The Segal conjecture for topological Hochschild homology of commutative S-algebras

Sverre Lunøe-Nielsen

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Foreword

This is my thesis for the degree of Dr.Scient. at the Institute of Mathematics, University of Oslo, funded by the Oslo Mathematics Doctoral Training Site (OMATS) program.

When I first applied for my doctoral fellowship during the summer of 2000, I consulted my Cand. Scient.-advisor Professor John Rognes for help on finding a suitable subject to study. During one afternoon he then presented four research projects that he thought would be interesting and approachable within a four-year period.

After what must have been three or four hours I was standing on the subway station on my way home, feeling quite dizzy, with some 15 sheets of paper, each sheet packed with diagrams, arrows and symbols. I remember feelings of doubt about which of the four projects to pick, as they all seemed intriguing - and, at the time, totally incomprehensible. I am not sure now what made me choose to spend over four years dealing with the Steenrod algebra, but as I am submitting this thesis to be printed, I feel very privileged to have been given this assignment. I want to express my deepest gratitude to my advisor John Rognes for his great intuition, insightful help and for being very, very patient.

Still on the level of the present thesis, I would like to thank Professor Robert Bruner for some very inspiring discussions during his visit in Oslo in the winter of 2004.

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I am fortunate enough to have good friends and family who's support have been a great help and encouragement: I would like to mention in particular my brother Håvard, my parents Sverre and Liv, my grandfather Erik Lunøe-Nielsen as well as my old friend Håkon Repstad. Their help and genuine interest in my studies through the years have made everything easier for me.

In closing, I send my warmest appreciation to my beautiful wife Kari and my son Erik. They have provided invaluable support and this thesis could not have been finished without their help. Well, to be honest, my son may not have been much help, but I am still grateful to him.

Contents

Introduction

Let B be a commutative S-algebra. Relating the algebraic K-theory of B to its topological cyclic homology by trace methods [16] is very useful in order to understand $K(B)$, as the cyclic homology is more accessible to calculations.

The study of the topological cyclic homology spectrum $TC(B)$ [6] requires knowledge of the equivariant structure of Bökstedt's $THH(B)$, the topological Hochschild homology spectrum of B. These are spectrum analogues of Hochschild complexes for commutative rings in the algebraic world and come equipped with a natural action of the circle group of complex units.

In our context we will work with equivariant $\mathbb{T}$-spectra where $\mathbb{T} = S^1$ is the circle group of complex units. Our spectra will be *cyclotomic*, a class of $\mathbb{T}$-spectra containing all spectra arising as topological Hochschild spectra of some *Functor with Smash Products*. For such spectra T one defines the topological cyclic homology as a certain inverse limit where the spectra in the limit system consist of fixed point spectra of T with actions of the various finite cyclic p-subgroups of $\mathbb{T}$. With the goal of computing the topological cyclic homology, we are then faced with the problem of getting a grip on the building blocks of the pro-spectrum defining it.

Generally, for G a finite group and X a G-spectrum, it is a hard problem to calculate the homotopy type of the fixed point spectrum of X under the action of G. However, it is in many cases possible to calculate the homotopy version of these fixed points by a standard spectral sequence. There is a functorially defined comparison map $\Gamma : X^G \to X^{hG}$ between the fixed points and the homotopy fixed points and one can ask the question if this map is a homotopy equivalence, possibly after some appropriate completion.

In the case of $X = T = THH(B)$ and $G = C_{p^n}$ cyclic of order p^n, this comparison map sits in a commutative diagram

$$\begin{array}{ccccc} T_{hC_{p^n}} & \xrightarrow{N} & T^{C_{p^n}} & \xrightarrow{R} & T^{C_{p^{n-1}}} \\ \| & & \downarrow{\Gamma_n} & & \downarrow{\hat{\Gamma}_n} \\ T_{hC_{p^n}} & \xrightarrow{N^h} & T^{hC_{p^n}} & \xrightarrow{R^h} & T^{tC_{p^n}} \end{array} \tag{0.0.1}$$

where the rows are homotopy cofiber sequences. Here $T_{hC_{p^n}}$ is the homotopy orbits-construction on T with respect to the action of C_{p^n}. The quotient $T^{tC_{p^n}}$ in the lower right corner is the *Tate construction* on T with respect to the action of C_{p^n}. Note in particular that the fibers on the left are equivalent and so the right hand square is homotopy Cartesian. Hence, we may rephrase the question of comparing fixed points with homotopy fixed points by the canonical map Γ_n into the equivalent question of comparing the spectra on the right by the vertical map $\hat{\Gamma}_n$.

We say that a map of spectra $f : X \to Y$ is k-co-connected if it induces a weak equivalence in degrees $* \geq k$. The following theorem by Tsalidis leads to an inductive approach to the problem.

Theorem 0.0.1 (Tsalidis [27]). *If the map $\hat{\Gamma}_1$ is k-co-connected for some k after p-completion, then $\hat{\Gamma}_n$ is k-co-connected after p-completion for all $n \geq 1$.*

In light of this theorem, we may concentrate on the case $n = 1$ and the map $\hat{\Gamma}_1 : T \to T^{tC_p}$. We will in the following suppress the prime p and denote this map by γ.

Previous and new results

In the following we will consider $T(B) = THH(B)$ where B will be one of the spectra $S, MU, BP, BP\langle m-1\rangle$ for $m \geq 0$. This family of spectra are connected by maps $S \to MU \to BP \to BP\langle m-1\rangle$. The main diagram to have in mind when we arrive at computations in chapter 8 is the following:

$$\begin{array}{ccccccccc} T(S) & \longrightarrow & T(MU) & \longrightarrow & T(BP) & \longrightarrow & \cdots & \longrightarrow & T(BP\langle m-1\rangle) \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} & & & & \downarrow{\scriptstyle\gamma} \\ T(S)^{tC_p} & \longrightarrow & T(MU)^{tC_p} & \longrightarrow & T(BP)^{tC_p} & \longrightarrow & \cdots & \longrightarrow & T(BP\langle m-1\rangle)^{tC_p} \end{array}$$

Working homotopically, Hesselholt-Madsen [20], Bökstedt-Madsen [8] and Ausoni-Rognes [3] have done calculations in the cases $m = 0, 1, 2$, respectively, showing co-connectivity results for the map γ in order to make TC-calculations. These calculations rely on the existence of Smith-Toda ring spectra $V(m)$ of chromatic type $m+1$, which are conjectured not to exist for $m \geq 4$. Hence, new strategies for calculation are sought in order to tackle the entire tower of spectra.

In the other end, in the case of the sphere spectrum, the map γ is known to be a p-adic equivalence. This is an equivalent formulation of the affirmed Burnside ring-conjecture of Segal for the cyclic groups C_p.

The proof of the Segal conjecture for groups of prime order is of special interest as it serves as a recipe for our approach to the rest of the intermediate cases in the diagram above. The main ideas are the following: The Tate construction is equivalent to a homotopy inverse limit of *truncated Tate spectra* $\underset{n\to-\infty}{\text{holim}} T^{tC_p}[n]$. If the spectrum T is a bounded below spectrum of finite type over $\mathbb{F}_p$, then so are the all the truncated Tate spectra in the tower defining the inverse limit. In other words, the Tate spectrum T^{tC_p} is equivalent to a homotopy inverse limit of bounded below spectra of finite type over $\mathbb{F}_p$.

For any such tower of spectra $X \to \dots X_{k-1} \to X_k \to \dots$, we may apply cohomology with $\mathbb{F}_p$-coefficients and form the sequential direct limit of cohomology groups. This limit is seldom equal to the cohomology of the inverse limit of spectra but is still of interest. We denote this colimit by $H^*_c(X;\mathbb{F}_p)$, the *continuous cohomology* of X.

Theorem 0.0.2 (Caruso-May-Priddy [15]). *If $X = \underset{n\in\mathbb{Z}}{\text{holim}} X_n$ is a homotopy inverse limit where X_n is bounded below and of finite type over $\mathbb{F}_p$ for each n, then there is a spectral sequence*

$$\begin{aligned} E_2^{*,*} \cong\ & \text{Ext}_A^{*,*}(H_c^*(X;\mathbb{F}_p),\mathbb{F}_p) \\ & \Rightarrow \pi_*(X)\hat{}_p \end{aligned} \qquad (0.0.2)$$

converging strongly to the p-completed homotopy of X. The Ext*'s are taken over modules of the $\mathbb{F}_p$-Steenrod algebra A.*

In order to prove co-connectivity results for the comparison map $\gamma : THH(S) \to THH(S)^{tC_p}$ in the case of the Segal conjecture one resolves the target by an inverse system of truncated Tate-spectra. One then calculates the colimit of the associated cohomology groups with $\mathbb{F}_p$-coefficients in order to provide input for the spectral sequence in the theorem above.

The key point in the proof of the Segal conjecture for cyclic groups of prime order is that the map γ induces an isomorphism of Ext-groups and thus an isomorphism of Caruso-May-Priddy spectral sequences.

This fact was originally shown to be true by calculations of Lin for $p = 2$ and Gunawardena for $p \neq 2$. See [23] and [1] for published accounts on this. The proofs depend on identifying the structure of the continuous cohomology as a module over the Steenrod algebra. Indeed, Singer defined an endofunctor $R_+(-)$ on the category of modules over the Steenrod algebra. For any A-module M, there is a natural evaluation morphism $\epsilon : R_+M \to M$ of A-modules. The key property of this map is that it induces an isomorphism

$\mathrm{Ext}_A(M, \mathbb{F}_p) \to \mathrm{Ext}_A(R_+M, \mathbb{F}_p)$ of Ext-groups. The Segal conjecture follows from the fact that the continuous cohomology of $THH(S)^{tC_p}$ is isomorphic as an A-module to $R_+\mathbb{F}_p$ and the map γ induces the evaluation map $\epsilon : R_+\mathbb{F}_p \to \mathbb{F}_p$.

The present work follow this *homological approach* to showing co-connectivity for γ by replacing the equivariant sphere spectrum $S_{C_p} = THH(S)$ by the spectra $THH(BP)$ and $THH(BP\langle m-1\rangle)$ for all $m \geq 0$ at the prime $p = 2$.

Theorem 0.0.3. *For $T(BP\langle m-1\rangle)$ the map Γ_1 is a p-adic equivalence in sufficiently high degrees after smashing with a suitable finite CW-complex. If we assume that BP is coherent enough so that $H_*(T(BP); \mathbb{F}_2)$ admits an action by the Dyer-Lashof algebra, then the homotopy fixed points and the strict fixed points of $T(BP)$ are p-adically equivalent.*

By saying that Γ_1 is an *equivalence in sufficiently high degrees*, we simply mean that there exists an integer k such that Γ_1 is k-co-connective. An upper bound for the lowest such k is given in theorem 11.3.4.

We remark that the co-connectivity of Γ_1 for $B = BP\langle m\rangle$ will be shown to increase with m. It is therefore surprising that the infinite case has the optimal co-connectivity property.

New methods

In order to compare Caruso-May-Priddy spectral sequences, we need to know the continuous cohomology groups $H_c^*(T(B)^{tC_2})$ as modules over the Steenrod algebra. We do this by working with the dual object, namely the *continuous homology* $H_*^c(T(B)^{tC_2})$. To this extent there is a homological Tate-spectral sequence converging to $H_*^cT(B)^{tC_2}$. This spectral sequence is an A_*-comodule algebra spectral sequence, and we will utilize all of the structure provided when doing calculations.

The first step is to analyze the differentials of the Tate spectral sequence converging to these homology groups thus giving the additive structure. This is manageable mainly because of the algebra structure inherited from the product of the Tate construction. Work by Bruner and Rognes [12] enables us to identify infinite cycles and determine the E^∞-terms of these spectral sequences.

We then proceed by analyzing the A_*-comodule extensions in the homological Tate spectral sequence. This is possible by using naturality of the action of the Steenrod algebra applied to a certain map $\Psi : R_+(B) \to T(B)^{tC_2}$ we introduce in chapter 6.

The continuous homology of the spectrum in the source is then completely identified as an A_*-comodule and by naturality we are able to identify much of the A_*-module structure of $H^c_* T(B)^{tC_2}$. Not all A_*-comodule structure is detected, but the missing parts will come from the map $\gamma : T(B) \to T(B)^{tC_2}$. The following diagram will be helpful to have in mind in chapters 8-11:

$$\begin{array}{ccc} & & T(B) \\ & & \downarrow{\scriptstyle \gamma} \\ R_+(B) & \xrightarrow{\Psi} & T(B)^{tC_2} \end{array} \tag{0.0.3}$$

The spectrum $R_+(B)$ realizes the Singer construction on the A-module $H^*(B; \mathbb{F}_2)$. By that we mean that $H^*_c(R_+(B)) \cong R_+(H^*B)$ as an A-module. In our applications the dual map in homology will be injective, capturing much of the hidden A_*-comodule extensions in the homological Tate spectral sequence converging to $H^c_*(T(B)^{tC_2})$.

The calculations proceed analogous to the Segal conjecture for C_p. In the case of $THH(BP)$ we are able to show that the continuous cohomology of its Tate-construction is isomorphic to $R_+ H^* T(BP)$ and that the map γ^* under this identification is equal to the evaluation map ϵ. Also in the case of the truncated $BP\langle m-1 \rangle$-spectra we are able to describe the continuous cohomology in a convenient closed form using the Singer functor. The map γ has in these cases a non-trivial cokernel which we are able to describe as well. This gives us complete control on the map of Caruso-May-Priddy spectral sequences and enables us to prove theorem 0.0.3.

Chapter 1

Limits of spectra

We introduce our first definitions regarding towers of spectra and their associated homology groups and cohomology groups. These towers will consist of spectra such that the singular $\mathbb{F}_p$ (co-)homology at each stage will be bounded below and of finite type over $\mathbb{F}_p$.

The motivation for this definition is a result by Caruso-May-Priddy saying that that there is an inverse limit of Adams spectral sequences arising from such towers, calculating the homotopy of the inverse limit spectrum.

The input for this inverse limit of Adams spectral sequences will give us the definitions of the continuous (co-)homology groups.

1.1 Inverse limit of Adams spectral sequences

We will work at a fixed prime p. In applications this will be $p = 2$ but some of the general theory will be stated for any prime.

Let $\{Y_n\}_{n\in\mathbb{Z}}$ be a collection of spectra with stable maps $f_n : Y_n \to Y_{n+1}$ for all n and let Y be the homotopy inverse limit over this system.

$$Y \to \ldots \to Y_n \stackrel{f_n}{\to} Y_{n+1} \stackrel{f_{n+1}}{\to} Y_{n+2} \to \ldots \tag{1.1.1}$$

Definition 1.1.1. *Fix a prime p. We say that a spectrum Y is bounded below and of finite type over $\mathbb{F}_p$ if the singular homology of Y with $\mathbb{F}_p$-coefficients $H_*(Y;\mathbb{F}_p)$ is bounded below and of finite type over $\mathbb{F}_p$.*

When p is understood we will write $H(-)$ for $H(-;\mathbb{F}_p)$ and say that a spectrum Y is bounded below and of finite type without reference to the prime p.

We assume for the rest of this chapter that Y is the inverse limit of a tower of spectra (1.1.1) such that each Y_n is bounded below and of finite type over $\mathbb{F}_p$.

For each n there is an Adams spectral sequence $\{E_r(Y_n)\}_r$ with E_2-term

$$E_2^{s,t}(Y_n) = \mathrm{Ext}_A^{s,t}(H^*(Y_n;\mathbb{F}_p),\mathbb{F}_p) \Rightarrow \pi_{t-s}(Y_n)^\wedge_p$$

converging strongly to the p-completed homotopy groups of Y_n. Here A denotes the mod p Steenrod algebra.

The tower (1.1.1) induces maps of Adams spectral sequences $f_{n*} : \{E_r(Y_n)\} \to \{E_r(Y_{n+1})\}$. For every r let $E_r(\underline{Y}) = \lim_{n\to-\infty} E^r(Y_n)$ denote the inverse limit as n goes to $-\infty$. Under the assumption that each Y_n is bounded below and of finite type, it was proven in [15] that $\{E_r(\underline{Y})\}_r$ is a spectral sequence converging strongly to $\pi_* Y^\wedge_p$. It is clear that the E_2-term of this *inverse limit of Adams spectral sequences* is given by

$$E_2^{s,t}(\underline{Y}) \cong \mathrm{Ext}_A^{s,t}(\operatorname*{colim}_{n\to-\infty} H^*(Y;\mathbb{F}_p),\mathbb{F}_p) \tag{1.1.2}$$

1.2 Continuous (co-)homology

The spectral sequence (1.1.2) is central to the proof of the Segal conjecture for groups of prime order and will be the foundation for the present work. To emphasize the role of the colimit of cohomology groups, we make the following definition.

Definition 1.2.1. *Let p be any prime. Let Y be the homotopy inverse limit of a tower of spectra as in* (1.1.1) *with each Y_n bounded below and of finite type over $\mathbb{F}_p$. Then define the continuous cohomology of Y with $\mathbb{F}_p$-coefficients as the colimit*

$$H_c^*(Y;\mathbb{F}_p) = \operatorname*{colim}_{n\to-\infty} H^*(Y_n;\mathbb{F}_p)\,.$$

Dually, define the continuous homology of Y with $\mathbb{F}_p$-coefficients as the inverse limit

$$H_*^c(Y,\mathbb{F}_p) = \lim_{n\to-\infty} H_*(Y_n;\mathbb{F}_p)\,.$$

We choose to suppress from the notation the tower of which Y is a homotopy inverse limit. In case confusion could appear, we will emphasize the tower of spectra explicitly.

Since we are considering field coefficients and each spectrum Y_n is assumed to be of finite type over $\mathbb{F}_p$, we get that the hom-dual of H_c^*Y is isomorphic to H_*^cY. The continuous homology will typically be an infinite dimensional vectorspace over $\mathbb{F}_p$ in each degree, so the double dual is generally not isomorphic to the continuous cohomology.

Note that the continuous cohomology is a direct limit of bounded below A-modules. The direct limit may of course not exist in the category of bounded below modules, but we do get a natural A-module structure on $H_c^*(Y;\mathbb{F}_p)$ in the category of A-modules with no boundedness restrictions.

Dually, the continuous homology is an inverse limit of bounded below A_*-comodules, but the inverse limit may not be neither bounded below nor an algebraic A_*-comodule. In general we get a completed coaction of A_*

$$H_*^cY \to A_*\hat{\otimes} H_*^cY$$

where $-\hat{\otimes}-$ denotes the completed tensor product.

Let $\{Y_n, f_n\}$ and $\{Z_n, g_n\}$ be towers of bounded below spectra of finite type, and assume that we have maps $i_n : Y_n \to Z_n$ such that $g_n \circ i_n = i_{n+1} \circ f_n$ for each n. Then we get a tower of spectra $\{Z_n/Y_n\}_n$ where Z_n/Y_n is the homotopy cofiber of i_n. Thus we have a commutative diagram of spectra in the stable category

$$\begin{array}{ccccccccc}
Y & \longrightarrow & \cdots & \longrightarrow & Y_n & \xrightarrow{f_n} & Y_{n+1} & \xrightarrow{f_{n+1}} & \cdots \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_{n+1}} & & \\
Z & \longrightarrow & \cdots & \longrightarrow & Z_n & \xrightarrow{g_n} & Z_{n+1} & \xrightarrow{g_{n+1}} & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \\
Z/Y & \longrightarrow & \cdots & \longrightarrow & Z_n/Y_n & \longrightarrow & Z_{n+1}/Y_{n+1} & \longrightarrow & \cdots
\end{array}$$

where $Y \xrightarrow{i} Z \to Z/Y$ is a stable (co)-fibration sequence. From the long exact sequence in homology for each cofiber sequence $Y_n \to Z_n \to Z_n/Y_n$ we see that each Z_n/Y_n is bounded below and of finite type. Moreover, we get a long exact sequence

$$\ldots \to H_c^{*-1}Y \to H_c^*Z/Y \to H_c^*Z \xrightarrow{i^*} H_c^*Y \to \ldots \qquad (1.2.1)$$

of continuous cohomology groups.

Chapter 2

Equivariant spectra

We review some notions from stable equivariant homotopy theory. The important tool for us will be the construction of the extended powers of spectra. In chapter 6 we will show that this construction will supply the bridge between the algebraic Singer construction and the topological Tate construction.

2.1 Equivariant spectra

We work within the framework of [22]. Let X be an equivariant G-spectrum indexed on a complete G-universe U. Let $i : U^G \to U$ be the inclusion of the trivial G-universe. The forgetful functor $i^* : G\mathscr{S}\mathscr{U} \to G\mathscr{S}\mathscr{U}^G$ has a left adjoint i_*. For any genuine G-spectrum $X \in G\mathscr{S}\mathscr{U}$, the counit

$$\epsilon : i_* i^* X \to X$$

induces a non-equivariant equivalence. Hence, if X is a free G-CW spectrum then ϵ is a G-equivalence.

When considering any of the functors $(-)^{hG}$, $(-)_{hG}$ or $(-)^{tG}$ from G-spectra to spectra, we need only the naive definition of G-spectra. Indeed, suppose $X \in G\mathscr{S}\mathscr{U}$. Then the composite

$$i_* EG_+ \wedge i^* X \simeq EG_+ \wedge i_* i^* X \to EG_+ \wedge X \tag{2.1.1}$$

is an equivalence in $G\mathscr{S}\mathscr{U}$. When taking orbits or fixed points we first restrict to naive spectra, so for example in the case of orbits we have $(EG_+ \wedge X)/G \simeq (i^* i_* EG_+ \wedge i^* X)/G = EG_+ \wedge_G i^* X$.

Let $N \subset G$ be a subgroup and let $X \in G\mathscr{S}\mathscr{U}^G$ be a naive G-spectrum that is N-free. Then there is a transfer isomorphism of G/N-spectra

$$\tau : X/N \xrightarrow{\simeq} \left(\Sigma^{-adN} i_* X\right)^N . \tag{2.1.2}$$

Here adG is the adjoint representation of N.

In our applications, $G = N$ and G will be a closed subgroup of $\mathbb{T}$. In particular G will be abelian, so the adjoint representation will be trivial of dimension 1 if and only if $G = \mathbb{T}$ and zero otherwise. In this situation the transfer equivalence takes its most simple form; for a G-free naive G-spectrum X there is an isomorphism of non-equivariant spectra

$$X/G \xrightarrow{\simeq} \Sigma^{-dimG} X^G .$$

2.2 The equivariant half-smash functor

We recall the definition and chain level description of the extended power construction from [11], chapter I. We then review the relevant definitions and facts about Dyer-Lashof operations in the $\mathbb{F}_2$-homology of an H_∞-ring spectrum.

In this and the following section, π will be a subgroup of Σ_j the group of permutations on j letters.

Let G be a compact Lie group and let $\mathscr{U}$ and $\mathscr{U}'$ be G-universes. For a G-CW complex W one then has the equivariant half-smash functor

$$W \ltimes (-) : hG\mathscr{S}\mathscr{U} \to hG\mathscr{S}\mathscr{U}'$$

generalizing the functor $W_+ \wedge (-)$ on the category of based G-spaces. Indeed, for Y any G-space, there is a natural isomorphism of spectra ([11], proposition 1.1)

$$W \ltimes \Sigma^\infty Y \cong \Sigma^\infty (W_+ \wedge Y) .$$

The half-smash functor has a right-adjoint functor $F[W, -)$, analogous to the adjoint pair $W_+ \wedge (-)$, $\mathrm{Map}(W_+, -)$ on the level of G-spaces.

When W is a free π-CW complex and X is a G-CW spectrum, then $W \ltimes X$ is a π-free $(\pi \times G)$-CW spectrum.

Let X, Y be G-spectra indexed on a G-universe $\mathscr{U}$. Then define the external smash product $X \wedge Y$ as the G-prespectrum indexed on $\mathscr{U}^{(2)}$ by

$$(X \wedge Y)(U \oplus V) = X(U) \wedge X(V) .$$

When $X = Y$ we denote their external smash product by $X^{(2)}$. This is a $\Sigma_2 \times G$-spectrum indexed on the universe $\mathscr{U}^{(2)}$ where G acts diagonally and Σ_2 acts by permuting the factors.

Definition 2.2.1. *Let X be a G-spectrum. The nth extended power construction on X, denoted by D_nX, is defined by*

$$D_nX := E\Sigma_n \ltimes_{\Sigma_n} X^{(n)} .$$

When $n = 2$, it is customary to call $D_2(X)$ the *quadratic construction on X.*

2.3 Homology of the extended power construction

Let W be a free π-CW complex and let X be a CW spectrum with cellular action of π. Then by theorem I.1.3 in [11] we have that $W \ltimes_\pi X$ is a CW spectrum with cellular chains

$$C_*(W \ltimes_\pi X) \cong C_*W \otimes_\pi C_*X \tag{2.3.1}$$

The following is an immediate corollary.

Corollary 2.3.1. *Let π be a finite group and let X be as above. Then if X is bounded below and of finite type, then so is $E\pi \ltimes_\pi X$.*

Proof. Since π is finite, we may choose a π-free CW complex W with finitely many π-cells in each degree. Thus, both W and X are bounded below and of finite type with cellular chains given by equation (2.3.1), so the claim follows. □

Let $\pi \subset \Sigma_j$. As stated in corollary 2.1 [11], we have that the external smash product $X^{(j)}$ is a CW spectrum with cellular chains $C_*X^{\otimes j}$. The action of π is permuting the tensor factors and we have

$$C_*(W \ltimes_\pi X^{(j)}) \cong C_*W \otimes_\pi C_*X^{\otimes j} . \tag{2.3.2}$$

Finally, when working over a field, there is a π-equivariant chain homotopy

$$H_*X^{\otimes j} \xrightarrow{\simeq} C_*X^{\otimes j}$$

given by choosing representatives for homology classes. Via this equivalence, (2.3.2) can be rewritten as

$$C_*(W \ltimes_\pi X^{(j)}) \cong C_*W \otimes_\pi H_*X^{\otimes j} . \tag{2.3.3}$$

Thus the homology of $D_\pi X$ is computed by group homology:

Corollary 2.3.2 ([11], Corollary 2.3).

$$H_*D_\pi X \cong H_*(\pi; H_*X^{\otimes j})$$

In our applications we will work over the field $\mathbb{F}_2$ and will only discuss the case $\pi = \Sigma_2 = \{1, T\}$. In this case the $\mathbb{F}_2$-homology of D_2X is particularly easy to describe by a standard calculation in group homology of Σ_2. In fact, choose a free resolution $W_* \to \mathbb{F}_2$ by letting $W_n = \mathbb{F}_2[\Sigma_2]\{e_n\}$ be the free $\mathbb{F}_2[\Sigma_2]$-module on a generator e_n in degree n and differentials $d_n : W_n \to W_{n-1}$ given by multiplication with the norm element $1+T$ for all n. Tensoring W_* with $H_*X^{\otimes 2}$ over $\mathbb{F}_2[\Sigma_2]$ produces a chain complex

$$\ldots \stackrel{1+T}{\to} H_*X^{\otimes 2}\{e_2\} \stackrel{1+T}{\to} H_*X^{\otimes 2}\{e_1\} \stackrel{1+T}{\to} H_*X^{\otimes 2}\{e_0\}\,.$$

From this we get that

$$H_*D_2X \quad = \mathbb{F}_2\{e_0 \otimes x_1 \otimes x_2\} \oplus \mathbb{F}_2\{e_i \otimes x \otimes x \big| i \geq 0\} \tag{2.3.4}$$

where the elements x_1, x_2 and x run through basis elements of H_*X such that $x_1 \neq x_2$, and $\{x_1 \otimes x_2, x \otimes x\}$ run through a Σ_2-basis for $H_*X^{\otimes 2}$.

2.4 Homology operations

We recall the construction of external and internal homology operations as defined in [11], chapter IX, §1. For any H_∞ ring spectrum E and any spectrum X one defines external homology operations $E_nX \to E_mD_pX$ indexed by $E_mD_pS^n$ as follows. Let $\alpha \in E_mD_pS^n$ be represented by a map $\alpha : S^m \to E \wedge D_pS^n$. Then, for any $x \in E_nX$ represented by $x : S^n \to E \wedge X$, let $Q_\alpha x \in E_mD_pX$ be the composite

$$\begin{aligned} S^m \stackrel{\alpha}{\to} E \wedge D_pS^n \quad &\stackrel{1\wedge D_px}{\to} E \wedge D_p(E \wedge X) \stackrel{1\wedge \delta_p}{\to} E \wedge D_pE \wedge D_pX \\ &\stackrel{1\wedge\xi_p\wedge 1}{\to} E \wedge E \wedge D_pX \stackrel{\phi\wedge 1}{\to} E \wedge D_pX \end{aligned} \tag{2.4.1}$$

If X happens to be an H_∞ ring spectrum itself, the composition by the map $1 \wedge \xi_p : E \wedge D_pX \to E \wedge X$ internalizes this operation.

For $E = S$ we get Bruner's homotopy operations and for $E = H\mathbb{F}_p$ we get the Dyer-Lashof operations Q^i in singular homology with mod p coefficients. For details on this last fact, see proposition IX.1.2 of [11].

When $p = 2$ and $E = H\mathbb{F}_2 = H$ is the $\mathbb{F}_2$ Eilenberg-MacLane spectrum, H_*D_2E was described in (2.3.4). Then for $x \in H_kX$ and $i \geq k$, the class $Q^ix \in H_{k+i}X$ is the value of $e_{i-k} \otimes x \otimes x$ under the map

$$\xi_{2*} : H_*D_2X \to H_*X\,.$$

We will sometimes use the lower index notation Q_ix to mean the value of $e_i \otimes x \otimes x$ under the map ξ_{2*}. For $x \in H_kX$, these two conventions relates by the formula $Q^ix = Q_{i-k}x$.

