ as defined in Definition 2.3.1. Then:*

(1) There exists a commutative diagram

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "L^{\otimes}"] \arrow[dr, "p"'] & & \mathcal{D}^{\otimes} \arrow[dl] \\ & \mathcal{O}^{\otimes}, & \end{tikzcd}$$

and a natural transformation $\alpha : id_{\mathcal{C}^{\otimes}} \to L^{\otimes}$ which exhibits $L^{\otimes}$ as a left adjoint to the inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ and such that $p(\alpha)$ is the identity natural transformation from p to itself, that is to say, the functor $L^{\otimes}$ and the inclusion $\mathcal{D}^{\otimes} \hookrightarrow \mathcal{C}^{\otimes}$ are adjoint functors relative to $\mathcal{O}^{\otimes}$ as defined in [Lur17, Def. 7.3.2.2].

(2) The composition $\mathcal{D}^{\otimes} \hookrightarrow \mathcal{C}^{\otimes} \xrightarrow{p} \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads.

(3) The inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a map of ∞-operads and the localization $L^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is an $\mathcal{O}$-monoidal functor.

In the 1-categorical setting, if $\mathcal{O}$ is a (single-colored) operad and X is an $\mathcal{O}$-algebra of topological spaces; then, the space of the path components $\pi_0(X)$ admits an $\mathcal{O}$-algebra structure such that the projection map $\pi_0 : X \to \pi_0(X)$ is a map of $\mathcal{O}$-algebras. Using Proposition 2.3.4 we can prove that, if $\mathcal{O}^{\otimes}$ is a single-colored ∞-operad, this result generalizes to the ∞-category of $\mathcal{O}$-algebras of the $\mathcal{O}$-monoidal category of Kan complexes $\mathcal{S}^{\otimes}$.

Proposition 2.3.5. *Let $\mathcal{O}^{\otimes}$ be a single-colored ∞-operad, and $X : \mathcal{O}^{\otimes} \to \mathcal{S}^{\otimes}$ be an $\mathcal{O}$-algebra of the $\mathcal{O}$-monoidal category $\mathcal{S}^{\otimes}$ associated to the Cartesian symmetric monoidal ∞-category $\mathcal{S}^{\times}$ as defined in Remark 2.2.8, Then:*

(1) There exists an $\mathcal{O}$-algebra $\pi_0(X) : \mathcal{O}^{\otimes} \to \mathcal{S}^{\otimes}$ such that the image of $\langle 1 \rangle$ by $\pi_0(X)$ is the Kan complex of the path components of $X(\langle 1 \rangle)$.

(2) There exists a morphism of $\mathcal{O}$-algebras $\pi_0 : X \to \pi_0(X)$ that induces the projection to the path components of $X(\langle 1 \rangle)$ on the underlying categories.

Proof. We consider the Cartesian symmetric monoidal category $\mathcal{S}^{\times}$ and the localization $L : \mathcal{S} \to \mathrm{Disc}$ that associates to a Kan complex its discrete subcategory, [Lur09, Remark 5.5.6.21]. From [Lur17, Example 2.2.1.7] we know that the localization L is compatible with the symmetric monoidal structure of $\mathcal{S}^{\times}$. Applying Proposition 2.3.4 we can define a symmetric monoidal functor

$$L^{\times} : \mathcal{S}^{\times} \to \mathrm{Disc}^{\times},$$

which is left adjoint to the inclusion $R^{\times} : \mathrm{Disc}^{\times} \hookrightarrow \mathcal{S}^{\times}$ relative to $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

We can now consider the pullback of $L^{\times}$ along the map $\mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$; we will denote this $\mathcal{O}$-monoidal map as

$$L^{\otimes} : \mathcal{S}^{\otimes} \to \mathrm{Disc}^{\otimes}.$$

From [Lur17, Prop. 7.3.2.5] follows that $L^{\otimes}$ is again left adjoint to the inclusion $R^{\otimes} : \mathrm{Disc}^{\otimes} \hookrightarrow \mathcal{S}^{\otimes}$. We consider the maps induced by the two adjoint functors on the ∞-categories of $\mathcal{O}$-algebra objects, from [Lur17, Remark 7.3.2.13] the maps are once again adjoint

$$\begin{tikzcd} \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S}) \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Disc}). \arrow[ll, bend left, "R"] \end{tikzcd} \qquad (\star)$$

Using the adjunction $(\star)$ we can prove implications (1) and (2). Let u be the unit transformation of the adjunction. The image of X by the composition of $R \circ L$ defines an $\mathcal{O}$-algebra $\pi_0(X)$ and the morphism defined by the natural transformation u on the object X gives the map $\pi_0 := u_X : X \to \pi_0(X)$ between $\mathcal{O}$-algebras of $\mathcal{S}^{\otimes}$. By evaluating π_0 on the object $\langle 1 \rangle$ of $\mathcal{O}$ we recover the usual projection on the path components as a morphism of the underlying category of $\mathcal{S}^{\otimes}$. $\square$

2.3.2 Overcategories

Other important 1-categorical constructions that, under reasonable conditions, extend to $\mathcal{O}$-monoidal categories are the undercategories and overcategories. In Section 2.2.2 of [Lur17] these constructions are presented with a high level of generality; for our applications that level of generality is not required, and, in order to improve the clarity of the exposition, we will focus only on the case of an ∞-overcategories over an $\mathcal{O}$-algebra object. This corresponds to considering the definitions and the results of Section 2.2.2 of [Lur17] with $K = \Delta^0$.

Definition 2.3.6. [Lur17, Def. 2.2.2.1] Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal category. Suppose that $A : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ is an $\mathcal{O}$-algebra of $\mathcal{C}^{\otimes}$. We define the simplicial set $\mathcal{C}^{\otimes}_{/A}$ equipped with a map $p' : \mathcal{C}^{\otimes}_{/A} \to \mathcal{O}^{\otimes}$ by the following universal property: for every map of simplicial sets $Y \to \mathcal{O}^{\otimes}$, there is a canonical bijection of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(Y, \mathcal{C}^{\otimes}_{/A})$ with the collection of diagrams

$$\begin{tikzcd} Y \arrow[r, "id \times \{1\}"] \arrow[d] & Y \times \Delta^1 \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathcal{O}^{\otimes} \arrow[r, "A"] & \mathcal{C}^{\otimes} \arrow[r, "p"] & \mathcal{O}^{\otimes}. \end{tikzcd}$$

We denote the simplicial set $\mathcal{C}^{\otimes}_{/A}$ as the $\mathcal{O}$-monoidal overcategory of $\mathcal{C}^{\otimes}$ over A.

The simplicial set $\mathcal{C}^{\otimes}_{/A}$ comes equipped with a natural forgetful functor $U : \mathcal{C}^{\otimes}_{/A} \to \mathcal{C}^{\otimes}$, and Theorem 2.2.2.4 of [Lur17] states that, as our notation suggests, this functor post-composed with the $\mathcal{O}$-monoidal structure of $\mathcal{C}^{\otimes}$ presents $\mathcal{C}^{\otimes}_{/A}$ as an $\mathcal{O}$-monoidal category.

Theorem 2.3.7. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal category and let $A : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ be an $\mathcal{O}$-algebra of $\mathcal{C}^{\otimes}$. Then:*

(1) A morphism in $\mathcal{C}^{\otimes}_{/A}$ is inert if and only if its image in $\mathcal{C}^{\otimes}$ by the forgetful functor $U : \mathcal{C}^{\otimes}_{/A} \to \mathcal{C}^{\otimes}$ is inert.

(2) The map $p' : \mathcal{C}^{\otimes}_{/A} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads.

By part (1) of Theorem 2.3.7 we have that the forgetful functor $U : \mathcal{C}^{\otimes}_{/A} \to \mathcal{C}^{\otimes}$ preserves and reflects inert morphisms, in particular, it is a lax $\mathcal{O}$-monoidal map. The following proposition will imply that the forgetful functor actually preserves and reflects any coCartesian morphisms, and is, therefore, an $\mathcal{O}$-monoidal map. The proposition is stated for relative colimit diagrams. It is sufficient to say that if $\mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal category then a diagram of the form

$$f : (\Delta^0)^{\triangleright} = \Delta^1 \to \mathcal{C}^{\otimes}$$

is a relative colimit diagram if and only if it corresponds to a p-coCartesian morphism of $\mathcal{C}^{\otimes}$ [Lur09, Example 4.3.1.4]. For the interested reader, the theory of relative colimits is discussed in [Lur09, Section 4.3.1].

Proposition 2.3.8. [Lur17, Prop. 2.2.2.9] *Suppose that we are given a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & \mathcal{C}^{\otimes}_{/A} \arrow[d, "p'"] \\ (Y)^{\triangleright} \arrow[r, "g"] \arrow[ur, dashed, "\bar{f}"] & \mathcal{O}^{\otimes}, \end{tikzcd}$$

satisfying the following condition: the composite map $Y \xrightarrow{f} \mathcal{C}^{\otimes}_{/A} \to \mathcal{C}^{\otimes}$ can be extended to a p-colimit diagram $g' : (Y)^{\triangleright} \to \mathcal{C}^{\otimes}$ lying over g.

Then:

(1) Let $\bar{f} : (Y)^{\triangleright} \to \mathcal{C}^{\otimes}_{/A}$ be a map rendering the diagram commutative. Then, $\bar{f}$ is a p'-colimit diagram if and only if the composite map $(Y)^{\triangleright} \xrightarrow{\bar{f}} \mathcal{C}^{\otimes}_{/A} \to \mathcal{C}^{\otimes}$ is a p-colimit diagram.

(2) There exists a map $\bar{f}$ satisfying the equivalent conditions of (1).

From Proposition 2.3.8 with $Y = \Delta^0$, combined with our previous observation regarding relative colimit diagrams, follows that a functor $f : \Delta^1 \to \mathcal{C}^{\otimes}_{/A}$ corresponds to a p'-coCartesian morphism if and only if the composition $U \circ f$ corresponds to a p-coCartesian morphism of $\mathcal{C}^{\otimes}$. In particular, $U : \mathcal{C}^{\otimes}_{/A} \to \mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal map.

2.4 Monoidal Grothendieck construction

In Section 3.2 of [Lur17] J. Lurie develops the ∞-categorical version of the Grothendieck construction, defining an equivalence between the ∞-category of coCartesian fibrations with base a small ∞-category $\mathcal{B}$ and the ∞-category of functors over $\mathcal{B}$ with target the ∞-category of small ∞-categories $\mathcal{C}at$

$$\mathrm{CoCart}(\mathcal{B}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{B}, \mathcal{C}at).$$

This functor is called the straightening functor and its inverse the unstraightening functor, and we denote them with $\mathrm{St}_{\mathcal{B}}$ and $\mathrm{Un}_{\mathcal{B}}$ respectively.

Informally the straightening assigns to a coCartesian fibration $\pi : \mathcal{A} \to \mathcal{B}$ a functor $\mathcal{B} \to \mathcal{C}at$ that maps an object $X \in \mathcal{B}$ to the small ∞-category $\mathcal{A}_X$ of the fiber over X of the fibration π; and maps a morphism $f : X \to Y$ of $\mathcal{B}$ to the functor $f_! : \mathcal{A}_X \to \mathcal{A}_Y$ defined, up to equivalence, by the fibration π.

In the special case where the coCartesian fibration is, in particular, a left fibration, then for each $X \in \mathcal{B}$ the fiber $\mathcal{A}_X$ is a Kan complex, and the equivalence specializes to an equivalence between the ∞-category of left fibrations with base $\mathcal{B}$ and the ∞-category of pre-sheaves over $\mathcal{B}$

$$\mathrm{LFib}(\mathcal{B}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{B}, \mathcal{S}).$$

In this section, we will see that the Grothendieck construction extends to $\mathcal{O}$-monoidal categories. A brief description of the monoidal Grothendieck construction is already present in Section 2.4.1 of [Lur17] where the author utilizes results involving a particular class of functors called lax Cartesian structures and the classical Grothendieck construction to define its monoidal version. A more explicit proof of the monoidal straightening and unstraightening equivalence can be found in the appendix of [Hin15].

Another interesting study on the subject that we would like to point out is [Ram22] by M. Ramzi. In this paper, the author defines the so-called metacosmic monoidal Grothendieck construction; producing the equivalence at the level of symmetric monoidal ∞-categories instead of an equivalence of their $\mathcal{O}$-algebras. Even if Ramzi's metacosmic version of the Grothendieck construction is never used directly in our arguments, we believe that his work might be relevant for future developments, since it opens the possibility of defining a metacosmic version of the iterated Thom spectrum.

We start by defining the analogue of a coCartesian fibration for the monoidal case.

Definition 2.4.1. [Ram22, Def. 1.11] An $\mathcal{O}$-monoidal map $\pi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a coCartesian $\mathcal{O}$-fibration if for each $X \in \mathcal{O}$ the map induced by π on the fibers $\pi_X : \mathcal{C}^{\otimes}_X \to \mathcal{D}^{\otimes}_X$ is a coCartesian fibration, and the $\mathcal{O}$-operations preserve π-coCartesian edges. That is to say, for each morphism $f : X \to Y$ of $\mathcal{O}^{\otimes}$ where $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_m)$ with $X_i, Y_j \in \mathcal{O}$, and each collection of morphisms $\{g_i\}_{1\leq i\leq n}$ where g_i is π_{X_i}-coCartesian, then the image of $\{g_i\}_{1\leq i\leq n}$ by the $\mathcal{O}$-operation induced by f corresponds to a collection $\{h_j\}_{1\leq j\leq m}$ of morphisms where h_j's are π_{Y_j}-coCartesian morphisms. We define $\mathrm{coCart}^{\mathcal{O}}(\mathcal{D})$ to be the full subcategory of the ∞-category $\mathrm{coCart}(\mathcal{D}^{\otimes})$ of coCartesian fibrations with base $\mathcal{D}^{\otimes}$ spanned by coCartesian $\mathcal{O}$-fibrations.

In the previous definition, in order to define the ∞-category of $\mathrm{coCart}^{\mathcal{O}}(\mathcal{D})$ as a full subcategory of $\mathrm{coCart}(\mathcal{D}^{\otimes})$ we implicitly used the fact that every coCartesian $\mathcal{O}$-fibration is, in particular, a coCartesian fibration of ∞-categories. This is the content of the following lemma.

Lemma 2.4.2. [Ram22, Lemma 1.10] *Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $\pi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a coCartesian $\mathcal{O}$-fibration. Then the functor $\pi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a coCartesian fibration of ∞-operads.*

We can finally introduce the monoidal version of the Grothendieck construction.

Proposition 2.4.3. [Hin15, Prop. A.2.1] *Let $\mathcal{O}^{\otimes}$ be an ∞-operad and $\mathcal{C}^{\otimes}$ and $\mathcal{O}$-monoidal category. There is an equivalence of ∞-categories*

$$coCart^{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{C}at)$$

between coCartesian $\mathcal{O}$-fibrations with base $\mathcal{C}^{\otimes}$ and lax $\mathcal{O}$-monoidal functors $\mathcal{C}^{\otimes} \to \mathcal{C}at^{\otimes}$. The equivalence induces the standard Grothendieck construction on the underlying categories.

Similar to the classical Grothendieck construction, the monoidal version specializes to an equivalence between the ∞-category of left $\mathcal{O}$-fibrations and the ∞-category of lax $\mathcal{O}$-monoidal pre-sheaves; where the definition of left $\mathcal{O}$-fibration is analogous to Definition 2.4.1.

Corollary 2.4.4. [Ram22, Corollary 4.8] *Let $\mathcal{O}^{\otimes}$ be an ∞-operad and $\mathcal{C}^{\otimes}$ and $\mathcal{O}$-monoidal category. There is an equivalence of ∞-categories*

$$LFib^{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{S})$$

between left $\mathcal{O}$-fibrations with base $\mathcal{C}^{\otimes}$ and lax $\mathcal{O}$-monoidal pre-sheaves $\mathcal{C}^{\otimes} \to \mathcal{S}^{\otimes}$.

The special case $\mathcal{C}^{\otimes} = \mathcal{O}^{\otimes}$ will play an important role in Chapter 5. We observe that by definition of $\mathcal{O}$-operations, a functor $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a coCartesian $\mathcal{O}$-fibration if and only if it is a coCartesian fibration of ∞-operads. Then, the ∞-category of left $\mathcal{O}$-fibrations with base $\mathcal{O}^{\otimes}$ is equivalent to the ∞-category of $\mathcal{O}$-monoidal categories $X^{\otimes} \to \mathcal{O}^{\otimes}$ such that the underlying category X is a Kan complex. We will call the objects of this ∞-

category $\mathcal{O}$-monoidal Kan complexes. By Applying Corollary 2.4.4 we obtain that the ∞-category of $\mathcal{O}$-monoidal Kan complexes is equivalent to the ∞-category of $\mathcal{O}$-algebras of $\mathcal{S}^{\otimes}$.

Informally, in this special case, the monoidal Grothendieck construction allows us to pass from statements regarding $\mathcal{O}$-algebras of $\mathcal{S}^{\otimes}$ to statements regarding $\mathcal{O}$-monoidal Kan complexes and vice versa. For example, we can use the monoidal straightening/unstraightening equivalence to rephrase Proposition 2.3.5 in terms of $\mathcal{O}$-monoidal Kan complexes.

Proposition 2.4.5. *Let $\mathcal{O}^{\otimes}$ be a single-colored ∞-operad, and $X^{\otimes}$ be an $\mathcal{O}$-monoidal Kan complex. Then:*

(1) There exists an $\mathcal{O}$-monoidal category $q : \pi_0(X)^{\otimes} \to \mathcal{O}^{\otimes}$ such that its underlying category is the Kan complex of the path components of X.

(2) There exists an $\mathcal{O}$-monoidal map $\pi_0 : X^{\otimes} \to \pi_0(X)^{\otimes}$ that induces the projection on the path components on the underlying categories.

(3) The map π_0 is a left $\mathcal{O}$-fibration.

Proof. For implication (1) we consider the monoidal unstraightening of the $\mathcal{O}$-algebra $\pi_0(X) : \mathcal{O}^{\otimes} \to \mathcal{S}^{\otimes}$ defined in Proposition 2.3.5. We obtain a left $\mathcal{O}$-fibration $\pi_0(X)^{\otimes} \to \mathcal{O}^{\otimes}$ that on the underlying categories corresponds to the unstraightening of the functor $(\pi_0)_{\langle 1 \rangle} : \{\langle 1 \rangle\} \to \mathcal{S}$, i.e., the left fibration $\pi_0(X) \to \{\langle 1 \rangle\}$.

For part (2) we consider the unstraightening of the morphism of $\mathcal{O}$-algebras $\pi_0 : X \to \pi_0(X)$ defined in Proposition 2.3.5. We obtain a morphism of $\mathrm{LFib}^{\mathcal{O}}(\mathcal{O})$ that, by abusing the notation, we will also denote by π_0. Since we defined the ∞-category $\mathrm{LFib}^{\mathcal{O}}(\mathcal{O})$ as the full subcategory of $\mathrm{LFib}(\mathcal{O}^{\otimes})$ spanned by left $\mathcal{O}$-fibrations, the morphism π_0 correspond to a functor

$$\begin{tikzcd} X^{\otimes} \arrow[rr, "\pi_0"] \arrow[dr] & & \pi_0(X)^{\otimes} \arrow[dl] \\ & \mathcal{O}^{\otimes} & \end{tikzcd}$$

that preserves the coCartesian morphism of the $\mathcal{O}$-monoidal structure of X, that is to say, the morphism π_0 correspond to an $\mathcal{O}$-monoidal map $\pi_0 : X^{\otimes} \to \pi_0(X)^{\otimes}$.

For part (3), we have to prove that the functor π_0 induces a left fibration on the underlying categories $\pi_0 : X \to \pi_0(X)$ and that the π_0-coCartesian morphisms are compatible with the $\mathcal{O}$-monoidal structure. By construction, the functor induced on the underlying categories is the usual projection on the path components of X, which is a left fibration. It only remains to prove the compatibility with the $\mathcal{O}$-operations. Let $\alpha : \langle r \rangle \to \langle 1 \rangle$ be an active morphism of $\mathcal{O}$ and let γ be a π_0-coCartesian morphism of $X^{\otimes}_{\langle r \rangle}$. The morphism α defines a functor $\otimes_{\alpha} : X^{\otimes}_{\langle r \rangle} \to X^{\otimes}_{\langle 1 \rangle}$, we wish to prove that

$\otimes_\alpha(\gamma)$ is a π_0-coCartesian morphism of $X = X^{\otimes}_{\langle 1\rangle}$. X is a Kan complex, therefore $\otimes_\alpha(\gamma)$ must be an equivalence and applying [Lur09, Prop. 2.4.1.5] we obtain that $\otimes_\alpha(\gamma)$ is π_0-coCartesian. $\square$

We now introduce a result that we will use multiple times in the last chapter to compute the straightening of pullbacks of coCartesian $\mathcal{O}$-fibrations along lax $\mathcal{O}$-monoidal maps.

Lemma 2.4.6. [Ram22, Lemma 3.12] *Let $\pi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a coCartesian $\mathcal{O}$-fibration, $\mathcal{O}'^{\otimes}$ be an ∞-operad over $\mathcal{O}^{\otimes}$, and let $f \in \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$. Then the following two constructions equip the pullback $\mathcal{C}_f := \mathcal{C} \underset{\mathcal{D}}{\times} \mathcal{O}'$ with an $\mathcal{O}'$-monoidal structure:*

- *Let $\psi : \mathcal{D}^{\otimes} \to \mathcal{C}at^{\otimes}$ be the lax $\mathcal{O}$-monoidal map that classifies π. The composition of lax $\mathcal{O}$-monoidal maps*
$$\mathcal{O}'^{\otimes} \to \mathcal{D}^{\otimes} \to \mathcal{C}at^{\otimes}$$
classifies a coCartesian $\mathcal{O}$-fibration $\pi' : \mathcal{C}_f^{\otimes} \to \mathcal{O}'^{\otimes}$, which by Lemmma 2.4.2 is in particular a coCartesian fibrations of ∞-operads.
- *Or we can take the pullback of ∞-operads*
$$\begin{tikzcd} \mathcal{C}_f^{\otimes} \arrow[r] \arrow[d, "\pi''"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}^{\otimes} \arrow[d, "\pi"] \\ \mathcal{O}'^{\otimes} \arrow[r, "f"'] & \mathcal{D}^{\otimes}. \end{tikzcd}$$
By [Ram22, Prop. 1.4] the map π'' is a coCartesian fibration of ∞-operads.

Then, the two $\mathcal{O}'$-monoidal structures $\pi' : \mathcal{C}_f^{\otimes} \to \mathcal{O}'^{\otimes}$ and $\pi'' : \mathcal{C}_f^{\otimes} \to \mathcal{O}'^{\otimes}$ are equivalent.

A natural question that arises from the monoidal Grothendieck construction is: under which conditions on the left $\mathcal{O}$-fibration $\pi : X^{\otimes} \to B^{\otimes}$ is the lax $\mathcal{O}$-monoidal functor $\psi : B^{\otimes} \to \mathcal{S}^{\otimes}$ given by its straightening not only lax $\mathcal{O}$-monoidal but $\mathcal{O}$-monoidal? We observe, however, that it is not reasonable to ask for the map ψ to be $\mathcal{O}$-monoidal, intuitively because it generally does not map the unit of $B^{\otimes}$ to the unit of $\mathcal{S}^{\otimes}$. We will see that the proper map to consider is instead the map induced by the lax $\mathcal{O}$-monoidal pre-sheaf ψ on the $\mathcal{O}$-monoidal categories of $\mathcal{O}$-modules that we will define in Section 3.2 of the next chapter.

Chapter 3

Modules of $\mathcal{O}$-monoidal categories

In the previous chapter, we introduced the $\mathcal{O}$-monoidal categories and their $\mathcal{O}$-algebras. The next natural step is to define the ∞-categories of modules of an $\mathcal{O}$-monoidal category. We will start with the associative monoidal case, where the construction of the ∞-categories of associative left modules and associative bimodules can be inferred from the 1-categorical case with a procedure similar to the one of our motivating example, Construction 2.0.1. These notions can be generalized to $\mathcal{O}$-monoidal categories, in particular, it is possible to define the ∞-category of $\mathcal{O}$-modules of an $\mathcal{O}$-monoidal category and, provided that the ∞-operad satisfies some technical conditions, this ∞-category admits the structure of an $\mathcal{O}$-monoidal category. In the last section of this chapter, we will define an important family of ∞-operads that generalizes to ∞-categories the little cubes operads originally defined in [BV68, Def. 5] by J.M. Boardman and R.M. Vogt.

3.1 Associative modules

We start by defining the ∞-operad $\mathrm{Assoc}^{\otimes}$ that, similarly to what we have seen for the ∞-operad $\mathrm{Comm}^{\otimes}$, will encode the structure of an associative monoidal ∞-category. We will then use the ∞-operad $\mathrm{Assoc}^{\otimes}$ to define the ∞-categories of associative left modules and associative bimodules.

3.1.1 Associative ∞-operad

In the beginning of Chapter 2, we have seen how it is possible, starting with a symmetric monoidal 1-category $\mathcal{C}$ to construct a 1-category $\mathcal{C}^{\otimes}$ equipped with a coCartesian forgetful functor $\pi : \mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ that reconstructs $\mathcal{C}$ and its product up to equivalence. It is natural to ask if a similar procedure exists for associative monoidal 1-categories too. Notably, the "prototype" symmetric monoidal 1-category $\mathcal{F}\mathrm{in}_*$ admits only one active morphism $\langle n \rangle \to \langle 1 \rangle$; which represents the product of n elements of $\mathcal{C}$. Therefore the coCartesian fibration π defines a unique product functor $\otimes : \mathcal{C}^n \to \mathcal{C}$. If we aim to generalize this construction to associative monoidal 1-categories we need to replace the 1-category $\mathcal{F}\text{in}_*$ with a 1-category that admits an active morphism $\langle n\rangle \to \langle 1\rangle$ for each possible linear ordering of $\langle n\rangle^\circ$; so that for each of these morphisms the coCartesian fibration will define an associative product $\otimes_\sigma : \mathcal{C}^n \to \mathcal{C}$ that will correspond to first permuting the n-tuple of objects of $\mathcal{C}$ following the linear ordering and then applying the associative product of $\mathcal{C}$. This observation suggests that the "prototype" associative monoidal 1-category is the 1-category where the objects are the same of $\mathcal{F}\text{in}_*$ and the morphisms are the morphisms of $\mathcal{F}\text{in}_*$ equipped with a linear ordering on each fiber.

Definition 3.1.1. [Lur17, Def. 4.1.1.1] We define the associative colored operad **Assoc** as the colored operad where:

(1) The set of colors consists of a single color $*$.

(2) For each finite set I, the set of morphisms $\mathrm{Mul}_{\mathbf{Assoc}}(\{*\}_{i\in I}, *)$ is given by the linear orderings of the set I.

(3) Let $f : I \to J$ be a map of finite sets with fibers $\{I_j\}_{j\in J}$. The composition map

$$\prod_{j\in J} \mathrm{Mul}_{\mathbf{Assoc}}(\{*\}_{i\in I_j}, *) \times \mathrm{Mul}_{\mathbf{Assoc}}(\{*\}_{j\in J}, *) \to \mathrm{Mul}_{\mathbf{Assoc}}(\{*\}_{i\in I}, *),$$

assigns the pair given by a collection of linear orderings $\{\preceq_j\}_{j\in J} \in \prod_{j\in J} \mathrm{Mul}_{\mathbf{Assoc}}(\{*\}_{i\in I_j}, *)$ and a linear ordering $\preceq' \in \mathrm{Mul}_{\mathbf{Assoc}}(\{*\}_{j\in J}, *)$ to the unique linear ordering $\preceq''$ of I that satisfies the following property: for each $x, y \in I$ such that $f(x) = f(y) = j$ then $x \preceq'' y$ if and only if $x \preceq_j y$, and for each $x, y \in I$ such that $f(x) \neq f(y)$ then $x \preceq'' y$ if and only if $f(x) \preceq' f(y)$.

Applying the Construction 2.1.1 to the single-colored operad **Assoc** we obtain the "prototype" associative monoidal 1-category.

Definition 3.1.2. We define the 1-category $\mathbf{Assoc}^{\otimes}$ as the 1-category where:

(1) The objects are the objects of $\mathcal{F}\text{in}_*$.

(2) Given a pair of objects $\langle m\rangle, \langle n\rangle \in \mathbf{Assoc}^{\otimes}$, a morphism with source $\langle m\rangle$ and target $\langle n\rangle$ consists of a pair $(\alpha, \{\preceq_i\}_{1\leq i\leq n})$ where α is a morphism of $\mathcal{F}\text{in}_*$ and $\preceq_i$ is a linear ordering of the fiber of α over $\{i\} \in \langle n\rangle^\circ$.

(3) The composition of a pair of morphisms

$$(\alpha, \{\preceq_i\}_{1\leq i\leq n}) : \langle m\rangle \to \langle n\rangle, \qquad (\beta, \{\preceq'_j\}_{1\leq j\leq l}) : \langle n\rangle \to \langle l\rangle$$

is the pair $(\beta\circ\alpha, \{\preceq''_j\}_{1\leq j\leq l})$, where the ordering $\preceq''_j$ is defined analogously to the ordering defined by the composition map of Definition 3.1.1.

We observe that the 1-category $\mathbf{Assoc}^{\otimes}$ comes equipped with a natural forgetful functor to $\mathcal{F}\mathrm{in}_*$, and that the nerve of this functor defines a new ∞-operad.

Definition 3.1.3. Let $\mathrm{Assoc}^{\otimes}$ be the ∞-operad defined by the nerve of the forgetful functor $\mathbf{Assoc}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. We will call this ∞-operad associative ∞-operad.

The previous discussion justifies our definition of an associative monoidal ∞-category.

Definition 3.1.4. An associative monoidal ∞-category is a coCartesian fibration of ∞-operads with target the ∞-operad $\mathrm{Assoc}^{\otimes}$.

Suppose that $\mathcal{O}^{\otimes}$ is an ∞-operad equipped with a map $q : \mathrm{Assoc}^{\otimes} \to \mathcal{O}^{\otimes}$ from the associative ∞-operad. This is the case for the commutative associative ∞-operad and the little cubes ∞-operads $\mathbb{E}_k^{\otimes}$ with $k \geq 1$. Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal category. We will use the term associative algebras of $\mathcal{C}^{\otimes}$ to refer to the ∞-category of Assoc-algebras $\mathrm{Alg}_{\,\mathrm{Assoc}/\mathcal{O}}(\mathcal{C})$. As we observed in Remark 2.2.8, this ∞-category is equivalent to the ∞-category $\mathrm{Alg}_{\,/\mathrm{Assoc}}(\mathcal{C}')$, where $\mathcal{C}'^{\,\otimes}$ is the associative monoidal ∞-category obtained by taking the pullback of the $\mathcal{O}$-monoidal structure p along the operadic map q. Once again we remind the reader that we sometimes will use this equivalence implicitly.

The associative ∞-operad enjoys a natural functor from the nerve of the simplicial ∞-category.

Construction 3.1.5. [Lur17, Construction 4.1.2.9] Given an object $[n] \in \Delta$ we define a cut in $[n]$ to be an equivalence relation on the set $[n]$ with at most two equivalence classes, and such that each subset of $[n]$ corresponding to an equivalence class is convex. We observe that the set of cuts is in bijection with the set of partitions of $[n]$ in two, possibly empty, convex sets (S_0, S_1), provided that we identify the two trivial partitions $([n], \emptyset) \simeq (\emptyset, [n])$. We will denote by $\mathbf{cut}([n])$ the set of cuts in $[n]$. This set admits a canonical bijection with the set $\langle n \rangle$

$$\begin{aligned} \langle n \rangle &\xrightarrow{\simeq} \mathbf{cut}([n]) \\ i &\longmapsto \begin{cases} (\{0,1,\dots,n\}, \emptyset) & \text{if } i = * \\ (\{0,1,\dots,i-1\}, \{i,\dots,n\}) & \text{otherwise.} \end{cases} \end{aligned}$$

By considering this bijection, we can construct a functor $\mathbf{cut} : \Delta^{\mathrm{op}} \to \mathbf{Assoc}^{\otimes}$ that on the objects is given by $[n] \mapsto \mathbf{cut}([n]) \simeq \langle n \rangle$. Explicitly, the functor $\mathbf{cut}$ is defined as follows:

(1) For each object $[n] \in \Delta$ we have $\mathbf{cut}([n]) = \langle n \rangle$.

(2) Let $\alpha : [n] \to [m]$ be a morphism of Δ. Then the morphism $\mathbf{cut}(\alpha) : \langle m \rangle \to \langle n \rangle$ of

$\mathbf{Assoc}^{\otimes}$ is given by the following formula

$$\mathbf{cut}(\alpha)(i) = \begin{cases} j & \text{if } \exists j \text{ s.t. } \alpha(j-1) \leq i \leq \alpha(j) \\ * & \text{otherwise,} \end{cases}$$

where to each fiber $\mathbf{cut}(\alpha)^{-1}\{j\}$ we assign the linear ordering induced by the natural linear ordering of $\langle n\rangle^{\circ}$ by the inclusion $\mathbf{cut}(\alpha)^{-1}\{j\} \subseteq \langle n\rangle^{\circ}$.

The nerve of the functor **cut** induces a functor of ∞-categories $\mathrm{cut} : \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathrm{N}(\mathbf{Assoc}^{\otimes}) = \mathrm{Assoc}^{\otimes}$.

3.1.2 Left modules

Let $\mathcal{C}^{\otimes}$ be an associative monoidal ∞-category and let $A \in \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})$ be an associative algebra of $\mathcal{C}^{\otimes}$; the goal of this section is to develop the theory of left A-modules of the associative monoidal ∞-category $\mathcal{C}^{\otimes}$.

We recall that if $\mathcal{C}$ is an associative monoidal 1-category and A is an associative algebra of $\mathcal{C}$, a left A-module M of $\mathcal{C}$ is an object of $\mathcal{C}$ equipped with a left A-action, or module structure, $\phi : A \otimes M \to M$ such that the following diagrams commute

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\phi"] & A \otimes M \arrow[d, "\phi"] \\ A \otimes M \arrow[r, "\phi"] & M, \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathcal{C}} \otimes M \arrow[rr, "\eta\otimes id"] \arrow[dr] & & A \otimes M \arrow[dl, "\phi"] \\ & M. & \end{tikzcd}$$

Similarly to what we have seen with the definition of monoidal categories and $\mathcal{O}$-algebras, before trying to generalize this notion to the ∞-categorical framework, we need to repackage the structures associated with a left A-module by constructing a colored operad that will encode this information with respect to the 1-category $\mathcal{C}^{\otimes}$ defined in Construction 2.0.1.

Definition 3.1.6. [Lur17, Def. 4.2.1.1] We define the colored operad **LM** as follows:

(1) The set of objects has two elements: the element $\mathfrak{a}$ which represents the associative algebra, and the element $\mathfrak{m}$ which represents the left module.

(2) Let $\{X_i\}_{i\in I}$ be a finite collection of objects of **LM** and Y another object of **LM**. Then:

- if $Y = \mathfrak{a}$, $\mathrm{Mul}_{\mathbf{LM}}(\{X_i\}_{i\in I}, Y)$ is the collection of all linear orderings of I provided that for each $i \in I$ we have $X_i = \mathfrak{a}$, and it is empty otherwise;
- if $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{LM}}(\{X_i\}_{i\in I}, Y)$ is the collection of all linear orderings

$\{i_1, \ldots, i_n\}$ on the set I such that $X_{i_n} = \mathfrak{m}$ and for each $j < n$ the object $X_{i_j} = \mathfrak{a}$, and is empty otherwise.

Here the two morphisms of $\mathrm{Mul}_{\mathbf{LM}}(\{\mathfrak{a}, \mathfrak{a}\}, \mathfrak{a})$ represent the associative algebraic structures of $\mathfrak{a}$, and the unique operation $\phi \in \mathrm{Mul}_{\mathbf{LM}}(\{\mathfrak{a}, \mathfrak{m}\}, \mathfrak{m})$ represent the module structure of $\mathfrak{m}$ as a left $\mathfrak{a}$-module.