There is a homotopy commutative diagram

$$\begin{array}{ccc} D_2X & \xrightarrow{\xi_2} & X \\ \uparrow & \nearrow_{\mu} & \\ X \wedge X & & \end{array}$$

where μ is the multiplication map and the vertical map is induced by the inclusion of the zero-skeleton of $E\Sigma_2$. Thus, the classes $e_0 \otimes x_1 \otimes x_2$ in equation (2.3.4) maps to their Pontryagin products $x_1 \cdot x_2 \in H_*X$ under the map ξ_{2*}.

Chapter 3

The Tate construction

We define the Tate construction and set up the relation with homotopy orbit- and homotopy fixed point spectra. We show that the Tate spectrum can be expressed as the homotopy inverse limit of bounded below spectra. In light of chapter 1 we will then focus on the continuous (co-)homology groups of the Tate construction.

The first section is concerned with the general setup of the Tate construction and its relatives; the homotopy orbit spectrum and the homotopy fix point spectrum.

We then move on to describing the homological Tate spectral sequences. There are two types, one converging to the continuous homology of the Tate construction and one converging to the continuous cohomology. These spectral sequences will be dual to each other but, as already noted in section 1.2, their abutments will generally not be dual.

Propositions 3.3.4, 3.3.5 and 3.3.6 state the properties of the (co-)homological Tate spectral sequences converging to the (co-)homology of the Tate construction of a G-spectrum X. We will revisit these results in the next chapter where $X = THH(B)$ will be the topological Hochschild homology spectrum of a commutative S-algebra B, and G will be a cyclic group of prime order.

3.1 Homotopy-orbits, -fixed point and the Tate construction

Let G be a compact Lie group and X a G-spectrum. We start by recalling the definitions of the homotopy-orbit, -fixed point spectrum of X together with their relative the Tate construction. For further details, see the introduction in [18].

Let EG be a free contractible G-space. Define $\widetilde{EG}$ to be the unreduced suspension of EG. This is a G-space with exactly two fixed points. Choosing one of these as a base point makes $\widetilde{EG}$ into a based G-space and we have a fundamental cofiber sequence of based G-spaces

$$EG_+ \to S^0 \to \widetilde{EG} \tag{3.1.1}$$

defined by letting the first map collapse EG onto the non-basepoint of S^0.

Definition 3.1.1. *For a genuine G-spectrum X indexed on a complete universe $\mathscr{U}$, we let*

$$\begin{array}{lll} X_{hG} & = EG_+ \wedge_G i^*X & \text{homotopy orbit spectrum of } X \\ X^{hG} & = \mathrm{Map}(EG_+, X)^G & \text{homotopy fixed point spectrum of } X \\ X^{tG} & = [\widetilde{EG} \wedge \mathrm{Map}(EG_+, X)]^G & \text{Tate spectrum of } X \end{array}$$

Here $i : \mathscr{U}^G \subset U$ is the inclusion of the G-fixed universe and i^*X is the underlying G-spectrum of X indexed on trivial G-representations only.

The projection $EG_+ \to S^0$ induces a map of G-spectra

$$X \to \mathrm{Map}(EG_+, X) \tag{3.1.2}$$

for any G-spectrum Y. Smashing the cofiber sequence (3.1.1) with the map (3.1.2) and taking G-fixed points, we get the following map of cofiber sequences:

$$\begin{array}{ccccc} [EG_+ \wedge X]^G & \longrightarrow & X^G & \longrightarrow & [\widetilde{EG} \wedge X]^G \\ \downarrow \simeq & & \downarrow & & \downarrow \\ [EG_+ \wedge \mathrm{Map}(EG_+, X)]^G & \longrightarrow & X^{hG} & \longrightarrow & X^{tG} \end{array} \tag{3.1.3}$$

The left vertical map is an equivalence by the equivariant Whitehead theorem since the map (3.1.2) is a non-equivariant equivalence.

In the following, G will be any closed subgroup of the circle group of complex units of norm 1. We denote the circle group by $\mathbb{T}$. In particular, G is commutative and the adjoint representation $ad(G)$ will be the trivial representation of dimension $\dim G$.

By the equivariant Whitehead theorem, the map $EG_+ \to S^0$ induces an equivalence of G-spectra $EG_+ \wedge \mathrm{Map}(EG_+, X) \leftarrow EG_+ \wedge \mathrm{Map}(S^0, X) \cong EG_+ \wedge X$. In addition, by the G-equivalence $i_*EG_+ \wedge i^*X \xrightarrow{\simeq} EG_+ \wedge X$ and the Adams transfer equivalence $\Sigma^{\dim G} EG_+ \wedge_G i^*X \xrightarrow{\simeq} [i_*EG_+ \wedge i^*X]^G$, we can rewrite (3.1.3) as the homotopy commutative diagram

$$\begin{array}{ccccc} \Sigma^{\dim G} X_{hG} & \longrightarrow & X^G & \longrightarrow & [\widetilde{EG} \wedge X]^G \\ \| & & \downarrow & & \downarrow \\ \Sigma^{\dim G} X_{hG} & \xrightarrow{N^h} & X^{hG} & \longrightarrow & X^{tG}\,. \end{array} \tag{3.1.4}$$

The Segal conjecture for finite groups can be interpreted as a homotopy limit problem showing that the middle map comparing G-fixed points and homotopy G-fixed points are equivalent. The above diagram formalizes the idea of breaking the homotopy limit problem into free and singular parts. This approach was successfully taken in Carlsson's proof of the Segal conjecture for finite p-groups [13].

The spectra in the lower row have been studied by means of spectral sequences converging to the their homotopy groups. These spectral sequences arise in the case of the homotopy orbit- and fixed point spectra by choosing a filtration of EG and by a filtration of $\widetilde{EG}$ introduced by Greenlees [17] in the case of the Tate spectrum X^{tG}.

We will in the end of this chapter return to the case of the Tate filtration and the resulting Tate spectral sequence, but we will be concerned with the spectral sequences that arises from applying homology with $\mathbb{F}_p$-coefficients instead of homotopy.

We end this section with some facts about the Tate construction.

Proposition 3.1.2 (Greenlees-May [18], proposition 2.6). *There is a natural equivalence $\Sigma\,\mathrm{Map}(\widetilde{EG}, EG_+ \wedge X) \xrightarrow{\simeq} \widetilde{EG} \wedge \mathrm{Map}(EG_+, X)$ of G-spectra.*

Proof. We have a commutative diagram of G-spectra

$$\begin{array}{ccccc} \mathrm{Map}(S^0, EG_+ \wedge X) & \longrightarrow & \mathrm{Map}(EG_+, EG_+ \wedge X) & \longrightarrow & \Sigma\,\mathrm{Map}(\widetilde{EG}, EG_+ \wedge X) \\ \uparrow \simeq & \nearrow & & & \\ EG_+ \wedge \mathrm{Map}(S^0, EG_+ \wedge X) & & \downarrow \simeq & & \downarrow \\ \downarrow \simeq & & & & \\ EG_+ \wedge \mathrm{Map}(S^0, X) & & & & \\ \downarrow \simeq & & & & \\ EG_+ \wedge \mathrm{Map}(EG_+, X) & \longrightarrow & \mathrm{Map}(EG_+, X) & \longrightarrow & \widetilde{EG} \wedge \mathrm{Map}(EG_+, X) \end{array} \tag{3.1.5}$$

in which the rows are (co-)fiber sequences arising from the fundamental cofiber sequence (3.1.1). ☐

Lemma 3.1.3. *Let $f : X \to Y$ be a map of G-spectra that is a non-equivariant homotopy equivalence. Then the induced map of Tate spectra $f^{tG} : X^{tG} \to Y^{tG}$ is a homotopy equivalence which is filtration preserving with respect to any filtration of $\widetilde{EG}$.*

Proof. The stable equivariant Whitehead theorem implies that the map $1 \wedge f : EG_+ \wedge X \to EG_+ \wedge Y$ is a G-homotopy equivalence. The map of Tate spectra is induced by applying the functor $\Sigma \operatorname{Map}(\widetilde{EG}, -)^G$ which takes G-equivalences to equivalences. ☐

Lemma 3.1.4. *If X is G-equivalent to $G_+ \wedge Y$ for some spectrum Y, then $X^{tG} \simeq *$ is contractible.*

Proof. Since G is compact we have the G-isomorphism

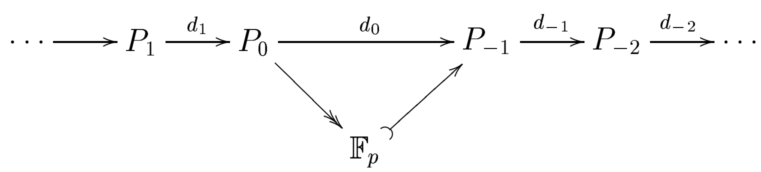
$$\widetilde{EG} \wedge \operatorname{Map}(EG_+, G_+ \wedge Y) \simeq \widetilde{EG} \wedge \operatorname{Map}(EG_+, Y) \wedge G_+ .$$

The projection $\widetilde{EG} \wedge G_+ \to *$ is a G-equivariant homotopy equivalence so the lemma follows. ☐

Remark: The same conclusion also holds if $X \simeq_G K \wedge Y$ for some finite free G-complex K. In our application, however, we will only encounter the case $K = G_+$.

3.2 Tate cohomology and the Greenlees filtration of $\widetilde{EG}$

We recall the definition of the Tate (co-)homology groups from [14]. Let G be a finite group and let $\{P_*, d_*\}$ be a complete resolution of the trivial $\mathbb{F}_pG$-module $\mathbb{F}_p$ by free $\mathbb{F}_pG$-modules. This is a diagram



where the P_k's are free $\mathbb{F}_pG$-modules and the horizontal sequence is exact.

Definition 3.2.1. *Given an $\mathbb{F}_pG$-module M the Tate homology and cohomology groups are defined by*

$$\widehat{H}_k(G;M) = H_k(P_* \otimes_{\mathbb{F}_pG} M)$$

and

$$\widehat{H}^k(G;M) = H^k(\mathrm{Hom}_{\mathbb{F}_pG}(P_*, M))$$

where $\{P_*\}$ is a complete $\mathbb{F}_pG$-resolution. These groups are independent of the chosen $\mathbb{F}_pG$-resolution and there are isomorphisms

$$\widehat{H}^k(G;M) \cong \widehat{H}_{-k-1}(G;M) \tag{3.2.1}$$

for all k.

Complete resolutions in algebra have topological analogues introduced by Greenlees [17]. We recall briefly the construction. Let EG be equipped with a G-CW structure and consider the associated skeleton filtration.

To form a complete resolution of $\widetilde{EG}$ we will have to work in the stable category of G-spectra. Whenever a space appears in the following, we will mean its suspension spectrum. Thus, $H_*(X;\mathbb{F}_p) = H_*(X)$ will mean spectrum homology with $\mathbb{F}_p$-coefficients of $\Sigma^\infty X$.

Let F_k be the cofiber of $EG_+^{(k-1)} \to S^0$ for $k \geq 0$ and $F_k = D(F_{-k})$, the Spanier-Whitehead dual of F_{-k}, for $k \leq 0$. The filtration quotients are recognized as $F_k/F_{k-1} \simeq \bigvee \Sigma^k G_+$ for all k, since EG is a free G-CW-complex. This holds for $k < 0$ as well because of the equivariant equivalence $D(G_+) \simeq G_+$.

This gives a sequential system of maps with non-equivariantly contractible homotopy inverse limit, and homotopy direct limit equivalent to $\widetilde{EG}$:

$$* \to \ldots \to F_{-2} \to F_{-1} \to F_0 = S^0 \to F_1 \to F_2 \to \ldots \to \widetilde{EG} \tag{3.2.2}$$

Applying $\mathbb{F}_p$-homology to the filtration $\{F_k\}$ gives a spectral sequence with $E^1_{st} = H_{s+t}(F_s/F_{s-1};\mathbb{F}_p)$ and differential $d^r_{st} : E^r_{st} \to E^r_{s-r,t-r+1}$ that converges to $H_*(\widetilde{EG};\mathbb{F}_p) = 0$. The spectral sequence collapses at the E^2-term since F_k/F_{k-1} is a wedge of k-spheres. Hence, we get a long exact sequence

$$\begin{array}{ccccccc} \cdots \longrightarrow H_2(F_2/F_1) & \longrightarrow & H_1(F_1/F_0) & \longrightarrow & H_0(F_0/F_{-1}) \longrightarrow \cdots \\ & & \downarrow & \nearrow & \\ & & H_0(S^0) \cong \mathbb{F}_p & & \end{array} \tag{3.2.3}$$

of free $\mathbb{F}_pG$-modules. Letting $P_k = H_{k+1}(F_{k+1}/F_k)$ for all k, yields a complete resolution of $\mathbb{F}_p$ by free $\mathbb{F}_pG$-modules.

3.3 Continuous homology of the Tate construction

Let $G \subset \mathbb{T}$ and let X be a G-spectrum which is bounded below and of finite type over $\mathbb{F}_p$.

By means of the Greenlees filtration (3.2.2), we may filter the Tate construction. For $n \in \mathbb{Z}$, let $\widetilde{EG}/F_{n-1}$ be the cofiber of the map $F_{n-1} \to \widetilde{EG}$, and let

$$X^{tG}[n] = [\widetilde{EG}/F_{n-1} \wedge \mathrm{Map}(EG_+, X)]^G . \tag{3.3.1}$$

For all n we have maps $X^{tG}[n] \to X^{tG}[n+1]$, and we will study the continuous (co-)homology of X^{tG} with respect to this filtration. To make sense of the continuous (co-)homology groups, we need the following.

Lemma 3.3.1. *The homotopy inverse limit of $X^{tG}[n]$ as $n \to -\infty$ is equivalent to X^{tG}. The homotopy colimit as $n \to \infty$ is contractible.*

Proof. Let $\{F_n\}_{n\in\mathbb{Z}}$ be the Greenlees filtration (3.2.2) of $\widetilde{EG}$. Consider the stable G-equivariant (co-)fibration sequence $F_n \to \widetilde{EG} \to \widetilde{EG}/F_n$ for $n \in \mathbb{Z}$. It is still a (co-)fibration sequence after smashing with $\mathrm{Map}(EG_+, X)$, taking fixed points and passing to the homotopy inverse limit over n. In other words we have a fibration sequence

$$\underset{n\to-\infty}{\mathrm{holim}}[F_n \wedge \mathrm{Map}(EG_+, X)]^G \to X^{tG} \to \underset{n\to-\infty}{\mathrm{holim}} X^{tG}[n] .$$

When n is negative, $F_n = D(F_{-n})$ is the Spanier-Whitehead dual of F_{-n}, so the fiber is equivalent to

$$\begin{aligned}\underset{n\to-\infty}{\mathrm{holim}}[D(F_{-n}) \wedge \mathrm{Map}(EG_+, X)]^G &\cong \underset{n\to-\infty}{\mathrm{holim}}\, \mathrm{Map}(F_{-n}, \mathrm{Map}(EG_+, X))^G \\ &\cong \mathrm{Map}(\underset{n\to\infty}{\mathrm{hocolim}} F_n, \mathrm{Map}(EG_+, X))^G \\ &\cong \mathrm{Map}(\widetilde{EG}, \mathrm{Map}(EG_+, X))^G \\ &\cong \mathrm{Map}(\widetilde{EG} \wedge EG_+, X)^G \\ &\simeq *\end{aligned} \tag{3.3.2}$$

Here we are using that F_n is dualizable in the stable category of G-spectra, and that $\widetilde{EG} \wedge EG_+$ is G-equivariantly contractible.

To show the last part of the lemma we use that, for $n \geq 0$, $\widetilde{EG}/F_n$ is a free G-CW complex. Indeed, for n positive,

$$\widetilde{EG}/F_n = \widetilde{EG}/\widetilde{EG}^{(n)} \simeq \Sigma EG/EG^{(n-2)} .$$

Thus, by the transfer equivalence we have

$$X^{tG}[n] = [\widetilde{EG}/F_n \wedge \mathrm{Map}(EG_+, X)]^G \simeq \Sigma^{\dim G}\widetilde{EG}/F_n \wedge_G i^* \mathrm{Map}(EG_+, X)\,.$$

Since colimits commute with orbits and smash products, the result follows since $\underset{n\to\infty}{\mathrm{hocolim}}\widetilde{EG}/\widetilde{EG}^{(n)}$ is G-equivariantly contractible. □

For $n \in \mathbb{Z}$, we let $X^{tG(n)} = [F_n \wedge \mathrm{Map}(EG_+, X)]^G$. We have the following diagram in which the columns are fibration sequences.

$$\begin{array}{ccccccccc}
* & \longrightarrow \cdots \longrightarrow & X^{tG(n-1)} & \longrightarrow & X^{tG(n)} & \longrightarrow \cdots \longrightarrow & X^{tG} \\
\downarrow & & \downarrow & & \downarrow & & \| \\
X^{tG} & = \cdots \longrightarrow & X^{tG} & = & X^{tG} & = \cdots \longrightarrow & X^{tG} \\
\| & & \downarrow & & \downarrow & & \downarrow \\
X^{tG} & \longrightarrow \cdots \longrightarrow & X^{tG}[n] & \longrightarrow & X^{tG}[n+1] & \longrightarrow \cdots \longrightarrow & *
\end{array} \qquad (3.3.3)$$

When X is bounded below and of finite type over $\mathbb{F}_p$ each of the spectra $X^{tG}[n]$ will be bounded below and of finite type as well. By lemma 3.3.1 it now makes sense to talk about the continuous homology and cohomology of X^{tG}, defined in chapter 1.1, with respect to the Tate-filtration $\{X^{tG}[n]\}_{n\in\mathbb{Z}}$.

In the case X and Y are both bounded below spectra of finite type, we can sharpen lemma 3.1.3 in terms of the continuous cohomology groups.

Lemma 3.3.2. *Let $f : X \to Y$ be a map of G-spectra such that f is a non-equivariant homotopy equivalence. Then the induced map of continuous cohomology groups of their associated Tate-spectra $f^{tG*} : H_c^* X^{tG} \to H_c^* Y^{tG}$ is an isomorphism.*

If X and Y are both bounded below and of finite type, it follows from this that f induces a p-adic equivalence.

Proof. Using the Greenlees filtration of X^{tG} and Y^{tG} we get a map of towers

$$\begin{array}{ccccccc}
X^{tG} & \longrightarrow \cdots \longrightarrow & X^{tG}[0] & \longrightarrow & X^{tG}[1] & \simeq & \Sigma X_{hG} \\
\downarrow{\scriptstyle f^{tG}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma f_{hG}} \\
Y^{tG} & \longrightarrow \cdots \longrightarrow & Y^{tG}[0] & \longrightarrow & Y^{tG}[1] & \simeq & \Sigma Y_{hG}
\end{array}$$

The map $f^{tG}[1] : X^{tG}[1] \to Y^{tG}[1]$ in the tower is identified with Σf_{hG} which is an equivalence by the equivariant Whitehead theorem. In addition, f^{tG}

induces $\Sigma^n f$ for $n \geq 0$ on the nth filtration layer, which is also an equivalence. The result follows by induction after applying cohomology.

The last statement follows since the inverse limit of Adams spectral sequences for X^{tG} and Y^{tG} are isomorphic on the E_2-term. □

We chose to express X^{tG} as the homotopy inverse limit of the tower in the lower row of diagram (3.3.3). In light of lemma 3.1.2 there is another obvious tower, namely

$$X^{tG} \to \ldots \to \Sigma \operatorname{Map}(\widetilde{EG}^{(n)}, EG_+ \wedge X)^G \to \Sigma \operatorname{Map}(\widetilde{EG}^{(n-1)}, EG_+ \wedge X)^G, . \tag{3.3.4}$$

This tower is again a tower of bounded below spectra of finite type, and the homotopy inverse limit of the tower is equivalent to the Tate construction on X.

We have not shown that the continuous (co-)homology groups are independent of the choice of tower, but the next comparison result says that the two towers introduced so far are equivalent.

Lemma 3.3.3. *For $n < 0$ there is a natural equivalence*

$$X^{tG}[n] \simeq \Sigma \operatorname{Map}(\widetilde{EG}^{(-n-1)}, EG_+ \wedge X)^G .$$

Proof. For $n \leq 0$, the cofiber spectrum $\widetilde{EG}/F_n$ is the cofiber of the composite $D(\widetilde{EG}^{(-n)}) \to D(S^0) \simeq S^0 \to \widetilde{EG}$, where $D(-)$ denotes the G-equivariant Spanier-Whitehead dual. We have a cofibration sequence of spectra

$$EG_+^{(-n-1)} \to S^0 \to \widetilde{EG}^{(-n)} .$$

Taking Spanier-Whitehead duals we get a stable homotopy cofibration sequence

$$F_n = D(\widetilde{EG}^{(-n)}) \to D(S^0) \to D(EG_+^{(-n-1)}) .$$

Hence, we get the following commutative diagram where the squares are stable pushout squares

$$\begin{array}{ccccc} F_n & \longrightarrow & D(S^0) = S^0 & \longrightarrow & \widetilde{EG} \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & D(EG_+^{(-n-1)}) & \longrightarrow & \widetilde{EG}/F_n . \end{array}$$

Thus, the homotopy fiber of $D(EG_+^{(-n-1)}) \to \widetilde{EG}/F_n$ is equivalent to the homotopy fiber of $S^0 \to \widetilde{EG}$, which is EG_+, so we have a fibration sequence

$$EG_+ \to D(EG_+^{(-n-1)}) \to \widetilde{EG}/F_n\,. \tag{3.3.5}$$

We then have the following analogue of diagram (3.1.5)

$$\begin{array}{ccc} EG_+ \wedge X & \longrightarrow & \mathrm{Map}(EG_+^{(-n-1)}, EG_+ \wedge X) \\ \downarrow \simeq & & \downarrow \simeq \\ EG_+ \wedge \mathrm{Map}(EG_+, X) & \longrightarrow & D(EG_+^{(-n-1)}) \wedge F(EG_+, X)\,. \end{array} \tag{3.3.6}$$

We get the upper row by applying $\mathrm{Map}(-, EG_+ \wedge X)$ to the cofibration sequence

$$\Sigma^{-1}\widetilde{EG}^{(-n)} \to EG_+^{(-n-1)} \to S^0 \to \widetilde{EG}^{(n)}$$

and the lower row by smashing the cofibration sequence (3.3.5) with $\mathrm{Map}(EG_+, X)$.

The two vertical maps are G-equivalences, so the map of cofibers

$$\Sigma\,\mathrm{Map}(\widetilde{EG}^{(-n)}, EG_+ \wedge X) \to \widetilde{EG}/F_n \wedge \mathrm{Map}(EG_+, X)$$

is a G-equivalence as well. The lemma now follows by taking G-fixed points. □

3.3.1 The homological Tate spectral sequence

We will show that the Tate filtration (3.3.1) gives rise to a spectral sequence converging to the continuous homology $H_*^c(X^{tG})$.

Applying homology with $\mathbb{F}_p$-coefficients to the lower tower of spectra in diagram (3.3.3), we get an inverse system of homology groups

$$H_*^c(X^{tG}) \to \ldots \to H_*(X^{tG}[n]) \to H_*(X^{tG}[n+1]) \to \ldots \tag{3.3.7}$$

with inverse limit equal, by definition 1.2.1, to the continuous homology of X^{tG} with respect to the tower $\{X^{tG}[n]\}_{n\in\mathbb{Z}}$. Note that, in general, homology and inverse limits do not commute, so the inverse limit is generally not the homology of X^{tG}.

Direct limits and homology do on the other hand commute, so by lemma 3.3.1, the colimit $H_*(X^{tG}[n])$ as $n \to \infty$ is trivial. Hence, by [5, Lemma 5.4

(b)], we get an exhaustive, complete Hausdorff filtration of $H^c_*(X^{tG})$ by the subgroups

$$F^n H^c_* X^{tG} = \ker[H^c_*(X^{tG}) \to H^c_*(X^{tG}[n+1])]\,. \tag{3.3.8}$$

For every n, we have a fiber sequence

$$[F_n/F_{n-1} \wedge \mathrm{Map}(EG_+, X)]^G \longrightarrow X^{tG}[n] \longrightarrow X^{tG}[n+1]\,. \tag{3.3.9}$$

Since F_n/F_{n-1} is a free G-spectrum, the fiber is equivalent to $[F_n/F_{n-1} \wedge X]^G$ by the map induced by the non-equivariant homotopy equivalence $EG_+ \to S^0$. Applying homology with $\mathbb{F}_p$-coefficients, these fibration sequences yield an unrolled exact couple

$$\begin{array}{ccc} H_*(X^{tG}[n]) & \xrightarrow{\ i\ } & H_*(X^{tG}[n+1]) \\ \uparrow{\scriptstyle k} & \swarrow{\scriptstyle j} & \\ H_*([F_n/F_{n-1} \wedge X]^G) & & \end{array} \tag{3.3.10}$$

which in turn gives a spectral sequence with $\hat{E}^1_{s,t} = H_{s+t}([F_s/F_{s-1} \wedge X]^G)$. As noted above, the colimit $H_*(X^{tG}[n])$ as $n \to \infty$ is trivial, so the spectral sequence converges conditionally in the sense of Boardman ([5] Definition 5.10) to the inverse limit $H^c_*(X^{tG}) = \lim_{n\to-\infty} H_*(X^{tG}[n])$.

To identify the $\hat{E}^2$-term, we use the natural isomorphisms

$$\begin{aligned} H_{s+t}([F_s/F_{s-1} \wedge X]^G) &\cong H_{s+t}(F_s/F_{s-1} \wedge_G X) \\ &\cong H_s(F_s/F_{s-1}) \otimes_{\mathbb{F}_pG} H_t(X)\,. \end{aligned} \tag{3.3.11}$$

With these identifications the d^1-differential is $d_{s-1} \otimes \mathrm{id} : H_s(F_s/F_{s-1}) \otimes_{\mathbb{F}_pG} H_t(X) \to H_{s-1}(F_{s-1}/F_{s-2}) \otimes_{\mathbb{F}_pG} H_t(X)$, where d_* is the differential in the complete resolution (3.2.3). Thus we get the homological Tate spectral sequence

$$\hat{E}^2_{s,t}(X) \cong \hat{H}_{s-1}(G; H_t(X)) \cong \hat{H}^{-s}(G; H_t(X)) \Rightarrow H^c_*(X^{tG}) \tag{3.3.12}$$

converging conditionally to the continuous homology of X^{tG} with $\mathbb{F}_p$-coefficients.

Since we are assuming that X has bounded below homology, the homological Tate spectral sequence is concentrated above some horizontal line.

Thus the spectral sequence is in one with entering differentials in the sense of Boardman.

Since X is of finite type, the $\hat{E}^2$-term is finitely generated in each bidegree so Boardman's derived limit RE^∞ vanishes as well and the spectral sequence converges strongly to its target. See [5, Theorem 7.1] for further details.

If we did not assume the finite type condition, we could get strong convergence if we knew that the spectral sequence collapsed at some finite stage.

The spectral sequence arises by applying homology with $\mathbb{F}_p$-coefficients. Hence, the exact couple (3.3.10) is equipped with a coaction of the dual mod p Steenrod algebra A_* and the resulting spectral sequence is an A_*-comodule spectral sequence. As noted in section 1.1, the continuous homology is a completed A_*-comodule. This coaction induces an A_* comodule structure on the associated graded with respect to the filtration (3.3.8) and convergence of the spectral sequence means that the associated graded is isomorphic to the $\hat{E}^\infty$-term as comodules over A_*.

We summarize the facts of the present section in the following proposition.

Proposition 3.3.4. *Let $G \subset \mathbb{T}$ be a finite subgroup and let X be a bounded below G-spectrum of finite type over $\mathbb{F}_p$.*

Then X^{tG} is equivalent to a homotopy inverse limit of a tower of bounded below spectra of finite type. There is an A_*-comodule homology type Tate spectral sequence converging strongly to the continuous homology of X^{tG} as a completed A_*-comodule. The homological Tate spectral sequence has $\hat{E}^2$-term

$$\hat{E}^2_{s,t}(X) = \widehat{H}^{-s}(G; H_t(X;\mathbb{F}_p)) \Rightarrow H^c_*(X^{tG};\mathbb{F}_p)\,. \tag{3.3.13}$$

We also get a cohomological Tate spectral sequence by dualizing the exact couple (3.3.10). This produces a filtration of the colimit cohomology groups, i.e. the continuous cohomology $H^*_c(X^{tG})$, by the image subgroups

$$F_n H^*_c(X^{tG}) = \mathrm{im}[H^*(X^{tG}[n]) \to H^*_c(X^{tG})]\,.$$

The filtration is by definition exhaustive, and is in addition both complete and Hausdorff because of the Milnor lim-lim^1 exact sequence

$$0 \to \mathrm{R}\lim_n H^{*-1}X^{tG}[n] \to H^*(\mathrm{hocolim}_n X^{tG}[n]) \to \lim_n H^* X^{tG}[n] \to 0\,. \tag{3.3.14}$$

We are using that the homotopy colimit of $X^{tG}[n]$ as $n \to \infty$ is homotopy trivial by lemma 3.3.1, so the middle term vanishes.

Thus the Tate filtration of X^{tG} gives rise to a conditionally convergent, cohomology type spectral sequence with target equal to the continuous cohomology $H_c^*(X^{tG})$.

Again, since X is assumed to be bounded below, we get a half-plane spectral sequence concentrated above some horizontal line. This is now a spectral sequence with exiting differentials and by [5, Theorem 6.1] the spectral sequence converges strongly to the continuous cohomology $H_c^*(X^{tG})$.

Since the cohomological exact couple comes from applying cohomology with $\mathbb{F}_p$-coefficients, we get a natural A-module structure on the spectral sequence with differentials being A-module homomorphisms.

Proposition 3.3.5. *Let $G \subset \mathbb{T}$ be a finite subgroup and let X be a bounded below G-spectrum of finite type over $\mathbb{F}_p$.*

Then X^{tG} is equivalent to a homotopy inverse limit of a tower of bounded below spectra of finite type. There is an A-module, cohomology type Tate spectral sequence converging strongly to the continuous cohomology of X^{tG} as an A-module. The cohomological Tate spectral sequence has $\hat{E}_2$-term

$$\hat{E}_2^{s,t}(X) = \widehat{H}_{-s}(G; H^t(X; \mathbb{F}_p)) \Rightarrow H_c^*(X^{tG}; \mathbb{F}_p)\,. \tag{3.3.15}$$

The cohomological Tate spectral sequence is dual to the homological Tate spectral sequence in the sense that $\hat{E}^r_{*,*}$ is dual to $\hat{E}_r^{*,*}$ in each bidegree for all r and that the cohomological differential $d_r : \hat{E}_r^{s,t} \to \hat{E}_r^{s+r,t-r+1}$ is dual to the homological differential $d^r : \hat{E}^r_{s+r,t-r+1} \to \hat{E}^r_{s,t}$ for all s, t and $r \geq 1$.

3.3.2 Multiplicative structure

Assume that X is a bounded below, finite type G-equivariant ring spectrum. We assume that the unit $\eta : S \to X$ and the multiplication map $\mu : X \wedge X \to X$ are equivariant with respect to the G-action.

By [18, Proposition 3.5] both the homotopy fixed point spectrum X^{hG} and the Tate spectrum X^{tG} are ring spectra. In the Tate case the product is defined in the following way: There is a composition

$$\begin{array}{c} [\widetilde{EG} \wedge \mathrm{Map}(EG_+, X)]^G \wedge [\widetilde{EG} \wedge \mathrm{Map}(EG_+, X)]^G \\ \downarrow \\ [\widetilde{EG} \wedge \widetilde{EG} \wedge \mathrm{Map}(EG_+ \wedge EG_+, X \wedge X)]^G \\ \downarrow{\scriptstyle 1\wedge 1\wedge \Delta^*} \\ [\widetilde{EG} \wedge \widetilde{EG} \wedge \mathrm{Map}(EG_+, X \wedge X)]^G \\ \downarrow{\scriptstyle 1\wedge 1\wedge \mu_*} \\ [\widetilde{EG} \wedge \widetilde{EG} \wedge \mathrm{Map}(EG_+, X)]^G \,. \end{array} \tag{3.3.16}$$

Up to homotopy, there is a unique G-equivalence $\widetilde{EG} \wedge \widetilde{EG} \xrightarrow{\simeq} \widetilde{EG}$. Taking the composition above followed by this homotopy equivalence, we get a product $X^{tG} \wedge X^{tG} \to X^{tG}$. The unit comes from the unit of X, together with the canonical map $S^0 \to \widetilde{EG}$ by the composition

$$S \xrightarrow{\eta} X \to F(EG_+, X) \to \widetilde{EG} \wedge F(EG_+, X)\,. \tag{3.3.17}$$

The homotopy fixed point spectrum also has a product coming from the product on X and the diagonal map $\Delta : EG_+ \to EG_+ \wedge EG_+$. The first two maps of (3.3.17) compose to give a unit for X^{hG} after taking G-fixed points. The ring structures are compatible in that the map $X^{hG} \to X^{tG}$ of (3.1.4) is a map of ring spectra.

Up to homotopy there is a unique homotopy equivalence $EG_+ \wedge EG_+ \xrightarrow{\simeq} EG_+$ as well. Using this we may define a product on the homotopy orbit spectrum X_{hG}. However, this spectrum lacks a unit, so it is not a ring spectrum.

The above facts can be found in [18, section §3].

Let H denote the $\mathbb{F}_p$ Eilenberg-MacLane spectrum. To discuss the multiplicative structure of the homological Tate spectral sequence, we note that the continuous homology of X^{tG} can be realized as the homotopy groups of the spectrum $(H \wedge X)^{tG}$. Indeed, for $n < 0$, we have

$$\begin{aligned} (H \wedge X)^{tG}[n] &\simeq \Sigma \,\mathrm{Map}(\widetilde{EG}^{(-n-1)}, EG_+ \wedge H \wedge X)^G \\ &\simeq \Sigma H \wedge \mathrm{Map}(\widetilde{EG}^{(-n-1)}, EG_+ \wedge X)^G \\ &\simeq \Sigma H \wedge X^{tG}[n]\,. \end{aligned}$$

The first and last equivalences follow from lemma 3.3.3 and the middle equivalence follows from the fact that $\widetilde{EG}^{(-n-1)}$ is dualizable. Thus, we have that $\pi_*(H \wedge X)^{tG}[n] \cong H_*(X^{tG}[n])$ for all $n < 0$. For a general G-spectrum X, we then have the following maps for any k:

$$\begin{aligned} \pi_k(H \wedge X)^{tG} &\to \lim_{n\to-\infty} \pi_k(H \wedge X)^{tG}[n] \\ &\cong \lim_{n\to-\infty} H_k(X^{tG}[n]) = H_k^c(X^{tG})\,. \end{aligned} \tag{3.3.18}$$

Since X was assumed to be bounded below and of finite type, then the groups in the inverse limit system are all of finite type, so $\lim^1$ vanishes and the first map in (3.3.18) is an isomorphism.

We may now give $(H \wedge X)^{tG}$ an increasing filtration by filtering $\widetilde{EG}$ by the Greenlees filtration (3.2.2). This produces a homotopical Tate spectral sequence with $\hat{E}^2$-term isomorphic to

$$\hat{E}^2_{s,t} \cong H^{-s}(G; \pi_t(H \wedge X)) \cong H^{-s}(G; H_t(X)) \tag{3.3.19}$$

converging to the homotopy $\pi_{s+t}(H \wedge X)^{tG} \cong H_*^c(X^{tG})$. See [20] for reference to this homotopical Tate spectral sequence.

When X is a G-equivariant ring spectrum, then so is $H \wedge X$ and we have seen that $(H \wedge X)^{tG}$ is also a ring spectrum. The induced ring structure on the homotopy of $(H \wedge X)^{tG}$ then gives a ring structure on the continuous homology by the isomorphism above.

Moreover, the homotopical Tate spectral sequence (3.3.19) is an algebra spectral sequence with differentials being derivations with respect to the product.

Proposition 3.3.6. *Let $G \subset \mathbb{T}$ be a finite subgroup and let X be a bounded below G-ring spectrum of finite type over $\mathbb{F}_p$.*

Then X^{tG} is equivalent to a homotopy inverse limit of a tower of bounded below spectra of finite type. There is an A_*-module algebra, homology type Tate spectral sequence converging strongly to the continuous homology of X^{tG} as an A_*-comodule algebra.

The homological Tate spectral sequence has $\hat{E}^2$-term

$$\hat{E}^2_{s,t}(X) = \widehat{H}^{-s}(G; H_t(X; \mathbb{F}_p)) \Rightarrow H_*^c(X^{tG}; \mathbb{F}_p) \tag{3.3.20}$$

and the differentials are derivations with respect to the product from the Tate cohomology groups.

3.4 A model for X^{tC_2}

For any G-representation V we denote the unit sphere of V by $S(V)$ and the one-point compactification of V by S^V. These are then (unbased) G-spaces, and by choosing a fixed point of S^V we get a cofiber sequence

$$S(V)_+ \to S^0 \to S^V \tag{3.4.1}$$

where the first map collapses $S(V)$ to the non-basepoint of S^0.

Let $L = \mathbb{R}$ with the antipodal action of C_2. We choose $S(\infty L)$ as a model for EC_2. For $V = \infty L$, the cofiber sequence (3.4.1) is a model for (3.1.1) and we have an explicit filtration of subskeleta

$$\begin{aligned} S(nL) &= EC_2^{(n)} \subset EC_2 \\ S^{nL} &= \widetilde{EC}_2^{(n)} \subset \widetilde{EC}_2 \,. \end{aligned}$$

Moreover, the cofiber sequence (3.4.1) respects this filtration, so for each $n > 0$ we have

$$\begin{array}{ccccc} EC_{2+}^{(n-1)} & \longrightarrow & S^0 & \longrightarrow & \widetilde{EC}_2^{(n)} \\ \| & & \| & & \| \\ S(nL)_+ & \longrightarrow & S^0 & \longrightarrow & S^{nL} \end{array} \tag{3.4.2}$$

There is a shearing isomorphism of C_2-spaces

$$\mathrm{sh}_n : \Sigma^n S^{nL} \to S^n \wedge S^n \tag{3.4.3}$$

where C_2 acts on the target by permuting the factors. The shearing map is defined by one-point compactifying the equivariant map $\mathbb{R}^n \oplus \mathbb{R}^{nL} \to \mathbb{R}^n \oplus \mathbb{R}^n$ sending $(x, y) \mapsto (x + y, x - y)$. Further, the diagonal map $\Delta : S^1 \to S^1 \wedge S^1$ defines a map

$$S^1 \wedge (S^{n-1})^{\wedge 2} \overset{\Delta \wedge 1 \wedge 1}{\to} S^1 \wedge S^1 \wedge (S^{n-1})^{\wedge 2} \cong (S^n)^{\wedge 2}$$

for each $n > 0$. By desuspending the composite above n times, we get a stable map

$$\Delta : \Sigma^{-(n-1)}(S^{n-1})^{\wedge 2} \to \Sigma^{-n}(S^n)^{\wedge 2} \,. \tag{3.4.4}$$

Remember that $(-)^{\wedge 2}$ denotes the external 2-fold smash product with equivariance given by permutation of the factors. Thus, we are looking at a directed system in the stable equivariant category

$$S^0 \to \ldots \to \Sigma^{-n}(S^n)^{\wedge 2} \to \Sigma^{-n-1}(S^{n+1})^{\wedge 2} \to \ldots . \tag{3.4.5}$$

We have the following compatibility result.

Lemma 3.4.1. *The increasing system of subskeleta $S^{nL} \subset S^{\infty L}$ corresponds via the shearing isomorphism* (3.4.3) *to the system* (3.4.5)*.*

Proof. The following diagram of C_2-equivariant spaces commutes

$$\begin{array}{ccc} \mathbb{R} \oplus \mathbb{R}^{n-1} \oplus \mathbb{R}^{(n-1)L} & \longrightarrow & \mathbb{R}^n \oplus \mathbb{R}^{nL} \\ {\scriptstyle 1\oplus \mathrm{sh}_{n-1}} \downarrow \cong & & {\scriptstyle \mathrm{sh}_n} \downarrow \cong \\ \mathbb{R} \oplus \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1} & \longrightarrow & \mathbb{R}^n \oplus \mathbb{R}^n \end{array} \tag{3.4.6}$$

where the maps are given by

$$\begin{array}{ccc} (x_0, x, y) & \longmapsto & (x_0, x, 0, y) \\ \downarrow & & \downarrow \\ (x_0, x+y, x-y) & \longmapsto & (x_0, x+y, x_0, x-y) . \end{array} \tag{3.4.7}$$

After one-point compactifying every space in diagram (3.4.6), we get

$$\begin{array}{ccc} \Sigma^n S^{(n-1)L} & \longrightarrow & \Sigma^n S^{nL} \\ \downarrow \cong & & \downarrow \cong \\ \Sigma S^{n-1} \wedge S^{n-1} & \xrightarrow{\Sigma^n \Delta} & S^n \wedge S^n \end{array} \tag{3.4.8}$$

which yields the following stable diagram after desuspending n times:

$$\begin{array}{ccc} S^{(n-1)L} & \longrightarrow & S^{nL} \\ \downarrow \cong & & \downarrow \cong \\ \Sigma^{-n+1}(S^{n-1})^{\wedge 2} & \xrightarrow{\Delta} & \Sigma^{-n}(S^n)^{\wedge 2} \end{array} \tag{3.4.9}$$

The upper arrow is the inclusion of the $(n-1)$th skeleton into the nth skeleton and the lemma follows by the commutativity of the diagram. □

We may then give a very concrete model for X^{tG} in the case $G = C_2$.

Corollary 3.4.2. *Let X be a genuine C_2-equivariant spectrum indexed on a complete C_2-universe $\mathscr{U}$ and let $i : \mathscr{U}^{C_2} \to \mathscr{U}$ be the inclusion of the C_2-trivial universe. Then X^{tC_2} is naturally equivalent to the homotopy inverse limit*

$$\operatorname*{holim}_{n\to\infty} \Sigma\Sigma^n EC_{2+} \wedge_{C_2} (S^{-n})^{\wedge 2} \wedge i^* X\,.$$

The maps in the inverse limit system are given by the stable diagonal maps (3.4.5).

Proof. For any compact Lie group G we may assume that the skeleta $EG^{(n)} \subset EG$, and thus the skeleta $\widetilde{EG}^{(n)} \subset \widetilde{EG}$, are finite CW-complexes. In particular, $\widetilde{EG}^{(n)}$ is dualizable and there are natural equivalences of G-spectra

$$\operatorname{Map}(\widetilde{EG}^{(n)}, EG_+ \wedge X) \simeq_G D(\widetilde{EG}^{(n)}) \wedge EG_+ \wedge X$$

for all $n \geq 0$. Thus,

$$\begin{aligned} \Sigma \operatorname{Map}(\widetilde{EG}, EG_+ \wedge X) &\cong \operatorname*{holim}_{n\to\infty} \Sigma \operatorname{Map}(\widetilde{EG}^{(n)}, EG_+ \wedge X) \\ &\cong \operatorname*{holim}_{n\to\infty} \Sigma D(\widetilde{EG}^{(n)}) \wedge EG_+ \wedge X \end{aligned}$$

The spectra in the inverse limit system are C_2-free, so by (2.1.1) and the Adams transfer isomorphism (2.1.2) we get that

$$\begin{aligned} X^{tG} &= \Sigma \operatorname{Map}(\widetilde{EG}, EG_+ \wedge X)^G \\ &\simeq \operatorname*{holim}_{n\to\infty} \Sigma D(\widetilde{EG}^{(n)}) \wedge_G EG_+ \wedge i^* X \end{aligned}$$

When $G = C_2$ and $EC_2 = S(\infty L)$ with the explicit skeleton filtration (3.4.2), the proof follows by applying lemma 3.4.1. Indeed, we have

$$\begin{aligned} D(\widetilde{EC}_2^{(n)}) &= D(S^{nL}) \xleftarrow{\cong} D(\Sigma^{-n} S^n \wedge S^n) \\ &\cong \Sigma^n D(S^n) \wedge D(S^n) = \Sigma^n S^{-n} \wedge S^{-n}\,. \end{aligned}$$

The arrow denotes the isomorphism induced by the shearing isomorphism of diagram (3.4.9) after taking its Spanier-Whitehead dual. $\square$

Chapter 4

THH and the Segal conjecture

We recall the definition of the topological Hochschild homology spectrum $T(B)$ for an FSP or symmetric spectrum defined on spheres B. We then follow [19] and discuss how the fundamental map of cofiber sequences (3.1.4) simplifies because of the cyclotomic structure of $T(B)$. By the resulting diagram (4.1.2) we can state our strategy to prove theorem 0.0.3. This will be done in section 4.2.

In the last part of the chapter we will specialize the (co-)homological Tate spectral sequences for the case when $X = T(B)$ and $G = C_2$.

4.1 Topological Hochschild homology

We review some of the basic definitions from [19, §2]. Let B be a functor with smash products. Then one defines a $\mathbb{T}$-prespectrum by letting the Vth space be the realization of the simplicial space $THH(B; S^V)_\bullet$ with k-simplices

$$THH(B; S^V)_k = \operatorname*{hocolim}_{(x_0,\ldots,x_k)\in I^{k+1}} \operatorname{Map}(S^{x_0} \wedge \ldots \wedge S^{x_k}, B(S^{x_0}) \wedge \ldots B(S^{x_k}) \wedge S^V)\,. \tag{4.1.1}$$

The homotopy colimit is taken over the $(k+1)$-fold product of the category I of finite sets and injective maps.

The $\mathbb{T}$-equivariance comes from the fact that $THH(B; S^V)_\bullet$ is a cyclic space in the sense of Connes, thus the realization can be given a natural action of the circle group $\mathbb{T}$. Passing from prespectra to spectra, we get a genuine $\mathbb{T}$-spectrum which we denote by $T(B)$.

For any FSP or symmetric spectrum defined on spheres B, $T(B)$ is a cyclotomic spectrum in the sense of Madsen [7, Proposition 1.3]. That is, there is a natural equivalence of $\mathbb{T}$-spectra $\rho^*_{C_p}\Phi^{C_p}T(B) \simeq T(B)$. Moreover, there is an equivalence $[\widetilde{EC}_p \wedge X]^{C_p} \simeq \Phi^{C_p} X$ for any $\mathbb{T}$-spectrum X. With these equivalences, we can write diagram (3.1.4) with $X = T(B)$ in the following form

$$\begin{array}{ccccc} T(B)_{hC_{p^n}} & \longrightarrow & T(B)^{C_{p^n}} & \longrightarrow & T(B)^{C_{p^{n-1}}} \\ \| & & \downarrow{\scriptstyle \Gamma_n} & & \downarrow{\scriptstyle \hat{\Gamma}_n} \\ T(B)_{hC_{p^n}} & \longrightarrow & T(B)^{hC_{p^n}} & \longrightarrow & T(B)^{tC_{p^n}} . \end{array} \tag{4.1.2}$$

When B is a commutative symmetric spectrum defined on spheres or a commutative FSP, $T(B)$ will be a ring spectrum and the maps Γ_n and $\hat{\Gamma}_n$ will be multiplicative.

Bökstedt introduced a homology type spectral sequence

$$E^2 \cong HH_*(H_*(B)) \Rightarrow H_*(T(B)) \tag{4.1.3}$$

starting with Hochschild homology and converging to the homology groups of $T(B)$. This is a first quadrant spectral sequence when B is bounded below. If B is of finite type, it follows that $T(B)$ is bounded below and of finite type as well.

4.2 The Segal conjecture for groups of prime order

The Segal conjecture for cyclic p-groups of order p^n can be formulated as a homotopy limit problem, namely showing that when $B = S$ the map Γ_n is a p-adic equivalence. The right hand square of diagram (4.1.2) is homotopy Cartesian, so this is equivalent to showing that $\hat{\Gamma}_n$ is a p-adic equivalence.

In the present work we will be concerned with the case $n = 1$ only. Thus, we will consider the diagram (4.1.2) when $n = 1$. Since this diagram is the most important diagram in this document, it deserves to be written explicitly. For brevity, we let $\gamma = \hat{\Gamma}_1$ and $\Gamma = \Gamma_1$.

$$\begin{array}{ccccc} T(B)_{hC_p} & \longrightarrow & T(B)^{C_p} & \longrightarrow & T(B) \\ \| & & \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \gamma} \\ T(B)_{hC_p} & \longrightarrow & T(B)^{hC_p} & \longrightarrow & T(B)^{tC_p} \end{array} \tag{4.2.1}$$

Since $T(B)$ is bounded below and of finite type, we may take the constant tower to express $T(B)$ as the homotopy inverse limit of bounded below spectra of finite type. In this case $H_*(T(B)) = H^c_*(T(B))$ and the map γ induces a map of continuous homology

$$\gamma_* : H_*(T(B)) \to H^c_*(T(B)^{tC_p}) \tag{4.2.2}$$

and continuous cohomology

$$\gamma^* : H^*_c(T(B)^{tC_p}) \to H^*(T(B))\,. \tag{4.2.3}$$

The Segal conjecture for cyclic groups of order p is solved in (at least) two steps. In this case, $B = S$, and the problem is to show that the map $\gamma : T(S) \to T(S)^{tC_p}$ is a p-adic equivalence. Then by the inverse limit of Adams spectral sequences (1.1.2), it is enough to show that γ^* induces an isomorphism

$$\gamma^* : \mathrm{Ext}_A(H^*(T(S)), \mathbb{F}_p) \to \mathrm{Ext}_A(H^*_c(T(S)^{tC_p}), \mathbb{F}_p) \tag{4.2.4}$$

of Ext-groups. This was accomplished by Lin for $p = 2$ [23] and Gunawardena for $p > 2$ [1] using calculations in the category of modules over the mod p Steenrod algebra.

The strategy for proving theorem 0.0.3 is to follow this strategy by replacing S by one of the S-algebras BP or $BP\langle m-1\rangle$.

It turns out that $H^*_c(T(S)^{tC_p})$ is isomorphic to the Singer construction on the A-module $\mathbb{F}_p$. We will come back to the isomorphism (4.2.4) when we introduce this construction in chapter 5.

For $n > 1$, the Segal conjecture was settled by Carlsson, using an inductive method in homotopy. This approach was generalized by Tsalidis [27].

4.3 The Tate spectral sequences for $THH(B)$

We specialise the spectral sequences from section 3.3.1 to the case when $X = T(B)$ is the topological Hochschild homology of an S-algebra B, and $G = C_2$, the cyclic group of order two. We assume that B is bounded below and of finite type over $\mathbb{F}_2$.

We start by identifying the $\hat{E}^2(T(B))$-terms. Since the C_2-action on $T(B)$ factors through the circle action

$$\lambda : \mathbb{T}_+ \wedge T(B) \to T(B)\,. \tag{4.3.1}$$

The induced action on homology $H_*(T(B))$ is trivial since $\mathbb{T}$ is path connected. Hence

$$\widehat{H}^{-*}(C_2; H_*(T(B)) \cong \widehat{H}^{-*}(C_2; \mathbb{F}_2) \otimes H_*(T(B)) . \qquad (4.3.2)$$

The Tate cohomology groups with trivial coefficients can be calculated explicitly [14]

$$\widehat{H}^*(C_2; \mathbb{F}_2) \cong P(u, u^{-1}) . \qquad (4.3.3)$$

This is a graded algebra with $deg(u) = -1$. See [14, p.250–252].

The spectral sequence $\hat{E}^r(T(S))$ is particularly simple, since $H_*(T(S)) \cong H_*(S)$ is concentrated in degree zero. Indeed, $\hat{E}^2(T(S)) \cong P(u, u^{-1})$ is concentrated on the horizontal axis. Thus, there is no room for further differentials and the spectral sequence collapses on the $\hat{E}^2$-term.

The unit $\eta : S \to B$ induces a map from the equivariant sphere spectrum $T(\eta) : S_{C_2} = T(S) \to T(B)$, and further a map

$$T(\eta)^{tC_2} : \hat{E}^r(T(S)) \to \hat{E}^r(T(B)) \qquad (4.3.4)$$

of spectral sequences. This map is given on the $\hat{E}^2$-terms as the map sending u^s in bidegree $(-s, 0)$ to the class $u^s \otimes 1$, for all $s \in \mathbb{Z}$.

By naturality of the map (4.3.4), this implies that the classes $u^s \otimes 1 \in \hat{E}^2_{-s,0}(T(B))$ are all infinite cycles, i.e. they can not support differentials.

We specialize propositions 3.3.4 and 3.3.6 in the case $X = T(B)$:

Proposition 4.3.1. *Let B be a bounded below S-algebra of finite type over $\mathbb{F}_2$.*

Then $T(B)^{tC_2}$ is equivalent to a homotopy inverse limit of a tower of bounded below spectra of finite type. There is an A_*-comodule, homology type Tate spectral sequence converging strongly to the continuous homology of $T(B)^{tC_2}$ as a completed A_*-comodule.

The homological Tate spectral sequence has $\hat{E}^2$-term

$$\hat{E}^2_{*,*}(T(B)) = P(u, u^{-1}) \otimes H_*(T(B); \mathbb{F}_2)) \Rightarrow H^c_*(T(B)^{tC_2}; \mathbb{F}_2) \qquad (4.3.5)$$

and the classes $u^s \otimes 1$ in bidegree $(-s, 0)$ are infinite cycles for all $s \in \mathbb{Z}$.

Further, if B is a commutative S-algebra, then $T(B)$ is an S-algebra and the spectral sequence becomes an A_*-comodule algebra spectral sequence. In this case the differentials are derivations with respect to the product from the Tate cohomology groups.

By the duality isomorphism (3.2.1) of Tate groups, we have that $\widehat{H}_{-*}(C_2;\mathbb{F}_2) \cong \widehat{H}^{*-1}(C_2;\mathbb{F}_2)$. By this isomorphism, we then let

$$H_{-*}(C_2;\mathbb{F}_2) \cong \Sigma P(v, v^{-1}) \qquad \deg(v) = 1$$

where Σv^{s-1} is dual to u^{-s}.

Dual to proposition 4.3.1, we have the following specialization of proposition 3.3.5:

Proposition 4.3.2. *Let B be a bounded below S-algebra of finite type over $\mathbb{F}_2$.*

Then $T(B)^{tC_2}$ is equivalent to a homotopy inverse limit of a tower of bounded below spectra of finite type. There is an A-module, cohomology type Tate spectral sequence converging strongly to the continuous cohomology of $T(B)^{tC_2}$ as an A-module. The cohomological Tate spectral sequence has $\hat{E}_2$-term

$$\hat{E}_2^{s,t}(T(B)) = \Sigma P(v, v^{-1}) \otimes H^*(T(B);\mathbb{F}_2) \Rightarrow H^*_c(T(B)^{tC_2};\mathbb{F}_2)\,. \tag{4.3.6}$$

The cohomological Tate spectral sequence is dual to the homological Tate spectral sequence in the sense that $\hat{E}^r_{*,*}$ is dual to $\hat{E}_r^{*,*}$ in each bidegree for all r and that the cohomological differential $d_r : \hat{E}_r^{s,t} \to \hat{E}_r^{s+r,t-r+1}$ is dual to the homological differential $d^r : \hat{E}^r_{s+r,t-r+1} \to \hat{E}^r_{s,t}$ for all s, t and $r \geq 1$.

In particular, none of the classes $\Sigma v^{s-1} \otimes 1$ in bidegree $(s, 0)$ are hit by any differentials.

We end this section with a note on the map γ.

Proposition 4.3.3. *The map $\gamma : T(S) \to T(S)^{tC_p}$ induces a non-trivial map of continuous homology groups taking the unit in $H_*(T(S))$ to the unit in $H^c_*(T(S)^{tC_p})$.*

Proof. Let S denote the equivariant sphere spectrum and let $C = C_p$ be the cyclic group of order p. We use that the equivariant sphere spectrum is split (see [18, §1]). This means that there is a map $s : S \to S^C$ such that s followed by the inclusion of the fixed points $F : S^C \to S$ is a non-equivariant equivalence.

The Frobenius map F can be factored as the composition $S^C \xrightarrow{\Gamma} S^{hC} \to S$ where $S^{hC} \to S$ is the map forgetting equivariance.

We get an extension of diagram (4.2.1)

$$\begin{array}{ccccccc}
 & & S & & & & \\
 & & \downarrow s & \searrow \simeq & & & \\
S_{hC} & \xrightarrow{R} & S^C & \longrightarrow & S & & \\
\| & & \downarrow \Gamma & & \downarrow \gamma & & \\
S_{hC} & \xrightarrow{N^h} & S^{hC} & \longrightarrow & S^{tC} & & \\
\| & & \downarrow & & \downarrow & & \\
S_{hC} & \xrightarrow{trf} & S & \xrightarrow{g} & S^{tC}[0] & \longrightarrow & \Sigma S_{hC}
\end{array}$$

where the middle vertical composite is an equivalence. We know from the homological Tate spectral sequence for $T(S)^{tC}$ that $H_0(S^{tC}[0]) \cong H_0^c(S^{tC}) \cong \mathbb{F}_p$. To show that the map γ_* is non-trivial on continuous homology groups, it suffices to show that the map g_* is surjective. This follows from the long exact sequence in homology since S_{hC} is 0-connected. □

4.4 The Tate spectral sequence for $G = \mathbb{T}$

In addition to what we have developed, there is also a Tate spectral sequence for the full circle group $\mathbb{T}$. We will state its properties and relate it to our case. This spectral sequence can be derived from a related homological homotopy fixed points spectral sequence discussed in [12].

Proposition 4.4.1. *Let B be a bounded below S-algebra of finite type over $\mathbb{F}_2$.*

Then $T(B)^{\mathbb{T}}$ is equivalent to a homotopy inverse limit of a tower of bounded below spectra of finite type. There is an A_*-module, homology type Tate spectral sequence converging strongly to the continuous homology of $T(B)^{t\mathbb{T}}$ as a completed A_*-comodule.

The homological Tate spectral sequence is concentrated in even columns and has $\hat{E}^2$-term

$$\hat{E}^2_{*,*}(T(B)) = P(t, t^{-1}) \otimes H_*(T(B); \mathbb{F}_2)) \Rightarrow H^c_*(T(B)^{t\mathbb{T}}; \mathbb{F}_2)\,. \qquad (4.4.1)$$

The classes $t^n \otimes 1$ have bidegree $(-2n, 0)$ and are infinite cycles for all $s \in \mathbb{Z}$.

Further, let $\sigma : H_*(T(B)) \to H_{*+1}(T(B))$ be the map sending $\alpha \in H_*(T(B))$ to the homology class $\lambda_*([e] \otimes \alpha)$, where λ is the circle action (4.3.1).

Then for $\alpha \in H_*(T(B))$, the d^2-differential is given by

$$d^2(t^n \otimes \alpha) = t^{n+1} \otimes \sigma(\alpha)\,. \tag{4.4.2}$$

If B is a commutative S-algebra, then $T(B)$ is an S-algebra and the spectral sequence becomes an A_*-comodule algebra spectral sequence. In this case the differentials are derivations with respect to the product from the Tate cohomology groups.