(3) The composition of **LM** is analogous to the composition of linear orderings that we have described in Definition 3.1.2.

We observe that the full suboperad spanned by the object $\mathfrak{a}$ is isomorphic to the colored operad **Assoc** that we defined in Definition 3.1.2. Furthermore, the colored operad **LM** admits an operadic map $\mathbf{LM} \to \mathbf{Assoc}$; the map is defined by assigning to each operation of $\mathrm{Mul}_{\mathbf{LM}}(\{X_i\}_{i \in I}, Y)$ its linear ordering of I. We defined $\mathbf{LM}^{\otimes}$ to be the 1-category obtained by applying the procedure described in Construction 2.1.3 to the colored operad **LM**.

Remark 3.1.7. Let $\mathcal{C}$ be a symmetric monoidal 1-category and let A be an associative algebra of $\mathcal{C}$. Starting from a left A-module M of $\mathcal{C}$ we can define an operadic map F from $\mathbf{LM}^{\otimes}$ to $\mathcal{C}^{\otimes}$ that represents the module M

$$\begin{tikzcd} \mathbf{LM}^{\otimes} \arrow[r, "F"] \arrow[dr] & \mathcal{C} \arrow[d] \\ & \mathcal{F}\mathrm{in}_*. \end{tikzcd}$$

We define the functor F as the functor that assigns to the object $\mathfrak{a}$ the algebra A, to the object $\mathfrak{m}$ the left module M, and to the morphisms of $\mathbf{LM}^{\otimes}$ the morphisms of $\mathcal{C}^{\otimes}$ given by compositions of the algebraic structures of A and the module structures of M accordingly. Here we are implicitly using that the Construction 2.1.3 is somehow functorial.

On the other hand, starting from an operadic map $F : \mathbf{LM}^{\otimes} \to \mathcal{C}^{\otimes}$ such that the composition with the inclusion $\mathbf{Assoc}^{\otimes} \hookrightarrow \mathbf{LM}^{\otimes}$ corresponds to the map given by an algebra $A = F(\mathfrak{a})$ as described in Remark 2.2.6, the image of the unique morphisms $\phi \in \mathrm{Map}_{\mathbf{LM}^{\otimes}}(\{\mathfrak{a}, \mathfrak{m}\}, \mathfrak{m})$ defines a left A-module structure on the object $M := F(\mathfrak{m})$. From the definition of the 1-category $\mathbf{LM}^{\otimes}$, it follows that the module structure of M is unital and compatible with the algebraic structure of A.

We are now ready to pass to the ∞-categorical case.

Definition 3.1.8. We define the ∞-operad $\mathcal{LM}^{\otimes}$ to be the nerve of the forgetful functor $\mathbf{LM}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ obtained by applying Construction 2.1.3 to **LM**. The nerve of the inclusion $\mathbf{Assoc} \to \mathbf{LM}$ defines a map of ∞-operads $\mathrm{Assoc}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$.

Remark 3.1.7 motivates our definition of left modules of an associative monoidal ∞-

category $\mathcal{C}^{\otimes}$.

Definition 3.1.9. [Lur17, Def. 4.2.1.13] We define the ∞-category $\mathrm{LMod}\,(\mathcal{C})$ of left modules of $\mathcal{C}^{\otimes}$ to be the ∞-category $\mathrm{Alg}_{\mathcal{LM}/\mathrm{Assoc}}(\mathcal{C})$ as defined in Definition 2.2.7. Precomposition with the functor $\mathrm{Assoc}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$ defines a forgetful functor $\mathrm{LMod}\,(\mathcal{C}) \to \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})$.

Let A be an associative algebra of $\mathcal{C}^{\otimes}$. We define the ∞-category $\mathrm{LMod}_A(\mathcal{C})$ of left A-modules of $\mathcal{C}$ to be the following pullback of ∞-categories

$$\begin{tikzcd} \mathrm{LMod}_A(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{LMod}\,(\mathcal{C}) \arrow[d] \\ \{A\} \arrow[r] & \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C}). \end{tikzcd}$$

In the 1-categorical framework, we know that if $f : A \to B$ is a morphism of associative algebras of a monoidal 1-category $\mathcal{C}$; then f induces a change of algebra functor

$$\mathrm{LMod}_B(\mathcal{C}) \to \mathrm{LMod}_A(\mathcal{C}).$$

The functor maps a left B-module $(N, \phi_B : B \otimes N \to N)$ to the left A-module with the same underlying object and with the left A-action given by

$$\phi_A : A \otimes N \xrightarrow{f \otimes id} B \otimes N \xrightarrow{\phi_B} N.$$

The following result recovers this association between morphisms of algebras and functors of ∞-categories of left modules in a functorial way.

Corollary 3.1.10. [Lur17, Corollary 4.2.3.3] *Let $\mathcal{C}^{\otimes}$ be an associative monoidal ∞-category and let $\theta : \mathrm{LMod}\,(\mathcal{C}) \to \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})$ be the forgetful functor defined above. Then θ is a Cartesian fibration.*

3.1.3 Bimodules

The definition of the ∞-category of left modules that we gave in the previous section extends without any particular effort to bimodules. We start by defining the analogue of the colored operad **LM** for bimodules.

Definition 3.1.11. [Lur17, Def. 4.2.1.1] We define the colored operad **BM** as follows:

(1) The set of objects has three elements: the element $\mathfrak{a}_-$ that represents the left algebra, the element $\mathfrak{a}_+$ that represents the right algebra, and the element $\mathfrak{m}$ that represents the $\mathfrak{a}_-$-$\mathfrak{a}_+$-bimodule.

(2) Let $\{X_i\}_{i \in I}$ be a finite collection of objects of **BM** and let Y be another object of

BM, then:

- If $Y = \mathfrak{a}_-(= \mathfrak{a}_+)$, $\mathrm{Mul}_{\mathbf{BM}}(\{X_i\}_{i\in I}, Y)$ is the collection of all linear orderings of I provided that for each $i \in I$ we have $X_i = \mathfrak{a}_-(= \mathfrak{a}_+)$, and it is empty otherwise;

- If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{BM}}(\{X_i\}_{i\in I}, Y)$ is the collection of all linear orderings $\{i_1, \dots, i_n\}$ on the set I such that there exists exactly one $i_j \in I$ such that $X_{i_j} = \mathfrak{m}$ and for each $k < j$ we have $X_{i_k} = \mathfrak{a}_-$ and for each $s > j$ we have $X_{i_s} = \mathfrak{a}_+$.

Here the two operations of $\mathrm{Mul}_{\mathbf{BM}}(\{\mathfrak{a}_-, \mathfrak{a}_-\}, \mathfrak{a}_-)$ represent the associative algebraic structure of $\mathfrak{a}_-$, the two operations of $\mathrm{Mul}_{\mathbf{BM}}(\{\mathfrak{a}_+, \mathfrak{a}_+\}, \mathfrak{a}_+)$ represent the associative algebraic structure of $\mathfrak{a}_+$, and the operations $\phi_- \in \mathrm{Mul}_{\mathbf{BM}}(\{\mathfrak{a}_-, \mathfrak{m}\}, \mathfrak{m})$ and $\phi_+ \in \mathrm{Mul}_{\mathbf{BM}}(\{\mathfrak{a}_+, \mathfrak{m}\}, \mathfrak{m})$ represent the module structures of $\mathfrak{m}$ as an $\mathfrak{a}_-$-$\mathfrak{a}_+$-bimodule.

(3) The composition of **BM** is analogous to the composition of linear orderings that we have described in Definition 3.1.2.

We observe that the full suboperad spanned by the object $\mathfrak{a}_-$ and the one spanned by the object $\mathfrak{a}_+$ are isomorphic to the colored operad **Assoc**; we will denote these two suboperads by $\mathbf{Assoc}_-$ and $\mathbf{Assoc}_+$ respectively. Furthermore, the suboperad spanned by $\mathfrak{a}_-$ and $\mathfrak{m}$ is isomorphic to the colored operad **LM**, similarly the suboperad spanned by $\mathfrak{a}_+$ and $\mathfrak{m}$ is isomorphic to **RM** which is the dual version of **LM**. Finally, the operad **BM** admits an operadic map to **Assoc** defined by assigning each operation of $\mathrm{Mul}_{\mathbf{BM}}(\{X_i\}_{i\in I}, Y)$ to its linear ordering of I.

Definition 3.1.12. We define the ∞-operad $\mathcal{BM}^{\otimes}$ to be the nerve of the forgetful functor of 1-categories $\mathbf{BM}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ obtained by applying Construction 2.1.3 to **BM**. We have two inclusions $\mathbf{Assoc} \hookrightarrow \mathbf{LM}$ given by the full suboperads spanned by $\mathfrak{a}_-$ and by the full suboperad $\mathfrak{a}_+$; the nerves of these inclusions define two maps of ∞-operads: $\mathrm{Assoc}_-^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$ and $\mathrm{Assoc}_+^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$.

The previous discussion motivates our definition of bimodules of an associative monoidal ∞-category $\mathcal{C}^{\otimes}$.

Definition 3.1.13. We define the ∞-category $\mathrm{BMod}(\mathcal{C})$ of bimodules of $\mathcal{C}^{\otimes}$ to be the ∞-category $\mathrm{Alg}_{\mathcal{BM}/\mathrm{Assoc}}(\mathcal{C})$. Precomposition with the functors $\mathrm{Assoc}_-^{\otimes} \hookrightarrow \mathcal{BM}^{\otimes}$ and $\mathrm{Assoc}_+^{\otimes} \hookrightarrow \mathcal{BM}^{\otimes}$ defines a forgetful functor $\mathrm{BMod}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{C}) \times \mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{C})$.

Let A and B be two associative algebras of $\mathcal{C}^{\otimes}$. We define the ∞-category ${}_A\mathrm{BMod}_B(\mathcal{C})$

of A-B-bimodules of $\mathcal{C}^{\otimes}$ to be the following pullback of ∞-categories

$$\begin{tikzcd}
{}_A\mathrm{BMod}_B(\mathcal{C}) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner"] & & \mathrm{BMod}(\mathcal{C}) \arrow[d] \\
\{A\} \times \{B\} \arrow[r] & \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C}) \times \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C}). &
\end{tikzcd}$$

For the reader who is keen to get a better intuition for the ∞-category of A-A-bimodules we suggest consulting Section A.1, where they will find a more in-depth treatment on the morphisms of the ∞-category ${}_A\mathrm{BMod}_A(\mathcal{C})$. In particular, in Section A.1 we will focus on explaining how it is possible to recover from a morphism of ${}_A\mathrm{BMod}_A(\mathcal{C})$ the familiar 1-categorical notion of a morphism of the underlying objects that commutes with the left and right A-operations of the modules.

3.1.4 Relative tensor product

The main feature that distinguishes associative bimodules from associative left modules is the existence of a natural relative tensor product. We will define this product and see that under reasonable conditions it equips the ∞-category ${}_A\mathrm{BMod}_A(\mathcal{C})$ with an associative monoidal structure.

Once again we will first look at the 1-categorical case to justify our construction of the relative tensor product. Let $\mathcal{C}$ be an associative monoidal 1-category with:

- A, B and C three associative algebras of $\mathcal{C}$;
- M an A-B-bimodule of $\mathcal{C}$;
- and N a B-C-bimodule of $\mathcal{C}$.

Then, under some reasonable hypotheses, for example, if the 1-category $\mathcal{C}$ is cocomplete, we can define the relative tensor product $M \otimes_B N$ of M and N as the object of A-C-bimodules that corepresents bilinear maps with source $M \otimes N$. One can check that the relative tensor product can be modeled by the reflexive coequalizer of the following diagram of $\mathcal{C}$

$$\begin{tikzcd}
M \otimes B \otimes N \arrow[r, shift left=2] \arrow[r, shift right=2] & M \otimes N \arrow[l] \arrow[r, dashed] & M \otimes_B N,
\end{tikzcd} \tag{$\star$}$$

where the two maps from left to right are the right B-action of M and the left B-action of N, and the map from right to left is the unit of B. If we aim to generalize this construction to the ∞-categorical framework we have to take into account that $(\star)$ is only the first of an infinite hierarchy of diagrams that $M \otimes_B N$ has to equalize up to coherent homotopy. For example, for each $n \geq 0$ we expect the following diagram to commute up to coherent homotopy

$$\begin{tikzcd}
M \otimes B^{n+1} \otimes N \arrow[r, shift left=3] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3] & M \otimes B^n \otimes N \arrow[r] & M \otimes_B N
\end{tikzcd} \tag{$\star\star$}$$

where the collection of morphisms is given by the right B-action of M, the algebraic structure of B, the left B-action of N and the unit of B. In the 1-categorical case, it is easy to prove that from the commutativity of diagram ($\star$) it follows that diagram ($\star\star$) commutes. In particular, if we repackage all these diagrams by forming a simplicial object of the 1-category $\mathcal{C}$, which is classically denoted as the bar construction of M and N

$$\mathrm{Bar}_B(M,N) := \Big[\ \dots \ \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \ M \otimes B \otimes N \ \substack{\longrightarrow\\\longrightarrow} \ M \otimes N \ \Big],$$

then its geometric realization is once again the object $M \otimes_B N$. In the ∞-categorical framework, however, this is not the case, and the simplicial object $\mathrm{Bar}_B(M,N)$ is necessary to express the commutativity of all the diagrams like ($\star\star$). This is why we will define the ∞-categorical version of the relative tensor product as the geometric realization of the bar construction rather than the reflexive coequalizer of the analogue of diagram ($\star$).

If we aim to construct the simplicial object $\mathrm{Bar}_B(M,N)$ in the ∞-category ${}_A\mathrm{BMod}_C(\mathcal{C})$ we need to take into consideration that we have to specify cells of $\mathcal{C}$ that express the commutativity of the face and degeneracy maps up to coherent homotopy. We aim to provide these structures by defining a (generalized) ∞-operad called $\mathrm{Tens}^{\otimes}_{\prec}$ which we can informally think of as the ∞-operad that encodes the structure of the bar construction together with the augmentation to its geometric realization.

In order to construct the ∞-operad $\mathrm{Tens}^{\otimes}_{\prec}$ we first have to define another ∞-operad called $\mathrm{Tens}^{\otimes}$.

Definition 3.1.14. [Lur17, Def. 4.4.1.1] We define the 1-category $\mathbf{Tens}^{\otimes}$ to be the 1-category where:

(1) An object of $\mathbf{Tens}^{\otimes}$ is given by:

- an object $\langle n \rangle \in \mathbf{Assoc}^{\otimes}$;
- an object $[k]$ of Δ^{op};
- and a pair of maps $c_-, c_+ : \langle n \rangle^{\circ} \to [k]$ such that for each $i \in \langle n \rangle^{\circ}$ we have $c_-(i) \leq c_+(i) \leq c_-(i+1)$.

(2) Let $(\langle n \rangle, [k], c_-, c_+)$ and $(\langle n' \rangle, [k'], c'_-, c'_+)$ be two objects of $\mathbf{Tens}^{\otimes}$. Then a morphism of $\mathbf{Tens}^{\otimes}$ between the two objects is given by:

- a morphism $\alpha : \langle n \rangle \to \langle n' \rangle$ of $\mathbf{Assoc}^{\otimes}$;
- a morphism $\lambda : [k'] \to [k]$ of Δ such that for every $j \in \langle n' \rangle^{\circ}$, with $\alpha^{-1}(j) = \{i_0 \prec i_1 \prec \dots \prec i_m\}$ we have

$$\lambda(c'_-(j)) = c_-(i_0), \quad \lambda(c'_+(j)) = c_+(i_m),$$

and

$$c_-(i_0) \leq c_+(i_0) = c_-(i_1) \leq c_+(i_1) = c_-(i_2) \leq \cdots \leq c_+(i_{m-1}) = c_-(i_m) \leq c_+(i_m).$$

We denote by $\mathrm{Tens}^\otimes$ the ∞-category given by the nerve of the 1-category $\mathbf{Tens}^\otimes$.

The ∞-category $\mathrm{Tens}^\otimes$ comes equipped with a forgetful functor $\mathrm{Tens}^\otimes \to \mathrm{N}(\Delta)^{\mathrm{op}} \times \mathrm{Assoc}^\otimes$; this forgetful functor exhibits $\mathrm{Tens}^\otimes$ as a generalized ∞-operad as defined in [Lur17, Def. 2.3.2.1]. Moreover, the forgetful functor allows us to define the generalized ∞-operad $\mathrm{Tens}^\otimes_\prec$ that we will use to model the bar construction.

Definition 3.1.15. [Lur17, Notation 4.4.2.1] The morphism $[1] \simeq \{0,2\} \hookrightarrow [2]$ defines a map of simplicial sets $\gamma : \Delta^1 \to \mathrm{N}(\Delta)^{\mathrm{op}}$. We define $\mathrm{Tens}^\otimes_\prec$ to be the pullback of the forgetful functor $\mathrm{Tens}^\otimes \to \mathrm{N}(\Delta)^{\mathrm{op}}$ along γ

$$\mathrm{Tens}^\otimes_\prec := \mathrm{Tens}^\otimes \underset{\mathrm{N}(\Delta)^{\mathrm{op}}}{\times} \Delta^1.$$

Before proceeding with the construction of the relative tensor product we take a moment to describe some of the structure of the generalized ∞-operad $\mathrm{Tens}^\otimes_\prec$ with the aim of providing some intuition for its role in the construction of the relative tensor product. The following observations are consequences of [Lur17, Prop. 4.4.1.11] and [Lur17, Remark 4.4.2.2].

Remark 3.1.16. The forgetful functor $\mathrm{Tens}^\otimes_\prec \to \Delta^1$ is a correspondence between the ∞-operads $\mathrm{Tens}^\otimes_{[2]} := \mathrm{Tens}^\otimes \underset{\mathrm{N}(\Delta)^{\mathrm{op}}}{\times} [2]$ and $\mathrm{Tens}^\otimes_{[1]} := \mathrm{Tens}^\otimes \underset{\mathrm{N}(\Delta)^{\mathrm{op}}}{\times} [1]$. Moreover, we have that:

- The ∞-operad $\mathrm{Tens}^\otimes_{[2]}$ is the pushout of the following diagram of ∞-operads

$$\begin{tikzcd}
\mathrm{Assoc}^\otimes \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathrm{Assoc}^\otimes_- \arrow[r, hook] & \mathcal{BM}^\otimes \arrow[dd] \\
\mathrm{Assoc}^\otimes_+ \arrow[d, hook] & \arrow[ul, phantom, "\ulcorner"] & \\
\mathcal{BM}^\otimes \arrow[rr] & & \mathrm{Tens}^\otimes_{[2]}.
\end{tikzcd}$$

  In particular, the underlying category of $\mathrm{Tens}^\otimes_{[2]}$ has three distinguished objects that represent the algebras:

  - the object $\mathfrak{a}_0 := (\langle 1\rangle, [2], c_- = 0, c_+ = 0)$ that models the left algebra;
  - the object $\mathfrak{a}_1 := (\langle 1\rangle, [2], c_- = 1, c_+ = 1)$ that models the middle algebra;

– and $\mathfrak{a}_2 := (\langle 1\rangle, [2], c_- = 2, c_+ = 2)$ that models the right algebra;

and two distinguished objects that represent the modules:

– the object $\mathfrak{m}_{01} := (\langle 1\rangle, [2], c_- = 0, c_+ = 1)$ that models the $\mathfrak{a}_0$-$\mathfrak{a}_1$-bimodule;

– the object $\mathfrak{m}_{12} := (\langle 1\rangle, [2], c_- = 1, c_+ = 2)$ that models the $\mathfrak{a}_1$-$\mathfrak{a}_2$-bimodule.

If $\mathcal{C}^{\otimes}$ is an associative monoidal ∞-category, we have the following equivalence

$$\mathrm{Alg}_{/\mathrm{Tens}_{[2]}}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{BMod}(\mathcal{C}) \underset{\mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})}{\times} \mathrm{BMod}(\mathcal{C}).$$

So, giving an operadic map $F : \mathrm{Tens}^{\otimes}_{[2]} \to \mathcal{C}^{\otimes}$ is equivalent to giving a pair of bimodules $M \in {}_A\mathrm{BMod}(\mathcal{C})_B$ and $N \in {}_B\mathrm{BMod}(\mathcal{C})_C$, where $M(\mathfrak{m}) = F(\mathfrak{m}_{01})$, $N(\mathfrak{m}) = F(\mathfrak{m}_{12})$, $M(\mathfrak{a}_-) = F(\mathfrak{a}_0)$, $M(\mathfrak{a}_+) = N(\mathfrak{a}_-) = F(\mathfrak{a}_1)$, and $N(\mathfrak{a}_+) = F(\mathfrak{a}_2)$.

- The ∞-operad $\mathrm{Tens}^{\otimes}_{[1]}$ is equivalent to the ∞-operad $\mathcal{BM}^{\otimes}$ defined in Definition 3.1.12. Therefore, its underlying category has three distinguished objects:

  – the object $\mathfrak{a}_- := (\langle 1\rangle, [1], c_- = 0, c_+ = 0)$ that models the left algebra;

  – the object $\mathfrak{a}_+ := (\langle 1\rangle, [1], c_- = 1, c_+ = 1)$ that models the right algebra;

  – and $\mathfrak{m} := (\langle 1\rangle, [1], c_- = 0, c_+ = 1)$ that, as we will see, corresponds to the relative tensor product of $\mathfrak{m}_{01}$ and $\mathfrak{m}_{12}$.

- There is a distinguished active morphism ϵ that represents $\mathfrak{m}$ as the augmentation of the bar construction to its geometric realization. Let $(\mathfrak{m}_{01}, \mathfrak{m}_{12}) \simeq (\langle 2\rangle, [2], c_-, c_+)$ be the object of $\mathrm{Tens}^{\otimes}_{[2]}$ where

$$c_-(1) = 0,\ c_-(2) = 1,\ \text{ and } c_+(1) = 1,\ c_+(2) = 2.$$

  Let β be the active morphism of $\mathrm{Assoc}^{\otimes}$ covering $\langle 2\rangle \to \langle 1\rangle$ equipped with the natural linear ordering of the fiber and let $\gamma : [2] \to [1]$ be the morphism of $\mathrm{N}(\Delta^{\mathrm{op}})$ defined in Definition 3.1.15. Then, the pair (β, γ) defines a morphism ϵ of $\mathrm{Tens}^{\otimes}_{\prec}$ from $(\mathfrak{m}_{01}, \mathfrak{m}_{12}) \simeq (\langle 2\rangle, [2], c_-, c_+)$ to $\mathfrak{m}$.

To introduce the simplicial objects in the picture, we need to define yet another ∞-category.

Definition 3.1.17. We denote by **Step** the full subcategory of $\mathrm{Fun}([1], \Delta)^{\mathrm{op}}$ spanned by those morphisms $f : [n] \to [k]$ in Δ such that for each $1 \leq i \leq n$ we have $f(i) \leq f(i-1) + 1$.

Remark 3.1.18. [Lur17, Notation 4.4.2.4] The 1-category **Step** admits a functor into the 1-category $\mathbf{Tens}^{\otimes}$ defined above. Let $\mathbf{cut} : \Delta^{\mathrm{op}} \to \mathbf{Assoc}^{\otimes}$ be the functor of Construction 3.1.5. We define the functor $\Phi : \mathbf{Step} \to \mathbf{Tens}^{\otimes}$ as follows:

- let $f : [n] \to [k]$ be an object of **Step**, then $\Phi(f) = (\mathbf{cut}(f), [k], c_-, c_+)$ where $c_-, c_+ : \langle n\rangle^\circ \to [k]$ are given by:

$$c_-(i) = f(i-1), \qquad c_+(i) = f(i);$$

- let $\alpha : f \to f'$ be a morphism of **Step**, which corresponds to a commutative diagram of the form

$$\begin{tikzcd} [n] \arrow[r, "f"] \arrow[d, "\alpha_0"] & [k] \arrow[d, "\alpha_1"] \\ [n'] \arrow[r, "f'"] & [k'] \end{tikzcd},$$

  then $\Phi(\alpha) = (\mathbf{cut}(\alpha_0), \alpha_1)$.

Furthermore, the 1-category **Step** admits a functor from the opposite simplicial category Δ^{op}. We define the functor $u : \Delta^{\mathrm{op}} \to \mathbf{Step}$ as follows: let $[n]$ be an object of Δ^{op}, then $u([n])$ is the morphism $f : [n+2] \to [2]$ given by

$$f(i) = \begin{cases} 0 & \text{if} \quad i = 0 \\ 1 & \text{if} \quad 0 < i < n+2 \\ 2 & \text{if} \quad i = n+2. \end{cases}$$

We can extend the functor u to a functor $u_+ : \Delta_+^{\mathrm{op}} \to \mathbf{Step}$ from the pointed opposite simplicial category by assigning to the object $[-1] := \{*\} \in \Delta_*^{\mathrm{op}}$ the object $\{id : [1] \to [1]\} \in \mathbf{Step}$.

Composing u and u_+ with the functor $\Phi : \mathbf{Step} \to \mathbf{Tens}^{\otimes}$ and then passing to nerves we can define:

- the simplicial object $U : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Tens}^{\otimes}_{[2]}$; which we can informally think of as the simplicial object

$$\cdots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} (\mathfrak{m}_{01}, \mathfrak{a}_1, \mathfrak{a}_1, \mathfrak{m}_{12}) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} (\mathfrak{m}_{01}, \mathfrak{a}_1, \mathfrak{m}_{12}) \rightrightarrows (\mathfrak{m}_{01}, \mathfrak{m}_{12});$$

- and the augmented simplicial object $U_+ : \mathrm{N}(\Delta_*)^{\mathrm{op}} \to \mathrm{Tens}^{\otimes}_{\prec}$; which can be informally described as

$$\cdots \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} (\mathfrak{m}_{01}, \mathfrak{a}_1, \mathfrak{a}_1, \mathfrak{m}_{12}) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} (\mathfrak{m}_{01}, \mathfrak{a}_1, \mathfrak{m}_{12}) \rightrightarrows (\mathfrak{m}_{01}, \mathfrak{m}_{12}) \xrightarrow{\epsilon} \mathfrak{m}.$$

In both cases the face maps are the morphisms of $\mathrm{Tens}^{\otimes}_{[2]}$ that represent the module structures of $\mathfrak{m}_{01}$ and $\mathfrak{m}_{12}$, and the algebraic structure of $\mathfrak{a}_1$. The augmentation is given by the morphism ϵ described in Remark 3.1.16.

If $\mathcal{C}^\otimes$ is an associative monoidal ∞-category, starting from an operadic map $F : \mathrm{Tens}^\otimes_\prec \to \mathcal{C}^\otimes$ and precomposing with the functor U_+ we can define the augmented simplicial abject $F \circ U_+ : \mathrm{N}(\Delta_*)^{\mathrm{op}} \to \mathcal{C}^\otimes$ of $\mathcal{C}^\otimes$; that, following the same notation that we used in Remark 3.1.16, we can informally describe as

$$\dots \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; (M, B, B, N) \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; (M, B, N) \;\substack{\longrightarrow\\[-2pt]\longrightarrow}\; (M, N) \longrightarrow X := F(\mathfrak{m}).$$

We can finally define the relative tensor product of $\mathrm{BMod}\,(\mathcal{C})$.

Definition 3.1.19. [Lur17, Def. 4.4.2.3] Let $q : \mathcal{C}^\otimes \to \mathrm{Assoc}^\otimes$ be an associative monoidal ∞-category and let $F : \mathrm{Tens}^\otimes_\prec \to \mathcal{C}^\otimes$ be a map of generalized ∞-operads such that $\mathrm{Tens}^\otimes_{[2]} \to \mathrm{Tens}^\otimes_\prec \xrightarrow{F} \mathcal{C}^\otimes$ determines a composable pair of bimodules $M, N \in \mathrm{BMod}\,(\mathcal{C})$ and $\mathrm{Tens}^\otimes_{[1]} \to \mathrm{Tens}^\otimes_\prec \xrightarrow{F} \mathcal{C}^\otimes$ defines a bimodule object $X \in \mathrm{BMod}\,(\mathcal{C})$. We will say that F exhibits X as the relative tensor product of M and N if F is an operadic q-colimit diagram as defined in [Lur17, Def. 3.1.1.2].

Instead of delving into the theory of operadic colimit diagrams, we will see that, under reasonable conditions, we can give an equivalent characterization of the relative tensor product via the bar construction of M and N that does not involve operadic colimits. But first, we have to define the bar construction of two bimodules.

Construction 3.1.20. [Lur17, Notation 4.4.2.4] Let $U_+ : \mathrm{N}(\Delta_*)^{\mathrm{op}} \to \mathrm{Tens}^\otimes_\prec$ be the augmented simplicial object defined in Remark 3.1.18 with augmentation the object $\mathfrak{m}$. By considering the augmentation as a constant simplicial object, U_+ defines a morphism $\beta : U \to U'$ of (non-augmented) simplicial objects of $\mathrm{Tens}^\otimes_\prec$ where U is the simplicial object $U : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Tens}^\otimes_{[2]} \hookrightarrow \mathrm{Tens}^\otimes_\prec$ defined above and U' is the constant simplicial object with constant value $\mathfrak{m} \in \mathrm{Tens}^\otimes_{[1]}$.

Now suppose that we have a commutative diagram of generalized ∞-operads

$$\begin{tikzcd} \mathrm{Tens}^\otimes_{[2]} \arrow[r, "F_0"] \arrow[d] & \mathcal{C}^\otimes \arrow[d, "q"] \\ \mathrm{Tens}^\otimes_\prec \arrow[r, "f"] & \mathrm{Assoc}^\otimes, \end{tikzcd}$$

where q is a coCartesian fibration of ∞-operads, i.e., $\mathcal{C}^\otimes$ is an associative monoidal ∞-category. The commutative diagram induces a commutative diagram on the ∞-categories of simplicial objects

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{Tens}^\otimes_{[2]}) \arrow[r, "(F_0\circ -)"] \arrow[d] & \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathcal{C}^\otimes) \arrow[d, "(q\circ -)"] \\ \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{Tens}^\otimes_\prec) \arrow[r, "(f\circ -)"] & \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{Assoc}^\otimes). \end{tikzcd}$$

Starting from the simplicial object $U \in \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{Tens}^{\otimes}_{[2]})$ and applying $(F_0 \circ -)$ we obtain the simplicial object $F_0 \circ U$ of $\mathcal{C}^{\otimes}$. On the other hand, postcomposing U with the inclusion in $\mathrm{Tens}^{\otimes}_{[2]} \to \mathrm{Tens}^{\otimes}_{\prec}$ and with the map f we obtain the simplicial object $f \circ U$ of $\mathrm{Assoc}^{\otimes}$. By the commutativity of the diagram, the object $F_0 \circ U$ covers the object $f \circ U$. Finally, the image of the morphism $\beta : U \to U'$ by the functor f defines a morphism $f \circ \beta : f \circ U \to f \circ U'$ between simplicial objects of $\mathrm{Assoc}^{\otimes}$.

Since q is coCartesian $q' := (q \circ -)$ is a coCartesian fibration too [Lur18, Theorem 5.2.1.1] and there exists a coCartesian morphism with source $F_0 \circ U$ covering the morphism $f \circ \beta$. The target of this morphism is a simplicial object of $\mathcal{C}^{\otimes}$ that we define to be the bar construction of M and N. We will denote this simplicial object with $\mathrm{Bar}_B(M, N)_\bullet$.

$$\begin{tikzcd}
\left[\; U \;\right] \arrow[r, mapsto, "(F_0\circ -)"] \arrow[d, mapsto] & \left[\; F_0 \circ U \dashrightarrow \mathrm{Bar}_B(M,N)_\bullet \;\right] \arrow[d, mapsto, "q'"] \\
\left[\; U \xrightarrow{\beta} U' \;\right] \arrow[r, mapsto, "(f\circ -)"] & \left[\; f \circ U \xrightarrow{f\circ\beta} f \circ U' \;\right].
\end{tikzcd}$$

Moreover, if we are given a functor F that extends the map F_0

$$\begin{tikzcd}
\mathrm{Tens}^{\otimes}_{[2]} \arrow[r, "F_0"] \arrow[d] & \mathcal{C}^{\otimes} \arrow[d] \\
\mathrm{Tens}^{\otimes}_{\prec} \arrow[r, "f"] \arrow[ur, dashed, "F"] & \mathrm{Assoc}^{\otimes},
\end{tikzcd}$$

the image of the morphism β by postcomposition with F defines yet another morphism $F \circ \beta : F_0 \circ U \to F \circ U'$ of simplicial objects of $\mathcal{C}^{\otimes}$ with source $F_0 \circ U$ and that covers the morphism $f \circ \beta$. By the universal property of coCartesian morphisms, we can fill the following diagram of simplicial objects of $\mathcal{C}^{\otimes}$ with the dashed arrow γ

$$\left[\begin{tikzcd}
F_0 \circ U \arrow[r] \arrow[dr, "F\circ\beta"'] & \mathrm{Bar}_B(M,N)_\bullet \arrow[d, dashed, "\gamma"] \\
 & F \circ U'
\end{tikzcd}\right].$$

Assuming that the ∞-category $\mathcal{C}^{\otimes}$ admits enough colimits, we can pass to the geometric realizations of the simplicial objects to obtain the following diagram of $\mathcal{C}^{\otimes}$

$$\left[\begin{tikzcd}
|F_0 \circ U| \arrow[r] \arrow[dr, "|F\circ\beta|"'] & |\mathrm{Bar}_B(M,N)_\bullet| \arrow[d, dashed, "|\gamma|"] \\
 & F(\mathfrak{m})
\end{tikzcd}\right].$$

It is natural to ask when the morphism $|\gamma|$ is an equivalence of $\mathcal{C}^{\otimes}$. In the next theorem we will see that, under some mild hypotheses, this is equivalent to asking if F exhibits X as the relative tensor product of M and N.

Theorem 3.1.21. [Lur17, Theorem 4.4.2.8] *Let $q : \mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ be a coCartesian fibration of ∞-operads which is compatible with $\mathrm{N}(\Delta)^{\mathrm{op}}$-indexed colimits, in the sense of [Lur17, Def. 3.1.1.8]. Suppose that we are given a commutative diagram of solid arrows of generalized ∞-operads*

$$\begin{tikzcd} \mathrm{Tens}^{\otimes}_{[2]} \arrow[r, "F_0"] \arrow[d] & \mathcal{C}^{\otimes} \arrow[d, "q"] \\ \mathrm{Tens}^{\otimes}_{\prec} \arrow[r, "f"] \arrow[ur, dashed, "F"] & \mathrm{Assoc}^{\otimes}, \end{tikzcd}$$

where F_0 corresponds to a pair of bimodules $M \in {}_A\mathrm{BMod}_B(\mathcal{C})$, $N \in {}_B\mathrm{BMod}_C(\mathcal{C})$. Then there exists an extension F of F_0 that fills the diagram, which exhibits $X = F_{|\mathrm{Tens}^{\otimes}_{[1]}} \in {}_{A'}\mathrm{BMod}_{C'}(\mathcal{C})$ as the relative tensor product of M and N. Moreover, if F is an arbitrary extension of F_0 making the above diagram commute, then F exhibits X as the relative tensor product of M and N if and only if the following condition are satisfied:

(1) The functor F induces q-coCartesian morphisms $A \to A'$, $B \to B'$.

(2) The functor F induces an equivalence

$$|\mathrm{Bar}_B(M,N)_{\bullet}| \xrightarrow{\simeq} F(\mathfrak{m}).$$

Is important to notice that the first condition is always satisfied if the morphisms of algebras induced by F are identities.