Let X be a $\mathbb{T}$-equivariant spectrum and let $C = C_2 \subset \mathbb{T}$. There is an equivalence $X^{tC} \simeq \mathrm{Map}(\mathbb{T}/C_+, \mathrm{Map}(\widetilde{E\mathbb{T}}, E\mathbb{T}_+ \wedge X))^{\mathbb{T}}$. The collapse map $\mathbb{T}/C \to \mathbb{T}/\mathbb{T}$ and the stable $\mathbb{T}$-transfer $\Sigma(\mathbb{T}/\mathbb{T}_+) \to \mathbb{T}/C_+$ define restriction and Verschiebung-maps of Tate-spectra $X^{t\mathbb{T}} \to X^{tC} \to \Sigma^{-1}X^{t\mathbb{T}}$.

On homological Tate spectral sequences for $X = T(B)$ this corresponds to the maps that respectively injects even columns and projects onto odd columns. Since there are no odd differentials in the $\mathbb{T}$-Tate spectral sequence, it follows from the fact that the differentials are derivations that there can be no odd differentials in the C_2-Tate spectral sequences either. Thus, we have a short exact sequence of continuous homology groups:

$$0 \to H_*^c X^{t\mathbb{T}} \to H_*^c X^{tC_2} \to \Sigma^{-1} H_*^c X^{t\mathbb{T}} \to 0 \tag{4.4.3}$$

We will see that this sequence is not split as completed A_*-comodules.

From proposition 4.4.1 and (4.4.3), we also get that the d^2-differentials in the homological C_2-Tate spectral sequence is given by

$$d^2(u^s \otimes \alpha) = u^{s+2} \otimes \sigma(\alpha)\,. \tag{4.4.4}$$

4.5 Homotopy fixed point spectral sequences

In addition to the Tate spectral sequence there is also a homological homotopy fixed point spectral sequence.

By the skeleton filtration of EG we get a tower of spectra

$$X^{hG} \to \ldots \to X^{hG}[n-1] \to X^{hG}[n] \to \ldots \to X^{hG}[0] \simeq X \tag{4.5.1}$$

expressing X^{hG} as the homotopy inverse limit. This tower and the associated *homological* homotopy fixed point spectral sequence was studied in [12] and relates by the norm sequence (4.2.1) to our case.

If X is a bounded below S-algebra of finite type, then it makes sense to talk about the continuous homology groups $H^c_*(X^{hG})$ and we have the following result from [12, Proposition 2.4]:

Proposition 4.5.1. *Let X be as above. Then there is a homological homotopy fixed point spectral sequence concentrated in the left half-plane above some horizontal line with*

$$E^2_{s,t} \cong H^{-s}(G; H_t(X)) \Rightarrow H^c_*(X^{hG})$$

converging strongly to the continuous homology of X^{hG}.

The norm map (4.2.1) *induces a map of spectral sequences that includes the half-plane into the whole plane.*

Chapter 5

The Singer construction

In section 5.2 we will review the definition and some important properties of the Singer construction at the prime $p = 2$. We will in section 5.1 briefly recall some facts about the mod 2 Steenrod algebra, its dual and the finite sub Hopf algebras A_m and E_m.

For any A-module M the Singer construction R_+M is an A-module resembling the A-module $A \otimes M$ in some ways. For one, there is an evaluation map $\epsilon : R_+M \to M$ analogous to the evaluation map $A \otimes M \to M$ sending $\mathrm{Sq}^n \otimes x \mapsto \mathrm{Sq}^n(x)$.

The Singer construction appeared originally in [25], but the work presented here concentrates on its relation to the work of Lin for the case of the Segal conjecture for the group C_2. A published account of this work is found in [23] where the calculation of Lin shows that the induced map $\epsilon^* : \mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}_A(R_+\mathbb{F}_2, \mathbb{F}_2)$ is an isomorphism. A further study appears in [1], where a more conceptual definition of the Singer construction is given.

For the purposes of chapter 10 and theorem 10.1.1, we need only the property that ϵ^* induces an Ext-isomorphism.

In section 5.4 we will use the language of [1] to restate the main technical lemma of [23] in a slightly more general way. Essentially, we will show that when M is a cyclic A_m-module, we have a short exact sequence of left A-modules

$$A \otimes_{A_m} K \to A \otimes_{A_m} R_+M \xrightarrow{q} \bigoplus_{j \geq 0} \Sigma^{j2^{m+1}} A \otimes_{A_{m-1}} M \qquad (5.0.1)$$

where the kernel $A \otimes_{A_m} K$ is generated over A by classes of negative degree. This result is a direct generalization of [23, Lemma 1.3], where the case $M = \mathbb{F}_2$ is considered. We will use the ideas of the cited paper to derive lemma 5.4.2.

This result will play a central role in chapter 11, where the quotient map q of (5.0.1) will arise in connection with the Segal conjecture for $T(BP\langle m\rangle)$. In fact, the map $\gamma^* : H_c^*T(BP\langle m\rangle)^{tC_2} \to H^*T(BP\langle m\rangle)$ will turn out to be built up from copies of the map q in diagram (5.0.1). At the end of chapter 11, we wish to state a co-connectivity result for $\pi_*(\gamma)$. This will be done by arguing in terms of Ext-groups, and section 5.5 will supply the technical tool needed for the proof of theorem 11.3.4.

To make the link with topology, we will in section 5.6 review a construction from [11] where the Singer construction on the $\mathbb{F}_p$-cohomology of a spectrum is given as a certain Tate construction involving the extended powers from definition 2.2.1.

Finally, in section 5.7 we will define a homological version of the Singer construction.

5.1 The mod 2 Steenrod algebra

Let A denote the mod 2 Steenrod algebra. We recall some facts from [24]. As an algebra, A is generated by the squaring operations Sq^n for all $n \geq 1$ modulo the Adem relations. There is also a coproduct $\mathrm{Sq}^n \mapsto \sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$, making A into a connected cocommutative Hopf algebra.

The dual Steenrod algebra A_* is a commutative Hopf algebra. As an algebra it is isomorphic to the polynomial algebra $P(\xi_n | n \geq 0)$ where ξ_n has degree $2^n - 1$. The coproduct is given by $\psi(\xi_n) = \sum_{i+j=n} \xi_i^{2^j} \otimes \xi_j$.

As a Hopf algebra, A is the union of an increasing system of finite sub Hopf algebras

$$A_0 \subset A_1 \subset \dots A_m \subset \dots \subset A$$

where A_m is generated as an algebra by the elements $\{\mathrm{Sq}^{2^n} | 0 \leq n \leq m\}$. The dual Hopf algebra A_{m*} is a quotient of A_*. Namely, let $I(m) \subset A_*$ be the Hopf ideal

$$I(m) = (\xi_1^{2^{m+1}}, \xi_2^{2^m}, \dots, \xi_m^4, \xi_{m+1}^2, \xi_k | k \geq m+1)\,. \tag{5.1.1}$$

Then $A_{m*} = A_*/I(m)$.

For each $m \geq 0$ we also have the sub Hopf algebra $E_m \subset A$, generated as an algebra by the Milnor primitives $\{Q_n | 0 \leq n \leq m\}$. The generators are given recursively by $Q_n = [\mathrm{Sq}^{2^n}, Q_{n-1}]$ and $Q_0 = \mathrm{Sq}^1$. Moreover, we have that $Q_n^2 = 0$, so E_m is an exterior algebra for each $m \geq 0$. Dually, E_{m*} is isomorphic to the quotient of A_* by the Hopf ideal

$$H(m) = (\xi_1^2, \dots, \xi_m^2, \xi_k | k \geq m+1)\,. \tag{5.1.2}$$

Let $B \subset A$ be a sub Hopf algebra. Regarding $\mathbb{F}_2$ as an B-module by the augmentation map $B \to \mathbb{F}_2$, we let $IB \subset B$ be the augmentation ideal. Then by $A/\!\!/B$ we mean the quotient $A/A \cdot IB = A \otimes_B \mathbb{F}_2$. In particular this is a left A-module.

5.2 Basic construction

The Singer construction is an endofunctor on the category of modules over the Steenrod algebra. In [11] the value of the functor on a module M is denoted $R_+(M)$ and is isomorphic as an $\mathbb{F}_2$-vector space to $\Sigma\mathbb{F}_2[v, v^{-1}] \otimes M$ where $\mathbb{F}_2[v, v^{-1}]$ is the ring of Laurent polynomials on a generator v of degree one.

The action of the Steenrod squares are given explicitly by

$$\mathrm{Sq}^s\Sigma(v^r \otimes x) = \sum_i \tbinom{r-i}{s-2i}\Sigma v^{r+s-i} \otimes \mathrm{Sq}^i x \tag{5.2.1}$$

for all $x \in M$.

An important property of this functor is that it comes equipped with a natural transformation $\epsilon : R_+(M) \to M$ of A-modules, given by the rule $\Sigma v^{r-1} \otimes x \mapsto \mathrm{Sq}^r x$.

Definition 5.2.1. *A map of A-modules $L \to M$ is a Tor-equivalence if the induced map*

$$\mathrm{Tor}^A_{**}(\mathbb{F}_p, L) \to \mathrm{Tor}^A_{**}(\mathbb{F}_p, M) \tag{5.2.2}$$

is an isomorphism.

The relevance of this is the following.

Proposition 5.2.2 ([1]). *If $L \to M$ is a Tor-equivalence then for every bounded above right A-module K the induced map*

$$\mathrm{Tor}^A_{**}(K, L) \to \mathrm{Tor}^A_{**}(K, M) \tag{5.2.3}$$

is an isomorphism. Moreover, for every bounded below left A-module N of finite type, the induced map

$$\mathrm{Ext}_A^{**}(M, N) \to \mathrm{Ext}_A^{**}(L, N) \tag{5.2.4}$$

is an isomorphism.

The Singer construction and the evaluation ϵ can be defined for odd primes as well, and the following important result holds for all primes:

Theorem 5.2.3 (Gunawardena, Miller [1]). *The evaluation map ϵ is a Tor-equivalence.*

Corollary 5.2.4. *Let M, N be any left A-modules such that N is bounded below and of finite type over $\mathbb{F}_p$. Then any A-linear map $f : R_+(M) \to N$ factors uniquely as $\bar{f} \circ \epsilon$ for some A-linear homomorphism $\bar{f} : M \to N$.*

Proof. Since N is bounded below and of finite type, a special case of theorem 5.2.3 and proposition 5.2.2 says that $\epsilon^* : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(R_+(M), N)$ is an isomorphism, and the corollary follows. □

Remark: A special case of this occurs when $N = M$ is a cyclic A-module. Then $\mathbb{F}_p \cong \mathrm{Hom}_A(M, M) \cong \mathrm{Hom}_A(R_+(M), M)$, so any A-linear map $R_+(M) \to M$ is an $\mathbb{F}_p$-multiple of ϵ.

5.3 The definition of Adams-Gunawardena-Miller

Whenever we have a left A_{m-1}-module M we may induce it up to an A-module by tensoring with A from the left over A_{m-1}. In [1], the Singer module is defined in a way resembling this construction. We will review the definition when $p = 2$. This case is omitted in the cited paper, but is included indirectly in the predecessor [23] where the ideas originally appeared.

Let A_* be the dual Steenrod algebra at the prime 2 and let $J(m)$ be the ideal

$$J(m) = (\xi_2^{2^m}, \xi_3^{2^{m-1}}, \ldots, \xi_m^4, \xi_{m+1}^2, \xi_{m+2}, \ldots)$$

of A_*. Define $B_{m*} = A_*/J(m)$ and let $B'_{m*} = B_{m*}[\xi_1^{-1}]$ be B_{m*} localized by inverting ξ_1. The quotient B_{m*} is a left A_{m*}-comodule and a right A_{m-1*}-comodule. The bi-comodule structure is inherited from the coproduct on A_*.

Multiplication by $\xi_1^{2^{m+1}}$ respects the bi-comodule structure and gives a morphism of bi-comodules $B_{m*} \to B_{m*}$.

The localized quotient B'_{m*} can be obtained as the colimit of the sequential limit system

$$B_{m*} \xrightarrow{\cdot \xi_1^{2^{m+1}}} B_{m*} \longrightarrow \cdots \longrightarrow B'_{m*} \tag{5.3.1}$$

where the maps are multiplication by $\xi_1^{2^{m+1}}$, and thus B'_{m*} inherits the structure of an A_{m*}-A_{m-1*} bi-comodule.

Both B_{m*} and B'_{m*} are of finite type, and their duals will be denoted by B_m and B'_m respectively. These will then be A_m-A_{m-1} bi-modules.

The canonical maps of ideals $(0) \to J(m+1) \to J(m)$ induce natural surjective maps

$$A_* \to B_{m+1*} \to B_{m*} \tag{5.3.2}$$

of A_{m*}-A_{m-1*} bi-comodules. Dually we get injections of A_m-A_{m-1} bimodules

$$B_m \hookrightarrow B_{m+1} \hookrightarrow A\,. \tag{5.3.3}$$

Inverting ξ_1, (5.3.2) gives the following commutative diagram of A_{m*}-A_{m-1*} bi-comodules for $m \geq 0$:

$$\begin{array}{ccc} B'_{m+1*} & \longrightarrow & B'_{m*} \\ \uparrow & & \uparrow \\ B_{m+1*} & \longrightarrow & B_{m*} \\ \uparrow & \nearrow & \\ A_* & & \end{array} \tag{5.3.4}$$

Dually, we get the following commutative diagram of A_m-A_{m-1} bimodules:

$$\begin{array}{ccc} B'_m & \longrightarrow & B'_{m+1} \\ \downarrow & & \downarrow \\ B_m & \longrightarrow & B_{m+1} \\ & \searrow & \downarrow \\ & & A \end{array} \tag{5.3.5}$$

For reasons to appear later, we give the top left vertical map of this diagram the name $q_0 : B'_m \to B_m$.

Let M be an A-module of finite type over $\mathbb{F}_2$. Then from the upper row of (5.3.5) we get maps of left A_m-modules

$$B'_m \otimes_{A_{m-1}} M \to B'_{m+1} \otimes_{A_{m-1}} M \to B'_{m+1} \otimes_{A_m} M \tag{5.3.6}$$

for all $m \geq 0$. The Singer construction on M is defined as the colimit over these maps as m tends to infinity:

$$R_+(M) = \operatorname*{colim}_m B'_m \otimes_{A_{m-1}} M\,. \tag{5.3.7}$$

The evaluation map $\epsilon : R_+(M) \to M$ is constructed from diagram (5.3.5) by taking the colimit

$$\epsilon : R_+(M) = \operatorname*{colim}_m B'_m \otimes_{A_{m-1}} M \to \operatorname*{colim}_m A \otimes_{A_{m-1}} M \cong M\,.$$

We end this section by emphasizing some facts about the bi-modules B'_m and B_m.

For $p \neq 2$, the following lemma is the content of [1, Lemma 2.1] and its proof for $p = 2$ follows directly from the proof in [1], making the standard modifications for the case $p = 2$. The proof of B'_m being a free right A_{m-1}-module in the case $p = 2$ can also be found in [23, proof of Lemma 2.4].

Lemma 5.3.1. *(i) B'_m is free as a left A_m-module. The elements dual to $\xi_1^{k2^{m+1}}$ for $k \in \mathbb{Z}$ may be taken as a base.*

(ii) B'_m is free as a right A_{m-1}-module. The elements dual to ξ_1^k for $k \in \mathbb{Z}$ may be taken as a base.

The second part of the lemma is equivalent to saying that the composite map $B'_0 \otimes A_{m-1} \to B'_m \otimes A_{m-1} \to B'_m$ is an isomorphism of right A_{m-1}-modules.

Following the discussion in [1, right below Lemma 2.1], we note that for M an A_m-module, the map

$$B'_m \otimes_{A_{m-1}} M \to B'_{m+1} \otimes_{A_m} M$$

is an isomorphism of left A_m-modules. This is true since, when considered as groups, both sides are isomorphic to $B(0) \otimes M$.

This means in particular that when considering the left A_m-module structure of $R_+(M)$, we can take $B'{}_m \otimes_{A_{m-1}} M$ as our model. Note that when M is just a left A_{m-1}-module, tensoring with $B'{}_m$ over A_{m-1} gives a functor from left A_{m-1}-modules to left A_m-modules, which we also denote by $R_+(-)$. This Singer construction on A_{m-1}-modules fits into the commutative diagram of module categories

$$\begin{array}{ccc} A - \mathrm{Mod} & \xrightarrow{R_+(-)} & A - \mathrm{Mod} \\ \downarrow & & \downarrow \\ A_{m-1} - \mathrm{Mod} & \xrightarrow{R_+(-)} & A_m - \mathrm{Mod} \end{array}$$

where the vertical arrows represent the obvious forgetful functors. The fact that the Singer construction takes A_{m-1}-modules to A_m-modules will play a crucial part in our estimations in chapter 11.

It is important to note that lemma 5.3.1 does not state that B'_m is free as an A_m-A_{m-1} bi-module. See the remark on page 55.

5.4 Splitting

In the original paper [23] the authors study the Singer construction on the trivial module $\mathbb{F}_2$, where the latter is made into a left A-module by the augmentation map $A \to \mathbb{F}_2$ coming from the Hopf-algebra structure on A. Recall that the Singer-module $R_+(\mathbb{F}_2) = \operatorname{colim}_m B'_m \otimes_{A_{m-1}} \mathbb{F}_2$ is isomorphic to $\Sigma P(v, v^{-1})$ with v of degree one and with the action of the Steenrod algebra as given by the formula (5.2.1).

Our notation differs from the one in the cited paper by a suspension. In the original context the Singer functor is defined to be a desuspension of our $R_+(-)$.

We will review some of the work done in [23]. Let $\Sigma P = R_+(\mathbb{F}_2) = \Sigma P(v, v^{-1})$. For a fixed m, one then studies the cohomological object P as the inverse limit of a tower of quotients

$$P \to \ldots \to P/F_{-k2^{m+1}-1,m} \to P/F_{(-k+1)2^{m+1}-1,m} \to \ldots \to 0 \qquad (5.4.1)$$

where $F_{r,m} \subset P$ is the left A_m submodule of P generated by classes of degree strictly less than r.

The main technical result in the cited paper is that after inducing up from left A_m-modules to left A-modules, each of the finite stages in the tower of quotients splits as a sum of cyclic A-modules:

Lemma 5.4.1. *([23], lemma 1.3) There is a splitting of left A-modules*

$$A \otimes_{A_m} P/F_{-1,m} \cong \bigoplus_{j \geq 0} \Sigma^{j2^{m+1}-1} A /\!\!/ A_{m-1}\,.$$

This lemma has the following generalization:

Proposition 5.4.2. *Let M be a left A_{m-1}-module of finite type. There is a short exact sequence of left A-modules*

$$0 \to A \otimes_{A_m} K \to A \otimes_{A_m} R_+(M) \to \bigoplus_{j \geq 0} \Sigma^{j2^{m+1}} A \otimes_{A_{m-1}} M \to 0\,.$$

The kernel is given by $K = \ker q_0 \otimes_{A_{m-1}} M$.

If M *is generated over* A_{m-1} *by classes of degree less than or equal to zero, then the kernel* $A \otimes_{A_m} K$ *is generated over* A *by classes in degree less than or equal to* -2^{m+1}.

The proof will given later.

The maps in the limit system (5.3.1) defining B'_{m*} are injections, and we get a filtration of the colimit consisting of A_{m*}-A_{m-1*} comodules $Q_{k*} \subset Q_{k-1*} \subset \ldots \subset Q_{0*} \subset B'_{m*}$ where Q_{k*} is explicitly given as the sub A_{m*}-A_{m-1*} bi-comodule $B_{m*}\{\xi_1^{k2^{m+1}}\} \cong \Sigma^{k2^{m+1}} B_{m*}$. Since we have inverted ξ_1, this filtration extends naturally to a bi-infinite filtration by also defining $Q_{k*} = B_{m*}\{\xi_1^{k2^{m+1}}\}$ for $k < 0$, so we have a limit system

$$0 \to \ldots \hookrightarrow Q_{2*} \hookrightarrow Q_{1*} \hookrightarrow Q_{0*} \hookrightarrow Q_{-1*} \hookrightarrow \ldots \hookrightarrow B'_{m*} \qquad (5.4.2)$$

of A_{m*}-A_{m-1*}-bicomodules with trivial inverse limit and B'_{m*} as its direct limit. We note that all the Q_{k*} are isomorphic as A_{m*}-A_{m-1*}-comodules, up to suspension.

The filtration quotients are recognized as suspensions of A_{m*}. Indeed the quotient Q_{0*}/Q_{1*} is isomorphic to the quotient in the short exact sequence

$$0 \to \Sigma^{2^{m+1}} A_*/J(m) \xrightarrow{\cdot \xi_1^{2^{m+1}}} A_*/J(m) \longrightarrow A_*/[J(m) + (\xi_1^{2^{m+1}})] \to 0$$

and $J(m) + (\xi_1^{2^{m+1}}) = I(m)$. Thus we have short exact sequences of A_{m*}-A_{m-1*} bi-comodules

$$0 \to Q_{k+1*} \to Q_{k*} \to \Sigma^{k2^{m+1}} A_{m*} \to 0 \qquad (5.4.3)$$

for all $k \in \mathbb{Z}$.

The colimit B'_{m*} is of finite type and we have a dual filtration

$$B'_m \to \ldots \to Q_{-1} \to Q_0 \to Q_1 \to Q_2 \to \ldots \to 0 \qquad (5.4.4)$$

of A_m-A_{m-1}-bimodules expressing B'_m as the inverse limit. This tower of quotients corresponds to the tower (5.4.1). In fact, there is an isomorphism of left A_m-modules $Q_k \otimes_{A_{m-1}} \mathbb{F}_2 \cong \Sigma P/F_{k2^{m+1}-1,m}$ for every $k \in \mathbb{Z}$.

Again we have $Q_k \cong \Sigma^{k2^{m+1}} B_m$ and the filtration kernels are given by the short exact sequences

$$0 \to \Sigma^{k2^{m+1}} A_m \to Q_k \to Q_{k+1} \to 0 \qquad (5.4.5)$$

for all $k \in \mathbb{Z}$.

Let M_* be the left A_{m-1*}-comodule dual to M. We may induce up the filtration (5.4.2) by cotensoring with A_* from the left using the left A_{m*}-comodule structure or cotensoring with M_* from the right using the right A_{m-1*}-comodule structure. Dually, we may tensor the filtration (5.4.4) with A from the left over A_m and M from the right over A_{m-1}. Any combination of these preserves exactness since Q_k is free as a right module over A_{m-1} for all k by lemma 5.3.1, and since A is flat over A_m.

Let $q_k : B'_m \to Q_k$ be the projection onto the kth stage in the tower (5.4.4). Then we have a short exact sequence of towers of A_m-A_{m-1} bi-modules:

$$\begin{array}{ccccccccc}
 & & \cdots \rightarrowtail & \ker(q_{-1}) & \rightarrowtail & \ker(q_0) & \rightarrowtail & \ker(q_1) & \rightarrowtail \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
B'_m & = & \cdots = & B'_m & = & B'_m & = & B'_m & = \cdots \\
\| & & & \downarrow{\scriptstyle q_{-1}} & & \downarrow{\scriptstyle q_0} & & \downarrow{\scriptstyle q_1} & \\
B'_m & \to & \cdots \to & Q_{-1} & \to & Q_0 & \to & Q_1 & \to \cdots
\end{array} \tag{5.4.6}$$

As noted, tensoring this diagram with M over A_{m-1} from the right preserves exactness, so we get a similar diagram of left A_m-modules:

$$\begin{array}{ccccc}
\cdots \rightarrowtail & \ker(q_{-1}) \otimes_{A_{m-1}} M & \rightarrowtail & \ker(q_0) \otimes_{A_{m-1}} M & \rightarrowtail \cdots \\
 & \downarrow & & \downarrow & \\
\cdots = & B'_m \otimes_{A_{m-1}} M & = & B'_m \otimes_{A_{m-1}} M & = \cdots \\
 & \downarrow{\scriptstyle q_{-1}\otimes 1} & & \downarrow{\scriptstyle q_0\otimes 1} & \\
\cdots \to & Q_{-1} \otimes_{A_{m-1}} M & \to & Q_0 \otimes_{A_{m-1}} M & \to \cdots
\end{array} \tag{5.4.7}$$

Note that the middle row is just $R_+(M)$ considered as a left A_m-module.

We will now look at the cohomological object $A \otimes_{A_m} B'_m$ as an A-A_{m-1}-bimodule.

Lemma 5.4.3. *The short exact sequences*

$$0 \to A_* \Box_{A_{m*}} Q_{k+1*} \to A_* \Box_{A_{m*}} Q_{k*} \to \Sigma^{k2^{m+1}} A_* \to 0$$

split as A_*-A_{m-1*} bi-comodules. Dually we have split short exact sequences of A-A_{m-1} bi-modules

$$0 \to \Sigma^{k2^{m+1}} A \xrightarrow{i} A \otimes_{A_m} Q_k \to A \otimes_{A_m} Q_{k+1} \to 0\,.$$

Proof. It suffices to prove the lemma in the cohomological situation. Moreover, it will be enough to consider the case $k = 0$ since the bi-modules Q_k are isomorphic up to suspension. Then $Q_0 = B_m$.

The inclusions of ideals $(0) \subset J(m) \subset I(m)$ give a surjection of A_{m*}-A_{m-1*} bi-comodules $A_* \to B_{m*} \to A_{m*}$. Dually, we have inclusions of A_m-A_{m-1} bi-modules $A_m \subset B_m \subset A$. These inclusions and the multiplication map ϕ on A give rise to the following commutative diagram of A-A_{m-1} bimodules

$$\begin{array}{ccc} A \otimes_{A_m} A_m & \xrightarrow{i} & A \otimes_{A_m} B_m \\ \downarrow{\cong} & & \downarrow \\ A & \xleftarrow{\phi} & A \otimes_{A_m} A\,. \end{array}$$

The top horizontal map is the injection in the short exact sequence of the lemma and the lower horizontal map is the multiplication. The lower right half of the diagram provides the needed splitting. □

Corollary 5.4.4. *For all $m \geq 0$ and all $k \in \mathbb{Z}$ there are isomorphisms of A-A_{m-1} bimodules*

$$A \otimes_{A_m} Q_k \cong \bigoplus_{j \geq k} \Sigma^{j2^{m+1}} A$$

The surjections $A \otimes_{A_m} Q_k \to A \otimes_{A_m} Q_{k+1}$ are compatible via these isomorphisms with the obvious projections, sending the summand $j = k$ to zero.

Proof. Using lemma 5.4.3 inductively l times, we get the splitting

$$A \otimes_{A_m} Q_n \cong \Big(\bigoplus_{j=n}^{n+l-1} \Sigma^{j2^{m+1}} A \Big) \oplus (A \otimes_{A_m} Q_{n+l})$$

Since A is connected and the connectivity of Q_{n+l} tends to infinity with l, this expression stabilizes in each degree at a finite stage, and we obtain the desired formula by passing to the colimit over l. This shows that each A-A_m-bimodule in the tower can be identified with a sum of suspensions of A.

The split in lemma 5.4.3 implies that there is a bimodule isomorphism of towers based at $L_n := A \otimes_{A_m} Q_n$

$$\begin{array}{ccccccc} \cdots \longrightarrow & A \otimes_{A_m} Q_{n-2} & \longrightarrow & A \otimes_{A_m} Q_{n-1} & \longrightarrow & L_n \\ & \downarrow{\cong} & & \downarrow{\cong} & & \| \\ \cdots \longrightarrow & \Big(\bigoplus_{n-2 \leq j < n} \Sigma^{j2^{m+1}} A \Big) \oplus L_n & \longrightarrow & \big(\Sigma^{(n-1)2^{m+1}} A \big) \oplus L_n & \longrightarrow & L_n \end{array}$$

Since the base L_n is isomorphic to $\bigoplus_{j \geq n} \Sigma^{j2^{m+1}} A$, the claim follows. □

Proof of Proposition 5.4.2. When considered as a left A_m-module we take $B'_m \otimes_{A_{m-1}} M$ as a model for $R_+(M)$.

Using the right A_{m-1}-module structure we get, by lemma 5.3.1, a short exact sequence of left A_m-modules

$$0 \to \ker(q_0) \otimes_{A_{m-1}} M \to B'_m \otimes_{A_{m-1}} M \overset{q_0 \otimes 1}{\to} Q_0 \otimes_{A_{m-1}} M \to 0 \qquad (5.4.8)$$

by tensoring with M from the right. The exact sequence in proposition 5.4.2 now follows by tensoring (5.4.8) with A from the left over A_m. This produces again an exact sequence since A is flat over A_m. By corollary 5.4.4, the quotient is isomorphic as a left A-module to the required sum of suspended copies of $A \otimes_{A_{m-1}} M$. The kernel is equal to $A \otimes_{A_m} K$ where $K = \ker(q_0) \otimes_{A_{m-1}} M$. We claim that K is generated as a left A_m-module from classes in degrees less than or equal to -2^{m+1}. From this the last statement of the lemma follows.