Example 3.1.22. [Lur17, Example 4.4.2.11] Let $q : \mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ be an associative monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realization of simplicial objects and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realization separately in each variable. This is equivalent to the condition of being compatible with $\mathrm{N}(\Delta)^{\mathrm{op}}$-indexed colimits of Theorem 3.1.21. Consider the forgetful functor $\mathrm{Tens}^{\otimes}_{\prec} \hookrightarrow \mathrm{Tens}^{\otimes} \to \mathrm{Assoc}^{\otimes}$. Then the relative tensor product defines a functor

$$\mathrm{BMod}(\mathcal{C}) \underset{\mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})}{\times} \mathrm{BMod}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathrm{Tens}_{[2]}/\mathrm{Assoc}}(\mathcal{C}) \to \mathrm{BMod}(\mathcal{C})$$

Moreover, from Theorem 3.1.21 it follows that for each triple of associative algebra objects $A, B, C \in \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})$, the functor restricts to a map

$${}_A\mathrm{BMod}_B(\mathcal{C}) \times {}_B\mathrm{BMod}_C(\mathcal{C}) \to {}_A\mathrm{BMod}_C(\mathcal{C}),$$

taking $A = B = C$ we obtain a product functor of the ∞-category ${}_A\mathrm{BMod}_A(\mathcal{C})$

$$\otimes_A : {}_A\mathrm{BMod}_A(\mathcal{C}) \times {}_A\mathrm{BMod}_A(\mathcal{C}) \to {}_A\mathrm{BMod}_A(\mathcal{C}).$$

In [Lur17, Section 4.4.3] it is proven that this product is unital and associative.

Even if we have defined an associative and unital product of the ∞-category ${}_A\mathrm{BMod}_A(\mathcal{C})$ we have not explicitly provided an Assoc-monoidal structure that presents the ∞-category ${}_A\mathrm{BMod}_A(\mathcal{C})$ as an associative monoidal ∞-category as defined in Definition 3.1.4. We will produce such structure in the next section as the special case $\mathcal{O}^\otimes = \mathrm{Assoc}^\otimes$ of a more general construction of $\mathcal{O}$-monoidal categories of $\mathcal{O}$-modules.

3.2 $\mathcal{O}$-modules

In the previous section, starting from an associative monoidal ∞-category $\mathcal{C}^\otimes$, we have defined the ∞-category of bimodules $\mathrm{BMod}(\mathcal{C})$ and proved that for any associative algebra A the ∞-category ${}_A\mathrm{BMod}_A(\mathcal{C})$ admits an associative and unital product functor given by the relative tensor product. In this section, we will generalize this construction to general $\mathcal{O}$-monoidal categories. Starting from an $\mathcal{O}$-monoidal ∞-category $\mathcal{C}^\otimes$ we will define the ∞-category of $\mathcal{O}$-modules $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^\otimes$ and we will see that, under reasonable conditions on the ∞-operad $\mathcal{O}^\otimes$ and the $\mathcal{O}$-monoidal category $\mathcal{C}^\otimes$, for each $\mathcal{O}$-algebra A of $\mathcal{C}^\otimes$ the ∞-category $\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C})^\otimes$ admits a natural $\mathcal{O}$-monoidal structure.

In the special case where $\mathcal{O}^\otimes = \mathrm{Assoc}^\otimes$ we have an equivalence between the underlying category of $\mathrm{Mod}^{\mathrm{Assoc}}_A(\mathcal{C})^\otimes$ and the ∞-category ${}_A\mathrm{BMod}_A(\mathcal{C})$; the associative product defined by the Assoc-monoidal structure of $\mathrm{Mod}^{\mathrm{Assoc}}_A(\mathcal{C})^\otimes$ will correspond, under this equivalence, to the relative tensor product defined at the end of the previous section.

Until this point, all the ∞-operads that we considered have been defined by the nerve of a 1-category, which has made it easy to motivate our constructions by first looking at the 1-categorical cases. Now we aim to define the ∞-category of $\mathcal{O}$-modules for a general ∞-operad $\mathcal{O}^\otimes$, which might not be the nerve of a 1-category. Therefore, in this case, we will not be able to draw an explicit connection with 1-categories.

We begin by introducing the notion of semi-inert morphisms of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and $\mathcal{O}^\otimes$.

Definition 3.2.1. A morphism $\alpha : \langle m\rangle \to \langle n\rangle$ of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is semi-inert if for each $i \in \langle n\rangle^\circ$ the set $\alpha^{-1}(i)$ has at most one element. We will say that α is null if it is the semi-inert morphism that carries $\langle m\rangle$ to the distinguished point $*$ of $\langle n\rangle$.

Let $p : \mathcal{O}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad and let $f : X \to Y$ be a morphism of $\mathcal{O}^\otimes$. We will say that f is semi-inert if the following conditions are satisfied:

(1) The image $p(f)$ is a semi-inert morphism of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

(2) For every inert morphism $g : Y \to Z$ of $\mathcal{O}^{\otimes}$, if $p(g \circ f)$ is an inert morphism of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, then $g \circ f$ is an inert morphism of $\mathcal{O}^{\otimes}$; in particular $g \circ f$ is p-coCartesian.

We say that a morphism f is null if its image $p(f)$ is null in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Definition 3.2.2. [Lur17, Notation 3.3.2.1] Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad, i.e., an ∞-operad such that the unique object $\emptyset \in \mathcal{O}^{\otimes}_{\langle 0 \rangle}$ is both initial and final. We denote by $\mathcal{K}_{\mathcal{O}}$ the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ spanned by the semi-inert morphisms of $\mathcal{O}^{\otimes}$; and by $e_0, e_1 : \mathcal{K}_{\mathcal{O}} \to \mathcal{O}^{\otimes}$ the evaluation maps given by evaluating on $\{0\}$ and $\{1\}$. We say that a morphism of $\mathcal{K}_{\mathcal{O}}$, which corresponds to a commutative square of $\mathcal{O}^{\otimes}$, is inert if its images under e_0 and e_1 are inert morphisms of $\mathcal{O}^{\otimes}$.

Suppose that the ∞-operad $\mathcal{O}^{\otimes}$ is unital and let Z be an object of $\mathcal{O}$. Then the fiber product $\{Z\} \underset{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})}{\times} \mathcal{K}_{\mathcal{O}}$ possess two sets of distinguished objects:

- the identity morphism $\{id_Z : Z \to Z\} \in \mathcal{K}_{\mathcal{O}}$, that informally represents the $\mathcal{O}$-module;
- and the set of null morphisms $\{\mathrm{Na}_Z : Z \to \emptyset \to Y\}_{Y \in \mathcal{O}^{\otimes}}$, which represents the $\mathcal{O}$-algebra.

Starting from a unital ∞-operad $\mathcal{O}^{\otimes}$ and an $\mathcal{O}$-monoidal category $\mathcal{C}^{\otimes}$, we will see how to construct the ∞-category $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes}$; and how to associate to each $\mathcal{O}$-algebra A an ∞-operad $\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C})^{\otimes}$.

Construction 3.2.3. [Lur17, Construction 3.3.3.1] We denote by $\widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ the simplicial set over $\mathcal{O}^{\otimes}$ defined by the following universal property: for every map of simplicial sets $X \to \mathcal{O}^{\otimes}$ there is a canonical bijection

$$\mathrm{Fun}_{\mathcal{O}^{\otimes}}(X, \widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}) \simeq \mathrm{Fun}_{\mathrm{Fun}(\{1\}, \mathcal{O}^{\otimes})}(X \underset{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})}{\times} \mathcal{K}_{\mathcal{O}}, \mathcal{C}^{\otimes}).$$

We denote by $\overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ the full simplicial subset of $\widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ spanned by the vertices $\bar{v}$ with the property that the functor

$$\{\bar{v}\} \underset{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})}{\times} \mathcal{K}_{\mathcal{O}} \to \mathcal{C}^{\otimes}$$

induced by the inclusion $\{\bar{v}\} \hookrightarrow \widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ carries inert morphisms to inert morphisms.

Similarly, if we denote by $\mathcal{K}^0_{\mathcal{O}}$ the full subcategory of $\mathcal{K}_{\mathcal{O}}$ spanned by the null morphisms of $\mathcal{O}^{\otimes}$; we can define the simplicial set $\widetilde{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C})$ over $\mathcal{O}^{\otimes}$ to be the simplicial set that satisfies the following universal property: for every map of simplicial sets $X \to \mathcal{O}^{\otimes}$ there is a canonical bijection

$$\mathrm{Fun}_{\mathcal{O}^{\otimes}}(X, \widetilde{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C})) \simeq \mathrm{Fun}_{\mathrm{Fun}(\{1\},\mathcal{O}^{\otimes})}(X \underset{\mathrm{Fun}(\{0\},\mathcal{O}^{\otimes})}{\times} \mathcal{K}^0_{\mathcal{O}}, \mathcal{C}^{\otimes}).$$

Let $\mathrm{Alg}^p_{/\mathcal{O}}(\mathcal{C})$ denote the full simplicial subset of $\widetilde{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C})$ spanned by the vertices $\bar{w}$ with the property that the functor

$$\{\bar{w}\} \underset{\mathrm{Fun}(\{0\},\mathcal{O}^{\otimes})}{\times} \mathcal{K}^0_{\mathcal{O}} \to \mathcal{C}^{\otimes}$$

induced by the inclusion $\{\bar{w}\} \hookrightarrow \widetilde{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C})$ carries inert morphisms to inert morphisms.

By definition, all the objects of $\mathcal{K}^0_{\mathcal{O}}$ correspond to morphisms of $\mathcal{O}^{\otimes}$ that are null, but since we assumed the ∞-operad $\mathcal{O}^{\otimes}$ to be unital, the null morphisms are uniquely identified by their target and their source. This observation is formalized in [Lur17, Lemma 3.3.3.3] where it is proven that the evaluation maps form a trivial Kan fibration between the ∞-category $\mathcal{K}^0_{\mathcal{O}}$ and the product $\mathcal{O}^{\otimes} \times \mathcal{O}^{\otimes}$.

We can use this result to obtain some intuition on the objects of the ∞-category $\mathrm{Alg}^p_{/\mathcal{O}}(\mathcal{C})$. An object $A \in \mathrm{Alg}^p_{/\mathcal{O}}(\mathcal{C})$ corresponds to an object $Z \in \mathcal{O}^{\otimes}$ and a functor over $\mathcal{O}^{\otimes}$ that preserves inert morphisms

$$F : \{Z\} \underset{\mathrm{Fun}(\{0\},\mathcal{O}^{\otimes})}{\times} \mathcal{K}^0_{\mathcal{O}} \to \mathcal{C}^{\otimes}.$$

Since the evaluation maps give a trivial Kan fibration $\mathcal{K}^0_{\mathcal{O}} \to \mathcal{O}^{\otimes} \times \mathcal{O}^{\otimes}$ the functor F defines a unique $\mathcal{O}$-algebra of $\mathcal{C}^{\otimes}$

$$\mathcal{O}^{\otimes} \simeq \{Z\} \underset{\mathrm{Fun}(\{0\},\mathcal{O}^{\otimes})}{\times} (\mathcal{O}^{\otimes} \times \mathcal{O}^{\otimes}) \xrightarrow{\simeq} \{Z\} \underset{\mathrm{Fun}(\{0\},\mathcal{O}^{\otimes})}{\times} \mathcal{K}^0_{\mathcal{O}} \xrightarrow{F} \mathcal{C}^{\otimes}.$$

Therefore, an object of $\mathrm{Alg}^p_{/\mathcal{O}}(\mathcal{C})$ corresponds to a pair given by an object of $\mathcal{O}^{\otimes}$ and an object of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. Formally, the evaluation maps induce a categorical equivalence $\mathcal{O}^{\otimes} \times \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}^p_{/\mathcal{O}}(\mathcal{C})$ [Lur17, Remark 3.3.3.7].

We can finally define the ∞-category of $\mathcal{O}$-modules of $\mathcal{C}^{\otimes}$.

Definition 3.2.4. [Lur17, Def. 3.3.3.8] Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad and $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ an $\mathcal{O}$-monoidal category. We define $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ to be the following pullback of simplicial sets

$$\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes} := \overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \underset{\mathrm{Alg}^p_{/\mathcal{O}}(\mathcal{C})}{\times} (\mathcal{O}^{\otimes} \times \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})).$$

Let A be an $\mathcal{O}$-algebra of $\mathcal{C}^{\otimes}$. We denote by $\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C})^{\otimes}$ the pullback

$$\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C})^{\otimes} := \overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \underset{\mathrm{Alg}^p_{/\mathcal{O}}(\mathcal{C})}{\times} (\mathcal{O}^{\otimes} \times \{A\}) \simeq \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes} \underset{\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})}{\times} (\{A\}).$$

For now, we have defined $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ only as a simplicial set, but we will see that under reasonable conditions it admits an $\mathcal{O}$-monoidal structure. Let us start by defining a sufficient condition on the ∞-operad $\mathcal{O}^{\otimes}$ which ensures that the simplicial set $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ is an ∞-operad.

Definition 3.2.5. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. We say that $\mathcal{O}^{\otimes}$ is coherent if:

(1) It is unital, meaning that the unique object $\emptyset \in \mathcal{O}^{\otimes}_{\langle 0\rangle}$ is both initial and final.

(2) The underlying category $\mathcal{O}$ is a Kan complex.

(3) The evaluation map $e_0 : \mathcal{K}_{\mathcal{O}} \to \mathcal{O}^{\otimes}$ is a flat categorical fibration as defined in [Lur17, Def. B.3.8].

In order to avoid the introduction of unnecessary material we have chosen to define a coherent ∞-operad using the characterization given in [Lur17, Theorem 3.3.2.2] instead of the classical definition [Lur17, Def. 3.3.1.9]. Condition (3) is a technical condition and it is sufficient to say that most of the ∞-operads we will consider, such as, for example, the little cubes ∞-operads that we will define in the next sections, are coherent ∞-operads.

Theorem 3.2.6. [Lur17, Theorem 3.3.3.9] *Let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal category, where $\mathcal{O}^{\otimes}$ is a coherent ∞-operad, and let A be an $\mathcal{O}$-algebra of $\mathcal{C}^{\otimes}$. Then, the induced map $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ is a map of ∞-operads.*

We can assume some additional properties on the $\mathcal{O}$-monoidal category $\mathcal{C}^{\otimes}$ that ensures that the induced map $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration, i.e., $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ is an $\mathcal{O}$-monoidal category.

Definition 3.2.7. Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal category. We say that $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a presentable $\mathcal{O}$-monoidal category if it satisfies the following conditions:

(1) The coCartesian fibration q is compatible with small colimits, [Lur17, Def. 3.1.1.18].

(2) For each Z in $\mathcal{O}$, the fiber $\mathcal{C}^{\otimes}_Z$ is a presentable ∞-category.

Theorem 3.2.8. [Lur17, Theorem 3.4.4.2] *Let $\mathcal{O}^{\otimes}$ be a small coherent ∞-operad, $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a presentable $\mathcal{O}$-monoidal ∞-category, and let $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be an $\mathcal{O}$-algebra object of $\mathcal{C}^{\otimes}$. Then, the induced map $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ exhibits $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ as a presentable $\mathcal{O}$-monoidal ∞-category.*

Remark 3.2.9. Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be two presentable $\mathcal{O}$-monoidal categories and let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a lax $\mathcal{O}$-monoidal map. Then, the map F induces a lax $\mathcal{O}$-monoidal functor F' on the ∞-categories of $\mathcal{O}$-modules. To define the functor F' we use the universal properties of the simplicial sets $\widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ and $\widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{D})^{\otimes}$. For each simplicial set X over $\mathcal{O}^{\otimes}$ we have

$$
\begin{aligned}
\mathrm{Fun}_{\mathcal{O}^{\otimes}}(X, \widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}) &\xrightarrow{\simeq} \\
&\xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{O}^{\otimes}}(X \underset{\mathrm{Fun}(\{0\},\mathcal{O}^{\otimes})}{\times} \mathcal{K}_{\mathcal{O}}, \mathcal{C}^{\otimes}) \\
&\xrightarrow{(F\circ -)} \mathrm{Fun}_{\mathcal{O}^{\otimes}}(X \underset{\mathrm{Fun}(\{0\},\mathcal{O}^{\otimes})}{\times} \mathcal{K}_{\mathcal{O}}, \mathcal{D}^{\otimes}) \\
&\xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{O}^{\otimes}}(X, \widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{D})^{\otimes}).
\end{aligned}
\tag{$\star$}
$$

by the ∞-categorical version of the Yoneda embedding, [Lur09, Prop. 5.1.3.1], the composition ($\star$) defines a unique, up to equivalence, functor

$$F' : \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathrm{Mod}^{\mathcal{O}}(\mathcal{D})^{\otimes}.$$

Let A be an $\mathcal{O}$-algebra object of $\mathcal{C}^{\otimes}$, and let B be the image in $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{D})$ of the algebra A by the functor F. Restricting F' to $\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C})^{\otimes}$ we obtain the following lax $\mathcal{O}$-monoidal map

$$F' : \mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C})^{\otimes} \to \mathrm{Mod}^{\mathcal{O}}_B(\mathcal{D})^{\otimes}.$$

Let us now go back to the question about $\mathcal{O}$-monoidal pre-sheaves that we asked at the end of Chapter 2. Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad, $\pi : X^{\otimes} \to B^{\otimes}$ be a left $\mathcal{O}$-fibration, and let $\psi : B^{\otimes} \to \mathcal{S}^{\otimes}$ be its straightening. As we mentioned in our previous discussion, the lax $\mathcal{O}$-monoidal map ψ does not generally preserve the unit, but we can consider instead the map induced by the lax $\mathcal{O}$-monoidal pre-sheaf on the ∞-categories of $\mathcal{O}$-modules

$$\psi' : \mathrm{Mod}^{\mathcal{O}}(B)^{\otimes} \to \mathrm{Mod}^{\mathcal{O}}(\mathcal{S})^{\otimes}.$$

Since $\mathcal{O}^{\otimes}$ is coherent the Kan complex $B^{\otimes}$ admits a trivial $\mathcal{O}$-algebra 1_B as defined in [Lur17, Section 3.2.1]. Restricting ψ' to the ∞-operad of 1_B-modules we obtain

$$\psi' : \mathrm{Mod}^{\mathcal{O}}_{1_B}(B)^{\otimes} \to \mathrm{Mod}^{\otimes}_{F}(\mathcal{S})^{\otimes},$$

where F is the $\mathcal{O}$-algebra induced by ψ from the trivial algebra 1_B. Applying Lemma 2.4.6 to the composition

$$F : \mathcal{O}^{\otimes} \xrightarrow{1_B} B^{\otimes} \xrightarrow{\psi} \mathcal{S}^{\otimes}$$

we obtain that under the equivalence described in Corollary 2.4.4, the algebra F corresponds to the $\mathcal{O}$-monoidal Kan complex $F^{\otimes}$ given by the pullback of π over the trivial algebra 1_B. We refer to the $\mathcal{O}$-monoidal Kan complex $F^{\otimes}$ as the fiber of π over the unit of $B^{\otimes}$.

$$\begin{tikzcd} F^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X^{\otimes} \arrow[d, "\pi"] \\ \mathcal{O}^{\otimes} \arrow[r, "1_B"] & B^{\otimes}. \end{tikzcd}$$

Now the map ψ' is a lax $\mathcal{O}$-monoidal map that preserves the trivial algebra and it is reasonable to ask under which conditions it is (strong) $\mathcal{O}$-monoidal. In the first sections of Chapter 5, we will provide an answer to this question for the special case where the ∞-operad $\mathcal{O}^{\otimes}$ belongs to the family of little cubes ∞-operads that we will introduce in the next section.

We end this section by showing that, under the sufficient conditions for the existence of the relative tensor product of Theorem 3.1.21, the ∞-category of Assoc-modules $\mathrm{Mod}_A^{\mathrm{Assoc}}(\mathcal{C})^{\otimes}$ is indeed a generalization of the associative monoidal ∞-category of bimodules defined in Example 3.1.22.

Proposition 3.2.10. [Lur17, Prop. 4.4.3.12., Theorem. 4.4.1.28] *Let $\mathcal{C}$ be an associative monoidal ∞-category and let A be an associative algebra of $\mathcal{C}$. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product of $\mathcal{C}$ preserves geometric realizations separately in each variable. Then:*

(1) The operadic map $\mathrm{Mod}_A^{\mathrm{Assoc}}(\mathcal{C})^{\otimes} \to \mathrm{Assoc}^{\otimes}$ is coCartesian, i.e., it exhibits $\mathrm{Mod}_A^{\mathrm{Assoc}}(\mathcal{C})$ as an associative monoidal ∞-category.

(2) The underlying category $\mathrm{Mod}_A^{\mathrm{Assoc}}(\mathcal{C})$ is equivalent to the ∞-category ${}_A\mathrm{BMod}_A(\mathcal{C})$.

(3) The tensor product of $\mathrm{Mod}_A^{\mathrm{Assoc}}(\mathcal{C})$ corresponds to the relative tensor product functor $\otimes_A : {}_A\mathrm{BMod}_A(\mathcal{C}) \times {}_A\mathrm{BMod}_A(\mathcal{C}) \to {}_A\mathrm{BMod}_A(\mathcal{C})$ defined in Example 3.1.4.

3.3 Little cubes ∞-operads and their modules

In this section we will introduce an infinite family of ∞-operads called little cubes ∞-operads. These are the ∞-categorical analogue of the little cubes operads originally introduced by J.M. Boardman and R. M. Vogt in [BV68, Def. 5]. Similar to the 1-categorical case, the little cubes ∞-operads can be arranged in a sequence of ∞-operads where each element encodes more monoidal structure than the previous one. In particular, they can be arranged in an infinite sequence of operadic maps

$$\mathbb{E}_0^{\otimes} \to \mathbb{E}_1^{\otimes} \to \mathbb{E}_2^{\otimes} \to \cdots \to \mathbb{E}_{\infty}^{\otimes},$$

where the ∞-operad $\mathbb{E}_1^{\otimes}$ is equivalent to the associative ∞-operad $\mathrm{Assoc}^{\otimes}$ that we described in Section 3.1.1, and the colimit of the sequence which is denoted by $\mathbb{E}_{\infty}^{\otimes}$ is equivalent to the commutative ∞-operad $\mathrm{Comm}^{\otimes}$.

We will start by defining the 1-categorical little cubes operads as single-colored topological operads, and then use the construction described in Construction 2.1.3 to define their ∞-categorical analogue. Let us begin with the definition of rectilinear embeddings.

Definition 3.3.1. Let $k \geq 0$. We denote by $\square^k := (-1,1)^k$ an open cube of dimension k. We say that a map $f : \square^k \to \square^k$ is a rectilinear embedding if there exist real constants $a_i > 0$ and b_i for $1 \leq i \leq k$ such that

$$f(x_1, \dots, x_k) = (a_1 x_1 + b_1, \dots, a_k x_k + b_k).$$

Let S be a finite set, we say that a map $\square^k \times S \to \square^k$ is a rectilinear embedding if for each $s \in S$ the restriction $\square^k \times \{s\} \to \square^k$ is a rectilinear embedding. We define $\mathrm{Rect}(\square^k \times S, \square^k)$ to be the set of rectilinear embeddings from $\square^k \times \{s\}$ to $\square^k$ equipped with the topology it inherits as an open subset of $(\mathbb{R}^{2k})^S$.

We now define the little cubes operad ${}^t\mathbb{E}_k$ for $k \geq 0$ as a single-colored topological operad, i.e., a colored operad where the sets of morphisms are equipped with a topology and the composition maps are continuous.

Definition 3.3.2. [Lur17, Def. 5.1.0.2] We define the colored operad ${}^t\mathbb{E}_k$ as follows:

(1) The set of objects has a single element $*$.

(2) Let S be a finite set, the space of S-indexed operations $\mathrm{Mul}_{{}^t\mathbb{E}_k}(\{*\}_{s \in S}, *)$ is the topological space of rectilinear embeddings $\mathrm{Rect}(\square^k \times S, \square^k)$.

(3) Composition of operations is defined by the composition of rectilinear embeddings.

With the usual procedure, we can define the ∞-categorical version of the little cubes operads.

Definition 3.3.3. Let ${}^t\mathbb{E}_k^{\otimes}$ be the topological category obtained from applying Construction 2.1.3 to the single-colored topological operad ${}^t\mathbb{E}_k$. Since ${}^t\mathbb{E}_k^{\otimes}$ is a topological category, it is, in particular, a fibrant simplicial category, and the coherent nerve of the forgetful functor ${}^t\mathbb{E}_k^{\otimes} \to \mathcal{F}\mathrm{in}_*$ defines the ∞-operad $\mathbb{E}_k^{\otimes} : \mathrm{N}({}^t\mathbb{E}_k^{\otimes}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

The little cubes ∞-operads enjoy many useful properties; first of all, they are coherent single-colored ∞-operads [Lur17, Theorem 5.1.1.1]. Moreover, they satisfy the ∞-categorical analogue of the Dunn additivity theorem [Lur17, Theorem 5.1.2.2].

Theorem 3.3.4 (Dunn Additivity Theorem)**.** *Let $k, k' \geq 0$ be non-negative integers. Then, there exists a bifunctor $\mathbb{E}_k^{\otimes} \times \mathbb{E}_{k'}^{\otimes} \to \mathbb{E}_{k+k'}^{\otimes}$, see [Lur17, Construction 5.1.2.1], that exhibits the ∞-operad $\mathbb{E}_{k+k'}^{\otimes}$ as a tensor product of the ∞-operads $\mathbb{E}_k^{\otimes}$ and $\mathbb{E}_{k'}^{\otimes}$.*

We will not discuss in detail the notion of tensor product of ∞-operads. We can briefly say that given two ∞-operads $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ it is possible to construct a new ∞-operad $\mathcal{O}^{\otimes} \otimes \mathcal{O}'^{\otimes}$ such that: if $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, then the two ∞-operads and their tensor product satisfy the following equivalence [Lur17, Section 2.2.5]

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})).$$

Once again, we point the reader who is interested in a more extensive discussion on the tensor product of ∞-operads to Section 2.2.5 of [Lur17] where the author defines the tensor product and presents some of its main properties.

Thanks to the Dunn additivity theorem, it is possible to prove many results involving the ∞-category $\mathbb{E}_k$-modules. The proof of the following two theorems is quite involved and requires some machinery that we have chosen to not include in this thesis, so without making any claim of completeness, we will present the statement of the theorems and then foreshadow how we will use these results during the construction of the Iterated Thom spectrum.

Theorem 3.3.5. [Lur17, Theorem 5.1.3.2] *Let $k \geq 1$ be an integer, $q : \mathcal{C}^{\otimes} \to \mathbb{E}_k^{\otimes}$ be an $\mathbb{E}_k$-monoidal category, and let $\iota : \mathbb{E}_1^{\otimes} \to \mathbb{E}_k^{\otimes}$ be a map of ∞-operads. Provided that $\mathcal{C}$ admits geometric realization of simplicial objects and that the tensor product functor, given by the map ι, preserves geometric realizations of simplicial objects. For each $\mathbb{E}_k$-algebra object A of $\mathcal{C}^{\otimes}$. Then:*

(1) The map $\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})^{\otimes} \to \mathbb{E}_k^{\otimes}$ is a coCartesian fibration of ∞-operads.

(2) Let $\mathcal{C}'^{\otimes}$ be the associative monoidal ∞-category given by the pullback of q along $\iota : \mathbb{E}_1^{\otimes} \to \mathbb{E}_k^{\otimes}$. And similarly let A' denote the associative algebra of $\mathcal{C}'^{\otimes}$ given by taking the pullback of A. Then there exists a functor

$$F : \mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})^{\otimes} \underset{\mathbb{E}_k^{\otimes}}{\times} \mathbb{E}_1^{\otimes} \to \mathrm{Mod}_{A'}^{\mathbb{E}_1}(\mathcal{C})^{\otimes}$$

which is associative monoidal. Here we are implicitly considering the equivalence $\mathbb{E}_1^{\otimes} \simeq \mathrm{Assoc}^{\otimes}$.

Let us go back to our motivating question with $\mathcal{O}^{\otimes} = \mathbb{E}_n^{\otimes}$. Let $\pi : X^{\otimes} \to B^{\otimes}$ be a left $\mathbb{E}_n$-fibration; in Chapter 2 we asked under which conditions its straightening $\psi : B^{\otimes} \to \mathcal{S}^{\otimes}$ is an $\mathbb{E}_n$-monoidal map and observed that since the pre-sheaf does not generally map the unit to the unit it was unreasonable to ask for strong monoidality. In the previous section, we solved this issue by considering instead the map induced by ψ on the ∞-categories of $\mathbb{E}_n$-modules

$$\psi' : \mathrm{Mod}_{1_B}^{\mathbb{E}_n}(B)^{\otimes} \to \mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S})^{\otimes}.$$

In Chapter 5 we will provide a sufficient condition on the fibration π, namely that it comes from an $\mathbb{E}_{n+1}$-fibration of grouplike Kan complexes, which ensures that the map ψ' is $\mathbb{E}_n$-monoidal. A key passage of the argument will be using Theorem 3.3.5 to reduce a statement regarding lax $\mathbb{E}_n$-monoidal maps to a statement regarding lax $\mathbb{E}_1$-monoidal maps; this will allow us to invoke Proposition 3.2.10 and check the monoidality for the more familiar ∞-categories of associative bimodules described in Section 3.1.3.

As we will see in the next chapter, the theory of Thom spectrum is defined for left modules, and until now, we focus instead on the theory of $\mathbb{E}_n$-modules; this is because, as explained above, working with $\mathbb{E}_n$-modules allows us to reduce the statements to the associative case, where we have a good description of the relative tensor product thanks to the bar construction. However, at some point, in order to recover the Thom functor, we will need to pass to the ∞-categories of left modules, and we will plan to do so by using the functor described in the following theorem.

Theorem 3.3.6. [Lur17, Theorem 5.1.4.10] *Let $k \geq 1$ and let $q : \mathcal{C}^{\otimes} \to \mathbb{E}_k^{\otimes}$ be an $\mathbb{E}_k$-monoidal category. Suppose that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product functor preserves geometric realizations of simplicial objects separately in each variable. Then, for each $\mathbb{E}_k$-algebra object A of $\mathcal{C}^{\otimes}$ there exists a functor*

$$\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})^{\otimes} \underset{\mathbb{E}_k^{\otimes}}{\times} \mathbb{E}_{k-1}^{\otimes} \to \mathrm{LMod}_A(\mathcal{C})^{\otimes}$$

which is $\mathbb{E}_{k-1}$-monoidal.

Chapter 4

Thom functor

Let us begin by providing a brief description of the 1-categorical Thom spectrum functor to give some motivation for its ∞-categorical generalization. The standard reference for the 1-categorical version is [LMS86] by L.G. Lewis, J.P. May, and M. Steinberger; otherwise one can consult [And+14b] by M. Ando et al. for a more modern approach to the subject.

Let R be an $\mathbb{E}_1$-ring spectrum. We define the topological space $\mathrm{GL}_1(R)$ as the pullback of the following diagram

$$\begin{tikzcd}
\mathrm{GL}_1(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Omega^\infty R \arrow[d] \\
\pi_0(\Omega^\infty R)^\times \arrow[r, "\xi"] & \pi_0(\Omega^\infty R),
\end{tikzcd}$$

where $\pi_0(\Omega^\infty R)^\times$ is the discrete space of units of the group $\pi_0(\Omega^\infty R)$. The $\mathbb{E}_1$-structure of R equips the space $\mathrm{GL}_1(R)$ with an $\mathbb{E}_1$-monoidal structure. Let us assume that $\mathrm{GL}_1(R)$ is not only an $\mathbb{E}_1$-algebra but a grouplike (strict) monoid; as explained in [And+14b] it is possible to avoid this assumption by working on the 1-category of $*$-modules, which is the space analogue of EKMM's spectra, where the monoids are exactly $\mathbb{E}_1$-algebras.

Since $\mathrm{GL}_1(R)$ is a grouplike monoid we can consider its universal principal $\mathrm{GL}_1(R)$-bundle $\mathrm{EGL}_1(R) \to \mathrm{BGL}_1(R)$. Suppose that we are given a system of invertible R-modules, i.e., a map $\xi : X \to \mathrm{BGL}_1(R)$, that we can informally think of as a pre-sheaf of free rank-one R-modules. The monoid $\mathrm{GL}_1(R)$ acts on the pullback of ξ along the universal $\mathrm{GL}_1(R)$-bundle

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{EGL}_1(R) \arrow[d] \\
X \arrow[r, "\xi"] & \mathrm{BGL}_1(R).
\end{tikzcd}$$

In particular, the spectrum $\Sigma^\infty_+ P$ admits the structure of a left $\Sigma^\infty_+\mathrm{GL}_1(R)$-module. We define the left R-module Thom spectrum of ξ as the following derived smash product

$$\mathrm{Th}_R(\xi) := R \underset{\Sigma^\infty_+\mathrm{GL}_1(R)}{\wedge} \Sigma^\infty_+ P.$$

It is possible to generalize this construction by considering instead a pre-sheaf of invertible R-modules; replacing the space $\mathrm{BGL}_1(R)$ with the geometric realization of the nerve of the full subcategory of left R-modules spanned by the invertible objects. This topological space is usually referred to as the Picard space of R and denoted by $\mathrm{Pic}(R)$.

We observe that the 1-categorical version presents an interaction between topological spaces and the 1-category of invertible left R-modules, and it relates those two different categories via the geometric realization of the nerve of the latter. However, the rigidness of 1-categories makes this passage somehow unnatural. Let us assume, for example, that the ring spectrum R is an $\mathbb{E}_m$-algebra with $m \geq 2$. Then the product of R equips the 1-category of invertible left R-modules with a relative tensor product that makes it a symmetric monoidal 1-category. From the coherence criterion [ML98, Theorem XI.3.1] we know that a symmetric monoidal 1-category is equivalent to a strictly symmetric monoidal 1-category. Therefore, by passing to the nerve the strict symmetric monoidal structure of the 1-category of invertible R-modules equips the space $\mathrm{Pic}(R)$ with the structure of an $\mathbb{E}_\infty$-algebra of topological space; effectively producing an $\mathbb{E}_\infty$-space from an $\mathbb{E}_2$-ring spectrum.

This is just one of the reasons suggesting that the framework of ∞-categories might be a more natural setting for the Thom functor. First of all, in ∞-categories we can model topological spaces with Kan complexes, so it is no longer necessary to consider the nerve of a category and we can directly define a system of invertible R-modules to be a functor $\xi : X \to \mathrm{Pic}(R)$ from a Kan complex to the ∞-categorical analogue of $\mathrm{Pic}(R)$. Furthermore, we will no longer encounter the rigidness problem described above since, if R is an $\mathbb{E}_m$-ring spectrum, the ∞-category of left R-modules admits, in general, only an $\mathbb{E}_{m-1}$-monoidal structure, [Lur17, Corollary 5.1.2.6].

Suppose that R is a commutative ring spectrum. One of the fundamental properties of the 1-categorical Thom spectrum is that it maps $\mathcal{O}$-algebras to $\mathcal{O}$-algebras. This suggests that we should exepect to being able to realize a monoidal version of the ∞-categorical Thom functor as a functor between the ∞-categories of $\mathcal{O}$-algebras

$$\mathrm{Th}_R : \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S}_{/\mathrm{Pic}(R)}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathrm{LMod}_R(\mathrm{Sp})). \qquad (\star)$$

As we will see in Section 4.2 the authors of [ABG18] have been able to realize a more refined version of $(\star)$. In particular, they defined the metacosmic version of the functor, that is to say, they produced a monoidal functor of symmetric monoidal ∞-categories

$$\mathrm{Th}_R : \mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)} \to \mathrm{LMod}_R(\mathrm{Sp})^{\otimes}$$

which induces on the ∞-categories of $\mathcal{O}$-algebras the functor ($\star$).

In Section 4.3, we will present the paper [ACB19] by O. Antolín-Camarena and T. Barthel where the authors developed a theory of lifts of lax $\mathcal{O}$-monoidal maps to $\mathcal{O}$-monoidal overcategories, which they then used to describe the monoidal Thom spectrum via a useful universal property.

From now on, we will use LMod_R instead of $\mathrm{LMod}_R(\mathrm{Sp})$ to denote the ∞-category of left R-modules.

4.1 Additive Thom functor

In [And+14a] the authors developed the general theory of Thom functor in the language of ∞-categories and they proved that the construction is compatible with their previous 1-categorical definition presented in [And+14b]. It is important to point out that in [And+14a] M.Ando et al. defined a system of invertible R-modules to be a map with source a Kan complex and with target the subcategory spanned by the R-modules that are equivalent to R. In this section, in order to improve the coherence of the exposition, we will present the results contained in [And+14a] by considering the target of the system of invertible R-modules to be the core of the subcategory spanned by the invertible R-modules, which we will denote also by $\mathrm{Pic}(R)$; so that the additive case will agree with the monoidal version that we will present in the next section.

We start by using the additive Grothendieck construction to produce an equivalence between the ∞-category of Kan complexes over X and the ∞-category of pre-sheaves over X.

Theorem 4.1.1. [Lur09, Theorem 2.2.1.2] *Let X be a Kan complex, then the Grothendieck construction defines an equivalence of ∞-categories*

$$\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{S}_{/X},$$

which sends a pre-sheaf over X to its colimit, regarded as a Kan complex over X.

We now formally define the ∞-categorical analogue of the 1-category $\mathrm{Pic}(R)$.

Definition 4.1.2. Let R be an $\mathbb{E}_m$-ring spectrum with $m \geq 2$, i.e., an $\mathbb{E}_m$-algebra of the symmetric monoidal stable ∞-category of spectra defined in [Lur17, Def. 7.1.0.1]. We consider the $\mathbb{E}_{m-1}$-monoidal ∞-category of left R-modules $\mathrm{LMod}_R^{\otimes}$, [Lur17, Def. 7.1.3.5], and define the ∞-category $\mathrm{Pic}(R)$ to be the core of the full subcategory of LMod_R

spanned by the invertible left R-modules. We recall that the core of an ∞-category $\mathcal{C}$, which we denote by $\mathcal{C}^{\simeq}$, is the Kan complex given by the following pullback of simplicial sets

$$\begin{tikzcd} \mathcal{C}^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d] \\ \mathrm{N}(\mathrm{h}\mathcal{C}^{\simeq}) \arrow[r] & \mathrm{N}(\mathrm{h}\mathcal{C}), \end{tikzcd}$$

where $\mathrm{h}\mathcal{C}^{\simeq}$ is the maximal subgroupoid of the 1-category $\mathrm{h}\mathcal{C}$.

We can now define the additive ∞-categorical version of the Thom functor.

Definition 4.1.3. [And+14a, Def. 2.20] The Thom R-module spectrum is the colimit preserving functor

$$\mathrm{Th}_R : \mathcal{S}_{/\mathrm{Pic}(R)} \to \mathrm{LMod}_R,$$

that maps a system of invertible R-modules $\xi : X \to \mathrm{Pic}(R)$ to the colimit of the following compostion of functors

$$X \xrightarrow{f} \mathrm{Pic}(R) \xhookrightarrow{\iota} \mathrm{LMod}_R.$$

4.2 Monoidal Thom functor

In [ABG18] M.Ando, A. Blumberg, and D. Gepner introduced a theory of parametrized objects which they have then used to formalize a generalized monoidal Thom spectrum functor. In particular, the functor is defined as the stabilization of the counit of an adjunction; this adjunction can be interpreted as the categorification of the adjunction between units and group rings.

For the rest of this section, we will fix a coherent ∞-operad $\mathcal{O}^{\otimes}$ equipped with a unit map $\eta : \mathbb{E}_1^{\otimes} \to \mathcal{O}^{\otimes}$. This is the same condition imposed in [ABG18], but, since we are interested in the particular case of $\mathcal{O}^{\otimes} = \mathbb{E}_n^{\otimes}$, in order to simplify the results we are willing to assume in addition that the ∞-operad is single-colored.

Let X be an $\mathcal{O}$-algebra of $\mathcal{S}^{\otimes}$. By the monoidal Grothendieck construction we know that X corresponds to an $\mathcal{O}$-monoidal Kan complex $X^{\otimes}$ with underlying category X. It is possible to prove that the ∞-category $\mathrm{Fun}(X, \mathcal{S})$ of pre-sheaves over $X^{\otimes}$ admits a natural $\mathcal{O}$-monoidal structure which can be informally interpreted as given by the Day convolution product.

Definition 4.2.1. [Lur17, Construction 2.2.6.7] Let X be an $\mathcal{O}$-algebra of $\mathcal{S}^{\otimes}$. There exists an $\mathcal{O}$-monoidal category that we will denote by $\mathrm{Fun}^{\mathcal{O}}(X, \mathcal{S})^{\otimes}$, with underlying category $\mathrm{Fun}(X, \mathcal{S})$. Moreover, as a consequence of the macrocosmic monoidal Grothendieck construction by M. Ramzi [Ram22, Corollary 4.9], we know that this

$\mathcal{O}$-monoidal category is equivalent to the overcategory $\mathcal{S}^{\otimes}_{/X}$ as defined in Section 2.3.6.

We will now introduce the two functors that realize the aforementioned adjunction. We start from the left adjoint functor $\mathrm{Pre} : \mathrm{Alg}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{LMod}_{\mathcal{S}}(\mathrm{Pr}^{L}))$. We will define the functor Pre as the monoidal version of the functor that assigns to a Kan complex X the ∞-category $\mathrm{Fun}(X, \mathcal{S})$ considered as an object of the ∞-category of presentable ∞-categories.

We first realize the additive version of the functor Pre. By abusing the notation we will use Pre to denote the additive functor, the monoidal functor, and the functor induced on the $\mathcal{O}$-algebras.

Proposition 4.2.2. [ABG18, Prop. 6.11] *There is a unique colimit-preserving functor*

$$\mathrm{Pre} : \mathcal{S} \to \mathrm{Pr}^{\boldsymbol{L}},$$

whose values at the space X is the ∞-category $\mathcal{S}_{/X}$ of pre-sheaves over X.

The authors then claim that, as a consequence of the properties of the Day convolution product, the functor extends to a symmetric monoidal functor once we equip $\mathcal{S}$ and Pr^{L} with the Cartesian monoidal structures described in [Lur17, Prop. 2.4.1.5].

Proposition 4.2.3. [ABG18, Prop. 6.12] *The functor* $\mathrm{Pre} : \mathcal{S} \to \mathrm{Pr}^{L}$ *extends to a symmetric monoidal functor*

$$\mathrm{Pre} : \mathcal{S}^{\times} \to (\mathrm{Pr}^{\boldsymbol{L}})^{\times}.$$

If we consider the functor induced by Pre on the $\mathcal{O}$-algebra objects we obtain the following maps between the ∞-categories of $\mathcal{O}$-algebras.

Corollary 4.2.4. [ABG18, Corollary 6.13] *The functor* $\mathrm{Pre} : \mathcal{S}^{\times} \to (\mathrm{Pr}^{L})^{\times}$ *induces on the ∞-category of $\mathcal{O}$-algebras the following functor*

$$\mathrm{Pre} : \mathrm{Alg}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\boldsymbol{L}}),$$

which assigns to an $\mathcal{O}$-algebra X of $\mathcal{S}^{\times}$ the $\mathcal{O}$-monoidal presentable ∞-category $\mathcal{S}^{\otimes}_{/X}$.

This last functor will be the left adjoint functor of the adjunction that we will use to realize the monoidal Thom spectrum.

Before defining the right adjoint functor $\mathrm{Pic} : \mathrm{Alg}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Alg}^{\mathrm{gp}}_{\mathcal{O}}(\mathcal{S})$, we need to introduce some preliminary notation. We start by defining the subcategory of grouplike $\mathcal{O}$-algebras of $\mathcal{S}^{\otimes}$.

Definition 4.2.5. We define $\mathrm{Alg}^{\mathrm{gp}}_{\mathcal{O}}(\mathcal{S})$ to be the full subcategory of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S})$ spanned

by the $\mathcal{O}$-algebra objects of $\mathcal{S}^{\times}$
$$\mathcal{O}^{\otimes} \xrightarrow{X} \mathcal{S}^{\times}$$
such that the associative algebra given by precomposing with the unit η
$$\mathbb{E}_1^{\otimes} \xrightarrow{\eta} \mathcal{O}^{\otimes} \xrightarrow{X} \mathcal{S}^{\times}$$
is a grouplike space as defined in [Lur17, Def. 5.2.6.2].

The functor Pic will assign to an $\mathcal{O}$-monoidal category $\mathcal{C}^{\otimes}$ its Picard ∞-groupoid, or Picard space, equipped with an $\mathcal{O}$-monoidal structure. We will first define the Picard space of an $\mathcal{O}$-monoidal category and then see that it admits a natural $\mathcal{O}$-monoidal structure.

Definition 4.2.6. [ABG18, Def. 1.4] Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal category. The unit $\eta : \mathbb{E}_1^{\otimes} \to \mathcal{O}^{\otimes}$ determines a distinguished active morphism $\beta : \langle 2 \rangle \to \langle 1 \rangle$ of $\mathcal{O}^{\otimes}$ and the $\mathcal{O}$-operation induced by β defines a product of the underlying category $\otimes_{\beta} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. We start by considering the full subcategory of $\mathcal{C}^{\otimes}$ spanned by the objects that are invertible under the product $\otimes_{\beta}$. We denote this subcategory by $\overline{\mathrm{Pic}}(\mathcal{C})$. The Picard space of $\mathcal{C}$, which we will denote by $\mathrm{Pic}(\mathcal{C})$, is the maximal subgroupoid, or core, of the ∞-category $\overline{\mathrm{Pic}}(\mathcal{C})$. If $\mathcal{C}^{\otimes} = \mathrm{LMod}_R^{\otimes}$ where R is an $\mathbb{E}_n$-ring spectrum with $n \geq 2$ we usually use $\mathrm{Pic}(R)$ instead of $\mathrm{Pic}(\mathrm{LMod}_R)$.

It is not hard to prove that the full subcategory $\overline{\mathrm{Pic}}(\mathcal{C})$ satisfies the hypothesis of Proposition 2.3.2 and therefore inherits the $\mathcal{O}$-monoidal structure of $\mathrm{LMod}_R^{\otimes}$. It is not immediate, however, that its core inherits the $\mathcal{O}$-monoidal product too. This is proven in the following proposition.

Proposition 4.2.7. [ACB19, Prop. 2.5] *Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal category. We define $\mathcal{C}^{\otimes}_{\mathrm{coCart}}$ to be the subcategory of $\mathcal{C}^{\otimes}$ spanned by q-coCartesian morphisms. The restriction of q to this subcategory $\tilde{q} : \mathcal{C}^{\otimes}_{\mathrm{coCart}} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads and for each $Z \in \mathcal{O}$, the underlying ∞-category $(\mathcal{C}^{\otimes}_{\mathrm{coCart}})_Z$ is the core of $\mathcal{C}_Z$. That is to say, if the ∞-operad $\mathcal{O}^{\otimes}$ is single-colored, the coCartesian fibration $\tilde{q} : \mathcal{C}^{\otimes}_{\mathrm{coCart}} \to \mathcal{O}^{\otimes}$ equips $\mathcal{C}^{\simeq}$ with an $\mathcal{O}$-monoidal structure.*

We can finally define the functor Pic as the functor that assigns to a presentable $\mathcal{O}$-monoidal category its Picard space equipped with its natural $\mathcal{O}$-monoidal structure.

Theorem 4.2.8. [ABG18, Theorem 7.7] *Let $\mathcal{O}^{\otimes}$ be a (single-colored) coherent ∞-operad equipped with a unit $\eta : \mathbb{E}_1^{\otimes} \to \mathcal{O}^{\otimes}$. Then the functor*
$$\mathrm{Pic} : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\boldsymbol{L}}) \to \mathrm{Alg}^{\mathrm{gp}}_{\mathcal{O}}(\mathcal{S})$$

is right adjoint to the functor

$$\mathrm{Pre} : \mathrm{Alg}^{\mathrm{gp}}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\boldsymbol{L}}).$$

Let $\mathcal{C}^{\otimes}$ be a presentable $\mathcal{O}$-monoidal category. The counit of the adjunction

$$\mathcal{S}^{\otimes}_{/\mathrm{Pic}(\mathcal{C})} \to \mathcal{C}^{\otimes}$$

is a morphism of $\mathcal{O}$-monoidal presentable category, i.e., a colimit preserving $\mathcal{O}$-monoidal functor. Passing to the stable setting, that is to say, replacing the Cartesian monoidal ∞-category $(\mathrm{Pr}^{\mathbf{L}})^{\times}$ with the Cartesian monoidal category of stable presentable ∞-categories $(\mathrm{Pr}^{\mathbf{L}}_{\mathrm{St}})^{\times}$, and repeating the procedure, we obtain the following adjunction

$$\begin{tikzcd} \mathrm{Alg}^{\mathrm{gp}}_{\mathcal{O}}(\mathcal{S}) \arrow[r, bend left, "\mathrm{Pre}_{\mathrm{St}}"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathbf{L}}_{\mathrm{St}}). \arrow[l, bend left, "\mathrm{Pic}"] \end{tikzcd}$$

For each stable presentable $\mathcal{O}$-monoidal category $\mathcal{R}^{\otimes} \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathbf{L}}_{\mathrm{St}})$ the counit of the adjunction defines the following morphism of presentable $\mathcal{O}$-monoidal stable categories

$$\mathrm{Sp}^{\otimes}_{/\mathrm{Pic}(\mathcal{R})} \to \mathcal{R}^{\otimes}.$$

We can finally define the $\mathcal{O}$-monoidal Thom spectrum functor.

Corollary 4.2.9. [ABG18, Corollary 8.1] *Let $\mathcal{R}^{\otimes}$ be a stable presentable $\mathcal{O}$-monoidal category. The composite functor*

$$\mathrm{Th}_{\mathcal{R}} : \mathcal{S}^{\otimes}_{/\mathrm{Pic}(\mathcal{R})} \to \mathrm{Sp}^{\otimes}_{/\mathrm{Pic}(\mathcal{R})} \to \mathcal{R}^{\otimes}$$

is a map of presentable $\mathcal{O}$-monoidal ∞-categories. We refer to this functor as the generalized monoidal Thom spectrum functor.

4.3 Universal property of Thom spectra

In [ACB19] the authors, with an approach similar to the one utilized in [And+14a] to define the additive version, realized a microcosmic version of the Thom spectrum functor. Starting from an $\mathbb{E}_n$-monoidal system of invertible R-modules $X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$, they considered the functor induced on the underlying categories $X \to \mathrm{Pic}(R)$ and its additive Thom spectrum as an object of LMod_R obtained by the following left Kan extension

$$\begin{tikzcd} X \arrow[r, "\xi"] \arrow[dr] & \mathrm{Pic}(R) \arrow[r, hook, "\iota"] \arrow[d, phantom, "\Downarrow"] & \mathrm{LMod}_R \\ & \Delta^0. \arrow[ur, dashed, "\mathrm{Th}_R(\xi)"'] & \end{tikzcd}$$

Then the authors proved that this colimit admits a natural $\mathbb{E}_n$-algebra structure provided by the $\mathbb{E}_n$-monoidal system of invertible R-modules. Similar to the additive case, this $\mathbb{E}_n$-algebra structure is defined by the operadic left Kan extension of the composition of $X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$ with $\mathrm{Pic}(R)^{\otimes} \hookrightarrow \mathrm{LMod}_R^{\otimes}$ along the $\mathbb{E}_n$-monoidal structure $p : X^{\otimes} \to \mathbb{E}_n^{\otimes}$.

$$\begin{tikzcd} X^{\otimes} \arrow[r, "\xi"] \arrow[d] & \mathrm{Pic}(R)^{\otimes} \arrow[r, hook, "\iota"] & \mathrm{LMod}_R^{\otimes} \arrow[d] \\ \mathbb{E}_n^{\otimes} \arrow[urr, dashed, "\mathrm{Th}_R(\xi)"] \arrow[rr, "id"] & & \mathbb{E}_n^{\otimes}. \end{tikzcd}$$

This association defines a functor

$$\mathrm{Th}_R : \mathrm{Alg}_{X/\mathbb{E}_n}(\mathrm{Pic}(R)) \to \mathrm{Alg}_{/\mathbb{E}_n}(\mathrm{LMod}_R).$$

Let us fix an ∞-operad $\mathcal{O}^{\otimes}$, a small $\mathcal{O}$-monoidal category $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and a cocomplete $\mathcal{O}$-monoidal category $\mathcal{D}^{\otimes}$. We start by defining the functor M that assigns to a lax $\mathcal{O}$-monoidal map $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a natural $\mathcal{O}$-algebra structure on the colimit of the map induced on the underlying categories.

Theorem 4.3.1. [ACB19, Theorem 2.8] *Let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a lax $\mathcal{O}$-monoidal map. Then, there exists an $\mathcal{O}$-algebra of $\mathcal{D}^{\otimes}$ given by a functor $MF : \mathcal{O}^{\otimes} \to \mathcal{D}^{\otimes}$ such that for every object $Z \in \mathcal{O}$ we have $MF(Z) = \mathrm{colim}\, F_Z : \mathcal{C}_Z \to \mathcal{D}_Z$.*

From the proof of [ACB19, Theorem 2.8] we can see that the $\mathcal{O}$-algebra MF corresponds to the operadic left Kan extension of F along the $\mathcal{O}$-monoidal structure $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$. Similar to the non-operadic case the operadic left Kan extension along an operadic map p is left adjoint to the functor given by precomposing with p.

Corollary 4.3.2. [ACB19, Corollary 2.11] *There exists a functor M left adjoint to*

$$(- \circ p) : \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}) \to \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}),$$

and the functor M assigns to each lax $\mathcal{O}$-monoidal map $F \in \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$ the colimit $\mathcal{O}$-algebra MF defined in Theorem 4.3.1.

Via the adjunction with the functor $(- \circ p)$, O. Antolín-Camarena and T. Barthel defined a universal property that relates the colimit $\mathcal{O}$-algebras to lax $\mathcal{O}$-monoidal lifts to overcategories.

Lemma 4.3.3. [ACB19, Lemma 2.12] *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be two $\mathcal{O}$-monoidal categories, A be an $\mathcal{O}$-algebra of $\mathcal{D}^{\otimes}$ and let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a lax $\mathcal{O}$-monoidal map. Then, lax $\mathcal{O}$-monoidal lifts of F along the projection $\mathcal{D}^{\otimes}_{/A} \to \mathcal{D}^{\otimes}$*

$$\begin{tikzcd} & \mathcal{D}^{\otimes}_{/A} \arrow[d] \\ \mathcal{C}^{\otimes} \arrow[ur, dashed] \arrow[r, "F"] & \mathcal{D}^{\otimes}, \end{tikzcd}$$

correspond to lax $\mathcal{O}$-monoidal natural transformations

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[r, bend left, "F"] \arrow[r, bend right, "A\circ p"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{D}^{\otimes}. \end{tikzcd}$$

More precisely, there is a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})}(F, A \circ p) \simeq \{F\} \underset{\mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})}{\times} \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}_{/A}).$$

In particular, the homotopy realizes an equivalence of ∞-categories

$$\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/A} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}_{/A}).$$

Combining Corollary 4.3.2 and Lemma 4.3.3 we obtain the following characterization of the colimit $\mathcal{O}$-algebra MF of a lax $\mathcal{O}$-monoidal map F.

Theorem 4.3.4. [ACB19, Theorem 2.13] *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be two $\mathcal{O}$-monoidal categories and let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a lax $\mathcal{O}$-monoidal map. Then, the $\mathcal{O}$-algebra MF of Theorem 4.3.1 is characterized by the following universal property: for each $\mathcal{O}$-algebra A of $\mathcal{D}^{\otimes}$, the space of $\mathcal{O}$-algebra maps $\mathrm{Map}_{\mathrm{Alg}_{/\mathcal{O}}}(\mathcal{D})(MF, A)$ is homotopy equivalent to the space of lax $\mathcal{O}$-monoidal lifts of F along the projection $\mathcal{D}^{\otimes}_{/A} \to \mathcal{D}^{\otimes}$*

$$\begin{tikzcd} & \mathcal{D}^{\otimes}_{/A} \arrow[d] \\ \mathcal{C}^{\otimes} \arrow[ur, dashed] \arrow[r, "F"] & \mathcal{D}^{\otimes}. \end{tikzcd}$$

Let us consider the case where $\mathcal{O}^{\otimes}$ is the little cubes ∞-operad $\mathbb{E}_n^{\otimes}$ with $n \geq 1$, the ∞-category $\mathcal{C}^{\otimes}$ is an $\mathbb{E}_n$-monoidal Kan complex, and the ∞-category $\mathcal{D}^{\otimes}$ is the $\mathbb{E}_n$-monoidal ∞-category of left R-modules where R is an $\mathbb{E}_{n+1}$-ring spectrum. Then, we can use the functor M to give an alternative presentation of the Thom spectrum functor. In particular, we will associate to an $\mathbb{E}_n$-monoidal system of invertible R-modules $\xi : X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$ the $\mathbb{E}_n$-algebra of $\mathrm{LMod}_R^{\otimes}$ given by applying the functor M to the composition of ξ with the inclusion $\iota : \mathrm{Pic}(R)^{\otimes} \hookrightarrow \mathrm{LMod}_R^{\otimes}$.

Definition 4.3.5. [ACB19, Def. 3.1] Let $\mathrm{Pic}(R)^{\otimes}$ be the $\mathbb{E}_n$-monoidal Kan complex as in Definition 4.2.6. We define the generalized Thom spectrum functor Th_R to be the following composition of functors

$$\begin{tikzcd} \mathrm{Alg}_{X/\mathbb{E}_n}(\mathrm{Pic}(R)) \arrow[r, "(\iota\circ -)"] \arrow[rr, bend right, "\mathrm{Th}_R"'] & \mathrm{Alg}_{X/\mathbb{E}_n}(\mathrm{LMod}_R) \arrow[r, "M"] & \mathrm{Alg}_{/E_n}(\mathrm{LMod}_R). \end{tikzcd}$$

It might look like we obtained something slightly more general than our initial goal since the Thom spectrum functor of Definition 4.3.5 is defined for any lax $\mathbb{E}_n$-monoidal systems instead of just for (strong) $\mathbb{E}_n$-monoidal systems. But this is not really the case. By definition, $\mathrm{Pic}(R)^{\otimes}$ is an $\mathbb{E}_n$-monoidal category where each morphism is coCartesian, so every lax $\mathbb{E}_n$-monoidal map $X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$ automatically maps coCartesian morphisms of $X^{\otimes}$ to coCartesian morphisms of $\mathrm{Pic}(R)^{\otimes}$ and it is, therefore, $\mathbb{E}_n$-monoidal.

4.4 Relative Thom spectra

We conclude the exposition of the preliminary material with a brief overview of J. Beardsley's construction of the relative Thom spectrum [Bea17]; outlining the main differences between the iterated Thom spectrum that we will present in the next chapter and the relative Thom spectrum.

Before presenting the results it is necessary to fix some notation. We will take the freedom of making some changes to J. Beardsley's notation in order to make it consistent with the notation that we will use in Chapter 5.

Let $\pi : X^{\otimes} \to B^{\otimes}$ be a left $\mathbb{E}_n$-fibration of $\mathbb{E}_n$-monoidal Kan complexes with $n \geq 2$. We denote by $F^{\otimes}$ the $\mathbb{E}_n$-monoidal Kan complex obtained by applying [Ram22, Prop. 1.4] to the following pullback diagram of ∞-operads

$$\begin{tikzcd} F^{\otimes} \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X^{\otimes} \arrow[d, "\pi"] \\ \mathbb{E}_n^{\otimes} \arrow[r, "1_B"] & B^{\otimes}, \end{tikzcd}$$

where 1_B is the trivial $\mathbb{E}_n$-algebra of $B^{\otimes}$ as defined in [Lur17, Section 3.2.1]. From Lemma 2.4.6 we know that under the equivalence given by the monoidal Grothendieck construction $\mathrm{LFib}^{\mathbb{E}_n}(\mathbb{E}_n) \simeq \mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{S})$, the $\mathbb{E}_n$-monoidal Kan complex $F^{\otimes}$ corresponds to the following $\mathbb{E}_n$-algebra of $\mathcal{S}^{\otimes}$

$$F : \mathbb{E}_n^{\otimes} \xrightarrow{1_B} B^{\otimes} \xrightarrow{\psi} \mathcal{S}^{\otimes},$$

were ψ is the straightening of the $\mathbb{E}_n$-fibration π. By definition, the $\mathbb{E}_n$-algebra F is the image of the trivial algebra 1_B by the functor induced by ψ on the ∞-categories of $\mathbb{E}_n$-algebras.

Let R be an $\mathbb{E}_m$-ring spectrum with $n \leq m$. Suppose that in addition to the $\mathbb{E}_n$-fibration π we are given a system of invertible R-modules $\xi : X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$. We define ξ_1 to be the lax $\mathbb{E}_n$-monoidal map given by the following composition

$$\xi_1 : F^{\otimes} \xrightarrow{i} X^{\otimes} \xrightarrow{\xi} \mathrm{Pic}(R).$$

Definition 4.4.1. We say that an ∞-category $\mathcal{C}$ is reduced if its 0-skeleton consists of a single vertex. Moreover, we say that an $\mathcal{O}$-monoidal category $\mathcal{C}^{\otimes}$ is reduced if its underlying category is reduced. It is not hard to prove that any connected Kan complex is equivalent to a reduced Kan complex.

We can now present the main theorem of [Bea17].

Theorem 4.4.2. [Bea17, Theorem 1] *Suppose $\pi : X^{\otimes} \to B^{\otimes}$ is a left $\mathbb{E}_n$-fibration of reduced Kan complexes for $n > 1$. Let $\xi : X^{\otimes} \to \mathrm{BGL}_1(R)^{\otimes}$ be an $\mathbb{E}_n$-monoidal system of free rank-one R-modules. Then, there is an $\mathbb{E}_{n-1}$-monoidal system of free rank-one $\mathrm{Th}_R(\xi_1)$-modules $B^{\otimes} \to BGL_1(\mathrm{Th}_R(\xi_1))^{\otimes}$ whose associated Thom spectrum is equivalent to $\mathrm{Th}_R(\xi)$.*

Applying the theory of orientations discussed in [ABG18, Corollary 1.8] to the system $B^{\otimes} \to \mathrm{BGL}_1(\mathrm{Th}_R(\xi_1))^{\otimes}$ J. Beardsley proved the following corollary.

Corollary 4.4.3. [Bea17, Corollary 4] *Given the assumptions of Theorem 4.4.2, there is a morphism of E_{n-1}-algebras of left R-modules $R \to \mathrm{Th}_R(\xi_1) \to \mathrm{Th}_R(\xi)$ which induces a Thom isomorphism of $\mathbb{E}_{n-2}$-algebras of left R-modules*

$$\mathrm{Th}_R(\xi) \wedge_{\mathrm{Th}_R(\xi_1)} \mathrm{Th}_R(\xi) \simeq \mathrm{Th}_R(\xi) \wedge_R R[B],$$

where $R[B] = R \wedge_{\mathbb{S}} \Sigma^{\infty}_{+} B$.

Theorem 4.4.2 is presented for systems of free rank-one R-modules but we believe that the result can be generalized without any major change in the argument to systems of invertible R-modules. The reduced condition, on the contrary, is crucial for the proof of Theorem 4.4.2. While J. Beardsley in [Bea17] presented many interesting applications for the relative Thom spectrum, the reduced condition prevents one from applying Theorem 4.4.2 to other interesting cases, for instance to the symmetric monoidal spherical fibration given by the group completion of the so-called J map $J_{\mathrm{gp}} : \mathbb{Z} \times \mathrm{BU} \to \mathrm{Pic}(\mathbb{S})$ [Hop18] along the symmetric left fibration given by the projection on the path components of $\mathbb{Z} \times \mathrm{BU}$.

In the next chapter, we will propose an alternative construction of the relative Thom spectrum. We will call this construction iterated Thom spectrum for reasons that will be clear to the reader. The advantage of our construction is that it replaces the reduced condition with the condition of having an additional $\mathbb{E}_1$-monoidal structure on the fibration, effectively replacing the left $\mathbb{E}_n$-fibration π with a left $\mathbb{E}_{n+1}$-fibration.

Chapter 5

Iterated Thom spectra

Now that we have introduced some basic ∞-categorical machinery, let us reformulate with the proper language the informal description of the iterated Thom spectrum that we have given during the introduction.

Let R be an $\mathbb{E}_m$-ring spectrum and let $\xi : X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$ be an $\mathbb{E}_n$-monoidal system of invertible R-modules over a grouplike Kan complex with $2 < n+1 \leq m$, suppose that in addition we are given an essentially surjective left $\mathbb{E}_n$-fibration $\pi : X^{\otimes} \to B^{\otimes}$ of grouplike Kan complexes. As we mentioned during the introduction, it is possible to use left Kan extensions to produce a functor on the underlying categories

$$\begin{tikzcd} X \arrow[r, "\xi"] \arrow[dr, "\pi"'] & \mathrm{Pic}(R) \arrow[r, hook] \arrow[d, phantom, "\Downarrow"] & \mathrm{LMod}_R \\ & B. \arrow[ur, dashed, "\mathrm{Th}_R(\xi)^B"'] & \end{tikzcd}$$

Utilizing either the theory of operadic left Kan extensions, as in Beardsley's work [Bea17], or the monoidal Grothendieck construction combined with the theory of lax $\mathbb{E}_n$-monoidal lifts, see Lemma 4.3.3, we can realize the previous functor as a lax $\mathbb{E}_n$-monoidal map

$$\mathrm{Th}_R(\xi)^B : B^{\otimes} \to \mathrm{LMod}_R^{\otimes}.$$

We now consider the map induced by $\mathrm{Th}_R(\xi)^B$ on the $\mathbb{E}_{n-1}$-monoidal categories of left modules, and, after pre and postcomposing with the appropriate equivalences we obtain the following lax $\mathbb{E}_{n-1}$-monoidal functor

$$\mathrm{Th}_R(\xi)^B : B^{\otimes} \simeq \mathrm{LMod}_{1_B}(B)^{\otimes} \to \mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}(\mathrm{LMod}_R)^{\otimes} \simeq \mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)},$$

where $\mathrm{Th}_R(\xi_1)$ is the image of the trivial algebra 1_B of $B^{\otimes}$. We can now ask if the resulting lax $\mathbb{E}_{n-1}$-monoidal functor $\mathrm{Th}_R(\xi)^B$ factors through the $\mathbb{E}_{n-1}$-monoidal subcategory $\mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes} \subseteq \mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}$, we will prove that this is the case if and only if the map $\mathrm{Th}_R(\xi)^B$ is $\mathbb{E}_{n-1}$-monoidal, and provide a sufficient condition on the fibration π that ensures that the map is $\mathbb{E}_{n-1}$-monoidal. In particular, we will prove that if the left $\mathbb{E}_n$-fibration π admits the structure of a left $\mathbb{E}_{n+1}$-fibration then $\mathrm{Th}_R(\xi)^B$ is $\mathbb{E}_{n-1}$-monoidal. Instead of attacking the problem directly, we will first prove that if $\psi : B^{\otimes} \to \mathcal{S}^{\otimes}$ is the lax $\mathbb{E}_{n+1}$-monoidal pre-sheaf given by the straightening of π, then the lax $\mathbb{E}_n$-monoidal functor induced by ψ on the category of $\mathbb{E}_n$-modules

$$\psi' : B^{\otimes} \simeq \mathrm{Mod}^{\mathbb{E}_n}_{1_B}(B)^{\otimes} \to \mathrm{Mod}^{\mathbb{E}_n}_{F}(\mathcal{S})^{\otimes}$$

is $\mathbb{E}_n$-monoidal, where F is the image of the trivial algebra 1_B of $B^{\otimes}$ by the map induced by ψ. We will then construct the functor $\mathrm{Th}_R(\xi)^B$ starting from the $\mathbb{E}_n$-monoidal map ψ' utilizing only operations that preserve the monoidality.

Once we have proved that the map $\mathrm{Th}_R(\xi)^B$ factors through the $\mathbb{E}_{n-1}$-monoidal category $\mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes}$ we can regard it as a system of invertible $\mathrm{Th}_R(\xi_1)$-modules and apply the monoidal Thom functor of Definition 4.3.5 to obtain an $\mathbb{E}_{n-1}$-monoidal left $\mathrm{Th}_R(\xi_1)$-module that we will denote as the iterated Thom spectrum of ξ along π. We will then prove that by considering the iterated Thom spectrum as an $\mathbb{E}_{n-1}$-algebra of left R-modules we can recover the original Thom spectrum $\mathrm{Th}_R(\xi)$ as an $\mathbb{E}_{n-1}$-algebra of $\mathrm{LMod}^{\otimes}_R$.

In this chapter, unless explicitly stated, with categories we will always mean ∞-categories, and with operads we will always mean ∞-operads.

5.1 $\mathbb{E}_n$-monoidal principal $G^{\otimes}$-bundles

In this section, we will propose a definition for $\mathbb{E}_n$-monoidal principal $G^{\otimes}$-bundles $X^{\otimes}$ over $B^{\otimes}$. Our main reference is the exposition of T. Nikolaus, U. Schreiber, and D. Stevenson on the general theory of principal G-bundles in the context of ∞-categories [NSS14]. (The authors of [NSS14] denote the base of a principal G-bundle with X; we have chosen to utilize B to denote the base and use X instead of P to denote the space over B. We hope that this change does not create any confusion in the reader.)

Let us first recall the 1-categorical definition of a principal G-bundle in the context of topological spaces. Let X and B be topological spaces, and G a topological group. Let $\rho : X \times G \to X$ be an action of G on X and $h : X \to B$ a continuous map compatible with the action ρ. We say that $h : X \to B$ is a principal G-bundle over B if:

- the action is principal, meaning that the shear map

$$\eta := (\rho, pr_1) : X \times G \to X \underset{B}{\times} X$$

is an isomorphism; which implies that the action is free and transitive over B;

- the bundle $X \to B$ is isomorphic to the quotient map $X \to X/G$;
- and the bundle is locally trivial.

Before presenting the definitions of ∞-action and ∞-principal G-bundle, we would like to point out that while in 1-categories for the map $\rho : X \times G \to X$ to be associative and compatible with the product of G are properties of the map ρ, in ∞-categories to be associative and compatible are instead structures associated to the map ρ. For example, for each diagram that encodes the notion of associativity and compatibility with the product of G, we have to specify two-cells that make the diagram commute and, as we will see in the definition, these structures are encoded with a functor from $\mathrm{N}(\Delta^{\mathrm{op}})$ to the ∞-category $\mathcal{S}$.

Definition 5.1.1. [NSS14, Def. 3.1] Let $X \in \mathcal{S}$ and G be a group object of $\mathcal{S}$ as defined in [NSS14, Def. 2.16]. A G-action on X is a groupoid object $(X//G)_\bullet$

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} X \times G \times G \substack{\rightarrow\\\rightarrow\\\rightarrow} X \times G \underset{d_1}{\overset{\rho = d_0}{\rightrightarrows}} X,$$

where $d_1 : X \times G \to X$ is the projection and such that the degree-wise projection is a morphism of simplicial objects

$$\begin{array}{ccccccc}
\cdots & \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} & X \times G \times G & \substack{\rightarrow\\\rightarrow\\\rightarrow} & X \times G & \rightrightarrows & X \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} & G \times G & \substack{\rightarrow\\\rightarrow\\\rightarrow} & G & \rightrightarrows & *.
\end{array}$$

The face map $d_0 : X \times G \to X$ is the morphism that corresponds to the 1-categorical notion of G-action, as explained in [NSS14, Remark 3.2].

Definition 5.1.2. [NSS14, Def. 3.4] Let $B \in \mathcal{S}$ and G be a group object of $\mathcal{S}$ as defined in [NSS14, Def. 2.16]. An ∞-principal G-bundle over B is:

- a morphism $X \to B$ in $\mathcal{S}$;
- a G-action on X;

such that the map $X \to B$ exhibits B as the geometric realization of the simplicial object $(X//G)_\bullet$.