Both B'_m and $Q_0 = B_m$ are free as left A_m-modules. Indeed, by lemma 5.3.1, we may identify $B'_{m*} \cong A_{m*}[\xi_1^{2^{m+1}}, \xi_1^{-2^{m+1}}]$ and $B_{m*} \cong A_{m*}[\xi_1^{2^{m+1}}]$ such that on duals $q_0 : B'_m \to B_m$ is identified with the obvious projection. Thus, we can identify the quotient map q_0 as canonical projection in the short exact sequence of left A_m-modules

$$0 \to \ker(q_0) \to \bigoplus_{j \in Z} \Sigma^{j2^{m+1}} A_m \to \bigoplus_{j \geq 0} \Sigma^{j2^{m+1}} A_m \to 0\,.$$

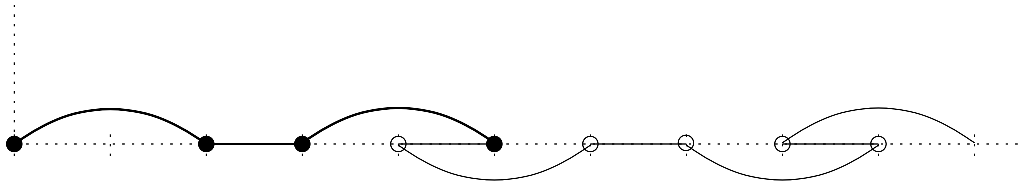
Thus $\ker(q_0) \cong \bigoplus_{j<0} \Sigma^{j2^{m+1}} A_m$ as a left A_m-module. In particular we see that $\ker(q_0)$ has all its left A_m-generators in degree less than or equal to -2^{m+1}.

When M is generated by classes of degree less than or equal to zero, $K = \ker(q_0) \otimes_{A_{m-1}} M$ is generated as a left A_m-module by classes in degrees less than or equal to -2^{m+1}. □

Remark: Note that we do not have the bimodule splitting of lemma 5.4.3 prior to inducing up to left A-modules. To see this, consider the case $m = 1$. Let $\mathbb{F}_2$ be the trivial left A_0-module concentrated in degree zero. Tensoring with $\mathbb{F}_2$ over A_0 from the right produces a short exact sequence

$$0 \to A_1/\!\!/A_0 \to B_1/\!\!/A_0 \to \Sigma^4 B_1/\!\!/A_0 \to 0\,.$$

As comodules, we have $B_{1*} \cong P(\xi_1) \otimes E(\xi_2)$, $A_{0*} \cong E(\xi_1)$ and $B_{1*}\Box_{A_{0*}}\mathbb{F}_2 \cong P(\xi_1^4) \otimes \mathbb{F}_2\{1, \xi_1^2, \xi_1^3 + \xi_2, \xi_2\xi_1^2 + \xi_1^5\}$. The picture of $B_1/\!\!/A_0$ in cohomology looks like this



where the $A_1/\!/A_0$ sub-module is drawn in bold-face. The inclusion of this sub module can not allow a retraction because of the non-trivial Sq^1 originating from degree 4.

We do not have an isomorphism between $A \otimes_{A_m} B'_m$ and $\lim_k A \otimes_{A_m} Q_k$, as tensor products and limits do not commute in this case. The problem arises because A is not finite.

We end this section by some remarks on the relationship between these objects. We will not need these remarks in the rest of our work, but we include them for the sake of completeness.

Corollary 5.4.5. *The inverse limit of the system* $\{A \otimes_{A_m} Q_k\}_k$ *as* $k \to -\infty$ *is isomorphic to the infinite product*

$$\prod_{j\in\mathbb{Z}} \Sigma^{j2^{m+1}} A$$

as an A-A_{m-1}*-bimodule.*

Proof. By corollary 5.4.4, the inverse limit system in question is isomorphic to the system where the kth bimodule is $\bigoplus_{j\geq k} \Sigma^{j2^{m+1}} A$. Since A is bounded below, this sum is naturally isomorphic to the product of suspensions of A. The map from the kth to the $(k+1)$th filtration sends the bottom factor $\Sigma^{k2^{m+1}} A$ to zero and is the identity on the remaining factors. Thus, the inverse limit is isomorphic to the product. □

There are natural homomorphisms $\kappa_k : A \otimes_{A_m} B'_m \to A \otimes_{A_m} Q_k$ for all k, compatible with the maps in the inverse limit system over k. Thus, there is a unique map

$$\kappa : A \otimes_{A_m} B'_m \to \lim_{k\to-\infty} A \otimes_{A_m} Q_k \cong \prod_{j\in\mathbb{Z}} \Sigma^{j2^{m+1}} A$$

compatible with the maps κ_k.

Lemma 5.4.6. *The map* κ *is an injection.*

Proof. The Steenrod algebra A is an increasing union of the finite sub Hopf-algebras A_r. For each $r \geq m$ and each $s \in \mathbb{Z}$ we have the following diagram:

$$\begin{array}{ccccc} A \otimes_{A_m} B'_m & \xrightarrow{\kappa} & \lim\limits_k A \otimes_{A_m} Q_k & \longrightarrow & A \otimes_{A_m} Q_s \\ \uparrow & & \uparrow & & \uparrow \\ A_r \otimes_{A_m} B'_m & \xrightarrow{\cong} & \lim\limits_k A_r \otimes_{A_m} Q_k & \longrightarrow & A_r \otimes_{A_m} Q_s \end{array}$$

The lower left horizontal map is an isomorphism since A_r is a finitely generated free right A_m-module and inverse limits commute with finite sums. The right vertical map is injective since Q_s is free as a left A_m-module. The upper horizontal composition is κ_s.

Any $x \in A \otimes_{A_m} B'_m$ can be lifted to $x' \in A_r \otimes_{A_m} B'_m$ for a sufficiently large r. Assume x is contained in the kernel of κ. This happens if and only if x maps to zero under κ_s for all s. Since the right vertical map is injective, the lifting x' maps trivially to $A_r \otimes_{A_m} Q_s$ for all s. Thus x' maps trivially to $\lim_k A_r \otimes_{A_m} Q_k$ via the isomorphism, and hence $x' = 0$. □

We are really showing that κ can be factored as

$$\begin{aligned} A \otimes_{A_m} \lim_k Q_k &\cong \operatorname*{colim}_r \lim_k A_r \otimes_{A_m} Q_k \\ &\to \lim_k \operatorname*{colim}_r A_r \otimes_{A_m} Q_k \\ &\cong \lim_k A \otimes_{A_m} Q_k \end{aligned}$$

where the arrow represents the canonical homomorphism commuting limits and colimits. The argument shows that this homomorphism is injective as long as the maps $A_r \otimes_{A_m} Q_s \to A_{r+1} \otimes_{A_m} Q_s$ are injective for all s.

Remark: One might wonder what the injective image of κ looks like inside the infinite product. Identifying the tower $\{A \otimes_{A_m} Q_k\}_{k\in\mathbb{Z}}$ with the tower $\{\bigoplus_{j\geq k} \Sigma^{j2^{m+1}} A\}_{k\in\mathbb{Z}}$ of corollary 5.4.4 we see that $\operatorname{im}\kappa$ must contain the infinite direct sum $\bigoplus_{j\in\mathbb{Z}} \Sigma^{j2^{m+2}} A$. This follows since when restricted to $\operatorname{im}\kappa$, the maps at every stage in the tower should be surjections.

However, by the definition of the splittings in lemma 5.4.3 it follows that the element $1\otimes 1 \in A\otimes_{A_m} B'_m$ maps non-trivially by the composition $r_k(1\otimes q_k)$ in diagram (5.4.9) for every k sufficiently small.

$$\begin{array}{ccc} & & A \otimes_{A_m} B'_m \\ & & \downarrow{\scriptstyle 1\otimes q_k} \\ \Sigma^{k2^{m+1}} A & \hookrightarrow & A \otimes_{A_m} Q_k \\ & \underset{r_k}{\longleftarrow} & \end{array} \tag{5.4.9}$$

Hence the image of $1 \otimes 1$ inside the infinite product must be represented by an infinite sequence $(x_k)_{k\in\mathbb{Z}}$ where $x_k \neq 0$ for infinitely many k. Such an element is not in the direct sum-sub bimodule.

Considered as a left A_m-module only, we know that B'_m is free on generators in degrees equal to zero modulo 2^{m+1}. Hence, $A \otimes_{A_m} B'_m \cong \bigoplus_{j\in\mathbb{Z}} \Sigma^{j2^{m+1}} A$ as a left A-module. The point is that this isomorphism is not compatible with the right action.

Even though the structure of $A \otimes_{A_m} R_+(M)$ seems a bit entangled, we shall see that it will behave cohomologically as if it were a direct sum, when M is bounded above.

5.5 Cohomological properties of $A \otimes_{A_m} R_+(M)$

In later applications we will study maps of A-modules $A \otimes_{A_m} R_+(M) \to N$ that are surjective, but not injective. We will describe the kernel of these maps by lemma 5.4.2. The maps will arise as maps of continuous cohomology groups and the following result will be of interest to us in connection with the Caruso-May-Priddy-spectral sequence (1.1.2). We use the notation of lemma 5.4.2.

Proposition 5.5.1. *Assume that M is a bounded above left A_{m-1}-module and let K be the left A_m-module $\ker(q_0) \otimes_{A_{m-1}} M$. For all s,t there is an isomorphism*

$$\mathrm{Ext}_A^{s,t}(A \otimes_{A_m} K, \mathbb{F}_2) \cong \mathrm{Ext}_A^{s,t}\Big(\bigoplus_{j<0} \Sigma^{j2^{m+1}} A \otimes_{A_{m-1}} M, \mathbb{F}_2\Big)\,.$$

Proof. For $k \leq 0$, let $K^k = \ker(q_k) \otimes_{A_{m-1}} M$. Then $K^0 = K$. From diagram (5.4.7) we have a descending chain of submodules

$$\ldots \subset K^{k-1} \subset K^k \subset \ldots \subset K$$

and for any $k < l \leq 0$ we have the following isomorphism of short exact sequences:

$$\begin{array}{ccccc}
A \otimes_{A_m} [K^l/K^k] & \longrightarrow & A \otimes_{A_m} [R_+(M)/K^k] & \longrightarrow & A \otimes_{A_m} [R_+(M)/K^l] \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\bigoplus_{k\leq j<l} \Sigma^{j2^{m+1}} A \otimes_{A_{m-1}} M & \longrightarrow & A \otimes_{A_m} Q_k \otimes_{A_{m-1}} M & \longrightarrow & A \otimes_{A_m} Q_l \otimes_{A_{m-1}} M
\end{array}$$

.

Hence for any $k \leq 0$ there is a short exact sequence

$$0 \to A \otimes_{A_m} K^k \to A \otimes_{A_m} K \to \bigoplus_{k \leq j < 0} \Sigma^{j2^{m+1}} A \otimes_{A_{m-1}} M \to 0 \tag{5.5.1}$$

Fixing s and t, we claim that $\mathrm{Ext}_A^{s,t}(K^k, \mathbb{F}_2) = 0$ for all integers $k \leq 0$ sufficiently small. Indeed, by a change of rings-isomorphism $\mathrm{Ext}_A^{s,t}(A \otimes_{A_m} K^k, \mathbb{F}_2) \cong \mathrm{Ext}_{A_m}^{s,t}(\ker(q_k) \otimes_{A_{m-1}} M, \mathbb{F}_2)$. As $k \to -\infty$, the maximal degree of $\ker(q_k)$ tends to $-\infty$ as k grows. The same holds for $\ker(q_k) \otimes_{A_{m-1}} M$ under the assumption that M is bounded above.

Let s, t be fixed. Since we are considering Ext over the finite algebra A_m, we can always choose an integer k small enough and a projective resolution $P_* \to \ker(q_k) \otimes_{A_{m-1}} M$ such that P_s vanishes in degree t. This implies that $\mathrm{Hom}_{A_m}^t(P_s, \mathbb{F}_2)$ vanishes and that $\mathrm{Ext}_{A_m}^{s,t}(\ker(q_k) \otimes_{A_{m-1}} M, \mathbb{F}_2) = 0$.

By the long exact sequence of Ext-groups induced by (5.5.1) we see that for a fixed pair s, t we have

$$\mathrm{Ext}_A^{s,t}(A \otimes_{A_m} K, \mathbb{F}_2) \cong \mathrm{Ext}_A^{s,t}\Big(\bigoplus_{-k \leq j < 0} \Sigma^{j2^{m+1}} A \otimes_{A_{m-1}} M, \mathbb{F}_2\Big)$$

for k small enough. Passing to the limit over smaller k, the right hand side does not change by the same argument as above, so the proposition follows. □

5.6 Topological model for the Singer construction

Let X be a non-equivariant bounded below spectrum of finite type. Then $H^*(X)$ is an A-module, and we may consider the Singer construction on $H^*(X)$. As before, we are considering the case $p = 2$.

Following [11, II §5], we will show how to realize $R_+(H^*(X))$ as the continuous cohomology of a certain Tate spectrum.

The system of stable diagonal maps (3.4.4) induces maps of spectra

$$\ldots \to \Sigma^{n+1} D_2(\Sigma^{-n-1} X) \xrightarrow{\Sigma^n \Delta} \Sigma^n D_2(\Sigma^{-n} X)\,. \tag{5.6.1}$$

This is a tower of bounded below spectra of finite type, so it makes sense to consider the continuous cohomology of its homotopy inverse limit.

Theorem 5.6.1 ([11, theorem 5.1]). *For any bounded below and finite type spectrum X, there is a natural isomorphism*

$$\omega : \operatorname*{colim}_{n\to\infty} \Sigma H^*(\Sigma^n D_2 \Sigma^{-n} X) \xrightarrow{\cong} \Sigma^{-1} R_+(H^*(X))\,.$$

Note that we have to introduce a single desuspension in the theorem in order to make the statement compatible with our definition of $R_+(-)$.

Proposition 5.6.2. *Let X be as above, and let $X^{\wedge 2}$ be the Σ_2-spectrum with Σ_2 acting by permuting the factors. Then there is a natural isomorphism*

$$H_c^*(X^{\wedge 2})^{t\Sigma_2} \cong R_+(H^*(X))\,.$$

Proof. By corollary 3.4.2 we have

$$(X^{\wedge 2})^{t\Sigma_2} \simeq \operatorname*{holim}_{n\to\infty} \Sigma\Sigma^n E\Sigma_{2+} \wedge_{\Sigma_2} (\Sigma^{-n} B)$$

By [22, VI.1.17] we have a natural isomorphism of Σ_2-spectra

$$E\Sigma_2 \ltimes (\Sigma^{-n} B) \cong E\Sigma_{2+} \wedge (\Sigma^{-n} B)\,.$$

The result now follows from theorem 5.6.1. □

In light of proposition 5.6.2 we make the following definition.

Definition 5.6.3. *Let X be a non-equivariant bounded below spectrum of finite type. Then*

$$R_+(X) = (X^{\wedge 2})^{t\Sigma_2}\,.$$

We note that the tower (5.6.1) was also used by Jones [21] to relate the root invariant and the quadratic construction.

We will in the next section look closer at the cohomological Tate spectral sequence converging to $R_+(H^*(X))$.

5.6.1 The Tate spectral sequence for R_+X

The tower of spectra (5.6.1) induces a direct limit system after applying cohomology with $\mathbb{F}_2$-coefficients:

$$\ldots \to H^*\Sigma^n D_2(\Sigma^{-n} X) \xrightarrow{\Sigma^n \Delta^*} H^*\Sigma^{n+1} D_2(\Sigma^{-n-1} X) \to \ldots \to \Sigma^{-1} R_+(H^*(X))\,. \tag{5.6.2}$$

By theorem 5.6.1, the direct limit is isomorphic to $\Sigma^{-1} R_+(H^*(X))$.

Still following [11, II §5], we describe this cohomological system explicitly. The homology of $D_2(\Sigma^{-n}X)$ is given by (2.3.4). Dualizing to cohomology, we denote the dual of e_i by w_j. Then for $x \in H^*(X)$, we have $\Sigma^n w_j \otimes (\Sigma^{-n}x)^{\otimes 2} \in H^*\Sigma^n D_2(\Sigma^{-n}X)$ and for $p = 2$, [11, Lemma II.5.6] states that

$$(\Sigma^n \Delta)^*(\Sigma^n w_j \otimes (\Sigma^{-n}x)^{\otimes 2}) = \Sigma^{n+1} w_{j+1} \otimes (\Sigma^{-n-1}x)^{\otimes 2} \tag{5.6.3}$$

and that the terms of the cohomology involving $w_j \otimes x_1 \otimes x_2$ lie in the kernel of $\Sigma^n \Delta^*$.

For $x \in H^q(X)$, the isomorphism of theorem 5.6.1 ([11, page 47, proof of II.5.1]) is given by

$$\omega(\Sigma^n w_{j+n} \otimes (\Sigma^{-n}x)^{\otimes 2}) = v^{j+q} \otimes x\,. \tag{5.6.4}$$

The cohomological Tate spectral sequence for $(X^{\wedge 2})^{t\Sigma_2}$ has $\hat{E}_2$-term

$$\begin{aligned} \hat{E}_2^{s,t} &= \widehat{H}_*(\Sigma_2; H^*(X)^{\otimes 2}) \\ &\cong \Sigma P(v, v^{-1}) \otimes \mathbb{F}_2\{x \otimes x\}_{x \in H^*(X)} \end{aligned} \tag{5.6.5}$$

where $x \in H^*(X)$ runs through an $\mathbb{F}_2$-basis of $H^*(X)$. By the explicit description (5.6.3) of the maps in (5.6.2), we see that the spectral sequence collapses at the $\hat{E}_2$-term.

Hence, we make the following connection between the Singer construction and the Tate spectral sequence.

Proposition 5.6.4. *Let X be a bounded below spectrum of finite type. The cohomological Tate spectral sequence $\hat{E}_r(R_+(X))$ converging to $H^*_c(R_+(X)) \cong R_+(H^*(X))$ has $\hat{E}_\infty$-term equal to*

$$\hat{E}_\infty^{*,*} \cong \Sigma P(v, v^{-1}) \otimes \mathbb{F}_2\{x \otimes x\}_{x \in H^*(X)}$$

where $x \in H^*(X)$ runs through an $\mathbb{F}_2$-basis of $H^*(X)$. For $x \in H^qX$, the element $\Sigma v^{q+r-1} \otimes x \in R_+(H^*(X))$ in the abutment is represented in the spectral sequence by the class $\Sigma v^{r-1} \otimes x^{\otimes 2}$ in filtration r.

Proof. We have already noted that the spectral sequence collapses.

By the isomorphism (5.6.4), the element $v^{q+r-1} \otimes x$ corresponds to $\Sigma^{1-r} w_0 \otimes (\Sigma^{r-1}x)^{\otimes 2} \in H^*_c(X^{\wedge 2})^{t\Sigma_2}$. As it cannot be pulled back further in the system (5.6.2), it is represented in the spectral sequence by the element $\Sigma v^{r-1} \otimes x^{\otimes 2}$. □

5.7 The dual Singer construction

Definition 5.7.1. *Let M_* be a bounded below left A_*-comodule of finite type with dual A-module M.*

The homological Singer construction $R_+(M_*)$ is defined as the dual of $R_+(M)$.

The discussion about duality in section 1.2 applies to the cohomological and homological Singer construction as well.

As an $\mathbb{F}_2$-vector space the Singer construction on M was given as the suspended tensor product $\Sigma P(v, v^{-1}) \otimes M$. Even if M is of finite type in each degree, this object will be infinite dimensional in each degree whenever M is unbounded above. In our case M will typically be the cohomology of some bounded below spectrum of finite type, hence M will typically be bounded below but not from above.

We let $P(u, u^{-1})$ be the dual of $\Sigma P(v, v^{-1})$ with u^n of degree $-n$ dual to Σv^{-n-1}. Then

$$R_+(M_*) = [R_+(M)]_*] \cong P(u, u^{-1}) \hat{\otimes} M_* \tag{5.7.1}$$

where $-\hat{\otimes}-$ is the completed tensor product.

The A-module action map $A \otimes R_+(M) \to R_+(M)$ for the cohomological Singer construction dualizes to a map

$$\nu : R_+(M_*) = [R_+(M)]_* \to [A \otimes R_+(M)]_* \cong A_* \hat{\otimes} R_+(M_*) \tag{5.7.2}$$

so $R_+(M_*)$ is a completed A_*-comodule.

Dualizing (5.2.1), we get that the dual Steenrod operations are given by

$$\mathrm{Sq}^s_*(u^n \otimes \alpha) = \textstyle\sum_i \binom{-1-n-s}{s-2i} u^{n+s-i} \otimes \mathrm{Sq}^i_* \alpha \tag{5.7.3}$$

for $\alpha \in M_*$. The sum (5.7.3) is finite since M_* is bounded below, so we have the following commutative diagram:

$$\begin{array}{ccc} R_+(M_*) & \xrightarrow{\nu} & A_* \hat{\otimes} R_+(M)_* \\ \uparrow & & \uparrow \\ P(u, u^{-1}) \otimes (M_*) & \longrightarrow & A_* \otimes P(u, u^{-1}) \otimes M_* \, . \end{array} \tag{5.7.4}$$

The dual of the evaluation map $\epsilon : R_+(M) \to M$ is given by the formula

$$\epsilon_*(\alpha) = \textstyle\sum_i u^{-i} \otimes \mathrm{Sq}^i_* \alpha \, . \tag{5.7.5}$$

With definition 5.7.1 and proposition 5.6.2 we get that for a bounded below spectrum X of finite type we have $R_+(H_*(X)) \cong H_*^c(R_+(X))$ as a completed A_*-comodule. This follows since $H_*^c(R_+(X))$ is dual to $H_c^*(R_+(X))$, as mentioned in section 1.2.

We have then the dual of proposition 5.6.4:

Proposition 5.7.2. *Let X be a bounded below spectrum of finite type. The homological Tate spectral sequence $\hat{E}^r(R_+(X))$ converging to $H_*^c(R_+(X)) \cong R_+(H_*(X))$ has $\hat{E}^\infty$-term equal to*

$$\hat{E}^\infty_{*,*} \cong P(u, u^{-1}) \otimes \mathbb{F}_2\{\alpha^{\otimes 2}\}_{\alpha \in H_*(X)}$$

where $\alpha \in H_*(X)$ runs through an $\mathbb{F}_2$-basis of $H_*(X)$. For $\alpha \in H_q(X)$, the element $u^{-q+n} \otimes \alpha \in R_+(H_*(X))$ in the abutment is represented in the spectral sequence by the class $u^n \otimes \alpha^{\otimes 2}$ in filtration $-n$.

Chapter 6

The Bökstedt map

At this point all the relevant topological and algebraic structure has been discussed and defined. In this chapter we set up the connection between the Singer construction and the continuous cohomology of $T(B)^G$ for B a commutative S-algebra. The main calculational result we will use in subsequent chapters is stated in theorem 6.1.1. The first section of this chapter will be concerned with the construction of the map Ψ and the proof of its properties.

In fact, the basic construction will give a map $[D_2^{(1)}B]^{tC_2} \to T(B)^{tC_2}$ of C_2-Tate spectra. Then, by the inclusion $B^{\wedge 2} = D_2^{(0)}B \subset D_2^{(1)}B$, we get a composite map of Tate spectra

$$\Psi : R_+B \to [D_2^{(1)}B]^{tC_2} \to T(B)^{tC_2} .$$

The construction of Ψ depends on $T(B)$ being an equivariant E_∞-ring spectrum. In chapter 7 we will use the simplicial structure of the topological Hochschild spectrum to reprove theorem 6.1.1 without using any E_∞-structure.

6.1 E_∞-structure

Let B be a commutative S-algebra of finite type and let $T(B)$ be the topological Hochschild homology spectrum of B. Then $T(B)$ is a $\mathbb{T}$-equivariant commutative S-algebra or, equivalently, a $\mathbb{T}$-equivariant E_∞-ring spectrum. We have structure maps $\xi_n : D_nT(B) \to T(B)$ for each $n \geq 0$ and they can be taken as maps in the category of $\mathbb{T}$-equivariant spectra. We will only use the map ξ_2 in the following.

The $\mathbb{T}$-equivariant map $\lambda : S^1 \ltimes B \to T(B)$ together with the second structure map ξ_2 of the E_∞-structure is used to define the following composite

$$\beta : D_2(S^1 \ltimes B) \xrightarrow{D_2\lambda} D_2T(B) \xrightarrow{\xi_2} T(B) . \qquad (6.1.1)$$

The composite β is a map of $\mathbb{T}$-spectra and thus induces a filtered map of Tate-spectra

$$\beta^{tG} : D_2(S^1 \ltimes B)^{tG} \to T(B)^{tG} \tag{6.1.2}$$

for any closed subgroup $G \subset \mathbb{T}$. In the remainder of this chapter we will discuss the structure of $D_2(S^1 \ltimes B)$ and its associated C_2-Tate spectrum. This is manageable mainly because the $\mathbb{T}$-equivariance of $D_2(S^1 \ltimes B)$ is 'concentrated' on the circle factors.

For this discussion we need not assume that B is bounded below and of finite type. For all applications, however, these assumptions will hold. In this case we have that $T(B)$ is connective and of finite type, and by corollary 2.3.1 we have that all three spectra in (6.1.1) are bounded below and of finite type.

The composite (6.1.1) appears in [10], where the author uses naturality to solve multiplicative extension problems in the Bökstedt spectral sequence for $T(\mathbb{F}_2)$ and $T(\mathbb{Z})$. In our context we will use naturality of the Tate construction to provide key input for determining the A_*-comodule structure of $H^c_*T(B)^{tC_2}$.

The following theorem will provide necessary input for our calculations in the chapters following chapter 8:

Theorem 6.1.1. *Let B be a commutative S-algebra. Then there is a map of non-equivariant spectra*

$$\Psi : R_+B = (B^{\wedge 2})^{tC_2} \to T(B)^{tC_2} \tag{6.1.3}$$

which induces the following map of $\hat{E}^2$-terms of homological Tate spectral sequences:

$$u^s \otimes \alpha^{\otimes 2} \mapsto u^s \otimes \alpha^2 \,.$$

Here $\alpha \in H_*B$ and $\alpha^2 = \mu_*(\iota \wedge \iota)_*(\alpha \otimes \alpha) \in H_*T(B)$ where $\iota : B \to T(B)$ is the inclusion of the zero-simplices and $\mu : T(B) \wedge T(B) \to T(B)$ is the multiplication map.

The proof of this theorem will be given at the end of the following section.

6.2 Approximating the extended power construction

In the following, we will consider different equivariant structures on the circle group. Let $\mathbb{T}$ denote the circle as a $\Sigma_2 \times \mathbb{T}$-space by letting $\mathbb{T}$ act by multiplication and Σ_2 act as the subgroup of order two. We write S^1 for the same underlying $\mathbb{T}$-space, but we consider S^1 as a $\Sigma_2 \times \mathbb{T}$-space by letting Σ_2 act trivially. Thirdly, we give the torus $S^1 \times S^1$ the structure of a $\Sigma_2 \times \mathbb{T}$-space by letting $\mathbb{T}$ act diagonally and let Σ_2 act by permuting the factors.

The twisted diagonal $\Delta^T : \mathbb{T} \to S^1 \times S^1$ that maps $x \mapsto (-x, x)$ is a $\Sigma_2 \times \mathbb{T}$-equivariant map. Remember that Σ_2 acts on the circle in the source as the subgroup of order two. This gives a map of $\mathbb{T} \times \Sigma_2$-spectra

$$\Delta^T : \mathbb{T} \ltimes B^{\wedge 2} \to (S^1 \ltimes B)^{\wedge 2} \tag{6.2.1}$$

which on the level of prespectra is given by

$$\mathbb{T}_+ \wedge B(V) \wedge B(W) \xrightarrow{(23)\circ(\Delta^T \wedge 1 \wedge 1)} S^1_+ \wedge B(V) \wedge S^1_+ \wedge B(W)$$

for representations $V, W \in U^{\mathbb{T}}$. The last map in the composite above, labelled (23), is the canonical isomorphism permuting the second and the third factors. Applying the half-smash functor $E\Sigma_2 \ltimes (-)$ on the map (6.2.1) produces a map of $\mathbb{T}$-spectra:

$$E\Sigma_2 \ltimes_{\Sigma_2} (\mathbb{T} \ltimes B^{\wedge 2}) \xrightarrow{E\Sigma_2 \ltimes_{\Sigma_2} \Delta^T} D_2(S^1 \ltimes B) \ . \tag{6.2.2}$$

The Σ_2-map $\pi : E\Sigma_2 \to *$ is a non-equivariant homotopy equivalence. The half-smash is a functor in both variables, so we get an induced map $E\Sigma_2 \ltimes \mathbb{T} \ltimes B^{\wedge 2} \to \mathbb{T} \ltimes B^{\wedge 2}$ of $\Sigma_2 \times \mathbb{T}$-spectra. Remember that $\mathbb{T}$ was given the antipodal action by Σ_2, so Σ_2 acts freely as the subgroup of $\mathbb{T}$ of order two. In particular, $\mathbb{T} \ltimes B^{\wedge 2}$ is a free Σ_2-spectrum and so by passing to Σ_2-orbits we get an equivalence of $\mathbb{T}$-spectra

$$\pi \ltimes 1 : E\Sigma_2 \ltimes_{\Sigma_2} (\mathbb{T} \ltimes B^{\wedge 2}) \to \mathbb{T} \ltimes_{\Sigma_2} B^{\wedge 2} .$$

Identifying $\mathbb{T}$ as the 1-skeleton of $E\Sigma_2$, we have $\mathbb{T} \ltimes_{\Sigma_2} B^{\wedge 2} = D_2^{(1)} B$. By choosing an inverse to the map $\pi \ltimes 1$ we define $\psi : D_2^{(1)} B \to T(B)$ to be the composite

$$\psi : D_2^{(1)} B \xrightarrow{(\pi \ltimes 1)^{-1}} E\Sigma_2 \ltimes_{\Sigma_2} \mathbb{T} \ltimes B^{\wedge 2} \xrightarrow{1 \ltimes \Delta^T} D_2(S^1 \ltimes B) \xrightarrow{\beta} T(B) \ . \tag{6.2.3}$$

This is a map of $\mathbb{T}$-spectra, so applying the C_2-Tate construction we get a map $\psi^{tC_2} : D_2^{(1)} B^{tC_2} \to T(B)^{tC_2}$, preserving the Tate-filtration. On $\hat{E}^1$-terms of the homological C_2-Tate spectral sequences, the map $\psi_*^{tC_2} : \hat{E}^1(D_2^{(1)}B) \to \hat{E}^1(T(B))$ is given by

$$\hat{C}^{-*}(C_2; \psi_*) : \hat{C}^{-*}(C_2; H_*D_2^{(1)}(B)) \to \hat{C}^{-*}(C_2; H_*T(B)) .$$

We will describe ψ_* using a chain level description of the spectra in diagram (6.2.3).

When B is a CW spectrum, we can give $\mathbb{T} \ltimes B^{\wedge 2}$ the structure of a CW spectrum with cellular action of the group Σ_2. Then by theorem [11, I.1.3] we have that $E\Sigma_2 \ltimes_{\Sigma_2} \mathbb{T} \ltimes B^{\wedge 2}$ is a CW spectrum with cellular chains

$$C_*E\Sigma_2 \ltimes_{\Sigma_2} (\mathbb{T} \ltimes B^{\wedge 2}) \cong C_*(E\Sigma_2) \otimes_{\Sigma_2} C_*(\mathbb{T} \ltimes B^{\wedge 2}) .$$

Give $\mathbb{T}$ the structure of a free Σ_2-CW complex with one free Σ_2-cell in dimensions 0,1, i.e. $C_*\mathbb{T} = \mathbb{F}_2[\Sigma_2]\{1, e\}$ where 1 has degree 0 and e has degree 1. The boundary map $\partial : C_1\mathbb{T} \to C_0\mathbb{T}$ is given by $\partial(Te) = \partial(e) = 1 + T$.

Let W_* be the chain complex of section 2.3, given by $W_n = \mathbb{F}_2[\Sigma_2]\{e_n\}$ in degree n. There is a Σ_2-chain equivalence $C_*(X^{\wedge 2}) \simeq H_*X^{\otimes 2}$ and we get

$$C_*E\Sigma_2 \ltimes_{\Sigma_2} (\mathbb{T} \ltimes B^{\wedge 2}) \cong W \otimes_{\Sigma_2} \mathbb{F}_2[\Sigma_2]\{1, e\} \otimes H_*X^{\otimes 2} . \tag{6.2.4}$$

Likewise for $D_2(S^1 \ltimes B) = E\Sigma_2 \ltimes_{\Sigma_2} (S^1 \ltimes B)^{\wedge 2}$ we have

$$C_*D_2(S^1 \ltimes B) = W \otimes_{\Sigma_2} C_*(S^1 \times S^1) \otimes H_*B^{\otimes 2} . \tag{6.2.5}$$

In order to make the map $E\Sigma_2 \ltimes_{\Sigma_2} \Delta^T$ into a Σ_2-cellular map, we need to choose a finer decomposition of the torus $S^1 \times S^1$ than the usual CW-structure with only one 2-cell. Indeed, we let

$$C_*(S^1 \times S^1) = \mathbb{F}_2[\Sigma_2]\{A, B, a, b, e, y\} \oplus \mathbb{F}_2\{d, x\} \tag{6.2.6}$$

with 0-cells x, y, 1-cells a, b, d, e and 2-cells A, B. If we let $\Sigma_2 = \{1, t\}$, this cell decomposition is depicted in figure 6.1.

The twisted diagonal map $\Delta^T : \mathbb{T} \to S^1 \times S^1$ is now a Σ_2-equivariant cellular map sending $1 \mapsto y$ and $e \mapsto e$. Thus $E\Sigma_2 \ltimes_{\Sigma_2} \Delta^T$ is determined by the formulas

$$\begin{aligned} e_i \otimes 1 \otimes \alpha \otimes \beta &\mapsto e_i \otimes y \otimes \alpha \otimes \beta \\ e_i \otimes e \otimes \alpha \otimes \beta &\mapsto e_i \otimes e \otimes \alpha \otimes \beta \end{aligned} \tag{6.2.7}$$

where $e_i \in W_i$ and $\alpha, \beta \in H_*X$.

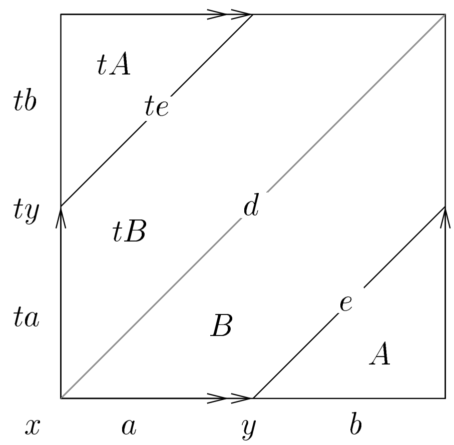


Figure 6.1: Σ_2-cell decomposition of $S^1 \times S^1$

The map $\pi \ltimes 1 : E\Sigma_2 \ltimes_{\Sigma_2} \mathbb{T} \ltimes B^{\wedge 2} \to D_2^{(1)}B$ was given by collapsing $E\Sigma_2$ to a point. On the chain level $\pi_* : W \to \mathbb{F}_2$ is given by

$$e_n \mapsto \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

and $\pi \ltimes 1 : W \otimes_{\Sigma_2} C_*(\mathbb{T}) \otimes H_*B^{\otimes 2} \to W^{(1)} \otimes_{\Sigma_2} H_*B^{\otimes 2}$ is given by

$$\begin{aligned} e_0 \otimes 1 \otimes \alpha \otimes \beta &\mapsto e_0 \otimes \alpha \otimes \beta \\ e_0 \otimes e \otimes \alpha \otimes \beta &\mapsto e_1 \otimes \alpha \otimes \beta \end{aligned} \tag{6.2.8}$$

for $\alpha, \beta \in H_*B$. Classes involving e_n for $n > 0$ lie in the kernel of $\pi \ltimes 1$.

With the above maps and concrete cellular models we can now prove our first result.

Proof of Theorem 6.1.1. The inclusion of the subgroup $C_2 \subset \mathbb{T}$ induces a map of C_2-spectra

$$i : C_2 \ltimes_{\Sigma_2} B^{\wedge 2} \subset \mathbb{T} \ltimes_{\Sigma_2} B^{\wedge 2} = D_2^{(1)}B\,.$$

Further there is an isomorphism $C_2 \ltimes_{\Sigma_2} B^{\wedge 2} \cong B^{\wedge 2}$ of C_2-spectra such that C_2 acts on $B^{\wedge 2}$ by permutation of the factors.

Hence, composing the inclusion i followed by ψ (6.2.3) and applying the C_2-Tate construction, we get the filtration preserving map

$$\Psi : (B^{\wedge 2})^{tC_2} \to T(B)^{tC_2}\,.$$

The source of this map is by definition R_+B.

The map of homological Tate spectral sequences is given on the $\hat{E}^2$-term by

$$\widehat{H}^{-*}(C_2; (\Psi \circ i)_*) : \widehat{H}^{-*}(C_2; H_*B^{\otimes 2}) \to \widehat{H}^{-*}(C_2; H_*T(B))\,.$$

Let $\alpha \in H_*B$. Then $i_*\alpha^{\otimes 2} = e_0 \otimes \alpha^{\otimes 2}$ is a cycle in $H_*\mathbb{T} \ltimes_{\Sigma_2} B^{\wedge 2}$ and by (6.2.8) and (6.2.7) we see that $((1 \ltimes \Delta) \circ (\pi \ltimes 1)^{-1} \circ i)_*(\alpha^{\otimes 2}) = e_0 \otimes y \otimes \alpha^{\otimes 2} \in C_*D_2(S^1 \ltimes B)$. This cycle is homologous to $e_0 \otimes x \otimes \alpha^{\otimes 2}$ since

$$\partial(e_0 \otimes a \otimes \alpha^{\otimes 2}) = e_0 \otimes (x + y) \otimes \alpha^{\otimes 2}\,.$$

In homology, β (6.1.1) sends this class to the product $\alpha^2 \in H_*T(B)$. □

Chapter 7

Inclusion of zero-simplices

In chapter 6 we introduced a map $R_+B \to T(B)^{tC_2}$ linking the Singer construction on the homology of B to the continuous homology of $T(B)^{tC_2}$. We will in the present chapter give another proof of theorem 6.1.1 by a different construction. To do this we must consider the topological Hochschild homology spectrum as a genuine C_2-spectrum indexed on a complete universe $\mathscr{U}$. Also, we assume that B is an FSP or a symmetric spectrum defined on spheres and use the definition of $T(B)$ given in [19].

In the last section we apply the construction in this chapter to show the Segal conjecture in its original form, i.e. for $B = S$.

7.1 Edgewise subdivision

Let $V \subset \mathscr{U}$ and recall from chapter 4 that the Vth space in the prespectrum defining $T(B)$ is given by the realization of the simplicial space $THH(B;S^V)_\bullet$. The p-fold edgewise subdivision functor $\mathrm{sd}_p(-)$ of Bökstedt-Hsiang-Madsen [6] comes equipped with a homeomorphism of C_p-spaces after realization

$$D_p : |\,\mathrm{sd}_p THH(B;S^V)_\bullet| \xrightarrow{\cong} |THH(B;S^V)_\bullet| \tag{7.1.1}$$

such that $C_p \subset S^1$ acts simplicially on the source.

For any simplicial space $X_\bullet$ we have a map $\iota : X_0 \hookrightarrow |X_\bullet|$ defined by inclusion of the zero-simplices. The p-fold edgewise subdivision of $THH(B;S^V)_\bullet$ has zero-simplices

$$\begin{aligned}[\mathrm{sd}_p THH(B;S^V)]_0 &= THH(B;S^V)_{p-1} \\ &= \operatorname*{hocolim}_{(x_0,\ldots,x_p)\in I^p} F(S^{x_0}\wedge\ldots\wedge S^{x_p}, B(S^{x_0})\wedge\ldots B(S^{x_p})\wedge S^V)\,.\end{aligned} \tag{7.1.2}$$

The generator of C_p acts by cyclically shifting the factors $(x_0, \dots, x_p)$ one position to the right. By the inclusion of the zero-simplices and the homeomorphism (7.1.1), we define ι^p as a map of prespectra that on V'th spaces is the composition

$$\iota^p : THH(B; S^V)_{p-1} \xrightarrow{\iota} |\,\mathrm{sd}_p THH(B; S^V)_\bullet| \xrightarrow[\cong]{D_p} |THH(B; S^V)_\bullet|\ .$$

After passing to spectra, ι^p induces a map of genuine C_p-spectra

$$\iota^p : B^{\wedge p} \to T(B)\,. \tag{7.1.3}$$

The following diagram commutes up to non-equivariant homotopy

$$\begin{array}{ccc} B^{\wedge p} & \longrightarrow & |\,\mathrm{sd}_p THH(B)_\bullet| \\ \downarrow & & \downarrow \cong \\ B & \longrightarrow & |THH(B)_\bullet| = T(B) \end{array} \tag{7.1.4}$$

where the horizontal maps are the inclusions of the zero-simplices, and the left vertical map is the iterated multiplication map of B. The composition along the upper horizontal map followed by the right vertical isomorphism is by definition ι^p. Note that the other maps of diagram (7.1.4) are not C_p-equivariant.

By the commutativity of diagram (7.1.4), we see that on homology ι^p is given by the iterated multiplication followed by the inclusion of zero-simplices $H_*B^{\otimes p} \to H_*B \to H_*T(B)$. Applying the Tate construction with respect to $C_p \subset \mathbb{T}$ we get a map $\hat{E}^r_{**}(B^{\wedge p}) \to \hat{E}^r_{**}(T(B))$ which is the iterated product in each column of the $\hat{E}^2$-term.

Theorem 7.1.1. *Let B be a commutative FSP or commutative symmetric spectrum defined on spheres. Then the following diagram exists and commutes up to homotopy*

$$\begin{array}{ccc} B & \xrightarrow{\iota} & T(B) \\ \downarrow & & \downarrow \gamma \\ R_+B & \xrightarrow{(\iota^p)^{tC_p}} & T(B)^{tC_p}\,. \end{array}$$

The upper horizontal map is the inclusion of the 0-simplices.

For $p = 2$, the map $(\iota^2)^{tC_2}$ induces the following map of $\hat{E}^2$-terms of homological Tate spectral sequences:

$$u^s \otimes \alpha^{\otimes 2} \mapsto u^s \otimes \alpha^2\,.$$

Here $\alpha \in H_*B$ and $\alpha^2 = \iota_*\mu_*(\alpha \otimes \alpha) \in H_*T(B)$ where $\iota : B \to T(B)$ is the inclusion of the zero-simplices and $\mu : B \wedge B \to B$ is the multiplication map.

Remark: The last part of the theorem can easily be extended to the case of odd primes. The restriction $p = 2$ appears because we have only given a definition of the Singer construction in this case.

Proof. Let p be a prime and let $C = C_p \subset \mathbb{T}$ be the cyclic subgroup of order p. For any genuine C-spectrum X we have an equivalence of spectra

$$[\widetilde{EC} \wedge X]^C \xrightarrow{\simeq} \Phi^C X .$$

The map is defined on V'th spaces as

$$\operatorname*{colim}_{W \subset \mathscr{U}} [\Omega^{W-V}(\widetilde{EC} \wedge X(W))]^C \to \operatorname*{colim}_{W \subset \mathscr{U}} \Omega^{W^C - V} X(W)^C$$

induced from the inclusion $W^C \subset W$ and the homeomorphisms $[\widetilde{EC} \wedge X(W)]^C \cong X(W)^C$. The proof of the fact that this induces an equivalence of spectra can be found in [19, page 34, the proof of Proposition 2.2].

From the cyclotomic structure (chapter 4) of THH we have an equivalence of $\mathbb{T}$-spectra $T(B) \simeq \rho_C^* \Phi^C T(B)$. Thus, we have a commutative diagram

$$\begin{array}{ccc}
B & \xrightarrow{\iota} & T(B) \\
\uparrow \simeq & & \uparrow \simeq \\
\Phi^C(B^{\wedge p}) & \xrightarrow{\Phi^C(\iota^p)} & \Phi^C T(B) \\
\uparrow \simeq & & \uparrow \simeq \\
[\widetilde{EC} \wedge B^{\wedge p}]^C & \xrightarrow{1 \wedge \iota^p} & [\widetilde{EC} \wedge T(B)]^C \\
\downarrow & & \downarrow \\
(B^{\wedge p})^{tC} & \xrightarrow{(\iota^p)^{tC}} & T(B)^{tC} .
\end{array} \qquad (7.1.5)$$

The lower left spectrum is by definition $R_+ B$. The upper square is gotten by the "simplicial cyclotomic structure equivalence", see [19]. Indeed, on Vth spaces there is a diagram

$$\begin{array}{ccc}
\operatorname*{colim}_{W \subset \mathscr{U}} \Omega^{W^C - V} THH(B; S^{W^C})_0 & \longrightarrow & \operatorname*{colim}_{W \subset \mathscr{U}} \Omega^{W^C - V} |THH(B; S^{W^C})| \\
\uparrow \simeq & & \uparrow \simeq \\
\operatorname*{colim}_{W \subset \mathscr{U}} [\Omega^{W^C - V} \operatorname{sd}_C THH(B; S^W)_0]^C & \longrightarrow & \operatorname*{colim}_{W \subset \mathscr{U}} \Omega^{W^C - V} |\operatorname{sd}_C THH(B; S^W)|^C
\end{array}$$

where the horizontal maps are given by inclusion of the 0-simplices into the realization. The vertical maps are induced by restriction to fixed point maps of the type $F(X,Y)^G \to F(X^G, Y^G)$. The diagram can be rewritten as

$$\begin{array}{ccc} B(V) & \longrightarrow & T(B)(V) \\ \uparrow \simeq & & \uparrow \simeq \\ (\Phi^C B^{\wedge p})(V) & \longrightarrow & (\Phi^C T(B))(V) \end{array}$$

which is exactly the upper square in (7.1.5) on V'th spaces.

By choosing inverses to the homotopy equivalences in diagram (7.1.5), the outer square gives the diagram in the proposition. □

7.2 The Segal conjecture for groups of prime order

When $B = S$ is the sphere spectrum, the inclusion $S \hookrightarrow T(S) = S_{C_p}$ and the iterated multiplication map $S^{\wedge p} \to S$ are both equivalences, thus by diagram (7.1.4) the C_p-equivariant map ι^p is a non-equivariant equivalence. Hence it induces an isomorphism of continuous cohomology groups of Tate spectra $H^*_c T(S)^{tC_p} \cong H^*_c S^{tC_p}_{C_p}$. The latter was identified in chapter 5 as the Singer construction on $\mathbb{F}_p$, so the map $\gamma : T(S) \to T(S)^{tC_p}$ induces an A-module homomorphism $R_+\mathbb{F}_p \to \mathbb{F}_p$ on continuous cohomology. By proposition 4.3.3, γ_* and hence γ^* is non-trivial and thus an Ext-isomorphism by corollary 5.2.4. The conclusion that γ and hence $\Gamma : S^{C_p}_{C_p} \to S^{hC_p}_{C_p}$ is a 2-adic equivalence follows by theorem 0.0.2 since γ^* induces an isomorphism of E_2-terms of the inverse limit of Adams spectral sequences.

Remark: Note that the isomorphism $H^*_c T(S)^{tC_2} \cong R_+\mathbb{F}_2$ implies that $T(S)^{tC_2} \simeq^\wedge_2 S$. This is not enough to prove the Segal conjecture, since we must know that the map γ induces this equivalence, i.e. we must know that γ induces an Ext-isomorphism.

Chapter 8

Computations

In this chapter we give the first step in the proof of theorem 0.0.3. Precisely, we will calculate the additive structure of $H^c_*(T(B)^{tC_2})$ for $B = BP\langle m-1\rangle$, BP and MU. The funny indexing of $BP\langle m-1\rangle$ will simplify later formulas.

These calculations can be found in [12] in the case of the homological homotopy fixed point spectral sequence in proposition 4.5.1. We will refer to the cited paper to show that our spectral sequences collapse at the $\hat{E}^3$-terms. This will be done by proposition 4.5.1 which supplies the bridge between the homological homotopy fixed point spectral sequence and the homological Tate spectral sequence.

Using either the inclusion of zero-simplices or chapter 6, we show in section 8.4 how to detect some of the hidden A_*-extensions in the homological Tate spectral sequence. The complete picture will be worked out the in the last chapters.

Our calculations in the case of BP will use that there is a map of S-algebras $\phi : MU \to BP$ which is surjective in homology. Because of this, even though the case $B = MU$ does not appear in theorem 0.0.3, it will play an important role in the proof.

In the remaining chapters, we will work at the prime $p = 2$ only.

8.1 Overview

To fix the setting, consider the following diagram:

$$
\begin{array}{ccccccc}
H_*MU & \longrightarrow & H_*BP & \longrightarrow & H_*BP\langle m-1\rangle & \longrightarrow & H_*H\mathbb{F}_2 \\
\downarrow \iota_* & & \downarrow \iota_* & & \downarrow \iota_* & & \downarrow \iota_* \\
H_*T(MU) & \longrightarrow & H_*T(BP) & \longrightarrow & H_*T(BP\langle m-1\rangle) & \longrightarrow & H_*T(\mathbb{F}_2) \\
\downarrow \gamma_* & & \downarrow \gamma_* & & \downarrow \gamma_* & & \downarrow \gamma_* \\
H^c_*T(MU)^{tC_2} & \longrightarrow & H^c_*T(BP)^{tC_2} & \longrightarrow & H^c_*T(BP\langle m-1\rangle)^{tC_2} & \longrightarrow & H^c_*T(\mathbb{F}_2)^{tC_2}
\end{array}
\tag{8.1.1}
$$

The maps ι_* come from the inclusion of the zero-simplices, and the maps γ_* come from the cyclotomic structure of $T(B)$.

We recall the structure of the upper row first. For the complex cobordism spectrum MU, the homology with $\mathbb{Z}$-coefficients is, by a result of Milnor, isomorphic to a polynomial algebra with generators in even degrees

$$H_*(MU) \cong \mathbb{Z}[b_k|k \geq 1] \qquad \deg(b_k) = 2k\,. \tag{8.1.2}$$

The polynomial generators come from unstable homology classes $\beta_{k+1} \in H_{2k+2}(BU(1);\mathbb{Z})$ by the homotopy equivalence $BU(1) \simeq MU(1)$ given by the zero-section in the canonical bundle over $BU(1)$.

We recall the A_*-comodule structure of H_*MU.

Theorem 8.1.1 ([26], theorem 20.10). *There is an isomorphism*

$$\kappa' : H_*MU \to D(A_*) \otimes P(v_i|i \neq 2^j - 1)$$

of graded A_*-comodules algebras, where the v_i's are all A_*-comodule primitives and their degrees are $\deg(v_i) = 2i$. *Here* $D(A_*) = P(\bar{\xi}_k^2|k \geq 0)$ *denotes the double of A_*.*

By $\bar{\xi}_k$ we mean $\chi(\xi_k)$, where χ is the involution coming from the Hopf algebra structure on A_*.

The Brown-Peterson spectrum BP at the prime two has homology groups

$$H_*(BP) \cong D(A_*) = P(\bar{\xi}_k^2 \mid k \geq 1) \tag{8.1.3}$$

and A_*-comodule structure inherited from the inclusion $D(A_*) \subset A_*$ into the dual Steenrod algebra. Dually, the cohomology is given by

$$H^*(BP) \cong A/\!\!/E \tag{8.1.4}$$

as the quotient of A by the exterior sub Hopf algebra E generated by the Milnor primitives $\{Q_n\}_{n\geq 0}$. In particular, $H^*(BP)$ is cyclic as a left A-module.

For $0 \leq m < \infty$, the connective Johnson-Wilson spectrum $BP\langle m-1\rangle$ at the prime two has homology

$$H_*BP\langle m-1\rangle \cong P(\bar\xi_1^2, \dots, \bar\xi_m^2, \bar\xi_k \mid k \geq m+1) \tag{8.1.5}$$

with A_*-comodule structure given by the obvious inclusion into the dual Steenrod algebra. Dually, the cohomology of $BP\langle m-1\rangle$ is

$$H^*(BP\langle m-1\rangle) \cong A/\!\!/E_{m-1}$$

where E_{m-1} is the exterior sub Hopf algebra generated by the elements $\{Q_n\}_{m>n\geq 0}$. For this to make sense when $m = 0$, we introduce the convention that $E_{-1} = \mathbb{F}_2$. In particular, $H^*(BP\langle m-1\rangle)$ is cyclic as a left A-module.

On homology, the map $\phi : MU \to BP$ induces the unique map of A_*-comodules sending 1 to 1 and sending v_i to zero. In particular ϕ_* is surjective.

On the other hand, the maps $H_*(BP) \to H_*(BP\langle m-1\rangle) \to H_*(BP\langle -1\rangle) = H_*(H\mathbb{F}_2) \cong A_*$ are all injections, identified by the isomorphisms (8.1.3) and (8.1.5) with the obvious inclusions.

All of the spectra MU, BP and $BP\langle m-1\rangle$ are S-algebras. Thus, we can consider their associated topological Hochschild homology spectra. The Bökstedt spectral sequence calculates the homology of $T(B)$ in these cases. The contents of the following proposition can be found in [2, Theorem 5.12].

Proposition 8.1.2. *For $B = MU$, BP, or $BP\langle m-1\rangle$ for $m \geq 0$, we have the following isomorphisms of A_*-comodule algebras over H_*B:*

$$\begin{aligned} H_*T(MU) &\cong H_*MU \otimes E(\sigma b_k \mid k \geq 1) \\ H_*T(BP) &\cong H_*BP \otimes E(\lambda_k \mid k \geq 1) \\ H_*T(BP\langle m-1\rangle) &\cong H_*BP\langle m-1\rangle \otimes E(\lambda_k \mid 1 \leq k \leq m) \otimes P(\mu_m) \end{aligned} \tag{8.1.6}$$

where $\lambda_k = \sigma\bar\xi_k^2$ and $\mu_m = \sigma\bar\xi_{m+1}$. For $B = BP, BP\langle m-1\rangle$ the exterior classes are all A_*-comodule primitives and for $BP\langle m-1\rangle$ there is a non-trivial Bockstein $\mathrm{Sq}^1_*\mu_m = \lambda_m$.

It is known that MU can be realized as a commutative S-algebra. For $BP\langle -1\rangle = H\mathbb{F}_2$, $BP\langle 0\rangle = H\mathbb{Z}_{(2)}$ and $BP\langle 1\rangle = ku_{(2)}$ this is also true, but it is not known to be true for $BP\langle m-1\rangle$ when $m > 2$. In our calculations concerning $m > 2$, we will assume that $BP\langle m-1\rangle$ has a commutative structure. See [4] for more information.

8.2 Differential graded algebras

In the next section we will be calculating the $\hat{E}^3$-term of the homological Tate spectral sequence for $T(B)$. We will use that the d^2-differential is given by the map in homology induced by the circle action (4.4.2).

Here, we will give a useful algebraic tool to make these calculations easier. The idea can be found in [12, proof of 6.1].

Let $K(x)$ be the differential graded algebra whose underlying $\mathbb{F}_2$-algebra is

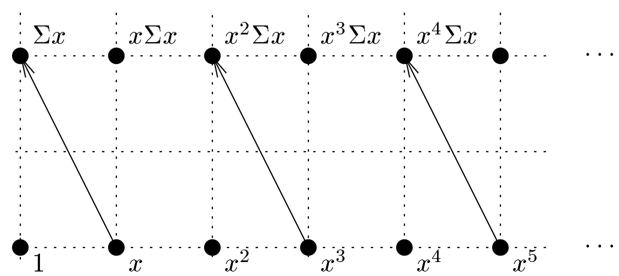
$$P(x) \otimes S(\Sigma x) = \begin{cases} P(x) \otimes P(\Sigma x) & \deg(x) \text{ odd} \\ P(x) \otimes E(\Sigma x) & \deg(x) \text{ even.} \end{cases}$$

The differential d is of degree 1 and is given by specifying that $d(x \otimes 1) = 1 \otimes \Sigma x$ and $d(1 \otimes \Sigma x) = 0$ and extending to a derivation. The homology of $K(x)$ with respect to the differential can be calculated using that d is a derivation as follows.

Lemma 8.2.1. *The homology of $K(x)$ with respect to the differential d is given by:*

$$H(K(x), d) \cong \begin{cases} P(x^2) & \deg(x) \textit{ odd} \\ P(x^2) \otimes E(x\Sigma x) & \deg(x) \textit{ even.} \end{cases}$$

Proof. If $\deg(x)$ is even, then $K(x) = P(x) \otimes E(\Sigma x)$ and since $d(x) = \Sigma x$ and $d(x^{2k}) = 0$ it follows that $d(x^{2k+1}) = x^{2k}\Sigma x$ for all k. The homology is therefore isomorphic to a polynomial algebra on the class x^2 tensor an exterior algebra on $x\Sigma x$.



When $\deg(x)$ is odd, then $K(x) = P(x) \otimes P(\Sigma x)$. This time, $d(x^{2k+1}(\Sigma x)^l) = x^{2k}(\Sigma x)^{l+1}$ for all $k, l \geq 0$, so the homology is polynomial on the class x^2. □

8.3 The homological Tate spectral sequence

Having described the homology of $T(B)$ for $B = MU, BP, BP\langle m-1\rangle$ as comodules over the dual Steenrod algebra, we are now in position to calculate the Tate spectral sequences converging to the continuous homology of $T(B)^{tC_2}$. The discussion splits into two parts, the first handling the infinite cases of MU and BP.

Since we are looking at topological Hochschild spectra, we have complete control on the circle action on homology groups. In fact, the descriptions of the homology groups in proposition 8.1.2 have this information built into them. In the calculations we are about to do we also use that the σ-operator is a derivation with respect to the algebra structure on homology. This last fact can be found in [2, Proposition 5.10]

We list the $\hat{E}^2$-terms of the homological Tate spectral sequences here for later reference.

$$\begin{aligned} \hat{E}^2_{*,*}(MU) &\cong P(u, u^{-1}) \otimes H_*MU \otimes E(\sigma b_k \mid k \geq 1) \\ \hat{E}^2_{*,*}(BP) &\cong P(u, u^{-1}) \otimes H_*BP \otimes E(\lambda_k \mid k \geq 1) \\ \hat{E}^2_{*,*}(BP\langle m-1\rangle) &\cong P(u, u^{-1}) \otimes H_*BP\langle m-1\rangle \otimes E(\lambda_k \mid 1 \leq k \leq m) \otimes P(\mu_m) \end{aligned} \tag{8.3.1}$$

8.3.1 The additive structure of $H_*^c(T(MU)^{tC_2}$

To calculate the $\hat{E}^3$-term of the homological Tate spectral sequence for $T(MU)^{tC_2}$, we recall from proposition 4.4.1 that the d^2-differential is given by the formula

$$d^2(u^s \otimes \alpha) = u^{s+2} \otimes \sigma(\alpha) \tag{8.3.2}$$

for $\alpha \in H_*(T(MU))$. Recall that $t = u^2$ when referring to 4.4.1.

Using the notation of section 8.2 and lemma 8.1.2, we have that $H_*(T(MU))$ together with the differential operator σ can be written as the differential graded algebra $K(b_k \mid k \geq 1)$. This uses that MU is a commutative S-algebra. Applying the Künneth formula and remembering that $\deg(b_k) = 2k$, we may calculate the homology of $H_*T(MU)$ with respect to the σ-operator as in section 8.2. We get that the $\hat{E}^3$-term of the C_2-Tate spectral sequence converging to the continuous homology of $T(MU)^{tC_2}$ is isomorphic to

$$\hat{E}^3_{*,*}(MU) \cong P(u, u^{-1}) \otimes P(b_k^2 \mid k \geq 1) \otimes E(b_l \sigma b_l \mid l \geq 1) \qquad (8.3.3)$$

The corresponding homological homotopy fixed point spectral sequence from proposition 4.5.1 for $T(MU)$ has $\hat{E}^3$-term isomorphic to

$$\hat{E}^3_{*,*}(MU) \cong P(u) \otimes P(b_k^2 \mid k \geq 1) \otimes E(b_l \sigma b_l \mid l \geq 1)$$

modulo some classes in filtration -1 and 0. This spectral sequence is considered by Bruner-Rognes [12, Theorem 6.4 a], where the authors show that the spectral sequence collapses at this stage.