Remark 5.1.3. In [NSS14, Prop. 3.13] the authors prove that there exists a universal principal G-bundle, $* \to \mathbf{B}G$, and every principal G-bundle can be constructed as the pullback of the universal one along a morphism $B \to \mathbf{B}G$. That is to say, there is an equivalence

$$\mathrm{GBund}(B) \simeq \mathrm{Map}_{\mathcal{S}}(B, \mathbf{B}G)$$

Remark 5.1.3 justifies our definition of $\mathbb{E}_n$-monoidal principal $G^{\otimes}$-bundle.

Definition 5.1.4. Let $B^{\otimes}$ be an $\mathbb{E}_n$-monoidal Kan complex and $G^{\otimes}$ a group object in the category of $\mathbb{E}_n$-monoidal Kan complexes as defined in [NSS14, Def. 2.16], i.e., an $\mathbb{E}_{n+1}$-monoidal grouplike Kan complex. An $\mathbb{E}_n$-monoidal principal $G^{\otimes}$-bundle is an $\mathbb{E}_n$-monoidal map

$$B^{\otimes} \to \mathbf{B}G^{\otimes},$$

where $\mathbf{B}G^{\otimes}$ is the $\mathbb{E}_n$-monoidal bar construction of $G^{\otimes}$ defined in [Lur17, Section 5.2].

Remark 5.1.5. Any $\mathbb{E}_n$-monoidal principal $G^{\otimes}$-bundle defines a principal G-bundle on the underlying categories. Let $h : B^{\otimes} \to \mathbf{B}G^{\otimes}$ be an $\mathbb{E}_n$-monoidal principal $G^{\otimes}$-bundle over $B^{\otimes}$. If we consider the functor induced by h on the underlying categories we obtain a map $B \to \mathbf{B}G$ and considering its pullback along the universal principal G-bundle we obtain a principal G-bundle over B.

Let us define the notion of the $\mathbb{E}_n$-monoidal shear map associated with an $\mathbb{E}_n$-monoidal principal $G^{\otimes}$-bundle.

Definition 5.1.6. Let $h : B^{\otimes} \to \mathbf{B}G^{\otimes}$ be an $\mathbb{E}_n$-monoidal principal $G^{\otimes}$-bundle over $B^{\otimes}$. By definition, $\mathbf{B}G^{\otimes}$ is the geometric realization of the simplicial object $\mathrm{Bar}(G)_\bullet$ defined in [Lur17, Section 5.2.2]. We now consider the following pullback square of simplicial objects of $\mathbb{E}_n$-monoidal categories

$$\begin{tikzcd}
B^{\otimes} \underset{\mathbf{B}G^{\otimes}}{\times} \mathrm{Bar}(G)_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B^{\otimes} \arrow[d, "h"] \\
\mathrm{Bar}(G)_\bullet \arrow[r] & \mathbf{B}G^{\otimes},
\end{tikzcd}$$

where we are considering $B^{\otimes}$ and $\mathbf{B}G^{\otimes}$ as constant simplicial objects with target $B^{\otimes}$ and $\mathbf{B}G^{\otimes}$ respectively. Let $X^{\otimes}$ be the $\mathbb{E}_n$-monoidal fiber of h over the unit; then the two face maps $d_0, d_1 : X^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} G^{\otimes} \to X^{\otimes}$ of the pullback will define an $\mathbb{E}_n$-monoidal shear map

$$\eta = (d_0, d_1) : X^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} G^{\otimes} \longrightarrow X^{\otimes} \underset{B^{\otimes}}{\times} X^{\otimes}.$$

Remark 5.1.7. The $\mathbb{E}_n$-monoidal principal bundles satisfy the principality condition, i.e., the shear maps are always equivalences. Let $h : B^{\otimes} \to \mathbf{B}G^{\otimes}$ be an $\mathbb{E}_n$-monoidal principal $G^{\otimes}$-bundle. From Corollary 2.2.13 we know that in order to prove that the $\mathbb{E}_n$-monoidal shear map is an equivalence it is sufficient to check that the functor induced on the underlying categories is an equivalence, but this is exactly the shear map of the principal G-action of the underlying categories described in Remark 5.1.5, and in [NSS14, Prop. 3.7] it is proven that principal G-bundles satisfy the principality condition.

The main examples of $\mathbb{E}_n$-monoidal principal bundles that we are interested in come from $\mathbb{E}_{n+1}$-monoidal essentially surjective maps between grouplike Kan complexes.

Proposition 5.1.8. *Let $\pi : X^{\otimes} \to B^{\otimes}$ be an essentially surjective morphism of $\mathbb{E}_{n+1}$-monoidal grouplike Kan complexes. Then $X^{\otimes} \underset{\mathbb{E}_{n+1}^{\otimes}}{\times} \mathbb{E}_n^{\otimes}$ naturally admits the structure of an $\mathbb{E}_n$-principal $F^{\otimes}$-bundle over $B^{\otimes} \underset{\mathbb{E}_{n+1}^{\otimes}}{\times} \mathbb{E}_n^{\otimes}$, where $F^{\otimes}$ is the $\mathbb{E}_{n+1}$-monoidal Kan complex defined by the following pullback*

$$\begin{tikzcd} F^{\otimes} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X^{\otimes} \arrow[d, "\pi"] \\ \mathbb{E}_{n+1}^{\otimes} \arrow[r, "1_B"'] & B^{\otimes}. \end{tikzcd}$$

Proof. From Dunn additivity theorem 3.3.4 the operad $\mathbb{E}_{n+1}^{\otimes}$ is equivalent to the tensor product of the operads $\mathbb{E}_1^{\otimes}$ and $\mathbb{E}_n^{\otimes}$, therefore there is an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathcal{S}) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S})).$$

Since the ∞-category of $\mathbb{E}_{n+1}$-monoidal Kan complexes is equivalent to the category $\mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathcal{S})$, we can consider $F^{\otimes}$, $X^{\otimes}$, and $B^{\otimes}$ as $\mathbb{E}_1$-algebra objects of the category of $\mathbb{E}_n$-monoidal Kan complexes, i.e., $\mathbb{E}_n$-monoidal Kan-complexes equipped with $\mathbb{E}_n$-monoidal associative products.

Now we consider the following sequence of $\mathbb{E}_n$-monoidal maps

$$\begin{tikzcd} F^{\otimes} \arrow[r, "\iota"] & X^{\otimes} \arrow[r, "\pi"] & B^{\otimes} \arrow[d, "h"] & & \\ & & \mathbf{B}F^{\otimes} \arrow[r, "\mathbf{B}\iota"] & \mathbf{B}X^{\otimes} \arrow[r, "\mathbf{B}\pi"] & \mathbf{B}B^{\otimes}. \end{tikzcd}$$

The map h defines the required $\mathbb{E}_n$-monoidal principal $F^{\otimes}$-bundle over $B^{\otimes}$. The sequence is given by considering the long fiber sequence associated with the fibration $\mathbf{B}F \to \mathbf{B}X \to \mathbf{B}B$ and applying [Lur17, Theorem 5.2.6.10] to recognize grouplike $\mathbb{E}_k$-algebras of $\mathcal{S}^{\otimes}$ as k-fold loop spaces. $\square$

5.2 $\mathbb{E}_n$-monoidal pre-sheaves from $\mathbb{E}_n$-monoidal principal bundles

Let $\pi : X^{\otimes} \to B^{\otimes}$ be an essentially surjective left $\mathbb{E}_{n+1}$-fibration of grouplike Kan complexes. We denote by $\psi : B^{\otimes} \to \mathcal{S}^{\otimes}$ the lax $\mathbb{E}_n$-monoidal pre-sheaf which classifies π, i.e., the operadic map given by the straightening of π. The goal of this section is to prove that we can factor the pre-sheaf ψ through a (strong) $\mathbb{E}_n$-monoidal map $\psi' : B^{\otimes} \to \mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S})^{\otimes}$ using the principal $\mathbb{E}_n$-monoidal action of the fiber $F^{\otimes}$ on $X^{\otimes}$ described in Proposition 5.1.8.

Suppose that we are given:

- $p' : X^{\otimes} \to \mathbb{E}_{n+1}^{\otimes}$ and $p : B^{\otimes} \to \mathbb{E}_{n+1}^{\otimes}$ two $\mathbb{E}_{n+1}$-monoidal grouplike Kan complexes, where $n \geq 2$;
- and an essentially surjective left $\mathbb{E}_{n+1}$-fibration $\pi : X^{\otimes} \to B^{\otimes}$, as defined in Definition 2.4.1.

Let 1_X be the trivial $\mathbb{E}_{n+1}$-algebra of $X^{\otimes}$ as defined in [Lur17, Section 3.2.1]. We denote by 1_B the image of 1_X by the map induced by π on the $\mathbb{E}_{n+1}$-algebras. Let $F^{\otimes}$ be the fiber of π over 1_B. If it is clear from context we will omit the subscripts B and X from the trivial algebras.

Using the equivalence $\mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathcal{S}) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S}))$, we can consider $X^{\otimes}$ and $B^{\otimes}$ as $\mathbb{E}_n$-monoidal Kan complexes equipped with $\mathbb{E}_n$-monoidal associative products. From Proposition 5.1.8 we know that $X^{\otimes}$ admits the structure of an $\mathbb{E}_n$-monoidal principal $F^{\otimes}$-bundle over $B^{\otimes}$. In particular, the $\mathbb{E}_n$-monoidal shear map η is an equivalence of left $\mathbb{E}_n$-fibrations

$$\begin{tikzcd} X^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} F^{\otimes} \arrow[rr, "\eta", "\simeq"'] \arrow[dr] & & X^{\otimes} \underset{B^{\otimes}}{\times} X^{\otimes} \arrow[dl] \\ & X^{\otimes}. & \end{tikzcd}$$

We start by applying the $\mathbb{E}_n$-monoidal straightening functor [Ram22, Cor. 4.8] to the left $\mathbb{E}_n$-fibration π to obtain a lax $\mathbb{E}_n$-monoidal map, or $\mathbb{E}_n$-monoidal pre-sheaf, that classifies π

$$\psi : B^{\otimes} \to \mathcal{S}^{\otimes}.$$

Where $q : \mathcal{S}^{\otimes} \to \mathbb{E}_n^{\otimes}$ is the $\mathbb{E}_n$-monoidal category obtained by taking the pullback of the Cartesian structure $\mathcal{S}^{\times} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, see Example 2.2.5, along the map $\mathbb{E}_n^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ as described in Remark 2.2.8.

In order to produce a (strong) $\mathbb{E}_n$-monoidal functor we consider the map induced by ψ

on the categories of $\mathbb{E}_n$-modules over the trivial algebra 1_B

$$\begin{tikzcd}
\mathrm{Mod}_{1_B}^{\mathbb{E}_n}(B)^{\otimes} \arrow[r, "\psi'"] \arrow[d] & \mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S})^{\otimes} \arrow[d] \\
B^{\otimes} \arrow[r, "\psi"] & \mathcal{S}^{\otimes}.
\end{tikzcd}$$

Where the vertical maps are forgetful functors and the $\mathbb{E}_n$-algebra F is the $\mathbb{E}_n$-algebra of $\mathcal{S}^{\otimes}$ induced by the trivial $\mathbb{E}_n$-algebra 1_B and the map ψ. As we have seen at the end of Chapter 4, the $\mathbb{E}_n$-algebra F classifies the $\mathbb{E}_n$-monoidal Kan complex $F^{\otimes}$ defined in Proposition 5.1.8. We claim that the map ψ' is $\mathbb{E}_n$-monoidal, and the rest of this section will be dedicated to proving this claim.

Since $1 := 1_B$ is a trivial algebra of $B^{\otimes}$, the forgetful functor $\mathrm{Mod}_1^{\mathbb{E}_n}(B)^{\otimes} \to B^{\otimes}$ is an equivalence of ∞-operads [Lur17, Prop. 3.4.2.1] therefore by Proposition 2.2.13 we know that it admits an $\mathbb{E}_n$-monoidal inverse. We fix an $\mathbb{E}_n$-monoidal inverse $B^{\otimes} \xrightarrow{\simeq} \mathrm{Mod}_1^{\mathbb{E}_n}(B)^{\otimes}$ and consider the map

$$B^{\otimes} \xrightarrow{\simeq} \mathrm{Mod}_1^{\mathbb{E}_n}(B)^{\otimes} \xrightarrow{\psi'} \mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S})^{\otimes}.$$

If our claim that ψ' is $\mathbb{E}_n$-monoidal is true, then this composition is also $\mathbb{E}_n$-monoidal. Before proving the claim, we need two technical lemmas. The following two results are almost direct consequences of a combination of results from [Lur09] and [Lur17].

Lemma 5.2.1. *Let $\mathcal{A}^{\otimes}$, $\mathcal{B}^{\otimes}$ and $\mathcal{C}^{\otimes}$ be three $\mathcal{O}$-monoidal categories and suppose that we have the following diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{A}^{\otimes} \arrow[r, "F"] \arrow[dr, "p"'] & \mathcal{B}^{\otimes} \arrow[r, "G"] \arrow[d, "p'"] & \mathcal{C}^{\otimes} \arrow[dl, "p''"] \\
& \mathcal{O}^{\otimes}, &
\end{tikzcd}$$

where G is an $\mathcal{O}$-monoidal map that induces a conservative functor on the underlying categories, i.e., a functor that reflects equivalences. Then, F is $\mathcal{O}$-monoidal if and only if $G \circ F$ is $\mathcal{O}$-monoidal.

Proof. The "only if" implication is trivial.

For the "if" implication, let us suppose that the composition $G \circ F$ is $\mathcal{O}$-monoidal. Let $g : A \to A'$ be a p-coCartesian morphism of $\mathcal{A}^{\otimes}$ covering the morphism $\alpha : X \to X'$ of $\mathcal{O}^{\otimes}$; we wish to show that the morphism $F(g) : B \to B'$ is p'-coCartesian. Since $\mathcal{A}^{\otimes}$ is $\mathcal{O}$-monoidal we can assume without loss of generality that $X' \in \mathcal{O}$.

The category $\mathcal{B}^{\otimes}$ is $\mathcal{O}$-monoidal, so there exists a p'-coCartesian morphism $h : B \to B''$

covering α with source B. By the universal property of coCartesian morphisms, there exists a unique, up to a contractible space of choices, morphism $\ell : B'' \to B'$ covering the identity that fits in the following commutative diagram of $\mathcal{B}^{\otimes}$

$$\begin{tikzcd} B \arrow[r, "F(g)"] \arrow[dr, "h"'] & B' \\ & B''. \arrow[u, "\ell"'] \end{tikzcd} \tag{$\star$}$$

Now we consider the image of the diagram $(\star)$ by the functor G.

$$\begin{tikzcd} \left[A \xrightarrow{g} A' \right] \arrow[r, mapsto, "F"] \arrow[dr, "p"'] & \left[\begin{array}{cc} B \xrightarrow{F(g)} & B' \\ \searrow^{h} & \uparrow \ell \\ & B'' \end{array} \right] \arrow[r, mapsto, "G"] \arrow[d, mapsto, "p'"] & \left[\begin{array}{cc} C \xrightarrow{(G\circ F)(g)} & C' \\ \searrow^{G(h)} & \simeq \uparrow G(\ell) \\ & C'' \end{array} \right] \arrow[dl, mapsto, "p''"] \\ & \left[\begin{array}{cc} X \xrightarrow{\alpha} & X' \\ \searrow^{\alpha} & \uparrow id \\ & X' \end{array} \right]. & \end{tikzcd}$$

The functors G and $G \circ F$ are both $\mathcal{O}$-monoidal, i.e., they preserve coCartesian morphisms, so the morphisms $(G \circ F)(g)$ and $G(h)$ are p''-coCartesian. Since $(G \circ F)(g)$ and $G(h)$ are coCartesian morphisms covering the same morphism of $\mathcal{O}^{\otimes}$ and having the same source, the morphism $G(\ell)$ must be an equivalence, but by hypothesis, the functor G reflects equivalences of the underlying categories, so ℓ is an equivalence of $\mathcal{B}$. Since all equivalences are coCartesian morphisms, we can use the dual version of [Lur17, Prop. 2.4.1.7] to conclude that $F(g)$ is p'-coCartesian. $\square$

Lemma 5.2.2. *Let $\mathcal{C}^{\otimes}$ be an $\mathbb{E}_n$-monoidal category and $A \in \mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{C})$. The $\mathbb{E}_1$-monoidal functor*

$$\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \mathbb{E}_1^{\otimes} \to \mathrm{Mod}_{A'}^{\mathbb{E}_1}(\mathcal{C}')^{\otimes}$$

defined in [Lur17, Construction 5.1.3.1] induces a conservative functor on the underlying categories, i.e., the functor reflects the equivalences of the underlying categories. Here $\mathcal{C}'^{\otimes} = \mathcal{C}^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \mathbb{E}_1^{\otimes}$ and $A' \in \mathrm{Alg}_{/\mathbb{E}_1}(\mathcal{C}')$ is the image of A by the forgetful functor $\mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{C}) \to \mathrm{Alg}_{/\mathbb{E}_1}(\mathcal{C}')$.

Proof. We claim that the forgetful functor $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C}) \to \mathcal{C}$ is conservative. Let us suppose that this claim is true. Since the inclusion $\mathbb{E}_1^{\otimes} \to \mathbb{E}_n^{\otimes}$ induces an equivalence on the underlying categories, the pullback of $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C}) \to \mathcal{C}$ along $\mathbb{E}_1 \to \mathbb{E}_n$ must be conservative too. From the commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C}) \underset{\mathbb{E}_n}{\times} \mathbb{E}_1 \arrow[rr] \arrow[dr] & & \mathrm{Mod}_{A'}^{\mathbb{E}_1}(\mathcal{C}') \arrow[dl] \\
& \mathcal{C}', &
\end{tikzcd}$$

it follows that the functor $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C}) \underset{\mathbb{E}_n}{\times} \mathbb{E}_1 \to \mathrm{Mod}_{A'}^{\mathbb{E}_1}(\mathcal{C}')$ is conservative.

We now prove the claim. [Lur17, Corollary 3.4.3.4] states that the map $\phi : \mathrm{Mod}^{\mathbb{E}_n}(\mathcal{C})^{\otimes} \to \mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{C})$ is a coCartesian fibration and a morphism of $\mathrm{Mod}^{\mathbb{E}_n}(\mathcal{C})^{\otimes}$ is ϕ-coCartesian if and only if its image in $\mathcal{C}^{\otimes}$ is an equivalence. Let f be a morphism of $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^{\otimes}$ such that its image in $\mathcal{C}^{\otimes}$ is an equivalence; we wish to prove that f is an equivalence as well. We aim to do so by proving that f is a coCartesian morphism covering an equivalence and applying [Lur09, Prop. 2.4.1.3]. By definition, the category $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^{\otimes}$ fits in the following pullback diagram

$$\begin{tikzcd}
\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^{\otimes} \arrow[r] \arrow[d, "\phi_A"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Mod}^{\mathbb{E}_n}(\mathcal{C})^{\otimes} \arrow[d, "\phi"] \\
\{A\} \arrow[r, hook] & \mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{C}).
\end{tikzcd}$$

From [Lur09, Prop. 2.4.2.3] we know that coCartesian fibrations are stable under pullbacks. Moreover, the map $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^{\otimes} \to \mathrm{Mod}^{\mathbb{E}_n}(\mathcal{C})^{\otimes}$ reflects and preserves coCartesian morphisms. So f is a ϕ_A-coCartesian morphism if and only if its image $\bar{f}$ is ϕ-coCartesian. Since the following diagram commutes

$$\begin{tikzcd}
\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^{\otimes} \arrow[rr] \arrow[dr] & & \mathrm{Mod}^{\mathbb{E}_n}(\mathcal{C})^{\otimes} \arrow[dl] \\
& \mathcal{C}^{\otimes} &
\end{tikzcd}$$

the image in $\mathcal{C}^{\otimes}$ of $\bar{f}$ is an equivalence. Hence from [Lur17, Corollary 3.4.3.4] we know that $\bar{f}$ is a ϕ-coCartesian morphism, this implies that f is a ϕ_A-coCartesian morphism covering the identity of $\{A\}$ and by [Lur09, Prop. 2.4.1.3] it is in particular an equivalence. We have proven that the functor $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ is conservative. It immediately follows that the functor induced on the underlying categories is conservative as well. $\square$

Proposition 5.2.3. *The map* $\psi' : \mathrm{Mod}_1^{\mathbb{E}_n}(B)^{\otimes} \to \mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S})^{\otimes}$ *is* $\mathbb{E}_n$*-monoidal.*

Proof. We consider the lax $\mathbb{E}_n$-monoidal map ϕ' given by the following composition

$$
\begin{tikzcd}
\mathrm{Mod}_1^{\mathbb{E}_n}(X)^{\otimes} \arrow[r, "\pi'"] \arrow[rr, bend right, "\phi'"'] & \mathrm{Mod}_1^{\mathbb{E}_n}(B)^{\otimes} \arrow[r, "\psi'"] & \mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S})^{\otimes}
\end{tikzcd}.
$$

Since π' is an essentially surjective $\mathbb{E}_n$-monoidal map, from Proposition 2.2.12 we know that ψ' is $\mathbb{E}_n$-monoidal if and only if the composition $\phi' = \psi' \circ \pi'$ is $\mathbb{E}_n$-monoidal. Therefore, it is sufficient to prove that ϕ' is $\mathbb{E}_n$-monoidal.

Let $\overline{\gamma}_n$ be a p'-coCartesian morphism of $\mathrm{Mod}_1^{\mathbb{E}_n}(X)^{\otimes}$ covering the morphism $\gamma_n : \langle r \rangle \to \langle m \rangle$ of $\mathbb{E}_n^{\otimes}$. We wish to show that $\phi'(\overline{\gamma}_n)$ is a q'-coCartesian morphism of $\mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S})^{\otimes}$, where q' is the $\mathbb{E}_n$-monoidal structure of $\mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S})^{\otimes}$ induced by the $\mathbb{E}_n$-monoidal structure $q : \mathcal{S}^{\otimes} \to \mathbb{E}_n^{\otimes}$ as described in Theorem 3.2.8.

Since $n \geq 1$, $\mathbb{E}_n^{\otimes}$ is equivalent to the tensor product of n copies of $\mathbb{E}_1^{\otimes}$ there exists an operad map $\mathbb{E}_1^{\otimes} \to \mathbb{E}_n^{\otimes}$, such that γ_n is the image of a morphism γ_1 of $\mathbb{E}_1^{\otimes}$. Then, the morphism $\overline{\gamma}_n$ must be the image of a morphism $\overline{\gamma}_1$ of the $\mathbb{E}_1$-monoidal category $\mathrm{Mod}_1^{\mathbb{E}_n}(X)^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \mathbb{E}_1^{\otimes}$.

We claim that to prove that the image of the morphism $\overline{\gamma}_n$ by the map ϕ' is q'-coCartesian it is sufficient to prove that the map ϕ'' induced by ϕ' on the pullbacks along $\mathbb{E}_1^{\otimes} \to \mathbb{E}_n^{\otimes}$ is $\mathbb{E}_1$-monoidal

$$
\begin{tikzcd}
\mathrm{Mod}_1^{\mathbb{E}_n}(X)^{\otimes} \arrow[r, "\phi'"] & \mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S})^{\otimes} \\
\mathrm{Mod}_1^{\mathbb{E}_n}(X)^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \mathbb{E}_1^{\otimes} \arrow[u] \arrow[r, "\phi''"] & \mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S})^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \mathbb{E}_1^{\otimes} \arrow[u]
\end{tikzcd}.
$$

Let us prove the claim. Assuming that ϕ'' is $\mathbb{E}_1$-monoidal, we wish to prove that the morphism $\phi'(\overline{\gamma}_n)$ is coCartesian. From [Lur09, Prop. 2.4.1.3] we know that since $\overline{\gamma}_n$ is coCartesian so is $\overline{\gamma}_1$. We assumed that the functor ϕ'' is $\mathbb{E}_1$-monoidal so the image $\phi''(\overline{\gamma}_1)$ is coCartesian and, in particular, it is locally coCartesian. Combining [Lur09, Remark 2.4.1.12] and [Lur09, Prop. 2.4.2.8] we obtain that $\phi'(\overline{\gamma}_n)$ is coCartesian.

By Theorem 3.3.5 we know that the map of ∞-operads $\mathbb{E}_1^{\otimes} \to \mathbb{E}_n^{\otimes}$ induces $\mathbb{E}_1$-monoidal functors H and G that fit in the following diagram

$$\begin{tikzcd}
\mathrm{Mod}_1^{\mathbb{E}_n}(X)^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \mathbb{E}_1^{\otimes} \arrow[r, "\phi''"] \arrow[d, "H"'] & \mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S})^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \mathbb{E}_1^{\otimes} \arrow[d, "G"] \\
\mathrm{Mod}_{1_X}^{\mathbb{E}_1}(X')^{\otimes} \arrow[r, "\phi'''"] & \mathrm{Mod}_F^{\mathbb{E}_1}(\mathcal{S}')^{\otimes},
\end{tikzcd}$$

where $X'^{\otimes} = X^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \mathbb{E}_1^{\otimes}$ and $\mathcal{S}'^{\otimes} = \mathcal{S}^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \mathbb{E}_1^{\otimes}$.

In Lemma 5.2.2 we have seen that G induces a conservative functor on the underlying categories, so, by Lemma 5.2.1, in order to prove that ϕ'' is $\mathbb{E}_1$-monoidal it is sufficient to prove that the map ϕ''' is $\mathbb{E}_1$-monoidal.

Since $\mathbb{E}_1^{\otimes} \simeq \mathrm{Assoc}^{\otimes}$, we have effectively reduced the proof to the case where $X^{\otimes}$ and $B^{\otimes}$ are associative monoidal grouplike Kan complexes with a principal associative monoidal action of $F^{\otimes}$ on $X^{\otimes}$. The action is the one defined by the associative monoidal principal $F^{\otimes}$-bundle obtained by taking the pullback of the map $B^{\otimes} \to \mathbf{B}F^{\otimes}$ along the functor $\mathrm{Assoc}^{\otimes} \to \mathbb{E}_n^{\otimes}$. In order to simplify the notation, we will refer to the map ϕ''' by ϕ', as if we were considering from the beginning associative monoidal categories.

Let $\bar{\beta}$ be a coCartesian morphism of $\mathrm{Mod}_1^{\mathrm{Assoc}}(X)^{\otimes}$ covering a morphism β of $\mathrm{Assoc}^{\otimes}$; we have to prove that $\phi'(\bar{\beta})$ is a coCartesian morphism of $\mathrm{Mod}_F^{\mathrm{Assoc}}(\mathcal{S})^{\otimes}$. Without loss of generality, we can assume that β is the unique active morphism with source $\langle 2\rangle$ and target $\langle 1\rangle$ given by the natural linear ordering of $\langle 2\rangle$, see proof of [Lur17, Theorem 5.1.3.2].

From Proposition 3.2.10 it follows that the monoidal categories $\mathrm{Mod}_1^{\mathrm{Assoc}}(X)^{\otimes}$ and $\mathrm{Mod}_F^{\mathrm{Assoc}}(\mathcal{S})^{\otimes}$ are equivalent to the categories ${}_1\mathrm{BMod}_1(X)$ and ${}_F\mathrm{BMod}_F(\mathcal{S})$ equipped with the relative tensor product defined in Section 3.1.4. Then there exist elements $x, y \in \mathrm{Mod}_1^{\mathrm{Assoc}}(X) \simeq {}_1\mathrm{BMod}_1(X) \simeq X$ such that

$$\bar{\beta} : (x, y) \to xy,$$

where xy is the product of x and y. Since the map ϕ' is lax Assoc-monoidal we know that there exist a unique coCartesian morphism $\tilde{\beta}$ covering β and a morphism ℓ covering the identity that fit in the following diagram of $\mathrm{Mod}_F^{\mathrm{Assoc}}(\mathcal{S})^{\otimes}$

$$\begin{tikzcd}
(\phi'(x), \phi'(y)) \arrow[r, "\phi'(\bar{\beta})"] \arrow[dr, "\tilde{\beta}"'] & \phi'(xy) \\
& \phi'(x) \otimes_F \phi'(y). \arrow[u, "\ell"']
\end{tikzcd}$$

In order to prove that the morphism $\phi'(\bar{\beta})$ is coCartesian it is sufficient to prove that the morphism ℓ is an equivalence. From Proposition 3.2.10 we know that we can describe the product of $\mathrm{Mod}_F^{\mathrm{Assoc}}(\mathcal{S})^{\otimes}$ via the bar construction. Let $\bar{H} : \mathrm{Tens}_{\prec}^{\otimes} \to X^{\otimes}$ be the map that presents xy as the relative tensor product of x and y

$$\begin{tikzcd}
\mathrm{Tens}^{\otimes}_{[2]} \arrow[r, "H"] \arrow[d] & X^{\otimes} \arrow[d, "p"] \\
\mathrm{Tens}^{\otimes}_{\prec} \arrow[r, "h"] \arrow[ur, "\overline{H}"'] & \mathrm{Assoc}^{\otimes},
\end{tikzcd}$$

where the bottom horizontal map is the one defined in [Lur17, Def. 4.4.1.1]

$$h : \mathrm{Tens}^{\otimes}_{\prec} \hookrightarrow \mathrm{Tens}^{\otimes} \to \mathrm{Assoc}^{\otimes}.$$

Postcomposing with the operadic map ϕ' we obtain the following commutative diagram

$$\begin{tikzcd}
\mathrm{Tens}^{\otimes}_{[2]} \arrow[r, "H"] \arrow[d] & X^{\otimes} \arrow[r, "\phi'"] & \mathcal{S}^{\otimes} \arrow[d, "q"] \\
\mathrm{Tens}^{\otimes}_{\prec} \arrow[rr] \arrow[ur, "\overline{H}"'] & & \mathrm{Assoc}^{\otimes}.
\end{tikzcd}$$

Then the morphism ℓ, which we claim to be an equivalence, corresponds to the morphism from the geometric realization of the bar construction of $\phi'(x)$ and $\phi'(y)$ to the image of $\mathfrak{m}$ by the map $\phi' \circ \bar{H}$

$$\phi'(x) \otimes_F \phi'(y) \simeq |\mathrm{Bar}_F(\phi'(x), \phi'(y))_\bullet| \to \phi'(|\mathrm{Bar}_1(x, y)_\bullet|) \simeq \phi' \circ \bar{H}(\mathfrak{m}) = \phi'(xy). \qquad (\star)$$

We refer to the morphism $(\star)$ as the morphism induced by the lax monoidal structure of ϕ' on x and y; the rest of the proof consists of proving that this morphism is an equivalence.

Let us consider the morphism of associative monoidal left fibrations η given by the shear map of the monoidal principal action of $F^{\otimes}$ on $X^{\otimes}$ over $B^{\otimes}$

$$\begin{tikzcd}
X^{\otimes} \underset{\mathrm{Assoc}^{\otimes}}{\times} F^{\otimes} \arrow[rr, "\eta", "\simeq"'] \arrow[dr, "\nu"'] & & X^{\otimes} \underset{B^{\otimes}}{\times} X^{\otimes} \arrow[dl, "\mu"] \\
& X^{\otimes}. &
\end{tikzcd}$$

Applying the monoidal straightening functor to η we obtain a natural equivalence between the lax associative monoidal pre-sheaves that classify the two left Assoc-fibrations ν and μ. Since, in both cases, the fibrations are defined from pullbacks, we can use Lemma 2.4.6 to compute their classifying operadic maps. The lax monoidal pre-sheaves that classify the left fibrations are the pre-sheaf $\phi = \psi \circ \pi$, and the constant pre-sheaf that maps the category $X^{\otimes}$ to the algebra object F

$$\left[\begin{tikzcd} X^{\otimes} \underset{B^{\otimes}}{\otimes} X^{\otimes} \arrow[r] \arrow[d, "\mu"'] \arrow[dr, phantom, "\lrcorner"] & X^{\otimes} \arrow[d, "\pi"] \\ X^{\otimes} \arrow[r, "\pi"] & B^{\otimes} \end{tikzcd}\right] \overset{\mathrm{St}_X}{\longmapsto} \left[\begin{tikzcd} X^{\otimes} \arrow[r, "\pi"] \arrow[rr, bend right, "\mathrm{St}_X(\mu)"'] & B^{\otimes} \arrow[r, "\psi"] & \mathcal{S}^{\otimes} \end{tikzcd}\right]$$

$$\left[\begin{tikzcd} X^{\otimes} \underset{\mathrm{Assoc}^{\otimes}}{\times} F^{\otimes} \arrow[r] \arrow[d, "\nu"] \arrow[dr, phantom, "\lrcorner"] & F^{\otimes} \arrow[d] \\ X^{\otimes} \arrow[r, "p"] & \mathrm{Assoc}^{\otimes} \end{tikzcd}\right] \overset{\mathrm{St}_X}{\longmapsto} \left[\begin{tikzcd} X^{\otimes} \arrow[r, "p"] \arrow[rr, bend right, "\mathrm{St}_X(\nu)"'] & \mathrm{Assoc}^{\otimes} \arrow[r, "F"] & \mathcal{S}^{\otimes} \end{tikzcd}\right].$$

Where we are abusing notation by denoting by F the associative algebra of $\mathcal{S}^{\otimes}$, the associative algebra considered as an F-bimodule, and F considered as an object of $\mathcal{S}$. Applying the monoidal straightening functor to the equivalence of associative monoidal left fibrations η, we obtain a natural equivalence of lax associative monoidal pre-sheaves

$$\begin{tikzcd} & B^{\otimes} \arrow[dr, bend left, "\psi"] & \\ X^{\otimes} \arrow[ur, bend left, "\pi"] \arrow[dr, bend right, "p"'] & \simeq\Uparrow\eta & \mathcal{S}^{\otimes}. \\ & \mathrm{Assoc}^{\otimes} \arrow[ur, bend right, "F"'] & \end{tikzcd}$$

Let x be an object of X and let $x : \mathcal{BM}^{\otimes} \to X^{\otimes}$ be the operadic map that describes x as a 1_X-bimodule. We denote by F and X_a, where $a = \pi(x)$, the F-bimodules obtained by postcomposing x with the operadic maps $F \circ p$ and ϕ respectively. Postcomposition with the natural equivalence η

$$\eta_x : \mathcal{BM}^{\otimes} \times \mathrm{J} \xrightarrow{x \times id} X^{\otimes} \times \mathrm{J} \xrightarrow{\eta} \mathcal{S}^{\otimes},$$

where J is the nerve of the free groupoid of the direct 1-category $[1]$, defines an equivalence η_x of F-bimodules with source F and target X_a

$$\begin{tikzcd} & & B^{\otimes} \arrow[dr, bend left, "\psi"] & \\ \mathcal{BM}^{\otimes} \arrow[r, "x"] \arrow[ddr] & X^{\otimes} \arrow[ur, bend left, "\pi"] \arrow[dr, bend right, "p"'] \arrow[dd, "p"] & \simeq\Uparrow\eta & \mathcal{S}^{\otimes} \arrow[dd, "q"] \\ & & \mathrm{Assoc}^{\otimes} \arrow[ur, bend right, "F"'] & \\ & \mathrm{Assoc}^{\otimes} \arrow[rr, "id"] & & \mathrm{Assoc}^{\otimes}. \end{tikzcd}$$

To have a better intuition for the morphism η_x we suggest consulting Section A.2 where we describe how it is possible to recover from the map $\eta_x : \mathcal{BM}^{\otimes} \times \mathrm{J} \to \mathcal{S}^{\otimes}$ the classical notion of a morphism of the underlying objects that is compatible with the left and right F-action.

The next step consists of using the natural transformation η to produce the following commutative diagram

$$\begin{tikzcd} F \arrow[d, "\eta_{xy}", "\simeq"'] & F \underset{F}{\otimes} F \arrow[l, "\simeq"'] \arrow[d, "\eta_x \underset{F}{\otimes} \eta_y", "\simeq"'] \\ X_{ab} & X_a \underset{F}{\otimes} X_b. \arrow[l] \end{tikzcd}$$

Where the horizontal morphisms are the ones defined by the lax monoidal structures of the pre-sheaves $F \circ p$ and $\phi = \psi \circ \pi$ on x and y, and $b = \pi(y)$. By the two-out-of-three rule, we will get that the bottom horizontal map is an equivalence as well.