Thus, it follows from proposition 4.5.1 and the fact that the differentials in the homological Tate spectral sequence are derivations, that the spectral sequence (8.3.3) collapses on the $\hat{E}^3$-term as well.

Proposition 8.3.1. *The homological Tate spectral $\hat{E}^r(T(MU))$ collapses at the $\hat{E}^3$-term. Hence additively,*

$$H^c_* T(MU)^{tC_2} \cong P(u, u^{-1}) \hat{\otimes} [P(b_k^2 \mid k \geq 1) \otimes E(b_l \sigma b_l \mid l \geq 1)]$$

8.3.2 The additive structure of $H^c_*(T(BP)^{tC_2}$

We now do the calculation for BP. To determine the $\hat{E}^3$-term of the Tate spectral sequence, we write $H_*BP = K(\bar{\xi}_k^2 \mid k \geq 1)$, analogously to the case of MU, and derive that

$$\hat{E}^3_{*,*}(BP) \cong P(u, u^{-1}) \otimes P(\bar{\xi}_k^4 \mid k \geq 1) \otimes E(\bar{\xi}_l^2 \sigma \bar{\xi}_l^2 \mid l \geq 1). \qquad (8.3.4)$$

We will now use that we have a surjection on homology $H_*MU \to H_*BP$ to determine the $\hat{E}^\infty$-term of the Tate spectral sequence.

Proposition 8.3.2. *The homological Tate spectral $\hat{E}^r(T(BP))$ collapses at the $\hat{E}^3$-term. Hence additively,*

$$H^c_* T(MU)^{tC_2} \cong P(u, u^{-1}) \hat{\otimes} [P(\bar{\xi}_k^4 \mid k \geq 1) \otimes E(\bar{\xi}_l^2 \sigma \bar{\xi}_l^2 \mid l \geq 1)]$$

Proof. The map $T(\phi) : T(MU) \to T(BP)$ induces a surjective map on homology groups. The induced map of homological Tate spectral sequences is also surjective on $\hat{E}^2$-terms given by

$$1 \otimes T(\phi)_* : P(u, u^{-1}) \otimes H_*(T(MU)) \to P(u, u^{-1}) \otimes H_*(T(BP)) \,.$$

Therefore, by proposition 8.3.1, the spectral sequence $\hat{E}^r(T(BP)^{tC_2})$ collapses at the $\hat{E}^3$-term. □

From the A_*-comodule structure of $H_*(T(MU))$ and $H_*(T(BP))$, propositions 8.3.1 and 8.3.2 give partial information about the completed A_*-comodule structure of $H_*^c(T(MU)^{tC_2})$ and $H_*^c(T(BP)^{tC_2})$, namely the A_*-comodule structure on the associated graded with respect to the Tate filtration.

We spell out what this means in the general case of $\hat{E}^r(X^{tG})$ (so X could be $T(MU)$ or $T(BP)$). Let $\bar{\alpha} \in H_*^c(X^{tG})$ be a class in the abutment represented in the spectral sequence by $u^s \otimes \alpha \in \hat{E}^\infty(X^{tG})$ in Tate filtration $-s$. Then for $n \geq 0$, the value of the dual Steenrod operation $\mathrm{Sq}_*^n\,\bar{\alpha} \in H_{t-n}^c(X^{tG})$ will be represented in the homological Tate spectral sequence in filtration less than or equal to $-s$. If $\mathrm{Sq}_*^n\bar{\alpha}$ is in fact represented in filtration $-s$, then its representative is $u^s \otimes \mathrm{Sq}_*^n(\alpha)$. In the case $\mathrm{Sq}_*^n(\alpha) = 0$, we can only conclude that $\mathrm{Sq}_*^n(\bar{\alpha})$ has filtration lower than $-s$. We will refer to the A_*-comodule structure on the associated graded comodules as the vertical A_*-structure.

To gain full control of the A_*-comodule structure on the abutment, we will need to study the map $\gamma_* : T(B) \to T(B)^{tC_2}$. Together with theorem 6.1.1, we will in chapter 10 establish the completed A_*-comodule structure on $H_c^*(T(BP)^{tC_2})$ and prove the first part of theorem 0.0.3.

The next section deals with the case $B = BP\langle m-1\rangle$ for $\infty > m \geq 0$. In these cases we will also record explicitly the vertical A_*-comodule structure. The resulting formulas will be used in chapter 11.

8.3.3 The additive structure of $H_*^c(T(BP\langle m-1\rangle)^{tC_2}$

For this section, we assume that $BP\langle m-1\rangle$ has the structure of a commutative S-algebra, so that $T(BP\langle m-1\rangle)$ is an S-algebra. We are considering the cases $\infty > m \geq 0$, so the family $\{BP\langle m-1\rangle\}$ includes the Eilenberg-MacLane spectra $BP\langle -1\rangle = H\mathbb{F}_2$ and $BP\langle 0\rangle = H\mathbb{Z}$ together with the 2-local connective complex K-theory spectrum ku. These three spectra are known to be commutative S-algebras.

In order to calculate the $\hat{E}^3$-term of the Tate spectral sequence, we rewrite the homology of $T(BP\langle m-1\rangle)$ so that the 0th column in the homological Tate spectral sequence, $\hat{E}^2_{0,*}(BP\langle m-1\rangle)$, is isomorphic to

$$P(\bar{\xi}_1^2, \ldots, \bar{\xi}_m^2, \bar{\xi}_{m+1}, \xi_k' \mid k \geq m+2) \otimes E(\lambda_k \mid m \geq k \geq 1) \otimes P(\mu_m) \quad (8.3.5)$$

where ξ_k' is the homogeneous class $\bar{\xi}_k + \bar{\xi}_{k-1}\sigma\bar{\xi}_{k-1} = \bar{\xi}_k + \bar{\xi}_{k-1}\mu_m^{2^{k-m-2}}$. We have chosen ξ_k' such that it is a cycle with respect to the derivation σ. Indeed, we have $\sigma(\xi'_{m+2+r}) = \sigma\bar{\xi}_{m+2+r} + \sigma(\bar{\xi}_{m+1+r}\sigma\bar{\xi}_{m+1+r}) = \mu_m^{2^{r+1}} + (\mu_m^{2^r})^2 \equiv 0$ for all $r \geq 0$. This trick of rewriting can be found in [3] and [12].

Using the notation of section 8.2, we may now identify this algebra, together with its differential operator σ, as $K(\bar{\xi}_1^2, \ldots, \bar{\xi}_m^2, \bar{\xi}_{m+1}) \otimes P(\xi'_k \mid k \geq m+2)$. As mentioned, all the ξ'-classes are cycles, so applying the Künneth-formula we get the following expression for $\hat{E}^3_{0,*}(BP\langle m-1\rangle)$:

$$P(\bar{\xi}_1^4, \ldots, \bar{\xi}_m^4, \bar{\xi}_{m+1}^2, \xi'_k \mid k \geq m+2) \otimes E(\bar{\xi}_l^2\sigma\bar{\xi}_l^2 \mid m \geq l \geq 1)\,. \tag{8.3.6}$$

We will now record the vertical A_*-comodule structure on $\hat{E}^3(T(BP\langle m-1\rangle)^{tC_2})$.

Lemma 8.3.3. *The A_*-comodule structure inherited from the comodule structure on the $\hat{E}^3$-term* (8.3.6) *is given by the formula*

$$\nu(\xi'_k) = \textstyle\sum_{i=0}^k \bar{\xi}_i \otimes (\xi'_{k-i})^{2^i} \tag{8.3.7}$$

for $k \geq m+3$. In addition

$$\nu(\xi'_{m+2}) \quad = \bar{\xi}_1^2 \otimes \bar{\xi}_m^2\sigma\bar{\xi}_m^2 + \sum_{i\geq 0}\bar{\xi}_i \otimes \xi'^{2^i}_{m+2-i} \tag{8.3.8}$$

We interpret $\xi'_k = \bar{\xi}_k$ for $m+1 \geq k \geq 1$ and $\bar{\xi}_0 = 1$ for the formula to make sense.

In addition, all the classes $\bar{\xi}_l^2\sigma\bar{\xi}_l^2$ are A_*-comodule primitives.

Proof. We verify this formula by using that the coaction map $\nu : H_*T(BP\langle m-1\rangle) \to A_* \otimes H_*T(BP\langle m-1\rangle)$ is a map of algebras and that it commutes with the derivation σ. Indeed, for $k \geq m+2$, $\nu(\sigma\bar{\xi}_k) = 1\otimes\sigma(\nu\bar{\xi}_k) = \sum_i \bar{\xi}_i\otimes\sigma\bar{\xi}_{k-i}^{2^i} = 1 \otimes \sigma\bar{\xi}_k \quad (= 1 \otimes \mu_m^{2^{k-m-1}})$. When $k \geq m+2$, all $\bar{\xi}_{k-i}^{2^i}$ are squares for $i \neq 0$, so σ vanishes on these classes. Then, for $k \geq m+3$, we have the following congruences modulo $A_* \otimes \mathrm{im}(\sigma)$

$$\begin{aligned}
\nu(\bar{\xi}_{k-1})\nu(\sigma\bar{\xi}_{k-1}) &= \sum_i \bar{\xi}_i \otimes \bar{\xi}_{k-1-i}^{2^i}\sigma\bar{\xi}_{k-1} \\
&= \sum_{i\leq k-m-2} \bar{\xi}_i \otimes \bar{\xi}_{k-1-i}^{2^i}(\sigma\bar{\xi}_{k-1-i})^{2^i} + \sum_{k-m-1\leq j} \bar{\xi}_j \otimes \bar{\xi}_{k-1-j}^{2^j}\sigma\bar{\xi}_{k-1} \\
&= \sum_{i\leq k-m-2} \bar{\xi}_i \otimes \left(\bar{\xi}_{k-1-i}\sigma\bar{\xi}_{k-1-i}\right)^{2^i} + \sum_{k-m-1\leq j} \bar{\xi}_j \otimes \sigma(\bar{\xi}_{k-1-j}^{2^j}\bar{\xi}_{k-1}) \\
&\equiv \sum_{i\leq k-m-2} \bar{\xi}_i \otimes \left(\bar{\xi}_{k-1-i}\sigma\bar{\xi}_{k-1-i}\right)^{2^i}
\end{aligned} \tag{8.3.9}$$

since the j-indexed terms are zero on $\hat{E}^3$. Then we have

$$\begin{aligned} \nu(\xi'_k) &= \nu(\bar{\xi}_k) + \nu(\bar{\xi}_{k-1})\nu(\sigma\bar{\xi}_{k-1}) \\ &\equiv \sum_i \bar{\xi}_i \otimes \bar{\xi}_{k-i}^{2^i} + \sum_{i \le k-m-2} \bar{\xi}_i \otimes \left(\bar{\xi}_{k-1-i}\sigma\bar{\xi}_{k-1-i}\right)^{2^i} \\ &= \sum_{i \le k-m-2} \bar{\xi}_i \otimes \left(\bar{\xi}_{k-i} + \bar{\xi}_{k-1-i}\sigma\bar{\xi}_{k-1-i}\right)^{2^i} + \sum_{k-m-1 \le j} \bar{\xi}_j \otimes \bar{\xi}_{k-j}^{2^j} \\ &= \sum_{i=0}^{k} \bar{\xi}_i \otimes (\xi'_{k-i})^{2^i} \end{aligned} \tag{8.3.10}$$

The case $k = m+2$ differs a bit. Now $\nu\sigma\bar{\xi}_{k-1} = 1 \otimes \sigma\bar{\xi}_{m+1} + \bar{\xi}_1 \otimes \sigma\bar{\xi}_m^2$. The last term does not disappear in this case since $\bar{\xi}_m^2$ is not a square in $H_*T(BP\langle m-1\rangle)$. A straightforward calculation like (8.3.9) gives that

$$\begin{aligned} \nu(\bar{\xi}_{m+1})\nu(\sigma\bar{\xi}_{m+1}) &\equiv 1 \otimes \bar{\xi}_{m+1}\sigma\bar{\xi}_{m+1} + \bar{\xi}_1 \otimes (\bar{\xi}_m^2\sigma\bar{\xi}_{m+1} + \bar{\xi}_{m+1}\sigma\bar{\xi}_m^2) + \bar{\xi}_1^2 \otimes \bar{\xi}_m^2\sigma\bar{\xi}_m^2 \\ &\equiv 1 \otimes \bar{\xi}_{m+1}\sigma\bar{\xi}_{m+1} + \bar{\xi}_1^2 \otimes \bar{\xi}_m^2\sigma\bar{\xi}_m^2 \end{aligned}$$

The middle term $\bar{\xi}_1 \otimes (-)$ disappears modulo $A_* \otimes \mathrm{im}(\sigma)$ since σ is a derivation, i.e. $x\sigma y + y\sigma x = \sigma(xy)$. Using this we get

$$\begin{aligned} \nu(\xi'_{m+2}) &= \nu(\bar{\xi}_{m+2}) + \nu(\bar{\xi}_{m+1}\sigma\bar{\xi}_{m+1}) \\ &\equiv \sum_i \bar{\xi}_i \otimes \bar{\xi}_{m+2-i}^{2^i} + 1 \otimes \bar{\xi}_{m+1}\sigma\bar{\xi}_{m+1} + \bar{\xi}_1^2 \otimes \bar{\xi}_m^2\sigma\bar{\xi}_m^2 \\ &= 1 \otimes \xi'_{m+2} + \bar{\xi}_1^2 \otimes \bar{\xi}_m^2\sigma\bar{\xi}_m^2 + \sum_{i \ge 1} \bar{\xi}_i \otimes \bar{\xi}_{m+2-i}^{2^i} \\ &= \bar{\xi}_1^2 \otimes \bar{\xi}_m^2\sigma\bar{\xi}_m^2 + \sum_{i \ge 0} \bar{\xi}_i \otimes {\xi'}_{m+2-i}^{2^i} \end{aligned}$$

Again,we set $\xi'_r = \bar{\xi}_r$ when $r \le m+1$ to shorten notation.

Finally, we check that the exterior classes are A_*-comodule primitives. Let $1 \le k \le m$. Then

$$\begin{aligned} \nu(\bar{\xi}_k^2\sigma\bar{\xi}_k^2) &= \nu(\bar{\xi}_k^2)\sum_i \bar{\xi}_i^2 \otimes \sigma(\bar{\xi}_{k-i}^{2^{i+1}}) = \nu(\bar{\xi}_k^2)(1 \otimes \sigma\bar{\xi}_k^2) \\ &= \sum_i \bar{\xi}_i^2 \otimes \bar{\xi}_{k-i}^{2^{i+1}}\sigma\bar{\xi}_k^2 \equiv 1 \otimes \bar{\xi}_k^2\sigma\bar{\xi}_k^2 \end{aligned}$$

□

We summarize this in the following proposition.

Proposition 8.3.4. *The homological C_2-Tate spectral sequence for $B = BP\langle m{-}1\rangle$ collapses at the $\hat{E}^3$-stage and thus we get that $\hat{E}^\infty_{*,*}(BP\langle m{-}1\rangle)$ is isomorphic to*

$$P(u, u^{-1}) \otimes P(\bar{\xi}_1^4, \dots, \bar{\xi}_m^4, \bar{\xi}_{m+1}^2, \xi'_k \mid k \geq m+2) \otimes E(\bar{\xi}_l^2 \sigma \bar{\xi}_l^2 \mid m \geq l \geq 1)$$

as an A_*-comodule algebra. The vertical A_*-comodule structure is determined by the fact that the exterior generators are A_*-comodule primitives and that there is a short exact sequence of A_*-comodules

$$(\bar{\xi}_l^2 \sigma \bar{\xi}_l^2 \mid m \geq k \geq 1) \subset \hat{E}^\infty_{0,*}(m-1) \to P(\bar{\xi}_1^4, \dots, \bar{\xi}_m^4, \bar{\xi}_{m+1}^2, \bar{\xi}_k \mid k \geq m+1)\,.$$

The kernel is the ideal generated by the exterior generators ν_k and the quotient map is defined by sending $\bar{\xi}_k^?$ to the classes with the same name for $k \leq m{+}1$ and sending ξ'_k to $\bar{\xi}_k$ for $k \geq m+2$. The quotient is canonically a sub A_*-comodule algebra of A_*.

There is an extension $\mathrm{Sq}^2_* \bar{\xi}_{m+2} = \nu_m$, so the sequence does not split as A_*-comodules. By (8.3.8)*, this is the only extension.*

Proof. Again referring to [12, Proposition 6.1], we have the corresponding homological homotopy fixed point spectral sequence with $\hat{E}^3$-term

$$\begin{aligned}&\hat{E}^3_{*,*}(BP\langle m{-}1\rangle) \cong \\ &\quad P(u) \otimes P(\bar{\xi}_1^4, \dots, \bar{\xi}_m^4, \bar{\xi}_{m+1}^2, \xi'_k \mid k \geq m+2) \otimes E(\bar{\xi}_l^2 \sigma \bar{\xi}_l^2 \mid m \geq l \geq 1)\end{aligned}$$

modulo some classes in filtration -1 and 0. As in the case of $T(MU)$, this spectral sequence is shown to collapse at this stage so again by proposition 4.5.1 and the fact that the differentials in the homological Tate spectral sequence are derivations, the spectral sequence $\hat{E}^r(BP\langle m{-}1\rangle^{tC_2})$ collapses on the $\hat{E}^3$-term as well. □

8.4 A_*-comodule structure

Recall by theorem 6.1.1 that for $B = MU$, BP or any of our $BP\langle m{-}1\rangle$, we have a map of Tate spectra

$$\Psi : R_+ B \to T(B)^{tC_2}\,.$$

Note that for $B = BP$ we must apply chapter 7 instead of the theory in chpater 6 since BP is not known to be commutative. Alternatively, we may produce the results in this section for $B = BP$ using the map $\phi : MU \to BP$.

From proposition 5.6.2 we have that $H^c_*(R_+B) \cong R_+(H_*B)$ as a completed A_*-comodule. The map Ψ induces the map

$$u^s \otimes \alpha^{\otimes 2} \mapsto u^s \otimes \alpha^2 \,. \tag{8.4.1}$$

of $\hat{E}^2$-terms of homological Tate spectral sequences. Here $\alpha \in H_*B$ and $\alpha^2 = \iota_*\mu_*(\alpha \otimes \alpha) \in H_*T(B)$ where $\iota : B \to T(B)$ is the inclusion of the zero-simplices and $\mu : B \wedge B \to B$ is the multiplication map.

Recall the dual Steenrod operations in the dual Singer construction from (5.7.3):

$$\mathrm{Sq}^s_*(u^n \otimes \alpha) = \textstyle\sum_i \binom{-1-n-s}{s-2i} u^{n+s-i} \otimes \mathrm{Sq}^i_*\alpha \tag{8.4.2}$$

By this formula and proposition 5.7.2 we get the following theorem by using naturality with respect to the map (8.4.1):

Theorem 8.4.1. *Let B be a bounded below commutative S-algebra of finite type.*

For $\alpha \in H_q(B)$ and $s \in \mathbb{Z}$, let $u^{-q+n} \otimes \alpha \in R_+(H_*(B))$. Then $\Psi_*(u^{-q+n} \otimes \alpha) \in H_*(T(B))$ is represented in the homological Tate spectral sequence by the element $u^n \otimes \alpha^2$ in filtration $-n$.

This gives a choice of classes in the abutment represented by the classes $u^n \otimes \alpha^2$. With this choice of representatives, the dual Steenrod operations are given by

$$\mathrm{Sq}^s_*(u^n \otimes \alpha^2) = \textstyle\sum_i \binom{-1-n+q-s}{s-2i} u^{n+s-2i} \otimes (\mathrm{Sq}^s_*\alpha)^2 \tag{8.4.3}$$

for all $n \in \mathbb{Z}$ and $\alpha \in H_qB$.

Chapter 9

Low degree calculations

To prepare for chapter 10, we need some low-degree facts about the map γ_* in the case of $B = BP\langle -1\rangle = H\mathbb{F}_2$ and $BP\langle 0\rangle = H\mathbb{Z}$.

Indeed, we will show that the map $\gamma_* : H_*(T(\mathbb{Z})) \to H^c_*(T(\mathbb{Z})^{tC_2})$ sends the class λ_1 non-trivially to the class represented in the homological Tate spectral sequence by $u^2 \otimes \bar{\xi}_1^2 \sigma \bar{\xi}_1^2$. This fact was conjectured by Bökstedt and Rognes in 1994.

9.1 The case $BP\langle -1\rangle$

Our discussion starts by a theorem of Hesselholt and Madsen:

Theorem 9.1.1 ([19], proposition 5.3). *The map $\gamma : T(\mathbb{F}_p) \to T(\mathbb{F}_p)^{tC_p}$ induces a p-adic equivalence of connective covers.*

From this we derive

Corollary 9.1.2. *The image of $\mu_0 = \sigma\bar{\xi}_1$ under the map $\gamma_* : H_*T(\mathbb{F}_2) \to H^c_*T(\mathbb{F}_2)^{tC_2}$ is represented in Tate filtration 2 by $[u^{-2}] \in \hat{E}^{\infty}_{2,0}(\mathbb{F}_2)$.*

Proof. Let $T = T(\mathbb{F}_2)$. From Bökstedt's calculations [9] we know that T is homotopy equivalent to a wedge of even suspensions of $H\mathbb{F}_2$. We have $\pi_*T \cong P(\mu_0)$ where μ_0 has degree 2. The Hurewicz map into $\mathbb{F}_2$-homology is injective and maps μ_0 to the class with the same name in H_*T. This follows since $\mu_0 = \sigma\bar{\xi}_1$ is A_*-comodule primitive.

Let $\partial : T \to \Sigma T_{hC_2}$ be the boundary map in the Norm-restriction sequence. The lower half of diagram (9.1.1) comes from the fundamental square

(3.1.4).

$$\begin{array}{ccc} \pi_2 T & \xrightarrow[\cong]{\partial_*} & \pi_1(T_{hC_2}) \\ \downarrow{\scriptstyle \mathrm{Hur}} & & \downarrow{\scriptstyle \mathrm{Hur}} \\ H_2 T & \xrightarrow{\partial_*} & H_1(T_{hC_2}) \\ \downarrow{\scriptstyle \gamma_*} & \nearrow{\scriptstyle \partial_*^h} & \\ H_2^c(T^{tC_2}) & & \end{array} \tag{9.1.1}$$

The proof of theorem 9.1.1 shows that the map $\partial_* : \pi_2 T \to \pi_1 T_{hC_2}$ of homotopy groups is an isomorphism. Hence, $\partial_* \mu_0$ must be represented by a non-zero class in the homotopical homotopy orbit spectral sequence. As fig-

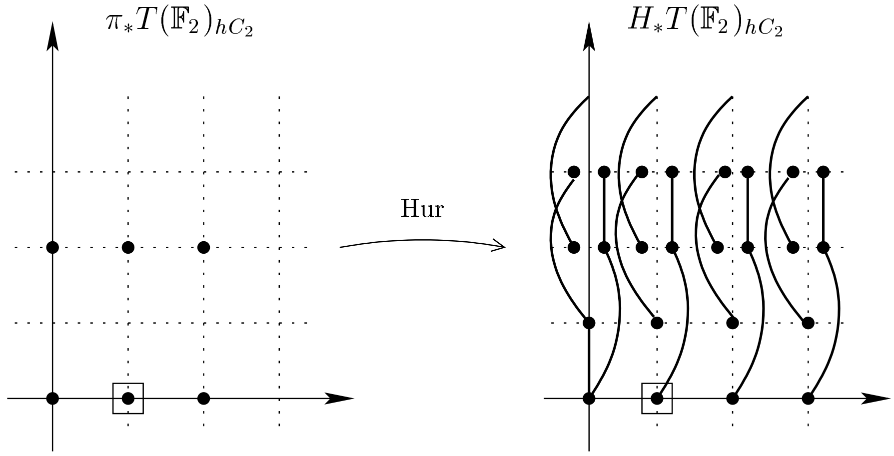


Figure 9.1: The E^2-terms of the homotopical and homological homotopy orbit spectral sequences

ure 9.1 shows, there is no room for differentials in the (both homotopical and homological) homotopy orbit spectral sequence affecting any of the classes in bidegree $(0,0)$ or $(1,0)$. Thus, $\pi_1 T_{hC_2}$ is cyclic with generator of filtration 1. Since $\partial_* \mu_0$ is non-trivial it must be represented in filtration 1 indicated by a box in the left figure.

The Hurewicz map $\pi_* T_{hC_2} \to H_*(T_{hC_2}; \mathbb{F}_2)$ induces a map of homotopy orbit spectral sequences which is an isomorphism on E^2-terms in vertical degree 0. Hence the $\mathbb{F}_2$-Hurewicz image of $\partial_* \mu$ must be non-trivial and must be represented in bidegree $(1,0)$ in the homological homotopy orbit spectral sequence.

The result follows since the map ∂_*^h induces a filtration shifting isomorphism $\hat{E}^\infty_{2,0}(\mathbb{F}_2) \cong E^\infty_{1,0}(\Sigma T_{hC_2})$. $\square$

9.2 The integers

Recall that $H_*T(\mathbb{Z}) \cong H_*H\mathbb{Z} \otimes E(\lambda_1) \otimes P(\mu_1)$ and that there is a non-trivial homology Bockstein relation $\mathrm{Sq}^1_*\mu_1 = \lambda_1$. All higher dual Steenrod squares vanish on μ_1.

We calculated the $\hat{E}^\infty$-term of the Tate spectral sequence converging to the continuous homology of $H^c_*T(\mathbb{Z})^{tC_2}$ in proposition 8.3.4. Additively it was given by $\hat{E}^\infty_{*,*}(\mathbb{Z}) \cong P(u, u^{-1}) \otimes P(\bar{\xi}_1^4, \bar{\xi}_2^2, \xi'_k | k \geq 3) \otimes E(\nu_1)$. Here we let $\nu_1 = \bar{\xi}_1^2\sigma\bar{\xi}_1^2$ to shorten notation. Note that ν_1 is not the coaction of some A_*-comodule.

To make our next calculations easier, we will want to work modulo a certain A_{1*}-sub comodule. Let M_* be an increasing filtered graded comodule. Any nonzero class $x \in M_*$ has a well defined filtration $\phi(x)$. For any $k \in \mathbb{Z}$, consider the sub-vector space $M_*\langle k\rangle$ defined degreewise by

$$M_n\langle k\rangle = \{x \in M_n | \deg(x) - \phi(x) \geq k\}$$

When X is a G-spectrum and $M_* = H^c_*X^{tG}$, the subspace $M\langle k\rangle$ is simply spanned by those classes represented in the homological Tate spectral sequence by an element of bidegree (s,t) where $t \geq k$.

Lemma 9.2.1. *The sub-vector space $H^c_*T(\mathbb{Z})^{tC_2}\langle 8\rangle \subset H^c_*T(\mathbb{Z})^{tC_2}$ is a A_{1*}sub-comodule.*

Proof. We will show that Sq^1_* and Sq^2_* applied to any class in $H^c_*T(\mathbb{Z})^{tC_2}\langle 8\rangle$ will be represented in vertical degrees ≥ 8.

Any class x with representative $[x]$ in vertical degree strictly greater than 9 must have $[\mathrm{Sq}^2_*x]$ and $[\mathrm{Sq}^1_*x]$ of vertical degree at least 8.

We must check the classes represented in vertical degree 8 and 9. We have $\hat{E}^\infty_{*,*}(\mathbb{Z}) \cong P(u, u^{-1}) \otimes P(\bar{\xi}_1^4, \bar{\xi}_2^2, \xi'_k | k \geq 3) \otimes E(\nu_1)$. Hence, $\hat{E}^\infty_{*,8}(\mathbb{Z}) \cong P(u, u^{-1}) \otimes \mathbb{F}_2\{\bar{\xi}_1^8\}$ and $\hat{E}^\infty_{*,9}(\mathbb{Z}) \cong P(u, u^{-1}) \otimes \mathbb{F}_2\{\nu_1\bar{\xi}_1^4\}$. See figure 9.2.

By theorem 8.4.1 we have chosen a representative for $u^r\bar{\xi}_1^8$ so that $\mathrm{Sq}^1_*u^r\bar{\xi}_1^8 = \binom{-r}{1}u^{r+1}\bar{\xi}_1^8$ and $\mathrm{Sq}^2_*u^r\bar{\xi}_1^8 = \binom{-r+1}{2}u^{r+2}\bar{\xi}_1^8$. Both are represented in vertical degree 8.

The last possibility would be a non-trivial vertical $\mathrm{Sq}^2_*\nu_1\bar{\xi}_1^4$, but the vertical A_*-comodule structure of proposition 8.3.4 tells us that $\mathrm{Sq}^2_*\nu_1\bar{\xi}_1^4 = 0$ modulo filtrations and must therefore be represented in vertical degree strictly greater than 8. □

In the following we will work with $H^c_*(T(\mathbb{Z})^{tC_2})$ modulo the A_{1*}-sub-comodule $H^c_*T(\mathbb{Z})^{tC_2}\langle 8\rangle$ of classes of vertical degree greater than 8. The reader might find it helpful to consult figure 9.2 in the discussion that follows.