Let us consider once again the map $\overline{H}$ that presents xy as the relative tensor product of x and y

$$\begin{tikzcd} \mathrm{Tens}^{\otimes}_{[2]} \arrow[r, "H"] \arrow[d] & X^{\otimes} \arrow[d, "p"] \\ \mathrm{Tens}^{\otimes}_{\prec} \arrow[r, "h"] \arrow[ur, "\overline{H}"'] & \mathrm{Assoc}^{\otimes}. \end{tikzcd}$$

By considering the diagram given by taking the Cartesian product of the functors with the identity of J and then postcomposing with the natural transformation η we obtain the following commutative diagram

$$\begin{tikzcd} \mathrm{Tens}^{\otimes}_{[2]} \times \mathrm{J} \arrow[r, "H"] \arrow[d] & X^{\otimes} \times \mathrm{J} \arrow[r, "\eta"] & \mathcal{S}^{\otimes} \arrow[d, "q"] \\ \mathrm{Tens}^{\otimes}_{\prec} \times \mathrm{J} \arrow[rr, "h \circ pr_1"] \arrow[ur, "\overline{H}"'] & & \mathrm{Assoc}^{\otimes}. \end{tikzcd}$$

Restricting the map $(\eta \circ \overline{H})$ to $\mathrm{Tens}^{\otimes}_{\prec} \times \{0, 1\}$ we can define two maps of ∞-operads from $\mathrm{Tens}^{\otimes}_{\prec}$ to the category $\mathcal{S}^{\otimes}$:

- the map of ∞-operads

$$(\eta \circ \overline{H})_{|\{0\}} = (F \circ p \circ \overline{H}) : \mathrm{Tens}^{\otimes}_{\prec} \longrightarrow \mathcal{S}^{\otimes},$$

  which, after precomposing with the functor $U_+ : \mathrm{N}(\Delta^{\mathrm{op}}_+) \to \mathrm{N}(\mathrm{Step}) \to \mathrm{Tens}^{\otimes}_{\prec}$ described in Remark 3.1.18, defines the augmented simplicial object $(F \circ p \circ \overline{H} \circ U_+)_\bullet$ with augmentation F. We can informally think of $(F \circ p \circ \overline{H} \circ U_+)_\bullet$ as the augmented simplicial object

$$\cdots \Rrightarrow (F, F, F) \rightrightarrows (F, F) \longrightarrow F;$$

- and the map of ∞-operads

$$(\eta \circ \overline{H})_{|\{1\}} = (\phi \circ \overline{H}) : \mathrm{Tens}^{\otimes}_{\prec} \longrightarrow \mathcal{S}^{\otimes},$$

  which defines the augmented simplicial object $(\phi \circ \overline{H} \circ U_+)_\bullet$ with augmentation X_{ab}, which can informally be described as

$$\cdots \Rrightarrow (X_a, F, X_b) \rightrightarrows (X_a, X_b) \longrightarrow X_{ab}.$$

Precomposing $(\eta \circ \overline{H})$ with U_+ we obtain an equivalence of augmented simplicial objects

$$(F \circ p \circ \overline{H} \circ U_+)_\bullet \xrightarrow{\simeq} (\phi \circ \overline{H} \circ U_+)_\bullet,$$

or, equivalently, a commutative square of (non-augmented) simplicial objects of $\mathcal{S}^{\otimes}$. (The commutative square is produced by precomposing the functor that classifies the previous morphism with the nerve of the standard projection from the cylinder category of Δ^{op} to the category Δ^{op}_*.)

$$\begin{array}{ccc}
(F \circ p \circ \overline{H} \circ U)_\bullet & \longrightarrow & F \\
\simeq \downarrow \eta_\bullet & & \simeq \downarrow \eta_{xy} \\
(\phi \circ \overline{H} \circ U)_\bullet & \longrightarrow & X_{ab},
\end{array}$$

where we are considering F and X_{ab} as constant simplicial objects with values F and X_{ab}, and U is the functor described in Remark 3.1.18.

By [Lur18, Theorem 5.2.1.1] we know that postcomposition with a coCartesian fibration is coCartesian, in particular, the functor

$$\mathrm{Fun}(\mathrm{N}(\Delta^{\mathrm{op}}), \mathcal{S}^{\otimes}) \xrightarrow{(q \circ -)} \mathrm{Fun}(\mathrm{N}(\Delta^{\mathrm{op}}), \mathrm{Assoc}^{\otimes})$$

is coCartesian. Let β be the morphism of $\mathrm{Fun}(\mathrm{N}(\Delta^{\mathrm{op}}), \mathrm{Assoc}^{\otimes})$ defined by the augmented simplicial object $\mathrm{N}(\Delta^{\mathrm{op}}_+) \xrightarrow{U_+} \mathrm{Tens}^{\otimes}_{\prec} \xrightarrow{h} \mathrm{Assoc}^{\otimes}$.

We consider the following diagram of $\mathrm{Fun}(\mathrm{N}(\Delta^{\mathrm{op}}), \mathcal{S}^{\otimes})$, where the morphisms $\beta_!$ are the coCartesian morphisms covering β and the objects $\mathrm{Bar}_F(F, F)_\bullet$ and $\mathrm{Bar}_F(X_a, X_b)_\bullet$ are defined as in Construction 3.1.20

$$
\begin{tikzcd}
 & & \mathrm{Bar}_F(F,F)_\bullet \\
(F \circ p \circ \overline{H} \circ U)_\bullet \arrow[rru, bend left, "\beta_!"] \arrow[r] \arrow[d, "\simeq"', "\eta_\bullet"] & F \arrow[d, "\simeq"', "\eta_{xy}"] & \\
(\phi \circ \overline{H} \circ U)_\bullet \arrow[r] \arrow[rrd, bend right, "\beta_!"'] & X_{ab} & \\
 & & \mathrm{Bar}_F(X_a, X_b)_\bullet .
\end{tikzcd}
$$

With the same procedure as the one described in Section A.1, starting from the diagram of solid arrows, we can use the universal property of the coCartesian morphisms to obtain the following commutative diagram of $\mathrm{Fun}(\mathrm{N}(\Delta^{\mathrm{op}}), \mathcal{S}^{\otimes})$

$$
\begin{tikzcd}
 & & \mathrm{Bar}_F(F,F)_\bullet \arrow[ld, dashed] \arrow[dd, dashed, "\simeq"', "\mathrm{Bar}_F(\eta_x,\eta_y)"] \\
(F \circ p \circ \overline{H} \circ U)_\bullet \arrow[rru, bend left, "\beta_!"] \arrow[r] \arrow[d, "\simeq"', "\eta_\bullet"] & F \arrow[d, "\simeq"', "\eta_{xy}"] & \\
(\phi \circ \overline{H} \circ U)_\bullet \arrow[r] \arrow[rrd, bend right, "\beta_!"'] & X_{ab} & \\
 & & \mathrm{Bar}_F(X_a, X_b)_\bullet . \arrow[lu, dashed]
\end{tikzcd}
$$

The right-most commutative square can be informally described as

$$
\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & F \times F \times F \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\simeq"', "\eta_x \times \eta_1 \times \eta_y"] & F \times F \arrow[r, dashed] \arrow[d, "\simeq"', "\eta_x \times \eta_y"] & F \arrow[d, dashed, "\simeq"', "\eta_{xy}"] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_a \times F \times X_b \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_a \times X_b \arrow[r, dashed] & X_{ab}.
\end{tikzcd}
$$

We now focus on the geometric realization of the right-most square

$$
\begin{tikzcd}
F \arrow[d, "\simeq"'] & {|\mathrm{Bar}_F(F,F)_\bullet|} \arrow[l] \arrow[d, "\simeq"'] \\
X_{ab} & {|\mathrm{Bar}_F(X_a,X_b)_\bullet|}. \arrow[l]
\end{tikzcd}
$$

where the bottom and the top horizontal morphisms are the morphisms defined by the lax monoidal structures of the maps ϕ and $F \circ p$. By unitality of the relative tensor product [Lur17, Prop. 4.4.3.16] the top horizontal morphism is an equivalence. By the two-out-of-three rule, the bottom map must be an equivalence too. $\square$

5.3 Construction of the iterated Thom spectrum

In this section, we will apply a series of operations, namely postcomposition with monoidal functors and base changes, to the $\mathbb{E}_n$-monoidal functor

$$\psi' : B^{\otimes} \to \mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S})^{\otimes}$$

introduced in the previous section to obtain the system of invertible $\mathrm{Th}_R(\xi_1)$-modules $\mathrm{Th}_R(\xi)^B$ as an $\mathbb{E}_{n-1}$-monoidal map. Once we have obtained the functor $\mathrm{Th}_R(\xi)^B$ we will apply to it O. Antolín-Camarena and T. Barthel's version of the monoidal Thom functor to define the iterated Thom spectrum $\mathrm{ThTh}^{\pi}(\xi)$ of ξ along π as an $\mathbb{E}_{n-1}$-algebra of $\mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}$.

Our initial hypotheses are the following:

- an $\mathbb{E}_m$-ring spectrum R;
- an essentially surjective left $\mathbb{E}_{n+1}$-fibration $\pi : X^{\otimes} \to B^{\otimes}$ of grouplike Kan complexes with $2 < n+1 \leq m$;
- and an $\mathbb{E}_n$-monoidal system of invertible R-modules $\xi : X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$, i.e., a morphism of $\mathbb{E}_n$-monoidal Kan complexes where $\mathrm{Pic}(R)^{\otimes}$ is the category defined in Definition 4.2.6.

As usual, we will use the equivalence $\mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathcal{S}) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S}))$ to consider $X^{\otimes}$, $B^{\otimes}$ and $\mathrm{Pic}(R)^{\otimes}$ as $\mathbb{E}_n$-monoidal categories equipped with associative $\mathbb{E}_n$-monoidal products. We start by producing a lax $\mathbb{E}_n$-monoidal version of the map $b \mapsto \xi_b$.

Construction 5.3.1. The system $\xi : X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$ defines the following morphism of left $\mathbb{E}_n$-fibrations

$$\begin{tikzcd} X^{\otimes} \arrow[rr, "(\xi,\pi)"] \arrow[dr, "\pi"'] & & \mathrm{Pic}(R)^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} B^{\otimes} \arrow[dl, "pr_2"] \\ & B^{\otimes}. & \end{tikzcd}$$

Applying the $\mathbb{E}_n$-monoidal straightening functor to the morphism (ξ,π), we obtain a natural transformation between the $\mathbb{E}_n$-monoidal pre-sheaves that classify the two fibrations

$$\begin{tikzcd} B^{\otimes} \arrow[rr, "\psi"] \arrow[dr, "p"'] & \arrow[d, phantom, "\Downarrow(\xi,\pi)"] & \mathcal{S}^{\otimes}. \\ & \mathbb{E}_n^{\otimes} \arrow[ur, "\mathrm{Pic}(R)"'] & \end{tikzcd}$$

In view of Lemma 4.3.3 we know that the natural transformation (ξ,π) defines a lift $\xi^B : B^{\otimes} \to \mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)}$ of the lax $\mathbb{E}_n$-monoidal map ψ to the overcategory $\mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)}$ as defined in Definition 2.3.6

$$\begin{tikzcd} & \mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)} \arrow[d, "U"] \\ B^{\otimes} \arrow[ur, "\xi^B"] \arrow[r, "\psi"'] & \mathcal{S}^{\otimes}. \end{tikzcd} \tag{$*$}$$

We can now consider the diagram induced by $(*)$ on the categories of $\mathbb{E}_n$-modules

$$\begin{tikzcd} & \mathrm{Mod}^{\mathbb{E}_n}_{\xi_1}(\mathcal{S}_{/\mathrm{Pic}(R)})^{\otimes} \arrow[d, "U'"] \\ B^{\otimes} \simeq \mathrm{Mod}^{\mathbb{E}_n}_{1}(B)^{\otimes} \arrow[ur, "\xi^B"] \arrow[r, "\psi'"'] & \mathrm{Mod}^{\mathbb{E}_n}_{F}(\mathcal{S})^{\otimes}. \end{tikzcd}$$

From Proposition 5.2.3 we know that the map ψ' is $\mathbb{E}_n$-monoidal, and to prove that its lift ξ^B is $\mathbb{E}_n$-monoidal we plan to apply Lemma 5.2.1.

Proposition 5.3.2. *The map $\xi^B : B^{\otimes} \to \mathrm{Mod}^{\mathbb{E}_n}_{\xi_1}(\mathcal{S}_{/\mathrm{Pic}(R)})^{\otimes}$ defined above is $\mathbb{E}_n$-monoidal.*

Proof. First, we prove that the map $U' : \mathrm{Mod}^{\mathbb{E}_n}_{\xi_1}(\mathcal{S}_{/\mathrm{Pic}(R)})^{\otimes} \to \mathrm{Mod}^{\mathbb{E}_n}_{F}(\mathcal{S})^{\otimes}$ induced by the forgetful functor is conservative and $\mathbb{E}_n$-monoidal:

- We start by proving that it is $\mathbb{E}_n$-monoidal. Applying the same argument as for Proposition 5.2.3 we can reduce the problem to $n = 1$.

  Let $\bar{\beta}$ be a coCartesian morphism of $\mathrm{Mod}^{\mathrm{Assoc}}_{\xi_1}(\mathcal{S}_{/\mathrm{Pic}(R)})^{\otimes}$ covering a morphism β of $\mathrm{Assoc}^{\otimes}$; we have to prove that $U'(\bar{\beta})$ is a coCartesian morphism of $\mathrm{Mod}^{\mathrm{Assoc}}_{F}(\mathcal{S})^{\otimes}$. We can assume without loss of generality that β is the unique active morphism covering $\langle 2\rangle \to \langle 1\rangle$ given by the natural linear ordering of $\langle 2\rangle$. Then there exist elements $f, g \in \mathrm{Mod}^{\mathrm{Assoc}}_{\xi_1}(\mathcal{S}_{/\mathrm{Pic}(R)}) \simeq {}_{\xi_1}\mathrm{BMod}_{\xi_1}(\mathcal{S}_{/\mathrm{Pic}(R)})$ such that

  $$\bar{\beta} : (f, g) \to f \otimes_{\xi_1} g.$$

  The map U' is lax Assoc-monoidal, hence there exists a unique coCartesian morphism $\tilde{\beta}$ covering β and a morphism ℓ covering the identity that fit in the following diagram of $\mathrm{Mod}^{\mathrm{Assoc}}_{F}(\mathcal{S})^{\otimes}$

  $$\begin{tikzcd} (U'(f), U'(g)) \arrow[r, "U'(\bar{\beta})"] \arrow[dr, "\tilde{\beta}"'] & U'(f \otimes_{\xi_1} g) \\ & U'(f) \otimes_F U'(g). \arrow[u, "\ell"'] \end{tikzcd}$$

  In order to prove that the morphism $U'(\bar{\beta})$ is coCartesian it is sufficient to prove that the morphism ℓ is an equivalence. From Proposition 3.2.10 we know that the product of $\mathrm{Mod}_F^{\mathrm{Assoc}}(\mathcal{S})^{\otimes}$ can be modeled by the bar construction. Let $\bar{H} : \mathrm{Tens}_{\prec}^{\otimes} \to \mathcal{S}_{/\mathrm{Pic}(R)}^{\otimes}$ be the map that presents $f \otimes_{\xi_1} g$ as the relative tensor product of f and g

$$\begin{tikzcd}
\mathrm{Tens}_{[2]}^{\otimes} \arrow[r, "H"] \arrow[d] & \mathcal{S}_{/\mathrm{Pic}(R)}^{\otimes} \arrow[d, "p"] \\
\mathrm{Tens}_{\prec}^{\otimes} \arrow[r, "h"] \arrow[ur, "\overline{H}"'] & \mathrm{Assoc}^{\otimes}.
\end{tikzcd}$$

Postcomposing with the operadic map U we obtain the following commutative diagram

$$\begin{tikzcd}
\mathrm{Tens}_{[2]}^{\otimes} \arrow[r, "H"] \arrow[d] & (\mathcal{S}_{/\mathrm{Pic}(R)})^{\otimes} \arrow[r, "U"] & \mathcal{S}^{\otimes} \arrow[d, "q"] \\
\mathrm{Tens}_{\prec}^{\otimes} \arrow[rr, "h"] \arrow[ur, "\overline{H}"'] & & \mathrm{Assoc}^{\otimes}.
\end{tikzcd}$$

By Remark 3.1.20 we know that the morphism ℓ is an equivalence if and only if $U \circ \bar{H}$ presents $U(f \otimes_{\xi_1} g)$ as the relative tensor product of $U(f)$ and $U(g)$. From the dual version of [Lur17, Lemma 2.2.2.9] it follows that the forgetful functor $U : \mathcal{S}_{/\mathrm{Pic}(R)}^{\otimes} \to \mathcal{S}^{\otimes}$ preserves colimits, therefore

$$U \circ \bar{H}(\mathfrak{m}) \simeq U(|\mathrm{Bar}_{\xi_1}(f, g)|) \simeq |U(\mathrm{Bar}_{\xi_1}(f, g))| \simeq |\mathrm{Bar}_F(U(f), U(g))|.$$

Applying Theorem 3.1.21 we conclude that $U \circ \bar{H}$ presents $U(f \otimes_{\xi_1} g)$ as the relative tensor product of $U(f)$ and $U(g)$ and therefore ℓ is an equivalence.

- We will now prove that the map U' induces a conservative functor on the underlying categories. From the proof of Lemma 5.2.2 we know that the vertical arrows of the following commutative diagram are conservative

$$\begin{tikzcd}
\mathrm{Mod}_{\xi_1}^{\mathbb{E}_n}(\mathcal{S}_{/\mathrm{Pic}(R)}) \arrow[r, "U'"] \arrow[d] & \mathrm{Mod}_F^{\mathbb{E}_n}(\mathcal{S}) \arrow[d] \\
\mathcal{S}_{/\mathrm{Pic}(R)} \arrow[r, "U"] & \mathcal{S}.
\end{tikzcd}$$

  Then, in order to prove that U' is conservative, is sufficient to prove that functor U is conservative, but U is a right fibration [Lur17, Corollary 2.1.2.2] and from the dual of [Lur17, Prop. 2.1.1.5] it follows that every right fibration is conservative.

To conclude the proof we apply Lemma 5.2.1 with $F = \xi^B$, $G = U'$, and $F \circ G = \psi'$. $\square$

The theory of Thom spectra is defined for the monoidal categories of left modules instead of the categories of bimodules. Until now we have stated our arguments in the categories of $\mathbb{E}_n$-bimodules, $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^{\otimes}$, because thanks to results like Theorem 3.3.5 we have a better description of the coCartesian morphisms of these $\mathbb{E}_n$-monoidal categories. Since we no longer need to use these results, we can pass to the categories of left modules. We can do that by considering the pullback of ξ^B along $\mathbb{E}_{n-1}^{\otimes} \to \mathbb{E}_n^{\otimes}$ and postcompose it with the $\mathbb{E}_{n-1}$-monoidal functor described in Theorem 3.3.6

$$B^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \mathbb{E}_{n-1}^{\otimes} \xrightarrow{\xi^B} \mathrm{Mod}_{\xi_1}^{\mathbb{E}_n}(\mathcal{S}_{/\mathrm{Pic}(R)})^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \mathbb{E}_{n-1}^{\otimes} \longrightarrow \mathrm{LMod}_{\xi_1}(\mathcal{S}_{/\mathrm{Pic}(R)})^{\otimes};$$

Abusing the notation we will also denote this composition by ξ^B. Postcomposing ξ^B with the map induced by the monoidal Thom functor on the category of left modules, where we are considering the monoidal Thom functor defined in Theorem 4.2.8, and then postcomposing again with the equivalence $\mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}(\mathrm{LMod}_R) \xrightarrow{\simeq} \mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}$ defined in [Lur17, Corollary 7.1.3.4]; we obtain the $\mathbb{E}_{n-1}$-monoidal map

$$\begin{tikzcd} B^{\otimes} \arrow[r, "\xi^B"] \arrow[rrr, bend right, "\mathrm{Th}_R(\xi)^B"'] & \mathrm{LMod}_{\xi_1}(\mathcal{S}_{/\mathrm{Pic}(R)})^{\otimes} \arrow[r, "\mathrm{Th}'_R"] & \mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}(\mathrm{LMod}_R)^{\otimes} \arrow[r, "\simeq"] & \mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}. \end{tikzcd}$$

We refer to this composition as the $\mathbb{E}_{n-1}$-monoidal system of $\mathrm{Th}_R(\xi_1)$-modules over B associated to ξ along π.

We will now prove that since the category B is a grouplike Kan complex, the map $\mathrm{Th}_R(\xi)^B$ factors through the $\mathbb{E}_{n-1}$-monoidal Kan complex $\mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes}$.

Proposition 5.3.3. *The $\mathbb{E}_{n-1}$-monoidal map*

$$\mathrm{Th}_R(\xi)^B : B^{\otimes} \to \mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}$$

factors through the inclusion $\mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes} \hookrightarrow \mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}$.

Proof. We first prove that the map factors through the subcategory $(\mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)})_{\mathrm{coCart}}$ as defined in Proposition 4.2.7; we recall that this is the subcategory of $\mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}$ spanned by coCartesian morphisms. Since $\mathrm{Th}_R(\xi)^B$ is $\mathbb{E}_{n-1}$-monoidal, it maps coCartesian morphisms to coCartesian morphisms, hence it is sufficient to prove that every morphism of $B^{\otimes}$ is coCartesian.

Proposition [Lur09, Prop. 2.4.2.4] states that every edge of $B^{\otimes}$ is coCartesian if and only if each fiber $B^{\otimes}_{\langle k\rangle}$ of the coCartesian fibration $B^{\otimes} \to \mathbb{E}_{n-1}^{\otimes}$ is a Kan complex. Since $B^{\otimes}$ is $\mathbb{E}_{n-1}$-monoidal, the fiber over $\langle k\rangle$ is equivalent to the k-fold Cartesian product of B, Proposition 2.2.1; and, by hypothesis, B is a Kan complex.

Now we want to prove that the image of $\mathrm{Th}_R(\xi)^B$ is contained in the $\mathbb{E}_{n-1}$-monoidal full subcategory of $(\mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)})_{\mathrm{coCart}}$ spanned by invertible objects, i.e., $\mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes}$. Let $b \in B$, we wish to prove that $\mathrm{Th}_R(\xi)^B(b)$ is an invertible left $\mathrm{Th}_R(\xi_1)$-module. By hypothesis, B is grouplike, therefore there exists an element $\overline{b} \in B$ which is homotopy inverse to b, that is

$$b \cdot \bar{b} \simeq 1_B, \quad \bar{b} \cdot b \simeq 1_B. \tag{$*$}$$

Applying the $\mathbb{E}_{n-1}$-monoidal functor $\mathrm{Th}_R(\xi)^B$ to the equivalences $(*)$ we obtain

$$\mathrm{Th}_R(\xi)^B(b) \underset{\mathrm{Th}_R(\xi_1)}{\otimes} \mathrm{Th}_R(\xi)^B(\bar{b}) \simeq \mathrm{Th}_R(\xi)^B(b \cdot \bar{b}) \simeq \mathrm{Th}_R(\xi)^B(1_B) \simeq \mathrm{Th}_R(\xi_1),$$

$$\mathrm{Th}_R(\xi)^B(\bar{b}) \underset{\mathrm{Th}_R(\xi_1)}{\otimes} \mathrm{Th}_R(\xi)^B(b) \simeq \mathrm{Th}_R(\xi)^B(\bar{b} \cdot b) \simeq \mathrm{Th}_R(\xi)^B(1_B) \simeq \mathrm{Th}_R(\xi_1),$$

this proves that $\mathrm{Th}_R(\xi)^B(b)$ is invertible. □

Proposition 5.3.3 was the last result necessary for the construction of the $\mathbb{E}_{n-1}$-monoidal system of invertible $\mathrm{Th}_R(\xi_1)$-modules over B associated to ξ along π

$$\mathrm{Th}_R(\xi)^B : B^{\otimes} \to \mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes}.$$

Now that we have defined a system of invertible $\mathrm{Th}_R(\xi_1)$-modules over B we can apply the Thom functor defined in Definition 4.3.5 to obtain an $\mathbb{E}_{n-1}$-algebra of $\mathrm{Th}_R(\xi_1)$-modules.

Definition 5.3.4. Let $\mathrm{Th}_R(\xi)^B$ be the system of invertible $\mathrm{Th}_R(\xi_1)$-modules defined above. We define the iterated Thom spectrum of ξ along π to be the $\mathbb{E}_{n-1}$-algebra of left $\mathrm{Th}_R(\xi_1)$-modules obtained by applying the functor M described in Theorem 4.3.1 to the following operadic map

$$B^{\otimes} \xrightarrow{\mathrm{Th}_R(\xi)^B} \mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes} \hookrightarrow \mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}.$$

We will denote this algebra by $\mathrm{ThTh}^{\pi}_R(\xi)$. If it is clear from context we will omit the fibration π from the notation.

5.4 Compatibility of the Thom spectrum and the iterated Thom spectrum

In the previous section, we have defined the iterated Thom spectrum associated with the $\mathbb{E}_n$-monoidal system of invertible R-modules $\xi : X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$ along a left $\mathbb{E}_{n+1}$-fibration $\pi : X^{\otimes} \to B^{\otimes}$. The goal of this section is to determine to what extent it is possible to recover the left R-module $\mathbb{E}_n$-algebra $\mathrm{Th}_R(\xi)$ associated with the original system of invertible R-modules ξ from the iterated Thom spectrum $\mathrm{ThTh}_R(\xi)$ defined in Definition 5.3.4.

We first observe that the ∞-category $\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}$ admits, in general, only an $\mathbb{E}_{n-1}$-monoidal structure; therefore one should not expect to be able to reconstruct the $\mathbb{E}_n$-ring spectrum $\mathrm{Th}_R(\xi)$ from the $\mathbb{E}_{n-1}$-ring spectrum $\mathrm{ThTh}_R(\xi)$. However, the main result of this section states that if we consider $\mathrm{ThTh}_R(\xi)$ as an $\mathbb{E}_{n-1}$-algebra in left R-modules, then it is equivalent to $\mathrm{Th}_R(\xi)$ considered as an $\mathbb{E}_{n-1}$-algebra in left R-modules.

Starting from the lax $\mathbb{E}_{n+1}$-monoidal pre-sheaf $\psi : B^{\otimes} \to \mathcal{S}^{\otimes}$ it is clear that we can reconstruct the fibration $\pi : X^{\otimes} \to B^{\otimes}$ via the monoidal unstraightening. Alternately, and similarly to the non-monoidal case, we can recover the $\mathbb{E}_{n+1}$-monoidal Kan complex $X^{\otimes}$ as a certain colimit of the pre-sheaf ψ.

We will prove that the image of the pre-sheaf ψ by the functor M defined in Theorem 4.3.1 is equivalent to the $\mathbb{E}_{n+1}$-monoidal Kan complex $X^{\otimes}$ considered as $\mathbb{E}_{n+1}$-algebra of $\mathcal{S}^{\otimes}$. The following proposition is the result of a personal exchange with Maxime Ramzi; we would like to thank Maxime for the support given.

Proposition 5.4.1. [Ram24] *Let $\mathcal{O}^{\otimes}$ be an ∞-operad, $p : B^{\otimes} \to \mathcal{O}^{\otimes}$, $p' : X^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-*monoidal *Kan complexes and let $\pi : X^{\otimes} \to B^{\otimes}$ be a left $\mathcal{O}$-fibration. Let $\psi : B^{\otimes} \to \mathcal{S}^{\otimes}$ be the lax $\mathcal{O}$-monoidal pre-sheaf that classifies the left fibration π. Then, $M(\psi)$ and X are equivalent as $\mathcal{O}$-algebras of $\mathcal{S}^{\otimes}$, where X is the pre-sheaf $X : \mathcal{O}^{\otimes} \to \mathcal{S}^{\otimes}$ that classifies the operadic structure of $X^{\otimes}$.*

Proof. From Corollary 4.3.2 we know that the functor M is left adjoint to the functor $(- \circ p)$ given by precomposing with the structure map $p : B^{\otimes} \to \mathcal{O}^{\otimes}$

$$\begin{tikzcd} \mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S}) \arrow[r, bend left, "M"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S}). \arrow[l, bend left, "(-\circ p)"] \end{tikzcd}$$

We will define another functor $L : \mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S})$ such that the image of the pre-sheaf ψ by L is exactly the operad map $X : \mathcal{O}^{\otimes} \to \mathcal{S}^{\otimes}$ that classifies the $\mathcal{O}$-monoidal structure of $X^{\otimes}$. Then, we will prove that the functor L is left adjoint to $(- \circ p)$ too, and by the uniqueness of left adjoints we will obtain the equivalence

$$\{\mathcal{O}^{\otimes} \xrightarrow{X} \mathcal{S}^{\otimes}\} = L(\psi) \simeq M(\psi)$$

of objects of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{S})$.

We start by constructing the functor L as the following composition

$$\begin{tikzcd}
\mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S}) \arrow[r, dashed, "L"] \arrow[d, "\mathrm{Un}_B"'] & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S}) \\
\mathrm{LFib}^{\mathcal{O}}(B) \arrow[r, "p_*"] & \mathrm{LFib}^{\mathcal{O}}(\mathcal{O}) \arrow[u, "\mathrm{St}_{\mathcal{O}}"']
\end{tikzcd}$$

where St and Un are the functors defined by the monoidal Grothendieck construction, and p_* is the functor that postcomposes left $\mathcal{O}$-fibrations with base $B^{\otimes}$ with the map $p : B^{\otimes} \to \mathcal{O}^{\otimes}$. Here it is crucial that $B^{\otimes}$ is an $\mathcal{O}$-monoidal Kan complex, otherwise postcomposition with p would have only produced coCartesian $\mathcal{O}$-fibrations instead of a left $\mathcal{O}$-fibrations.

The adjunction between the functor L and the functor $(- \circ p)$ will follow from the adjunction

$$\begin{tikzcd}
\mathrm{LFib}^{\mathcal{O}}(B) \arrow[r, bend left, "p_*"] \arrow[r, phantom, "\perp"] & \mathrm{LFib}^{\mathcal{O}}(\mathcal{O}) \arrow[l, bend left, "p^*"]
\end{tikzcd} \tag{$\star$}$$

between the functors p_* and p^* induced by the structure map $p : B^{\otimes} \to \mathcal{O}^{\otimes}$ on the categories of left $\mathcal{O}$-fibrations. In particular, we will define the counit of the adjunction $L \dashv (- \circ p)$ using the straightening/unstraightening equivalence on the counit of the adjunction $p_* \dashv p^*$.

First, we claim that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Alg}_{/\mathcal{O}}(\mathcal{S}) \arrow[r, "(-\circ p)"] \arrow[d, "\mathrm{Un}_{\mathcal{O}}"'] & \mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S}) \arrow[d, "\mathrm{Un}_B"] \\
\mathrm{LFib}^{\mathcal{O}}(\mathcal{O}) \arrow[r, "p^*"] & \mathrm{LFib}^{\mathcal{O}}(B),
\end{tikzcd}$$

that is to say $p^* \circ \mathrm{Un}_{\mathcal{O}} \simeq \mathrm{Un}_B(- \circ p)$.

Let us assume the claim to be true, and let $h : p_*p^* \to id$ be the counit transformation of the adjunction $p_* \dashv p^*$. We define the natural transformation $\bar{h} : L(- \circ p) \to id$, which we will then prove to be the counit of the adjunction $L \dashv (- \circ p)$, to be the morphism given by the following composition

$$L(- \circ p) = (\mathrm{St}_{\mathcal{O}}\, p_*\, \mathrm{Un}_B)(- \circ p) \xrightarrow{\simeq} \mathrm{St}_{\mathcal{O}}\, p_*p^*\, \mathrm{Un}_{\mathcal{O}} \xrightarrow{\mathrm{St}\, h\, \mathrm{Un}} id.$$

To prove that L is left adjoint to $(- \circ p)$ we have to show that for each $\phi \in \mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S})$ and $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S})$ applying the functor L and then postcomposing with the morphism given by the natural transformation $\bar{h}$ we obtain an equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S})}(\phi, A\circ p) \simeq \mathrm{Map}_{\mathrm{Alg}_{/\mathcal{O}}(\mathcal{S})}(L(\phi), A).$$

To prove this we consider the following chain of equivalences

$$\begin{aligned}\mathrm{Map}_{\mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S})}(\phi, A\circ p) &\simeq \mathrm{Map}_{\mathrm{LFib}^{\mathcal{O}}(B)}(\mathrm{Un}_B(\phi), p^*\mathrm{Un}_{\mathcal{O}}(A))\\ &\simeq \mathrm{Map}_{\mathrm{LFib}^{\mathcal{O}}(\mathcal{O})}(p_*\circ \mathrm{Un}_B(\phi), \mathrm{Un}_{\mathcal{O}}(A))\\ &\simeq \mathrm{Map}_{\mathrm{Alg}_{/\mathcal{O}}(\mathcal{S})}(L(\phi), A).\end{aligned}$$

The first equivalence is given by applying the unstraightening functor and using the claim $\mathrm{Un}_B(-\circ p)\simeq p^*\mathrm{Un}_{\mathcal{O}}$. The second equivalence is given by applying the functor p_* and then postcomposing with the morphism given by the counit h defined above. Since the functors p_* and p^* are adjoint the resulting map is an equivalence of mapping spaces. Finally, the third equivalence is given by applying the straightening functor. Let $f:\phi\to(A\circ p)$ be a morphism of $\mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S})$. We now prove that the image of f by the chain of equivalences corresponds to the morphism obtained by applying the functor L to f and then postcomposing it with the morphism given by $\bar{h}$.

We start by applying the unstraightening functor to f

$$\mathrm{Un}_B\phi \xrightarrow{\mathrm{Un}_B f} \mathrm{Un}_B(A\circ p).$$

The equivalence $\mathrm{Un}_B(-\circ p)\simeq p^*\mathrm{Un}_{\mathcal{O}}$ defines an equivalence of $\mathrm{LFib}^{\mathcal{O}}(B)$ from $\mathrm{Un}_B(A\circ p)$ to $p_*\mathrm{Un}_{\mathcal{O}}A$; let us call it γ. The image of f by the first equivalence of the chain is the morphism

$$\mathrm{Un}_B\phi \xrightarrow{\mathrm{Un}_B f} \mathrm{Un}_B(A\circ p) \xrightarrow{\gamma} p_*\mathrm{Un}_{\mathcal{O}}A.$$

The second equivalence of the chain consists of applying p^* and then postcomposing with the counit h.

$$p^*\mathrm{Un}_B\phi \xrightarrow{p^*\mathrm{Un}_B f} p^*\mathrm{Un}_B(A\circ p) \xrightarrow{p^*\gamma} p^*p_*\mathrm{Un}_{\mathcal{O}}A \xrightarrow{h\mathrm{Un}_{\mathcal{O}}} \mathrm{Un}_{\mathcal{O}}A.$$

Finally, we apply the straightening functor; this passage completes the chain of equivalences.

$$\mathrm{St}_{\mathcal{O}}p^*\mathrm{Un}_B\phi \xrightarrow{\mathrm{St}_{\mathcal{O}}p^*\mathrm{Un}_B f} \mathrm{St}_{\mathcal{O}}p^*\mathrm{Un}_B(A\circ p) \xrightarrow{\mathrm{St}_{\mathcal{O}}p^*\gamma} \mathrm{St}_{\mathcal{O}}p^*p_*\mathrm{Un}_{\mathcal{O}}A \xrightarrow{\mathrm{St}_{\mathcal{O}}h\mathrm{Un}_{\mathcal{O}}} \mathrm{St}_{\mathcal{O}}\mathrm{Un}_{\mathcal{O}}A \xrightarrow{\simeq} A.$$

We recognize that by definition of L and $\bar{h}$ the resulting morphism is exactly

$$(\bar{h}\circ -)Lf : L\phi \xrightarrow{Lf} L(A\circ p) \xrightarrow{\bar{h}} A.$$

It only remains to prove the claim, but this follows directly from the fact that the straightening/unstraightening equivalence

$$\mathrm{LFib}^{\mathcal{O}}(Z^{\otimes}) \simeq \mathrm{Alg}_{Z/\mathcal{O}}(\mathcal{S})$$

is natural in $Z^{\otimes}$ [GHN17, Corollary A.32]. $\square$

We established that is possible to reconstruct the $\mathbb{E}_n$-monoidal Kan complex $X^{\otimes}$ from the lax $\mathbb{E}_n$-monoidal pre-sheaf ψ via the functor M. The next step is to prove that we can reconstruct the $\mathbb{E}_n$-monoidal map $\xi : X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$ from the lax $\mathbb{E}_n$-monoidal lift $\xi^B : B^{\otimes} \to \mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)}$ defined in Construction 5.3.1.

Proposition 5.4.2. *Under the same hypotheses as Proposition 5.4.1, let* $C \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S})$ *and let* $\xi : X \to C$ *be a morphism of* $\mathcal{O}$*-algebras. Let* $\xi^B : B^{\otimes} \to \mathcal{S}^{\otimes}_{/C}$ *be a lax* $\mathcal{O}$*-monoidal lift of* ψ*; here we are using the same notation as in Construction 5.3.1. Then, after identifying* $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{S}_{/C}) \simeq \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S})_{/C}$*, we have*

$$M(\xi^B) \simeq \xi$$

as elements of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{S}_{/C})$

Proof. The argument is analogous to the one of Proposition 5.4.1. We will define a functor L such that $L(\xi^B) \simeq \xi$ and then prove that L is left adjoint to the functor $(- \circ p)$. The only difference is that in this case, the operadic maps are no longer lax $\mathcal{O}$-monoidal pre-sheaves and in order to pass to the categories of left $\mathcal{O}$-fibrations and use the adjunction $p_* \dashv p^*$ we have to first apply the equivalence $F : \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S}_{/C}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S})_{/C}$ defined after Lemma 4.3.3. In this way, we are able to consider the operadic lifts from $B^{\otimes}$ to $\mathcal{S}^{\otimes}_{/C}$ as lax monoidal pre-sheaves with source $B^{\otimes}$ over the pre-sheaf defined by the composition $B^{\otimes} \xrightarrow{p} \mathcal{O}^{\otimes} \xrightarrow{C} \mathcal{S}^{\otimes}$.

We start by constructing the functor L as the following composition

$$\begin{tikzcd}
\mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S}_{/C}) \arrow[rr, dashed, "L"] \arrow[d, "F"'] & & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S}_{/C}) \\
\mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S})_{/C\circ p} \arrow[d, "\mathrm{Un}_B"'] & & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S})_{/C} \arrow[u, "F^{-1}"'] \\
\mathrm{LFib}^{\mathcal{O}}(B)_{/p^*\mathrm{Un}_{\mathcal{O}}(C)} \arrow[r, "p_*"] & \mathrm{LFib}^{\mathcal{O}}(\mathcal{O})_{/p_*p^*\mathrm{Un}_{\mathcal{O}}(C)} \arrow[r, "(h_{\mathrm{Un}(C)}\circ -)"] & \mathrm{LFib}^{\mathcal{O}}(\mathcal{O})_{/\mathrm{Un}_{\mathcal{O}}(C)}, \arrow[u, "\mathrm{St}_{\mathcal{O}}"']
\end{tikzcd}$$

where St and Un are the monoidal straightening and unstraightening functors, the functors p_* and p^* are the morphisms of left $\mathcal{O}$-fibrations induced by the map $p : B^{\otimes} \to \mathcal{O}^{\otimes}$, and the functor $(h_{\mathrm{Un}(C)} \circ -)$ is given by postcomposing the objects of $\mathrm{LFib}^{\mathcal{O}}(\mathcal{O})_{/p_*p^*\mathrm{Un}_{\mathcal{O}}(C)}$ with the morphism $h_{\mathrm{Un}(C)} : p_*p^*\mathrm{Un}_O(C) \to \mathrm{Un}_O(C)$ defined by the counit of the adjunction $p_* \dashv p^*$ on the object $\mathrm{Un}_{\mathcal{O}}(C)$.

We observe that the composition corresponding to the bottom row of the diagram

$$\mathrm{LFib}^{\mathcal{O}}(B)_{/p^*\mathrm{Un}_O(C)} \xrightarrow{p_*} \mathrm{LFib}^{\mathcal{O}}(\mathcal{O})_{/p_*p^*\mathrm{Un}_O(C)} \xrightarrow{h_{\mathrm{Un}(C)}\circ -} \mathrm{LFib}^{\mathcal{O}}(\mathcal{O})_{/\mathrm{Un}_O(C)}$$

is itself a left adjoint functor as described in [Lur09, Lemma 5.2.5.2]; in particular, its right adjoint is the functor induced by p^* on the overcategories over the left $\mathcal{O}$-fibration $\mathrm{Un}_{\mathcal{O}}(C)$.

With an argument similar to the one of Proposition 5.4.1, we will define a natural transformation $\bar{h} : L(- \circ p) \to id$ which we will then prove to be the counit of the adjunction $L \dashv (- \circ p)$. Let $\widehat{h} : (h_{\mathrm{Un}(C)} \circ -)p_*\, p^* \to id$ be the counit of the adjunction $(h_{\mathrm{Un}(C)} \circ -)p_* \dashv p^*$. We define the morphism $\bar{h}$ to be the following composition

$$\begin{tikzcd}
L(- \circ p) \arrow[rr, dashed, "\bar{h}"] \arrow[d, equal] & & id \\
(F^{-1}\,\mathrm{St}_{\mathcal{O}}\,(h_{\mathrm{Un}(C)} \circ -)\, p_*\, \mathrm{Un}_B\, F)(- \circ p) \arrow[rr, "\simeq"] & & F^{-1}\,\mathrm{St}_{\mathcal{O}}\,(h_{\mathrm{Un}(C)} \circ -)p_*p^*\,\mathrm{Un}_{\mathcal{O}}\,F. \arrow[u, "F^{-1}\,\mathrm{St}\,\widehat{h}\,\mathrm{Un}\,F"']
\end{tikzcd}$$

In order to prove that L is left adjoint to $(- \circ p)$ we have to prove that for each $\xi^B \in \mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S}_{/C})$ and $\phi \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{S}_{/C})$ applying the functor L and then postcomposing with the morphism given by the natural transformation $\bar{h}$ produces an equivalence between mapping spaces

$$\mathrm{Map}_{\mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S}_{/C})}(\xi^B, \phi \circ p) \simeq \mathrm{Map}_{\mathrm{Alg}_{/\mathcal{O}}(\mathcal{S}_{/C})}(L(\xi^B), \phi). \tag{$\star$}$$

Instead of introducing new notation, we are treating ξ^B as a general element of $\mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S}_{/C})$. We will denote the pre-sheaf given by postcomposing it with the forgetful functor and its monoidal unstraightening as usual. We still need to introduce some notation related to the operadic map ϕ; we will denote by $Y : \mathcal{O}^{\otimes} \to \mathcal{S}^{\otimes}$ its associated pre-sheaf, with $\nu : Y \to C$ the image of ϕ by the functor F, and with $Y^{\otimes}$ the monoidal unstraightening of the pre-sheaf Y. We will now prove ($\star$) by a chain of equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S}_{/C})}(\xi^B, \phi \circ p) &\simeq \mathrm{Map}_{\mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S})_{/C\circ p}}(\psi \xrightarrow{\xi} C \circ p, Y \circ p \xrightarrow{\nu\circ p} C \circ p) \\
&\simeq \mathrm{Map}_{\mathrm{LFib}^{\mathcal{O}}(B)_{/\mathrm{Un}_B(C\circ p)}}(X^{\otimes} \xrightarrow{\mathrm{Un}_B(\xi)} \mathrm{Un}_B(C \circ p), \mathrm{Un}_B(Y \circ p) \xrightarrow{\mathrm{Un}_B(\nu\circ p)} \mathrm{Un}_B(C \circ p)) \\
&\simeq \mathrm{Map}_{\mathrm{LFib}^{\mathcal{O}}(B)_{/p^*C^{\otimes}}}(X^{\otimes} \xrightarrow{\mathrm{Un}_B(\xi)} p^*C^{\otimes}, p^*Y^{\otimes} \xrightarrow{p^*\mathrm{Un}_{\mathcal{O}}(\nu)} p^*C^{\otimes}) \\
&\simeq \mathrm{Map}_{\mathrm{LFib}^{\mathcal{O}}(\mathcal{O})_{/C^{\otimes}}}(p_*X^{\otimes} \xrightarrow{h_{\mathrm{Un}(C)}\circ p_*\mathrm{Un}_B(\xi)} C^{\otimes}, Y^{\otimes} \xrightarrow{\mathrm{Un}_{\mathcal{O}}(\nu)} C^{\otimes}) \\
&\simeq \mathrm{Map}_{\mathrm{Alg}_{/\mathcal{O}}(\mathcal{S}_{/C})}(L(\xi^B), \phi).
\end{aligned}$$

The first two equivalences are given by applying the functors F and Un_B respectively. Both of those functors are categorical equivalences; thus they induce equivalences of mapping spaces. The third equivalence is given by postcomposing with the equivalence of $\mathrm{LFib}^{\mathcal{O}}(B)$ defined by $\mathrm{Un}_B(- \circ p) \simeq p^*\mathrm{Un}_{\mathcal{O}}$. The fourth equivalence is given by applying the functor $(h_{\mathrm{Un}(C)} \circ -)p_*$ and then postcomposing with the morphism given by the natural transformation $\widehat{h}$ defined above. Since $(h_{\mathrm{Un}(C)} \circ -)p_*$ and p^* are adjoint the resulting process is an equivalence. The final equivalence is given by applying the straightening functor $\mathrm{St}_{\mathcal{O}}$ and the functor F^{-1}.

Let $g : \xi^B \to (Y \circ p)$ be a morphism of $\mathrm{Alg}_{B/\mathcal{O}}(\mathcal{S}_{/C})$. We will now prove that the image of g by the chain of equivalences is the morphism obtained by applying the functor L to g and then postcomposing it with the counit $\bar{h}$ defined above.

Applying the functor F to g we obtain the morphism

$$\begin{tikzcd} \psi \arrow[rr, "Fg"] \arrow[dr, "\xi"'] & & Y \circ p \arrow[dl, "\nu \circ p"] \\ & C \circ p. & \end{tikzcd}$$

We are abusing the notation by referring to the image of g by F as a morphism of the category $\mathrm{LFib}^{\mathcal{O}}(B)$ rather than the two-cell that fills the diagram; which corresponds to the actual image of g, since it is a morphism of the overcategory. We believe that this abuse of notation is reasonable since it makes the argument much easier to follow. However, one should keep in mind that the morphisms of overcategories that we will discuss correspond to two-cells. The image of the morphism after the second equivalence of the chain is

$$\begin{tikzcd} X^{\otimes} \arrow[rr, "\mathrm{Un}_B Fg"] \arrow[dr, "\mathrm{Un}_B \xi"'] & & \mathrm{Un}_B(Y \circ p) \arrow[dl, "\mathrm{Un}_B(\nu \circ p)"] \\ & \mathrm{Un}_B(C \circ p). & \end{tikzcd}$$

Now we postcompose with the morphism induced by the equivalence $\mathrm{Un}_B(- \circ p) \xrightarrow{\simeq} p^*\mathrm{Un}_{\mathcal{O}}$ on the overcategories

$$\begin{tikzcd} X^{\otimes} \arrow[rr, "Fg"] \arrow[dr, "\mathrm{Un}_B \xi"'] & & \mathrm{Un}_B(Y \circ p) \arrow[rr, "\simeq"] \arrow[dl, "\mathrm{Un}_B(\nu \circ p)"] & & p^*Y^{\otimes} \arrow[dl, "p^*\mathrm{Un}_{\mathcal{O}}\nu"] \\ & \mathrm{Un}_B(C \circ p) \arrow[rr, "\simeq"'] & & p^*C^{\otimes}. & \end{tikzcd}$$

Applying the functors p_* and $(h_{C^\otimes} \circ -)$ we obtain

$$\begin{tikzcd}
p_*X^\otimes \arrow[rr, "(h_{C^\otimes}\circ -)p_*\mathrm{Un}_B Fg"] \arrow[dr, "p_*\mathrm{Un}_B\xi"'] & & p_*\mathrm{Un}_B(Y \circ p) \arrow[rr, "\simeq"] \arrow[dl, "p_*\mathrm{Un}_B(\nu\circ p)"] & & p_*p^*Y^\otimes \arrow[dl, "p_*p^*\mathrm{Un}_B\nu"'] \arrow[dr] & \\
& p_*\mathrm{Un}_B(C \circ p) \arrow[rr, "\simeq"'] & & p_*p^*C^\otimes \arrow[rr, "h\mathrm{Un}_{\mathcal{O}}F"'] & & C^\otimes.
\end{tikzcd}$$

Postcomposing with the counit $\widehat{h}$ we obtain the image of g after the fourth equivalence of the chain

$$\begin{tikzcd}
p_*X^\otimes \arrow[rr, "(h_{C^\otimes}\circ -)p_*\mathrm{Un}_B Fg"] \arrow[dr, "p_*\mathrm{Un}_B\xi"'] & & p_*\mathrm{Un}_B(Y \circ p) \arrow[rr, "\simeq"] \arrow[dl, "p_*\mathrm{Un}_B(\nu\circ p)"] & & p_*p^*Y^\otimes \arrow[dl, "p_*p^*\mathrm{Un}_B\nu"'] \arrow[dr] \arrow[rr, "h\mathrm{Un}_{\mathcal{O}}F"] & & Y^\otimes \arrow[dl, "\mathrm{Un}_{\mathcal{O}}\nu"] \\
& p_*\mathrm{Un}_B(C \circ p) \arrow[rr, "\simeq"'] & & p_*p^*C^\otimes \arrow[rr, "h\mathrm{Un}_{\mathcal{O}}F"'] & & C^\otimes. &
\end{tikzcd}$$

We apply the straightening functor $\mathrm{St}_{\mathcal{O}}$

$$\begin{tikzcd}
X \arrow[rr, "\mathrm{St}_{\mathcal{O}}(h_{C^\otimes}\circ -)p_*\mathrm{Un}_B Fg"] \arrow[dr, "\mathrm{St}_{\mathcal{O}}p_*\mathrm{Un}_B\xi"'] & & \mathrm{St}_{\mathcal{O}}p_*\mathrm{Un}_B(Y \circ p) \arrow[rr, "\simeq"] \arrow[dl, "\mathrm{St}_{\mathcal{O}}p_*\mathrm{Un}_B(\nu\circ p)"] & & \mathrm{St}_{\mathcal{O}}p_*p^*Y^\otimes \arrow[dl, "\mathrm{St}_{\mathcal{O}}p_*p^*\mathrm{Un}_B\nu"'] \arrow[rr, "\mathrm{St}_{\mathcal{O}}h\mathrm{Un}_{\mathcal{O}}F"] & & Y \arrow[dl, "\mathrm{St}_{\mathcal{O}}\mathrm{Un}_{\mathcal{O}}\nu"] \\
& \mathrm{St}_{\mathcal{O}}p_*\mathrm{Un}_B(C \circ p) \arrow[rr, "\simeq"'] & & \mathrm{St}_{\mathcal{O}}p_*p^*C^\otimes \arrow[rr, "St_{\mathcal{O}}h\mathrm{Un}_{\mathcal{O}}F"'] & & C. &
\end{tikzcd}$$

Finally, we apply the homotopy inverse of F to the diagram. The resulting morphism is the image of g by the chain of equivalences. We recognize the composition $F^{-1}\mathrm{St}_{\mathcal{O}}(h_{C^\otimes} \circ -)p_*\mathrm{Un}_B F$ as the functor L

$$X \xrightarrow{\ Lg\ } F^{-1}\mathrm{St}_{\mathcal{O}}p_*\mathrm{Un}_B(Y \circ p) \xrightarrow{\ \simeq\ } \mathrm{St}_{\mathcal{O}}p_*p^*Y^\otimes \xrightarrow{F^{-1}\mathrm{St}_{\mathcal{O}}h\mathrm{Un}_{\mathcal{O}}F} Y$$

and conclude that the image of g by the chain of equivalences is indeed the morphism obtained by applying the functor L to g and then postcomposing it with the counit $\bar{h}$.

It only remains to prove that the image of the lax $\mathbb{E}_n$-monoidal lift ξ^B defined in Construction 5.3.1 by the functor L is equivalent to the morphism of algebras $\xi : X \to \mathrm{Pic}(R)$ considered as an element of $\mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{S}_{/\mathrm{Pic}(R)})$; in this case $C = \mathrm{Pic}(R)$. We defined $\xi^B : B^\otimes \to \mathcal{S}^\otimes_{/\mathrm{Pic}(R)}$ to be the lift of the pre-sheaf $\psi : B^\otimes \to \mathcal{S}^\otimes$ obtained by applying F^{-1} to the straightening of the morphism of left $\mathbb{E}_n$-fibrations $(\xi,\pi) : X^\otimes \to \mathrm{Pic}(R)^\otimes \times_{\mathbb{E}_n^\otimes} B^\otimes$, so $\mathrm{Un}_B \circ F(\xi^B) \simeq (\xi,\pi)$ as objects of $\mathrm{LFib}^{\mathbb{E}_k}(B)_{/p^*\mathrm{Pic}(R)}$. The image of the object (ξ,π) by the functors p_* and $(h_{\mathrm{Un}(C)} \circ -)$ corresponds to the composition

$$\begin{tikzcd}
X^\otimes \arrow[r, "(\xi{,}\pi)"] \arrow[dr, "p\circ\pi"'] & \mathrm{Pic}(R) \times_{\mathbb{E}_n^\otimes} B^\otimes \arrow[r, "pr_1"] & \mathrm{Pic}(R)^\otimes \arrow[dl] \\
& \mathbb{E}_n^\otimes &
\end{tikzcd}$$

considered as an object of left $\mathbb{E}_n$-fibrations with base $\mathbb{E}_n^{\otimes}$ over the left $\mathbb{E}_n$-fibration $\mathrm{Pic}(R)^{\otimes} \to \mathbb{E}_n^{\otimes}$. This is exactly the system of invertible R-modules $\xi : X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$, hence $h_{\mathrm{Un}(C)} \circ p_*(\xi,\pi) \simeq \xi$. It only remains to apply the straightening functor and the map F^{-1} to consider ξ as an $\mathbb{E}_n$-algebra of the $\mathbb{E}_n$-monoidal category $\mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)}$.

$$\begin{tikzcd}
\left\{ B^{\otimes} \xrightarrow{\xi^B} \mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)} \right\} \arrow[rr, mapsto, "L"] \arrow[d, mapsto, "F"'] & & \left\{ \mathbb{E}_n^{\otimes} \xrightarrow{\xi} \mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)} \right\} \\
\left\{ B^{\otimes} \underset{\mathrm{Pic}(R)\circ p}{\overset{\psi}{\underset{\Downarrow(\xi,\pi)}{\rightrightarrows}}} \mathcal{S}^{\otimes} \right\} \arrow[d, mapsto, "\mathrm{Un}_B"'] & & \left\{ \mathbb{E}_n^{\otimes} \underset{\mathrm{Pic}(R)}{\overset{X}{\underset{\Downarrow\xi}{\rightrightarrows}}} \mathcal{S}^{\otimes} \right\} \arrow[u, mapsto, "F^{-1}"'] \\
\left\{ \begin{array}{c} X^{\otimes} \xrightarrow{(\xi,\pi)} \mathrm{Pic}(R)^{\otimes} \times_{\mathbb{E}_n^{\otimes}} B^{\otimes} \\ \pi \searrow \quad B^{\otimes} \quad \swarrow pr_2 \end{array} \right\} \arrow[dr, mapsto, "p_*"'] & & \left\{ \begin{array}{c} X^{\otimes} \xrightarrow{(\xi,\pi)} \mathrm{Pic}(R)^{\otimes} \times_{\mathbb{E}_n^{\otimes}} B^{\otimes} \xrightarrow{pr_1} \mathrm{Pic}(R)^{\otimes} \\ p\circ\pi \searrow \quad \mathbb{E}_n^{\otimes} \quad \swarrow \end{array} \right\} \arrow[u, mapsto, "\mathrm{St}_{\mathcal{O}}"'] \\
& \left\{ \begin{array}{c} X^{\otimes} \xrightarrow{(\xi,\pi)} \mathrm{Pic}(R)^{\otimes} \times_{\mathbb{E}_n^{\otimes}} B^{\otimes} \\ p\circ\pi \searrow \quad \mathbb{E}_n^{\otimes} \quad \swarrow p\circ pr_2 \end{array} \right\}. \arrow[ur, mapsto, "h\circ -"'] &
\end{tikzcd}$$

□

We can now prove the main result of this section, the spectra $\mathrm{ThTh}_R(\xi)$ and $\mathrm{Th}_R(\xi)$ are equivalent as $\mathbb{E}_{n-1}$-algebras of the category of left R-modules. We will see that, by combining the previous results, the statement will follow from the commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\xi_1}(\mathcal{S}_{/\mathrm{Pic}(R)})) \arrow[r, "\mathrm{Th}'_R"] \arrow[d] & \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}(\mathrm{LMod}_R)) \arrow[d] \arrow[dr, "\simeq"] & \\
\mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathcal{S}_{/\mathrm{Pic}(R)}) \arrow[r, "\mathrm{Th}_R"] \arrow[d, "M"] & \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_R) \arrow[d, "M"] & \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}) \arrow[l, "H"'] \arrow[d, "M"] \\
\mathrm{Alg}_{/\mathbb{E}_{n-1}}(\mathcal{S}_{/\mathrm{Pic}(R)}) \arrow[r, "\mathrm{Th}_R"] & \mathrm{Alg}_{/\mathbb{E}_{n-1}}(\mathrm{LMod}_R) & \mathrm{Alg}_{/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}), \arrow[l, "H"']
\end{tikzcd}$$

where the unlabeled arrows are forgetful functors and H is the functor induced on the $\mathbb{E}_{n-1}$-algebras by the change of algebra functor described in [Lur17, Section 7.1.3]. Before proving that the diagram commutes, let us justify why the result will follow from its commutativity.

In Construction 5.3.1 we defined the lax $\mathbb{E}_{n+1}$-monoidal lift $\xi^B : B^\otimes \to \mathcal{S}^\otimes_{/\mathrm{Pic}(R)}$. Considering the image of ξ^B by the pullback along $\mathbb{E}^\otimes_{n-1} \hookrightarrow \mathbb{E}^\otimes_{n+1}$ we obtain an object of the middle-left category, we will call this map also ξ^B. Then we have proven that the operadic map induced by ξ^B on the categories of left modules, an object of the top-left category, is an $\mathbb{E}_{n-1}$-monoidal map.

We defined the $\mathbb{E}_{n-1}$-monoidal map $\mathrm{Th}_R(\xi)^B$, an object of the middle-right category, by postcomposing the map ξ^B first with the map induced by the Thom functor Th_R on the category of left modules, and then with the equivalence $\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}(\mathrm{LMod}_R)^\otimes \to \mathrm{LMod}^\otimes_{\mathrm{Th}_R(\xi_1)}$. We have then proven that the map $\mathrm{Th}_R(\xi)^B$ factors through the subcategory $\mathrm{Pic}(\mathrm{Th}_R(\xi_1))^\otimes$.

Finally, we defined the iterated Thom spectrum $\mathrm{ThTh}_R(\xi)$, an object of the bottom-right category, by applying the functor M to the following composition of lax $\mathbb{E}_{n-1}$-monoidal maps

$$B^\otimes \xrightarrow{\mathrm{Th}_R(\xi)^B} \mathrm{Pic}(\mathrm{Th}_R(\xi_1))^\otimes \hookrightarrow \mathrm{LMod}^\otimes_{\mathrm{Th}_R(\xi_1)}.$$

Starting with the object $\xi^B \in \mathrm{Alg}_{B/\mathbb{E}_n}(\mathrm{LMod}_{\xi_1}(\mathcal{S}_{/\mathrm{Pic}(R)}))$, this process corresponds to following the top-most and then the right-most arrows of the diagram.

On the other hand, starting again from the map ξ^B, top-left category, we can compose it with the forgetful functor to obtain the lax $\mathbb{E}_{n-1}$-monoidal map $\xi^B : B^\otimes \to \mathcal{S}^\otimes_{/\mathrm{Pic}(R)}$, middle-left category. We have proven in Proposition 5.4.2 that by applying the functor M to the map ξ^B one recovers the original system of invertible R-modules ξ as an $\mathbb{E}_{n-1}$-algebra object of $\mathcal{S}^\otimes_{/\mathrm{Pic}(R)}$, bottom-left category. The Thom spectrum $\mathrm{Th}_R(\xi)$, an object of the bottom-middle category, can be obtained by applying the Thom functor to the algebra ξ. Starting again from the top-left category, this process corresponds to following the left-most and then the bottom-most arrows of the diagram.

We will use the following lemma to solve most of the commutative squares.

Lemma 5.4.3. *Let $\mathcal{B}^\otimes$, $\mathcal{C}^\otimes$ and $\mathcal{D}^\otimes$ be $\mathcal{O}$-monoidal categories. Suppose that we have two operadic maps L and R which are left and right adjoint relative over $\mathcal{O}^\otimes$ [Lur17, Def. 7.3.2.2]*

$$\begin{tikzcd} \mathcal{C}^\otimes \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] \arrow[dr] & & \mathcal{D}^\otimes \arrow[ll, bend left, "R"] \arrow[dl] \\ & \mathcal{O}^\otimes & \end{tikzcd}.$$

Then, the diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}) \arrow[r, "(L\circ -)"] \arrow[d, "M"] & \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{D}) \arrow[d, "M"] \\ \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \arrow[r, "(L\circ -)"] & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}) \end{tikzcd} \qquad (\star)$$

commutes. Where M is the functor defined in Theorem 4.3.1.

Proof. We first prove that the functors induced by L and R on the categories of $\mathcal{O}$-algebra objects and $\mathcal{B}$-algebra objects are adjoint. We will prove it for the $\mathcal{B}$-algebras, the procedure for the $\mathcal{O}$-algebras is analogous. Postcomposition with the functors L and R form an adjunction on the categories of functors over $\mathcal{O}^{\otimes}$

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{B}^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r, bend left, "(L\circ -)"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{B}^{\otimes}, \mathcal{D}^{\otimes}). \arrow[l, bend left, "(R\circ -)"] \end{tikzcd}$$

This follows from the fact that adjunctions are closed under exponentiations and then considering the adjunction induced on the overcategories of functors over $\mathcal{O}^{\otimes}$ [Lur09, Prop. 5.2.5.1].

We recall that the category $\mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{D})$ is the full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{B}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by the operadic maps. In particular, for each $\psi, \phi \in \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{D})$ we have

$$\mathrm{Map}_{\mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{D})}(\psi, \phi) \simeq \mathrm{Map}_{\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{B}^{\otimes}\mathcal{D}^{\otimes})}(\psi, \phi).$$

Hence it is immediate to check that the functors $(L \circ -)$ and $(R \circ -)$ restricted to the categories of $\mathcal{B}$-algebras are left and right adjoint. Let $\phi \in \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{D})$ and $\gamma \in \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C})$

$$\begin{aligned} \mathrm{Map}_{\mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{D})}(L \circ \gamma, \phi) &\simeq \mathrm{Map}_{\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{B}^{\otimes}, \mathcal{D}^{\otimes})}(L \circ \gamma, \phi) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{B}^{\otimes}, \mathcal{C}^{\otimes})}(\gamma, R \circ \phi) \\ &\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C})}(\gamma, R \circ \phi). \end{aligned}$$

Both the functors M and $(L \circ -)$ admit right adjoints, so the diagram $(\star)$ commutes if and only if the diagram of right adjoint functors commutes

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}) & \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{D}) \arrow[l, "(R\circ -)"'] \\ \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \arrow[u, "(-\circ p)"] & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}). \arrow[l, "(R\circ -)"] \arrow[u, "(-\circ p)"'] \end{tikzcd}$$

Since the horizontal functors of the diagram are given by postcomposing with the map $R : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ and the vertical functors are given by precomposing with the map $p : B^{\otimes} \to \mathcal{O}^{\otimes}$ the diagram commutes. $\square$

Theorem 5.4.4. *Let R be an $\mathbb{E}_m$-ring spectrum, $\pi : X^{\otimes} \to B^{\otimes}$ be an essentially surjective left $\mathbb{E}_{n+1}$-fibration of grouplike Kan complexes with $2 < n+1 \leq m$, and $\xi : X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$ be an $\mathbb{E}_n$-monoidal system of invertible R-modules. Then,*

(1) There exists an $\mathbb{E}_{n-1}$-monoidal system of invertible $\mathrm{Th}_R(\xi_1)$-modules $\mathrm{Th}_R(\xi)^B : B^{\otimes} \to \mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes}$.

(2) The Thom spectrum of the system $\mathrm{Th}_R(\xi)^B$ defines an $\mathbb{E}_{n-1}$-algebra $\mathrm{ThTh}_R(\xi)$ of $\mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}$ that we denote as the iterated Thom spectrum of ξ along π.

(3) The iterated Thom spectrum $\mathrm{ThTh}_R(\xi)$, as an $\mathbb{E}_{n-1}$-algebra of left R-modules, is equivalent to the $\mathrm{Th}_R(\xi)$ considered as an $\mathbb{E}_{n-1}$-algebra of left R-modules.

Proof. Implications (1) and (2) follow from the constructions of Section 5.3.

Part (3) follows from Proposition 5.4.2 and the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\xi_1}(\mathcal{S}_{/\mathrm{Pic}(R)})) \arrow[r, "\mathrm{Th}'_R"] \arrow[d] & \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}(\mathrm{LMod}_R)) \arrow[d] \arrow[dr, "\simeq"] & \\
\mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathcal{S}_{/\mathrm{Pic}(R)}) \arrow[r, "\mathrm{Th}_R"] \arrow[d, "M"] & \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_R) \arrow[d, "M"] & \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}) \arrow[l, "H"'] \arrow[d, "M"] \\
\mathrm{Alg}_{/\mathbb{E}_{n-1}}(\mathcal{S}_{/\mathrm{Pic}(R)}) \arrow[r, "\mathrm{Th}_R"] & \mathrm{Alg}_{/\mathbb{E}_{n-1}}(\mathrm{LMod}_R) & \mathrm{Alg}_{/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}). \arrow[l, "H"']
\end{tikzcd}$$

We will now prove square by square that the diagram commutes:

- The top-left square

$$\begin{tikzcd}
\mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\xi_1}(\mathcal{S}_{/\mathrm{Pic}(R)})) \arrow[r, "\mathrm{Th}'_R"] \arrow[d] & \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}(\mathrm{LMod}_R)) \arrow[d] \\
\mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathcal{S}_{/\mathrm{Pic}(R)}) \arrow[r, "\mathrm{Th}_R"] & \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_R)
\end{tikzcd}$$

  commutes since we have defined Th'_R to be the functor induced by Th_R on the categories of left modules.

- For the bottom-left square

$$\begin{tikzcd} \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathcal{S}_{/\mathrm{Pic}(R)}) \arrow[r, "\mathrm{Th}_R"] \arrow[d, "M"] & \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_R) \arrow[d, "M"] \\ \mathrm{Alg}_{/\mathbb{E}_{n-1}}(\mathcal{S}_{/\mathrm{Pic}(R)}) \arrow[r, "\mathrm{Th}_R"] & \mathrm{Alg}_{/\mathbb{E}_{n-1}}(\mathrm{LMod}_R), \end{tikzcd}$$

we observe that the $\mathbb{E}_{n-1}$-monoidal functor $\mathrm{Th}_R : \mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)} \to \mathrm{LMod}^{\otimes}_R$ induces a colimit preserving functor of presentable categories $(\mathrm{Th}_R)_{\langle 1\rangle} : \mathcal{S}_{/\mathrm{Pic}(R)} \to \mathrm{LMod}_R$ on the underlying categories. So there exists a functor $G_{\langle 1\rangle} : \mathrm{LMod}_R \to \mathcal{S}_{/\mathrm{Pic}(R)}$ which is right adjoint to $(\mathrm{Th}_R)_{\langle 1\rangle}$. Applying [Lur17, Corollary 7.3.2.7] we can extend the functor $G_{\langle 1\rangle}$ to an operadic map over $\mathbb{E}^{\otimes}_{n-1}$

$$\begin{tikzcd} \mathrm{LMod}^{\otimes}_R \arrow[rr, "G"] \arrow[dr] & & \mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)} \arrow[dl] \\ & \mathbb{E}^{\otimes}_{n-1} & \end{tikzcd}$$

which is right adjoint to Th_R relative to $\mathbb{E}^{\otimes}_{n-1}$. We can apply Lemma 5.4.3 to conclude that the square commutes.

- The top-right triangle

$$\begin{tikzcd} \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}(\mathrm{LMod}_R)) \arrow[d] \arrow[dr, "\simeq"] & \\ \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_R) & \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}) \arrow[l] \end{tikzcd}$$

commutes by definition of the change of algebra functor $\mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)} \to \mathrm{LMod}^{\otimes}_R$ [Lur17, Section 7.1.3];

- And finally, to prove the commutativity of the bottom-right square

$$\begin{tikzcd} \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_R) \arrow[d, "M"] & \mathrm{Alg}_{B/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}) \arrow[l, "H"'] \arrow[d, "M"] \\ \mathrm{Alg}_{/\mathbb{E}_{n-1}}(\mathrm{LMod}_R) & \mathrm{Alg}_{/\mathbb{E}_{n-1}}(\mathrm{LMod}_{\mathrm{Th}_R(\xi_1)}), \arrow[l, "H"'] \end{tikzcd}$$

we apply again Lemma 5.4.3. By [Lur17, Prop. 7.1.2.6] the change of algebra functor $\mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}\mathrm{LMod}^{\otimes}_R$ induces a small-colimit preserving functor on the underlying presentable categories. Hence, the functor induced on the underlying categories admits a right adjoint functor $G_{\langle 1\rangle}$. Applying [Lur17, Corollary 7.3.2.7] we extend the functor $G_{\langle 1\rangle}$ to an operadic map over $\mathbb{E}^{\otimes}_{n-1}$ which is right adjoint to the change of algebra functor. The commutativity of the square follows from Lemma 5.4.3.

□

As an immediate application for Theorem 5.4.4, we can consider the theory of orientations of Thom spectra discussed in [ACB19, Section 3.2] on the system $\mathrm{Th}_R(\xi)^B$ to recover the iterated Thom spectrum version of Corollary 4.4.3.

Corollary 5.4.5. *Under the hypotheses of Theorem 5.4.4, there exists an equivalence of* $\mathbb{E}_{n-1}$*-algebras of left* $\mathrm{Th}_R(\xi_1)$*-modules*

$$\mathrm{Ind}_{\mathrm{Th}_R(\xi_1)}^{\mathrm{Th}_R(\xi)}(\mathrm{Th}_R(\xi)) \simeq \mathrm{Ind}_{\mathbb{S}}^{\mathrm{Th}_R(\xi)}(\Sigma_+^\infty B),$$

where Ind *are the functors obtained from the map described in [Lur17, Prop. 7.1.2.6]. The equivalence induces on the underlying* $\mathrm{Th}_R(\xi_1)$*-modules the following equivalence*

$$\mathrm{Th}_R(\xi) \underset{\mathrm{Th}_R(\xi_1)}{\otimes} \mathrm{Th}_R(\xi) \simeq \mathrm{Th}_R(\xi) \underset{\mathbb{S}}{\otimes} \Sigma_+^\infty B.$$

Proof. Let $\mathrm{Th}_R(\xi)^B : B^\otimes \to \mathrm{Pic}(\mathrm{Th}_R(\xi_1))^\otimes$ be the $\mathbb{E}_{n-1}$-monoidal system of invertible $\mathrm{Th}_R(\xi_1)$-modules of Theorem 5.4.4. Since $B^\otimes$ is grouplike, the equivalence $\mathrm{Th}\mathrm{Th}_R(\xi) \to \mathrm{Th}_R(\xi)$ provides an $\mathbb{E}_{n-1}$-$\mathrm{Th}_R(\xi)$-orientation of the system $\mathrm{Th}_R(\xi)^B$. Applying [ACB19, Prop. 3.16] to the $\mathrm{Th}_R(\xi)$-orientation we obtain the following equivalence of $\mathbb{E}_{n-1}$-algebras of left $\mathrm{Th}_R(\xi)$-modules

$$\mathrm{Ind}_{\mathrm{Th}_R(\xi_1)}^{\mathrm{Th}_R(\xi)}(\mathrm{Th}_R(\xi)) \simeq \mathrm{Ind}_{\mathrm{Th}_R(\xi_1)}^{\mathrm{Th}_R(\xi)}(\mathrm{Th}\mathrm{Th}_R(\xi)) \simeq \mathrm{Ind}_{\mathbb{S}}^{\mathrm{Th}_R(\xi)}(\Sigma_+^\infty B).$$

Applying the change of algebras induced by the map of $\mathbb{E}_{n-1}$-algebras $\mathrm{Th}_R(\xi_1) \to \mathrm{Th}_R(\xi)$ we obtain the required equivalence of $\mathrm{Th}_R(\xi_1)$-modules. □

5.5 Examples

As we mentioned at the end of Chapter 4, the main advantage of our construction compared with Beardsley's relative Thom spectrum is that we are no longer assuming the Kan complexes to be reduced. Let $X^\otimes$ be an $\mathbb{E}_{n+1}$-monoidal Kan complex, in Proposition 2.4.5 we have proven that the projection on the path components of X admits the structure of an $\mathbb{E}_{n+1}$-monoidal map $\pi_0 : X^\otimes \to \pi_0(X)^\otimes$ that makes it a left $\mathbb{E}_{n+1}$-fibration. This allows us to produce many examples of iterated Thom spectra since we can associate to each $\mathbb{E}_n$-monoidal system of invertible R-modules $\xi : X^\otimes \to \mathrm{Pic}(R)^\otimes$ the system of invertible $\mathrm{Th}_R(\xi_1)$-modules over $\pi_0(X)$ of ξ along the projection on the path components π_0.

Corollary 5.5.1. *Let R be an $\mathbb{E}_m$-ring spectrum, $X^\otimes$ an $\mathbb{E}_{n+1}$-monoidal grouplike Kan complex with $2 < n+1 \leq m$, and $\xi : X^\otimes \to \mathrm{Pic}(R)^\otimes$ an $\mathbb{E}_n$-monoidal system of invertible R-modules. Then:*

(1) *There exists an* $\mathbb{E}_{n-1}$*-monoidal system of invertible* $\mathrm{Th}_R(\xi_1)$*-modules* $\mathrm{Th}_R(\xi)^{\pi_0(X)} : \pi_0(X)^{\otimes} \to \mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes}$.

(2) *The Thom spectrum of the system* $\mathrm{Th}_R(\xi)^{\pi_0(X)}$ *defines an* $\mathbb{E}_{n-1}$*-algebra* $\mathrm{ThTh}_R^{\pi_0}(\xi)$ *of* $\mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}$.

(3) *The iterated Thom spectrum* $\mathrm{ThTh}_R^{\pi_0}(\xi)$*, as an* $\mathbb{E}_{n-1}$*-algebra of left* R*-modules, is equivalent to the* $\mathrm{Th}_R(\xi)$ *considered as an* $\mathbb{E}_{n-1}$*-algebra of left* R*-modules.*

Proof. Let $\pi_0 : X^{\otimes} \to \pi_0(X)^{\otimes}$ be the left $\mathbb{E}_{n+1}$-fibration of grouplike Kan complexes defined in Proposition 2.4.5. The result follows directly from Theorem 5.4.4 applied to π_0. □

Example 5.5.2. The motivating example for the construction of the iterated Thom spectrum is the $\mathbb{E}_\infty$-monoidal spherical fibration $J_{\mathrm{gp}} : (\mathbb{Z} \times \mathrm{BU})^{\otimes} \to \mathrm{Pic}(\mathbb{S})^{\otimes}$ given by the group-completion of the so-called J map, [Hop18]. The spherical fibration J_{gp} presents the periodic complex cobordism spectrum $\mathrm{MUP} \simeq \bigvee_{n\in\mathbb{Z}} \Sigma^{2n}\mathrm{MU}$ as an $\mathbb{E}_\infty$-Thom spectrum. Since we can express the ∞-operad $\mathbb{E}_\infty^{\otimes}$ as the tensor product of $\mathbb{E}_\infty^{\otimes}$ and $\mathbb{E}_1^{\otimes}$, applying Corollary 5.5.1 to $J_{\mathrm{gp}} : (\mathbb{Z} \times \mathrm{BU})^{\otimes} \to \mathrm{Pic}(\mathbb{S})^{\otimes}$ along the projection on the path components $\pi_0 : (\mathbb{Z} \times \mathrm{BU})^{\otimes} \to \mathbb{Z}^{\otimes}$ we obtain:

- an $\mathbb{E}_\infty$-monoidal system of invertible MU-modules $\mathrm{Th}_{\mathbb{S}}(J_{\mathrm{gp}})^{\mathbb{Z}} : \mathbb{Z}^{\otimes} \to \mathrm{Pic}(\mathrm{MU})^{\otimes}$.
- and an $\mathbb{E}_\infty$-algebra of left MU-modules $\mathrm{ThTh}_{\mathbb{S}}^{\pi_0}(J_{\mathrm{gp}})$, such that considering its $\mathbb{E}_\infty$ underlying ring spectrum we recover the $\mathbb{E}_\infty$-ring spectrum $\mathrm{Th}_{\mathbb{S}}(J_{\mathrm{gp}}) \simeq \mathrm{MUP}$.

We do not necessarily need $X^{\otimes}$ to be a grouplike Kan complex to construct the iterated Thom spectrum along the projection on its path components. We will prove that Corollary 5.5.1 still holds if we assume that the Kan complex $X^{\otimes}$ is just a replete Kan complex. Having the possibility to apply the iterated Thom spectrum construction to replete Kan complexes is particularly relevant. This is because systems of invertible modules over replete Kan complexes play an important role in the latest developments of the theory of topological logarithmic structures.

Definition 5.5.3. Let $X^{\otimes}$ be an $\mathbb{E}_{n+1}$-monoidal Kan complex. We say that $X^{\otimes}$ is replete if the following square is a pullback square

$$\begin{tikzcd} X^{\otimes} \arrow[r, "\gamma"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner"] & X^{\otimes}_{\mathrm{gp}} \arrow[d, "\pi_{gp}"] \\ \pi_0(X)^{\otimes} \arrow[r, "\gamma"] & \pi_0(X)^{\otimes}_{\mathrm{gp}}, \end{tikzcd} \tag{$\star$}$$

where the horizontal arrows are the group completions of $X^{\otimes}$ and $\pi_0(X)^{\otimes}$.

A pre-log symmetric spectrum (A, M, α), as defined in [Rog09, Def. 7.1], consists of: a symmetric ring spectrum A, a M commutative monoid of the category of $\mathcal{J}$-spaces, and a map of commutative monoids $\alpha : M \to \Omega^{\bullet}_{\otimes} A$. We point out that specifying the map α is equivalent to specifying its adjoint $\mathbb{S}^{\mathcal{J}}[M] \to A$. For technical reasons, one usually assumes the space M to be replete. For an in-depth exposition of the subject, we refer the reader to [Rog09]; where J. Rognes originally defined topological logarithmic structures and log THH.

In recent years J. Rognes, S. Sagave, and C. Schlichtkrull published a series of two papers presenting many interesting results on log THH; [RSS18] and [RSS15]. In particular, in [RSS18] the authors computed log THH in some important examples. A key step of their argument is computing the homology of the spectrum $\mathbb{S}^{\mathcal{J}}[M]$ by recognizing it as the Thom spectrum of a system of invertible $\mathbb{S}$-modules of the form

$$M_{h\mathcal{J}} \to \mathrm{B}\mathcal{J} \xrightarrow{\simeq} Q\mathbb{S}^0 \to \mathbb{Z} \times \mathrm{BO} \to \mathrm{Pic}(\mathbb{S}).$$

Considering the effectiveness of this approach; the authors suggested extending the classical definition of pre-log symmetric spectrum by considering Thom spectra over M as the source of topological pre-log structures instead of $\mathbb{S}^{\mathcal{J}}[M]$.

Corollary 5.5.4. *Let R be an $\mathbb{E}_m$-ring spectrum, $X^{\otimes}$ an $\mathbb{E}_{n+1}$-monoidal replete Kan complex with $2 < n+1 \leq m$, and $\xi : X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$ an $\mathbb{E}_n$-monoidal system of invertible R-modules. Then:*

(1) There exists an $\mathbb{E}_{n-1}$-monoidal system of invertible $\mathrm{Th}_R(\xi_1)$-modules $\mathrm{Th}_R(\xi)^{\pi_0(X)} : \pi_0(X)^{\otimes} \to \mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes}$;

(2) The system $\mathrm{Th}_R(\xi)^{\pi_0(X)}$ defines an $\mathbb{E}_{n-1}$-algebra $\mathrm{ThTh}^{\pi_0}_R(\xi)$ of $\mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}$;

(3) The iterated Thom spectrum $\mathrm{ThTh}^{\pi_0}_R(\xi)$, as an $\mathbb{E}_{n-1}$-algebra of left R-modules, is equivalent to the $\mathrm{Th}_R(\xi)$ considered as an $\mathbb{E}_{n-1}$-algebra of left R-modules.

Proof. Let $X^{\otimes}$ be a replete $\mathbb{E}_{n+1}$-monoidal Kan complex and let $\xi : X^{\otimes} \to \mathrm{Pic}(R)^{\otimes}$ be an $\mathbb{E}_n$-monoidal system of invertible R-modules. Since $\mathrm{Pic}(R)^{\otimes}$ is grouplike the system ξ factors through the group completion $\gamma : X^{\otimes} \to X^{\otimes}_{\mathrm{gp}}$

$$\begin{tikzcd} X^{\otimes} \arrow[rr, "\xi"] \arrow[dr, "\gamma"'] & & \mathrm{Pic}(R)^{\otimes} \\ & X^{\otimes}_{\mathrm{gp}}. \arrow[ur, "\xi_{\mathrm{gp}}"'] & \end{tikzcd}$$

Let $\pi : X^{\otimes} \to \pi_0(X)^{\otimes}$ be the left $\mathbb{E}_{n-1}$-fibration defined in Proposition 2.4.5 and $\psi : X^{\otimes} \to \pi_0(X)^{\otimes}$ be the lax $\mathbb{E}_n$-monoidal pre-sheaf that classifies the left $\mathbb{E}_n$-fibration π. Since the square $(\star)$ is a pullback diagram, we can use Lemma 2.4.6 to prove that the pre-sheaf ψ is the composition of the group completion $\gamma : \pi_0(X)^{\otimes} \to \pi_0(X)^{\otimes}_{\mathrm{gp}}$ and the lax $\mathbb{E}_n$-monoidal pre-sheaf ψ_{gp} that classifies the left fibration $\pi_{\mathrm{gp}} : X^{\otimes}_{\mathrm{gp}} \to \pi_0(X)^{\otimes}_{\mathrm{gp}}$:

$$\begin{tikzcd}
\pi_0(X)^{\otimes} \arrow[rr, "\psi"] \arrow[dr, "\gamma"'] & & \mathcal{S}^{\otimes} \\
& \pi_0(X)^{\otimes}_{\mathrm{gp}}. \arrow[ur, "\psi_{\mathrm{gp}}"'] &
\end{tikzcd}$$

Now we can use the map ξ to produce an $\mathbb{E}_n$-monoidal lift of ψ to the category $\mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)}$ as described in Construction 5.3.1. We wish to show that the lift $\xi^{\pi_0(X)} : \pi_0(X)^{\otimes} \to \mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)}$ factors as the composition of the group completion γ and the lift $\xi^{\pi_0(X)_{\mathrm{gp}}}_{\mathrm{gp}} : \pi_0(X)^{\otimes}_{\mathrm{gp}} \to \mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)}$ given by the $\mathbb{E}_n$-monoidal map ξ_{gp}

$$\begin{tikzcd}
& & \mathcal{S}^{\otimes}_{/\mathrm{Pic}(R)} \arrow[d] \\
\pi_0(X)^{\otimes} \arrow[r, "\gamma"] \arrow[urr, bend left, "\xi^{\pi_0(X)}"] & \pi_0(X)^{\otimes}_{\mathrm{gp}} \arrow[r, "\psi_{\mathrm{gp}}"] \arrow[ur, "\xi^{\pi_0(X)_{\mathrm{gp}}}_{\mathrm{gp}}"] & \mathcal{S}^{\otimes}.
\end{tikzcd}$$

We observe that the two lifts $\xi^{\pi_0(X)}$ and $\xi^{\pi_0(X)_{\mathrm{gp}}}_{\mathrm{gp}} \circ \gamma$ are given by applying Lemma 4.3.3 to the straightenings of the following morphisms of left $\mathbb{E}_n$-fibrations with base $\pi_0(X)^{\otimes}$

$$\begin{tikzcd}
X^{\otimes} \arrow[rr, "{(\xi,\pi)}"] \arrow[dr, "\pi"'] & & \mathrm{Pic}(R)^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \pi_0(X)^{\otimes} \arrow[dl, "pr_2"] \\
& \pi_0(X)^{\otimes}, &
\end{tikzcd}
\qquad
\begin{tikzcd}
X^{\otimes} \arrow[rr, "{\gamma^*(\xi_{\mathrm{gp}},\pi_{\mathrm{gp}})}"] \arrow[dr, "\pi"'] & & \mathrm{Pic}(R)^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \pi_0(X)^{\otimes} \arrow[dl, "pr_2"] \\
& \pi_0(X)^{\otimes}. &
\end{tikzcd}$$

To show that the two lifts $\xi^{\pi_0(X)_{\mathrm{gp}}}_{\mathrm{gp}} \circ \gamma$ and $\xi^{\pi_0(X)}$ are equivalent is sufficient to prove that the following is a pullback diagram

$$\begin{tikzcd}
X^{\otimes} \arrow[r, "\gamma"] \arrow[d, "{(\xi,\pi)}"'] \arrow[dr, phantom, "\lrcorner"] & X^{\otimes}_{\mathrm{gp}} \arrow[d, "{(\xi_{\mathrm{gp}},\pi_{\mathrm{gp}})}"] \\
\mathrm{Pic}(R)^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \pi_0(X)^{\otimes} \arrow[r, "{(id,\gamma)}"] & \mathrm{Pic}(R)^{\otimes} \underset{\mathbb{E}_n^{\otimes}}{\times} \pi_0(X)^{\otimes}_{\mathrm{gp}};
\end{tikzcd}$$

but this is a consequence of the fact that $(\star)$ is a pullback diagram. Passing to the categories of $\mathbb{E}_n$-modules; as a consequence of our previous discussion, the map induced by the lift $\xi^{\pi_0(X)}$ is equivalent to the composition of the $\mathbb{E}_n$-monoidal map induced by $\xi^{\pi_0(X)_{\mathrm{gp}}}_{\mathrm{gp}}$ on the categories of $\mathbb{E}_n$-modules and the $\mathbb{E}_n$-monoidal map induced by the group completion γ. Applying Proposition 2.2.12 we conclude that $\xi^{\pi_0(X)} : \pi_0(X)^{\otimes} \to$

$\mathrm{Mod}_{\xi_1}^{\mathbb{E}_n}(\mathcal{S}_{/\mathrm{Pic}(R)})^{\otimes}$ is $\mathbb{E}_n$-monoidal

$$\begin{tikzcd} \pi_0(X)^{\otimes} \simeq \mathrm{Mod}_1^{\mathbb{E}_n}(\pi_0(X))^{\otimes} \arrow[rr, "\xi^{\pi_0(X)}"] \arrow[dr, "\gamma"'] & & \mathrm{Mod}_{\xi_1}^{\mathbb{E}_n}(\mathcal{S}_{/\mathrm{Pic}(R)})^{\otimes} \\ & \mathrm{Mod}_1^{\mathbb{E}_n}(\pi_0(X)_{\mathrm{gp}})^{\otimes}. \arrow[ur, "\xi_{\mathrm{gp}}^{\pi_0(X)_{\mathrm{gp}}}"'] & \end{tikzcd}$$

Following with the construction as in Section 5.3, we obtain the $\mathbb{E}_{n-1}$-monoidal system of $\mathrm{Th}_R(\xi_1)$-modules

$$\mathrm{Th}_R(\xi)^{\pi_0(X)} : \pi_0(X)^{\otimes} \to \mathrm{LMod}^{\otimes}_{\mathrm{Th}_R(\xi_1)}.$$

The construction of the map $\mathrm{Th}_R(\xi)^{\pi_0(X)}$ only involves taking pullbacks and postcomposing the $\mathbb{E}_{n-1}$-monoidal lift $\xi^{\pi_0(X)}$ with $\mathbb{E}_{n-1}$-monoidal functors. Therefore, from the previous discussion, follows that the map $\mathrm{Th}_R(\xi)^{\pi_0(X)}$ factors as the composition of the group completion γ and the map $\mathrm{Th}_R(\xi_{\mathrm{gp}})^{\pi_0(X)_{\mathrm{gp}}}$

$$\begin{tikzcd} \pi_0(X)^{\otimes} \arrow[rr, "\mathrm{Th}_R(\xi)^{\pi_0(X)}"] \arrow[dr, "\gamma"'] & & \mathrm{LMod}^{\otimes}_R \\ & \pi_0(X)^{\otimes}_{\mathrm{gp}} \arrow[ur, "\mathrm{Th}_R(\xi_{\mathrm{gp}})^{\pi_0(X)_{\mathrm{gp}}}"'] & \end{tikzcd}$$

Now $\pi_0(X)^{\otimes}_{\mathrm{gp}}$ is an $\mathbb{E}_{n+1}$-monoidal grouplike Kan complex; hence we can apply Proposition 5.3.3 to conclude that the map $\mathrm{Th}_R(\xi_{\mathrm{gp}})^{\pi_0(X)_{\mathrm{gp}}}$ factors through the full subcategory $\mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes}$. So the map $\mathrm{Th}_R(\xi)^{\pi_0(X)}$ must factor through $\mathrm{Pic}(\mathrm{Th}_R(\xi_1))^{\otimes}$ too. We have finally obtained the system of invertible $\mathrm{Th}_R(\xi_1)$-modules associated with ξ and π. Parts (2) and (3) follow from the construction of the iterated Thom spectrum $\mathrm{ThTh}_R(\xi)$ and the discussion on its compatibility with the Thom spectrum $\mathrm{Th}_R(\xi)$ without any variation from the arguments presented in Sections 5.3 and Section 5.4. □

Example 5.5.5. Corollary 5.5.4 allows us to apply the iterated Thom spectrum construction directly to the spherical fibration $J : (\mathbb{N} \times \mathrm{BU})^{\otimes} \to \mathrm{Pic}(\mathbb{S})^{\otimes}$. In this case, the spherical fibration J presents the spectrum $\mathrm{MUP}_{\geq 0} \simeq \bigvee_{n\in\mathbb{N}} \Sigma^{2n}\mathrm{MU}$ as an $\mathbb{E}_\infty$-Thom spectrum. Applying Corollary 5.5.4 to $J : (\mathbb{N} \times \mathrm{BU})^{\otimes} \to \mathrm{Pic}(\mathbb{S})^{\otimes}$ along the projection on the path components $\pi_0 : (\mathbb{N} \times \mathrm{BU})^{\otimes} \to \mathbb{N}^{\otimes}$ we obtain:

- an $\mathbb{E}_\infty$-monoidal system of invertible MU-modules $\mathrm{Th}_{\mathbb{S}}(J)^{\mathbb{N}} : \mathbb{N}^{\otimes} \to \mathrm{Pic}(\mathrm{MU})^{\otimes}$;
- and an $\mathbb{E}_\infty$-algebra of left MU-modules $\mathrm{ThTh}_{\mathbb{S}}^{\pi_0}(J)$, such that considering its underlying ring $\mathbb{E}_\infty$-spectrum we recover the $\mathbb{E}_\infty$-ring spectrum $\mathrm{Th}_{\mathbb{S}}(J) \simeq \mathrm{MUP}_{\geq 0}$.

Appendix A

Morphisms of bimodules

A.1 Morphism induced on the underlying objects

Let $\mathcal{C}^{\otimes}$ be an associative monoidal category and let $A \in \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})$ be an associative algebra of $\mathcal{C}$. In this section, we will try to give some intuition for the morphisms of the category ${}_A\mathrm{BMod}_A(\mathcal{C})$. In particular, we will see how it is possible to recover the classical notion of a morphism of $\mathcal{C}$ between the underlying objects that is compatible with the right and left A-actions.

By definition ${}_A\mathrm{BMod}_A(\mathcal{C})$, is the category

$$\{A\} \underset{\mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})}{\times} \mathrm{Alg}_{\mathcal{BM}/\mathrm{Assoc}}(\mathcal{C}) \underset{\mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})}{\times} \{A\},$$

where the maps $\mathrm{Alg}_{\mathcal{BM}/\mathrm{Assoc}}(\mathcal{C}) \to \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})$ are defined by precomposition with the embeddings $\mathrm{Assoc}_{-}^{\otimes} \hookrightarrow \mathcal{BM}^{\otimes}$ and $\mathrm{Assoc}_{+}^{\otimes} \hookrightarrow \mathcal{BM}^{\otimes}$.

Let $\mathbf{f} : \Delta^1 \to {}_A\mathrm{BMod}_A(\mathcal{C})$ be a morphism of ${}_A\mathrm{BMod}_A(\mathcal{C})$ between the bimodules $\mathbf{M} : \mathcal{BM}^{\otimes} \to \mathcal{C}^{\otimes}$ and $\mathbf{N} : \mathcal{BM}^{\otimes} \to \mathcal{C}^{\otimes}$. Since $\mathrm{Alg}_{\mathcal{BM}/\mathrm{Assoc}}(\mathcal{C})$ is the full subcategory of $\mathrm{Fun}_{\mathrm{Assoc}^{\otimes}}(\mathcal{BM}^{\otimes}, \mathcal{C}^{\otimes})$ spanned by the maps of ∞-operads over $\mathrm{Assoc}^{\otimes}$; the morphism f corresponds to a natural transformation between the operadic maps $\mathbf{M}$ and $\mathbf{N}$

$$\begin{tikzcd}
\mathcal{BM}^{\otimes} \arrow[r, bend left, "\mathbf{M}"{name=U}] \arrow[r, bend right, "\mathbf{N}"'{name=D}] \arrow[dr] & \mathcal{C}^{\otimes} \arrow[d, "p"] \arrow[from=U, to=D, phantom, "\Downarrow \mathbf{f}"] \\
 & \mathrm{Assoc}^{\otimes},
\end{tikzcd}$$

or alternatively

$$\begin{tikzcd}
\mathcal{BM}^{\otimes} \times \Delta^1 \arrow[r, "\mathbf{f}"] \arrow[dr] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\
 & \mathrm{Assoc}^{\otimes}.
\end{tikzcd}$$

We recall that the operad $\mathcal{BM}^{\otimes}$ has three distinguished objects: $\mathfrak{m}$ that represents the bimodule, $\mathfrak{a}_-$ that represents the left algebra, and $\mathfrak{a}_+$ that represents the right algebra.

We will denote by M and N the underlying objects of the category $\mathcal{C}$ defined by the operadic maps, i.e., the images $\mathbf{M}(\mathfrak{m})$ and $\mathbf{N}(\mathfrak{m})$. Similarly, we will denote by f the morphism of $\mathcal{C}$ between the underlying objects M and N defined by the natural transformation $\mathbf{f}$, i.e., $\mathbf{f}_{\mathfrak{m}} : M = \mathbf{M}(\mathfrak{m}) \to \mathbf{N}(\mathfrak{m}) = N$.

Let us see how we can recover a commutative diagram that expresses the compatibility of the morphism f with the bimodule structures of the objects M and N from the natural transformation $\mathbf{f}$. We will focus on the right action; the argument for the compatibility of the left action is analogous.

Let $\overline{\beta} : (\mathfrak{m}, \mathfrak{a}_+) \to \mathfrak{m}$ be the active morphism of $\mathcal{BM}^{\otimes}$ that represents the right action. We consider the following diagram of ∞-category; where the top-left diagram belongs to the category $\mathcal{BM}^{\otimes} \times \Delta^1$

$$
\begin{tikzcd}
\begin{matrix} \{0\} \\ \Delta^1 \downarrow \\ \{1\} \end{matrix}
\left[\begin{tikzcd}[ampersand replacement=\&] (\mathfrak{m}, \mathfrak{a}_+) \arrow[r, "\overline{\beta}"] \arrow[d] \& \mathfrak{m} \arrow[d] \\ (\mathfrak{m}, \mathfrak{a}_+) \arrow[r, "\overline{\beta}"] \& \mathfrak{m} \end{tikzcd}\right]
\arrow[rr, mapsto, "\mathbf{f}"] \arrow[dr, mapsto] & &
\left[\begin{tikzcd}[ampersand replacement=\&] (M, A) \arrow[r] \arrow[d, "{(f,id)}"] \& M \arrow[d, "f"] \\ (N, A) \arrow[r] \& N \end{tikzcd}\right]
\arrow[dl, mapsto, "p"] \\
& \left[\begin{tikzcd}[ampersand replacement=\&] \langle 2 \rangle \arrow[r, "\beta"] \arrow[d, "id"] \& \langle 1 \rangle \arrow[d, "id"] \\ \langle 2 \rangle \arrow[r, "\beta"] \& \langle 1 \rangle \end{tikzcd}\right]. &
\end{tikzcd}
$$

Let $\beta_!$ be the coCartesian morphisms of $\mathcal{C}^{\otimes}$ covering the morphism β of $\mathrm{Assoc}^{\otimes}$. Starting from the diagram of solid arrows

$$
\left[\begin{tikzcd} (\mathfrak{m}, \mathfrak{a}_+) \arrow[r, "\overline{\beta}"] \arrow[d] & \mathfrak{m} \arrow[d] \\ (\mathfrak{m}, \mathfrak{a}_+) \arrow[r, "\overline{\beta}"] & \mathfrak{m} \end{tikzcd}\right]
\overset{\mathbf{f}}{\longmapsto}
\left[\begin{tikzcd}
 & & M \otimes A \arrow[dl, dashed] \arrow[ddd, dashed, "f \otimes id"] \\
(M, A) \arrow[r] \arrow[d, "{(f,id)}"] \arrow[urr, bend left, "\beta_!"] & M \arrow[d, "f"] & \\
(N, A) \arrow[r] \arrow[drr, bend right, "\beta_!"'] & N & \\
 & & N \otimes A \arrow[ul, dashed]
\end{tikzcd}\right], \qquad (\star)
$$

our goal is to produce the dashed arrows and the commutativity of the right-most diagram.

In order to do so we need the following characterization of coCartesian morphisms; this is the dual version of [Lur09, Remark 2.4.1.4].

Proposition A.1.1. *Let $p : X \to S$ be an inner fibration of simplicial sets. An edge $\gamma : \Delta^1 \to X$ is p-coCartesian if and only if for every $n \geq 2$ and every commutative diagram of solid arrows*

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d] \arrow[dr, "\gamma"] & \\ \Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S, \end{tikzcd}$$

there exists a dashed arrow as indicated, rendering the diagram commutative.

We will produce the commutative diagram $(\star)$ in two steps, splitting the initial commutative square of $\mathcal{BM}^{\otimes} \times \Delta^1$ into two triangles. Starting from the diagram of solid arrows

$$\begin{tikzcd} \left[\begin{tikzcd}[ampersand replacement=\&] (\mathfrak{m}, \mathfrak{a}_+) \arrow[r, "\overline{\beta}"] \arrow[dr] \& \mathfrak{m} \arrow[d] \arrow[dl, phantom, "\tau \Nearrow"] \\ \& \mathfrak{m} \end{tikzcd}\right] \arrow[rr, mapsto, "\mathbf{f}"] \arrow[dr, mapsto] & & \left[\begin{tikzcd}[ampersand replacement=\&] \& \& M \otimes A \arrow[dl, dashed] \arrow[ddl, dashed, bend left] \\ (M, A) \arrow[r] \arrow[dr] \arrow[urr, bend left, "\beta_!"] \& M \arrow[d, "f"] \& \\ \& N \& \end{tikzcd}\right] \arrow[dl, mapsto, "p"] \\ & \left[\begin{tikzcd}[ampersand replacement=\&] \& \& \langle 1 \rangle \arrow[dl, "id"] \arrow[ddl, bend left, "id"] \\ \langle 2 \rangle \arrow[r, "\beta"] \arrow[dr, "\beta"'] \arrow[urr, bend left, "\beta"] \& \langle 1 \rangle \arrow[d, "id"] \& \\ \& \langle 1 \rangle \& \end{tikzcd}\right], & \end{tikzcd}$$

since $\beta_!$ is coCartesian we can use Proposition A.1.1 for $n = 2$ to first construct the dashed arrows and the two-cells $\Delta^{\{0,1,2\}}$ and $\Delta^{\{0,1,3\}}$, and then with $n = 3$ we can construct the two-cell $\Delta^{\{1,2,3\}}$. (The two-cell $\Delta^{\{0,2,3\}}$ is just the image of the two-cell τ by the functor $\mathbf{f}$, and the three-cell of $\mathrm{Assoc}^{\otimes}$ is just the degeneracy of the one cell that corresponds to the morphism β.)

Repeating the same process with the lower triangle of the commutative square of $\mathcal{BM}^{\otimes} \times \Delta^1$ we obtain

$$
\begin{tikzcd}
\left[\begin{tikzcd}
(\mathfrak{m}, \mathfrak{a}_+) \arrow[d] \arrow[dr] & \\
(\mathfrak{m}, \mathfrak{a}_+) \arrow[r, "\overline{\beta}"'] \arrow[ur, phantom, "\Swarrow\nu"] & \mathfrak{m}
\end{tikzcd}\right] \arrow[rr, mapsto, "\mathbf{f}"] \arrow[dr, mapsto] & &
\left[\begin{tikzcd}
 & & M \otimes A \arrow[dl] \arrow[ddd, dashed, "id\otimes f"] \\
(N, A) \arrow[urr, bend left, "\beta_!"] \arrow[d, "(f\,, id)"'] \arrow[r] & N & \\
(M, A) \arrow[drr, bend right, "\beta_!"'] & & \\
 & & N \otimes A \arrow[uul, dashed]
\end{tikzcd}\right] \arrow[dl, mapsto, "p"] \\
 & \left[\begin{tikzcd}
 & & \langle 1 \rangle \arrow[dl, "id"] \arrow[dd, "id"] \\
\langle 2 \rangle \arrow[urr, bend left, "\beta"] \arrow[r, "\beta"] \arrow[drr, "\beta"'] & \langle 1 \rangle \arrow[dr, "id"] & \\
 & & \langle 1 \rangle
\end{tikzcd}\right]. &
\end{tikzcd}
$$

Combining the two-cells we finally obtain the commutative diagram

$$
\begin{tikzcd}
M \arrow[d, "f"] & M \otimes A \arrow[l] \arrow[d, "f\otimes id"] \arrow[dl] \\
N & N \otimes A. \arrow[l]
\end{tikzcd}
$$

A.2 Compatibility of $\eta_{\mathbf{x}}$ with the left F-action

During the proof of Proposition 5.2.3 we have been able to produce a natural transformation η between the associative monoidal categories $X^{\otimes}$ and $\mathcal{S}^{\otimes}$; and then showed that for each $x \in X$ the natural transformation η defined a morphism η_x between objects of the category ${}_F\mathrm{BMod}_F(\mathcal{S})$. It might be counterintuitive that the morphism η_x is a morphism of F-bimodules since, objectwise, it corresponds to the multiplication by x on the left $\eta_x : f \mapsto x \cdot f$; which does not seem compatible with the left F-action of F and X_a. The compatibility here follows from the fact that we have defined the natural transformation η not only as a natural transformation between the underlying categories X and $\mathcal{S}$ but as a natural transformation between the associative monoidal categories.

Let $\mathbf{x} : \mathcal{BM}^{\otimes} \to X^{\otimes}$ be the operadic map that presents the element x as a 1_x-bimodule. We consider the following diagram, where $\overline{\beta}$ is the morphism of $\mathcal{BM}^{\otimes}$ that represents the left action

$$
\begin{tikzcd}
\begin{matrix}\{0\}\\ {\scriptstyle\Delta^1}\big\downarrow\\ \{1\}\end{matrix}
\left[\begin{matrix} (\mathfrak{a}_-,\mathfrak{m}) & \xrightarrow{\overline{\beta}} & \mathfrak{m} \\ \big\downarrow & & \big\downarrow \\ (\mathfrak{a}_-,\mathfrak{m}) & \xrightarrow{\overline{\beta}} & \mathfrak{m}\end{matrix}\right] \arrow[rr, mapsto, "\eta\circ\mathbf{x}"] \arrow[dr, mapsto] & & \left[\begin{matrix} (F,F) & \longrightarrow & F \\ \big\downarrow{\scriptstyle (id,\eta_x)} & & \big\downarrow{\scriptstyle \eta_x} \\ (F,X_a) & \longrightarrow & X_a\end{matrix}\right] \arrow[dl, mapsto, "q"] \\
& \left[\begin{matrix}\langle 2\rangle & \xrightarrow{\beta} & \langle 1\rangle \\ \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle id} \\ \langle 2\rangle & \xrightarrow{\beta} & \langle 1\rangle\end{matrix}\right]. &
\end{tikzcd}
$$

We can use the fact that $\mathcal{S}^{\otimes}$ is associative monoidal and the universal property of the coCartesian morphisms covering β to produce the dashed arrows starting from the solid diagram; this process is described with more details in Section A.1.

$$
\left[\begin{matrix} (\mathfrak{a}_-,\mathfrak{m}) & \xrightarrow{\overline{\beta}} & \mathfrak{m} \\ \big\downarrow & & \big\downarrow \\ (\mathfrak{a}_-,\mathfrak{m}) & \xrightarrow{\overline{\beta}} & \mathfrak{m}\end{matrix}\right] \overset{\eta\circ\mathbf{x}}{\longmapsto}
\left[\begin{tikzcd}
& & F \times F \arrow[dl, dashed] \arrow[ddd, dashed, "id\times\eta_x"] \\
(F,F) \arrow[r] \arrow[d, "{(id,\eta_x)}"] \arrow[urr, bend left, "\beta_!"] & F \arrow[d, "\eta_x"] & \\
(F,X_a) \arrow[r] \arrow[drr, bend right, "\beta_!"'] & X_a & \\
& & X_a \times F \arrow[ul, dashed]
\end{tikzcd}\right].
$$

The rightmost square describes the compatibility of η_x with the left F-action of F and X_a. Since the diagram is a diagram in $\mathcal{S}$, the commutativity of the square means that there exists a homotopy between the two compositions. We know that the map $id \times \eta_x$ is not unique; it is instead defined up to a contractible space of choice by the coCartesian morphism $\beta_!$. So for $id \times \eta_x$ instead of the natural choice $(f, f') \mapsto (f, \eta_x(f'))$, we can choose the map that applies η_x to the second component and conjugate by x the first component, $(f, f') \mapsto (x \cdot f \cdot \overline{x}, \eta_x(f'))$. It can be proven that this map is a candidate for $id \times \eta_x$ too since conjugation by x is homotopic to the identity. Now, since $x \cdot \overline{x} \simeq 1$, it is easier to see why such homotopy between the two compositions exists.

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