The class who's representative is $[u^r\xi_3']$ is well defined for any $r \in \mathbb{Z}$ since there is no ambiguity left in lower filtrations. Thus $\mathrm{Sq}_*^2\xi_3'$ is a well defined class and the vertical A_*-coaction says that $[\mathrm{Sq}_*^2 u^r\xi_3'] = [u^r\nu_1]$. We choose $\mathrm{Sq}_*^2\xi_3'$ to be represented by $[u^r\nu_1]$. Note that by these choices together with the choices coming from $R_+(H\mathbb{Z})$ via theorem 8.4.1, we have now chosen classes in the abutment for all representatives in $H_*^c(T(\mathbb{Z})^{tC_2})$ for all the classes in $\hat{E}^\infty_{s,t}(T(\mathbb{Z})^{tC_2})$ with $t < 8$.

Total degree 4 of $H_*^cT(\mathbb{Z})^{tC_2}$ modulo $H_*^cT(\mathbb{Z})^{tC_2}\langle 8\rangle$ is additively given as

$$\mathbb{F}_2\{u^{-4}, \bar{\xi}_1^4, u\nu_1, u^2\bar{\xi}_2^2, u^3\xi_3'\}\,.$$

Modulo Tate-filtrations < -4, total degree 2 is given as

$$\mathbb{F}_2\{u^{-2}, u^2\bar{\xi}_1^4, u^3\nu_1, u^4\bar{\xi}_2^2\}\,.$$

From the inclusion of zero-simplices, $R_+H\mathbb{Z} \to T(\mathbb{Z})^{tC_2}$, and the explicit formulas for the A_{1*}-comodule coaction in the Singer construction, we get the following formulas

$$\begin{aligned} \mathrm{Sq}_*^1 u^r\bar{\xi}_1^4 &= \tbinom{-r}{1} u^{r+1}\bar{\xi}_1^4 \\ \mathrm{Sq}_*^2 u^r\bar{\xi}_1^4 &= \tbinom{-r-1}{2} u^{r+2}\bar{\xi}_1^4 \\ \mathrm{Sq}_*^1 u^r\bar{\xi}_2^2 &= \tbinom{-r+1}{1} u^{r+1}\bar{\xi}_2^2 \\ \mathrm{Sq}_*^2 u^r\bar{\xi}_2^2 &= \tbinom{-r}{2} u^{r+2}\bar{\xi}_2^2 + u^r\bar{\xi}_1^4\,. \end{aligned} \tag{9.2.1}$$

These operations and the $\mathrm{Sq}_*^2\xi_3'$ are depicted with solid curves in figure 9.2.

We list some values of $\mathrm{Sq}_*^2 : H_4^cT(\mathbb{Z})^{tC_2} \to H_2^cT(\mathbb{Z})^{tC_2}$.

$$\begin{aligned} u^{-4} &\mapsto 0 \\ \bar{\xi}_1^4 &\mapsto u^2\bar{\xi}_1^4 \\ u^3\xi_3' &\mapsto u^3\nu_1 \\ u^2\bar{\xi}_2^2 &\mapsto u^2\bar{\xi}_1^4 + u^4\bar{\xi}_2^2\,. \end{aligned} \tag{9.2.2}$$

Lemma 9.2.2. $[\mathrm{Sq}_*^1 u^r\nu_1]$ *is well defined modulo filtrations* $< -r-1$ *and given by* $[\mathrm{Sq}_*^1 u^r\nu_1] = \binom{r}{1}[u^{r+1}\nu_1]$. *In figure 9.2, these dual operations are drawn with solid dashed lines.*

Proof. We start with $[\nu_1]$ in filtration zero. Again we refer to figure 9.2. The class $u\bar{\xi}_2^2$ has trivial Sq_*^1 modulo filtrations < -1. This follows directly from the vertical A_*-structure. There are only two classes representing $[\nu_1]$ modulo filtration < -1 and they differ by $u\bar{\xi}_2^2$. Since Sq_*^1 annihilates this ambiguity

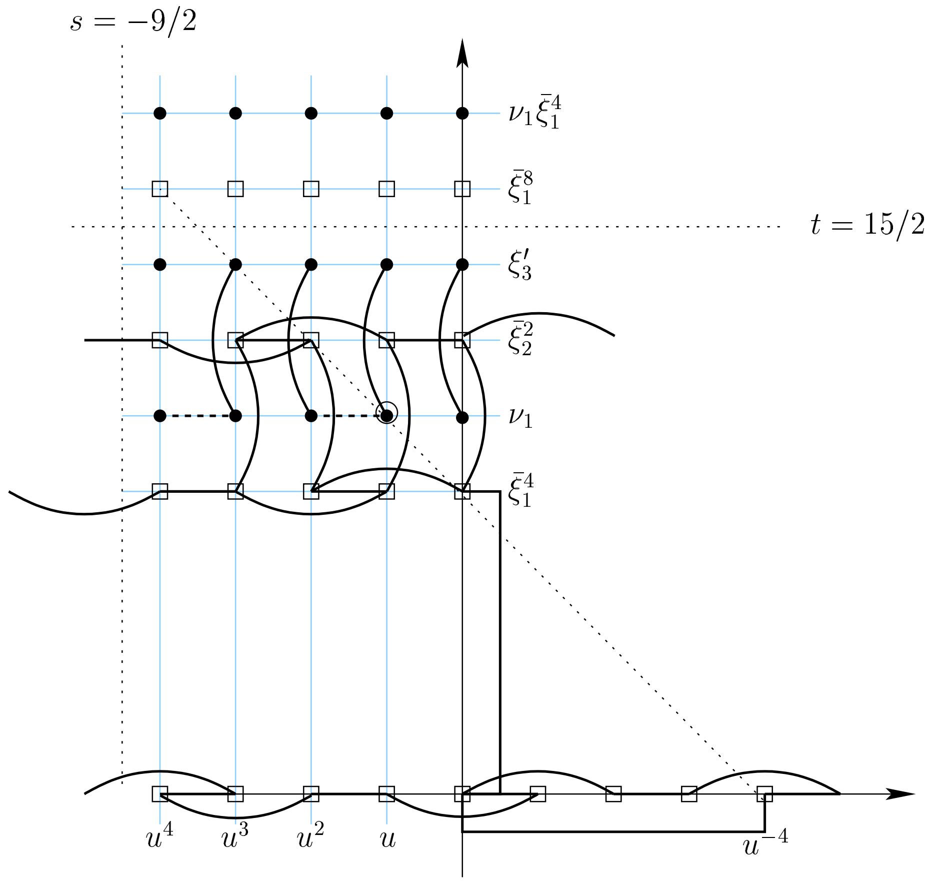


Figure 9.2: Some A-module structure in the case of the integers

modulo filtrations, the coaction of Sq^1_* on $H^c_*(T(\mathbb{Z})^{tC_2})$ is well defined modulo filtration < -1.

Let ν_1 be a representative for $[\nu_1]$. Recall from (4.4.3) that we have an extension of filtered A_*-comodules

$$0 \to H^c_*T(\mathbb{Z})^{t\mathbb{T}} \to H^c_*T(\mathbb{Z})^{tC_2} \to \Sigma^{-1}H^c_*T(\mathbb{Z})^{t\mathbb{T}} \to 0$$

such that the first map induces an injection of the even columns of Tate spectral sequences. Since we may alter the representative ν_1 by any element in filtration -1, we may assume that ν_1 lifts to $H^c_*T(\mathbb{Z})^{t\mathbb{T}}$. The $\hat{E}^\infty$-term of the $\mathbb{T}$-Tate spectral sequence for $T(\mathbb{Z})$ is isomorphic to the even columns of the C_2-Tate spectral sequence. Since $\mathrm{Sq}^1_*[\nu_1]$ is represented in filtration < 0 it follows that it is represented in filtration < -1. This shows that $\mathrm{Sq}^1_*[\nu_1] \in H^c_*T(\mathbb{Z})^{tC_2}$ is trivial modulo filtrations < -1.

The classes represented by $[u^r]$ on the horizontal axis have $\mathrm{Sq}^s_*u^r =$

$\binom{-1-r-s}{s}u^{r+s}$ for all $s \geq 0$. We know the A_{0*}-coaction on ν_1 modulo filtration < -1, so the A_{0*}-coaction on the product $u^r\nu_1$ is known modulo filtration $< -r-1$. It is given by the Cartan formula: $[\mathrm{Sq}^1_*(u^r\nu_1)] = [(\mathrm{Sq}^1_* u^r)\nu_1] + [u^r\,\mathrm{Sq}^1_*\nu_1] = [\binom{r}{1}u^{r+1}\nu_1]$. □

Theorem 9.2.3. *The classes λ_1 and μ_1 of degrees 3 and 4, have non-trivial images under the map $\gamma_* : H_*T(\mathbb{Z}) \to H^c_*T(\mathbb{Z})^{tC_2}$. Their values are represented in the spectral sequence by the classes*

$$\begin{aligned} [\gamma_*(\mu_1)] &= u^{-4} \\ [\gamma_*(\lambda_1)] &= u^2\nu_1 \end{aligned} \tag{9.2.3}$$

in Tate filtration 4 and -2 respectively.

Proof. We start by identifying the image of μ_1. The map $H\mathbb{Z} \to H\mathbb{F}_2$ is injective in homology, mapping $\bar{\xi}_2$ to the class with the same name in $H_*\mathbb{F}_2$. The σ-operator is natural with respect to maps $H\mathbb{Z} \to H\mathbb{F}_2$, so this implies that $\mu_1 = \sigma\bar{\xi}_2 \mapsto \sigma\bar{\xi}_2 = \mu_0^2 \in H_*T(\mathbb{F}_2)$. Consider the following commutative diagram:

$$\begin{array}{ccc} H_*T(\mathbb{Z}) & \longrightarrow & H_*T(\mathbb{F}_2) \\ \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} \\ H^c_*T(\mathbb{Z})^{tC_2} & \longrightarrow & H^c_*T(\mathbb{F}_2)^{tC_2}\,. \end{array} \tag{9.2.4}$$

Corollary 9.1.2 says that $[\gamma_*(\mu_0)] = [u^{-2}]$ in the Tate spectral sequence on the right. By the algebra structure, we get that $\gamma_*(\mu_0^2)$ is represented by $[u^{-4}]$ in filtration 4. Since the map of spectral sequences $\hat{E}^\infty_{s,t}(T(\mathbb{Z})^{tC_2}) \to \hat{E}^\infty_{s,t}(T(\mathbb{F}_2)^{tC_2})$ is an isomorphism in vertical degree $t = 0$, we must have that $\gamma_*(\mu_1)$ is represented by u^{-4} in $\hat{E}^\infty(T(\mathbb{Z})^{tC_2})$ as well and the first claim of the theorem follows.

The first ambiguity of $\gamma_*(\mu_1)$ lies in filtration 0 represented by $\bar{\xi}_1^4$. This class has non-trivial $\mathrm{Sq}^4_*\bar{\xi}_1^4 = 1$ modulo negative filtrations. By the fact that $\mathrm{Sq}^4_* u^{-4} = 1$ we see that $\gamma_*(\mu_1) = u^{-4} + \bar{\xi}_1^4$ modulo negative filtrations.

Since $\mathrm{Sq}^2_*\gamma_*(\mu_1)$ must be trivial in $H^c_*T(\mathbb{Z})^{tC_2}$ it must in particular be trivial modulo $H^c_*T(\mathbb{Z})^{tC_2}\langle 8\rangle$ and modulo Tate-filtration < -4. By the Sq^2_*-coactions listed in (9.2.2) the only possibility for this to happen is that $\gamma_*(\mu_1) = u^{-4} + \bar{\xi}_1^4 + u\nu_1$ modulo filtrations < -1. Indeed, the $\mathbb{F}_2$-linear map

$$\mathrm{Sq}^2_* : \mathbb{F}_2\{u^{-4}, \bar{\xi}_1^4, u\nu_1, u^2\bar{\xi}_2^2, u^3\xi_3'\} \to \mathbb{F}_2\{u^{-2}, u^2\bar{\xi}_1^4, u^3\nu_1, u^4\bar{\xi}_2^2\}$$

has image $\mathbb{F}_2\{u^2\bar{\xi}_1^4, u^3\nu_1, u^4\bar{\xi}_2^2\}$. The class u^{-4} lies in the kernel, so Sq_*^2 induces a surjective linear map

$$\mathrm{Sq}_*^2 : \mathbb{F}_2\{\bar{\xi}_1^4, u\nu_1, u^2\bar{\xi}_2^2, u^3\xi_3'\} \to \mathbb{F}_2\{u^2\bar{\xi}_1^4, u^3\nu_1, u^4\bar{\xi}_2^2\}\,.$$

The list of cooperations (9.2.2) implies that this map is an isomorphism when restricted to the subspace $\mathbb{F}_2\{\bar{\xi}_1^4, u^2\bar{\xi}_2^2, u^3\xi_3'\}$, implying that any class in the kernel of Sq_*^2 that can be written as a sum that includes $\bar{\xi}_1^4$ must also contain $u\nu_1$.

Lemma 9.2.2 now implies that $\gamma_*\lambda_1 = \mathrm{Sq}_*^1\gamma_*\mu_1 = \mathrm{Sq}_*^1(u^{-4} + \bar{\xi}_1^4 + u\nu_1) = u^2\nu_1$ modulo filtrations < -2, so $[\gamma_*\mu_1] = [u^2\nu_1]$. □

Theorem 9.2.3 enables us to completely determine the map $\gamma_* : H_*T(\mathbb{Z}) \to H_*^cT(\mathbb{Z})^{tC_2}$ in chapter 11.

Chapter 10

BP

In this chapter we finally come to the proof of the part of theorem 0.0.3 dealing with BP.

The proof will consist of describing the precise structure of $H^c_*(T(BP)^{tC_2})$ as a completed A_*-comodule. This involves the description of the map $\Psi_* : R_+(H_*(BP)) \to H^c_*(T(BP)^{tC_2})$ from theorem 8.4.1 and knowledge about the map $\gamma_* : H^c_*(T(BP)) \to H^C_*(T(BP)^{tC_2})$. Specifically, we will need to know where the exterior classes $\lambda_k \in H_*(T(B))$ map under γ_*. It will turn out that the images of γ_* and Ψ_* generate the entire continuous homology of $T(BP)^{tC_2}$.

As a technical point in the proof of proposition 4.3.3 we will have to assume that BP is coherent enough to give $H_*T(BP)$ a well defined action of the Dyer-Lashof operations.

10.1 Main theorem

The main theorem of this chapter is the following:

Theorem 10.1.1. *The continuous cohomology of $T(BP)^{tC_2}$ with respect to the Tate filtration is isomorphic to $R_+(H^*T(BP))$. Under this identification the map γ^* corresponds to the evaluation map $\epsilon : R_+(H^*T(BP)) \to H^*(T(BP))$ up to some non-zero scalar. In particular, γ^* is a* Tor*-equivalence.*

This shows the first part of theorem 0.0.3: Since γ^* induces an Ext-isomorphism, it induces an isomorphism of E_2-terms of the inverse limit of Adams spectral sequences, and we get that $\gamma : T(BP) \to T(BP)^{tC_2}$ is a 2-adic equivalence. By the homotopy Cartesian square in diagram (4.1.2), we get:

Theorem 10.1.2. *The map $\Gamma : T(BP)^{C_2} \to T(BP)^{hC_2}$ is a 2-adic equivalence.*

The proof of theorem 10.1.1 will be given in the end of section 10.2.

10.1.1 Homology operations

The Brown-Peterson spectrum BP is an S-algebra, but is not known to be coherent enough as to allow an H_∞-structure or even better; be a commutative S-algebra. However, the maps $MU \to BP \to H\mathbb{F}_2$ are maps of S-algebras such that in $\mathbb{F}_2$-homology, the first is surjective and the last is injective. Moreover, the composite is a map of commutative S-algebras. Hence, the Dyer-Lashof operations acting on $H_*H\mathbb{F}_2$ restrict to the injective image of H_*BP.

We state some basic properties for the σ-operator acting on the $\mathbb{F}_2$-homology of a commutative S-algebra. The next result and its proof can be found in [2, Propositions 5.9 and 5.10].

Proposition 10.1.3. *Let B be a commutative S-algebra. Then the σ-operator is a graded derivation with respect to the homology algebra multiplication and commutes with the Dyer-Lashof operations. That is, for $\alpha_1, \alpha_2 \in H_*(B;\mathbb{F}_2)$ and any integer k we have*

$$Q^k(\sigma\alpha_1) = \sigma Q^k(\alpha_1)$$

in addition to the Leibniz rule

$$\sigma(\alpha_1\alpha_2) = \sigma(\alpha_1)\alpha_2 + \alpha_1\sigma(\alpha_2)$$

We are working over $\mathbb{F}_2$, so the usual signs are not present.

In addition to the previous proposition we recollect some facts about the Dyer-Lashof operations. These operations acting on $H_*(H\mathbb{F}_2) \cong A_*$ are known and explicitly listed in [11, III theorem 2.2]. In particular we have that $Q^{2^k}(\bar{\xi}_k^2) = \bar{\xi}_{k+1}^2$ for all $k \geq 1$. Indeed, for any x we have $Q^{2n}x^2 = (Q^n x)^2$, and so $Q^{2^{k+1}}\sigma\bar{\xi}_k^2 = \sigma Q^{2^{k+1}}\bar{\xi}_k^2 = \sigma(Q^{2^k}\bar{\xi}_k)^2 = \sigma\bar{\xi}_{k+1}^2$.

The map $\phi_* : H_*MU \to H_*BP$ is surjective in homology. Choose a lifting of $\bar{\xi}_1^2 \in H_*BP$ and denote this lifting by $\bar{b}_1$. Then define $\bar{b}_{k+1} = Q^{2^k}\bar{b}_k$ for all $k \geq 2$. We then have $\phi_*(\bar{b}_k) = \bar{\xi}_k^2$. Moreover, the induced map $T(\phi)_* : H_*T(MU) \to H_*T(BP)$, maps $\sigma\bar{b}_k$ to $\sigma\bar{\xi}_k^2$ since Q^i commutes with the σ-operator. Denote the element $\sigma\bar{b}_k$ by $\bar{\lambda}_k$.

Assuming that BP is coherent enough to give $H_*T(BP)$ a well defined action of the Dyer-Lashof operations, we have:

Lemma 10.1.4. *The following is true for all $k \geq 0$ and all $\lambda_k \in H_*T(BP)$:*

(i) $Q^{odd}(\lambda_k) = 0$

(ii) $Q^{2^{k+1}}(\lambda_k) = \lambda_{k+1}$

(iii) $Q^{2^{k+1}+2}(\lambda_k) = 0$

Proof. (i) follows from the fact that any odd operation must act trivially on even-degree homology.

(ii) For any x we have $Q^{2n}x^2 = (Q^n x)^2$, and so $Q^{2^{k+1}}\sigma\bar{\xi}_k^2 = \sigma Q^{2^{k+1}}\bar{\xi}_k^2 = \sigma(Q^{2^k}\bar{\xi}_k)^2 = \sigma\bar{\xi}_{k+1}^2$.

From the explicit list of operations from [11, III, Theorem 2.2] one gets that $Q^{2^k+1}\bar{\xi}_k = 0$. From this, (iii) follows. Indeed, $Q^{2^{k+1}+2}\sigma\bar{\xi}_k^2 = \sigma(Q^{2(2^k+1)}\bar{\xi}_k^2) = \sigma(Q^{2^k+1}\bar{\xi}_k)^2 = 0$. □

Lemma 10.1.5. *Let B be a commutative S-algebra. Then for any class $x \in H_*T(B)$*

$$Q^{2r}(x\sigma x) \equiv Q^r(x) \cdot \sigma Q^r(x)$$

for all r, modulo classes in the image of σ.

Proof. We use the Cartan formula and the fact that σ is a derivation with respect to the algebra structure on $H_*T(B)$. Indeed

$$\begin{aligned} Q^{2r}(x\sigma x) &= \textstyle\sum_i Q^i(x)\sigma Q^{2r-i}(x) \\ &= Q^r(x)\sigma Q^r(x) + \textstyle\sum_{i<r} \sigma(Q^i(x)Q^{2r-i}(x)) \,. \end{aligned}$$

We are also using (10.1.3) saying that the σ-operator commutes with the Dyer-Lashof operations. □

The following lemma says that we have Dyer-Lashof operations acting on the 0th column of the Tate spectral sequence.

Lemma 10.1.6. *Let B be a commutative S-algebra. Suppose given a class $x \in H_nT(B)^{tC_2}$ such that $[x] \in \hat{E}^\infty_{0,n}$ is an infinite cycle in the homological Tate spectral sequence surviving to the $\hat{E}^\infty$-term. If $Q^i[x] \in \hat{E}^\infty_{0,n+i}$ is an infinite cycle, then this class represents $Q^i x$ in the spectral sequence.*

Proof. By the Norm-Restriction sequence and the fact that x has filtration zero, we conclude that the class x can be lifted to the homotopy fixed points of X and is represented in the homotopy fixed point spectral sequence by $[x]$ in filtration zero. The edge homomorphism $T(B)^{hC_2} \to T(B)$ takes all classes of negative filtration to zero and is an injection on classes represented in the 0th column. Thus, to determine where $Q^i(x)$ is represented in the Tate spectral sequence, we may lift x to the homotopy fixed points, push it into the homology of $T(B)$ and apply the Q^i-operation. □

10.2 Proof of the main theorem

In the proof of the next proposition, things would have been easier if BP were known to have an E_∞-structure. Even though the homology of BP allows the action of the Dyer-Lashof algebra, we can not conclude that the homology of the Tate-construction on $T(BP)$ carries the same structure. We will work around this, using that we still have a map from $T(MU)^{tC_2}$ which is surjective on $\hat{E}^{\infty}_{**}$. The trouble is that the map to $T(\mathbb{F}_2)^{tC_2}$ is no longer injective on continuous homology (the kernel is the ideal generated by the λ_k's) so we need to take a little more care.

Proposition 10.2.1. *Assume that BP is coherent enough to give $H_*T(BP)$ a well defined action of the Dyer-Lashof operations.*

For all k, the exterior class $\lambda_k \in H_*T(BP)$ is sent by γ_* to a non-zero class represented by $\lambda_k^c = u^{2(2^k-1)}\nu_k$ in the Tate spectral sequence.

Proof. We proceed by induction on k. Consider the following commutative diagram:

$$\begin{array}{ccc} H_*^cT(S)^{tC_2} \otimes H_*T(MU) & \xrightarrow{1\otimes\phi_*} & H_*^cS^{tC_2} \otimes H_*T(BP) \\ \downarrow{\scriptstyle (\eta\wedge\gamma)_*} & & \downarrow \\ H_*^cT(MU)^{tC_2} & \xrightarrow{\phi_*^{tC_2}} & H_*^cT(BP)^{tC_2}\,. \end{array} \tag{10.2.1}$$

The continuous homology of $T(S)^{tC_2}$ is as before identified with the Laurent polynomials $P(u,u^{-1})$, mapping u^s for all s by the unit $T(\eta)$ to the classes in $T(BP)^{tC_2}$ represented by the infinite cycles with the same name in the spectral sequence. The map $\eta\wedge\gamma$ on the left is the composition $S^{tC_2}\wedge T \to T^{tC_2}\wedge T^{tC_2} \to T^{tC_2}$ (for $T = T(MU)$. The corresponding map on the right is gotten by applying continuous homology to the composition $T(S)\wedge T(BP) \to T(S\wedge BP) \cong T(BP)$. The left vertical and the top horizontal map commute with the action of the Dyer-Lashof algebra.

By theorem 9.2.3 we already know by mapping down to $H\mathbb{Z}$ that the statement in the proposition is true for $k=1$. In other words we have that $u^{-2}\otimes\lambda_1 \in H_*^cS^{tC_2}\otimes H_*T(BP)$ maps to a class represented by ν_1 in filtration zero. To see this, note that the maps of spectral sequences on $\hat{E}^{\infty}$-terms, $\hat{E}^{\infty}_{*,*}(T(MU)) \to \hat{E}^{\infty}_{*,*}(T(BP)) \to \hat{E}^{\infty}_{*,*}(T(\mathbb{Z}))$, are isomorphisms in bidegrees $(*,t)$ for $t\le 5$. The differences at the $\hat{E}^2$-terms are cancelled by the d^2-differential.

Since the image of $u^{-2}\otimes\sigma\bar{b}_1$ maps down to a class in $H_*^cT(BP)^{tC_2}$ represented by $[\nu_1]$ and the map $\phi_{*,*}^{tC_2}$ of spectral sequences is an isomorphism in vertical degrees ≤ 5, we get that $(\eta \wedge \gamma)_*(u^{-2} \otimes \sigma\bar{b}_1)$ is represented by $[\bar{b}_1\sigma\bar{b}_1] \in \hat{E}^\infty_{0,5}(T(MU))$.

For each $k \geq 1$, define $\beta_k \in H^c_*T(MU)^{tC_2}$ by $\beta_{k+1} = Q^{2^{k+2}}\beta_k$ and $\beta_1 = \bar{b}_1\sigma\bar{b}_1$. Then we have that β_k maps to a class in $H^c_*T(BP)^{tC_2}$ represented by $\nu_k \in \hat{E}^\infty_{0,2^{k+2}-3}$. This is OK for $k = 1$ and by induction on k one derives that in fact

$$\beta_k \equiv (Q^{2^k} \cdots Q^{2^2}\bar{b}_1)\sigma(Q^{2^k} \cdots Q^{2^2}\bar{b}_1) = \bar{b}_k\sigma\bar{b}_k \tag{10.2.2}$$

modulo the image of σ. We are using lemma 10.1.5 and that the Q^i's commute with the σ-operator.

Similarly, define $\alpha_k \in H^c_*S^{tC_2} \otimes H_*T(MU)$ by the same recipe; $\alpha_{k+1} = Q^{2^{k+2}}\alpha_k$ and $\alpha_1 = u^{-2} \otimes \sigma\bar{b}_1$. By definition of $\bar{b}_1$, we have that $\alpha_k \mapsto u^{-2(2^k-1)} \otimes \lambda_k$ for $k = 1$. Assume by induction this also holds for $k \leq n$. Since the map $(1 \otimes \phi)_*$ commutes with the Dyer-Lashof operations, the Cartan formula then gives

$$\begin{aligned}(1 \otimes \phi)_*\alpha_{n+1} &= Q^{2^{n+2}}(u^{-2(2^n-1)} \otimes \lambda_n) \\ &= \textstyle\sum_i Q^i(u^{-2(2^n-1)}) \otimes Q^{2^{n+2}-i}(\lambda_n)\,.\end{aligned} \tag{10.2.3}$$

The terms in the sum vanish unless $i \geq |u^{-2(2^n-1)}| = 2^{n+1} - 2$ and $2^{n+2} - i \geq |\lambda_n| = 2^{n+1} - 1$, that is $2^{n+1} - 2 \leq i \leq 2^{n+1} + 1$. This reduces the sum to four terms:

$$\textstyle\sum_{-2 \leq i \leq 1} Q^{2^{n+1}+i}(u^{-2(2^n-1)}) \otimes Q^{2^{n+1}-i}(\lambda_n)\,. \tag{10.2.4}$$

Moreover, the terms with i odd vanish by (i) in lemma 10.1.4. Together with lemma 10.1.4 (iii), we get that

$$\begin{aligned}(1 \otimes \phi)_*\alpha_{n+1} = \ & Q^{2^{n+1}-2}(u^{-2(2^n-1)}) \otimes Q^{2^{n+1}+2}(\lambda_n) \\ & + Q^{2^{n+1}}(u^{-2(2^n-1)}) \otimes Q^{2^{n+1}}(\lambda_n) \\ = \ & Q^{2^{n+1}}(u^{-2(2^n-1)}) \otimes \lambda_{n+1} \\ = \ & u^{-2(2^{n+1}-1)} \otimes \lambda_{n+1}\,.\end{aligned} \tag{10.2.5}$$

The last equality follows from the Nishida relations; $\mathrm{Sq}^2_*Q^0(u^2) = Q^{-2}(u^2) + Q^{-1}(\mathrm{Sq}^1_*u^2) = Q^{-2}(u^2) = u^4$, which implies that $Q^0u^2 = u^2$. (Remember that $\deg(u^2) = -2$, so Q^0u^2 may be non-zero.) By the Cartan formula $Q^{|x|}(xy) = x^2 \cdot Q^0(y)$ for all x and y, so when $x = u^{-2(2^n)}$ and $y = u^2$ we get the desired equality.

So $u^{-2(2^k-1)} \otimes \lambda_k$ maps to ν_k for $k \geq 1$. Then it follows that the product $1 \otimes \lambda_k$ maps to $u^{2(2^k-1)} \otimes \nu_k$, and we are done. □

By the preceding proposition the classes $\lambda_k^c = u^{2(2^k-1)}\nu_k$ represent the images of λ_k under γ_*. From proposition 8.3.4 we know that the classes λ_k are A_*-comodule primitives, hence their images under γ_* are A_*-comodule primitives as well.

We know that the map $\phi : MU \to BP$ induces a surjection $\phi_*^c : H_*^c(T(MU)^{tC_2}) \to H_*^c(T(BP)^{tC_2})$. Then for each k, choose $\bar{\lambda}_k^c$ in the preimage of λ_k^c. The ideal generated by the classes $\bar{\lambda}_k^c$ for $k \geq 1$ is a sub A_*-comodule, and we will refer to the image of this ideal in $H_*^c(T(BP)^{tC_2})$ as the ideal generated by the classes λ_k^c for $k \geq 1$ and denote it by $(\lambda_k^c \mid k \geq 1)$. Since each class λ_k^c is A_*-comodule primitive, the ideal generated by these classes is an A_*-sub comodule.

The A_*-comodule quotient of $H_*^c(T(BP)^{tC_2})$ by the ideal generated by the ideal (λ_k^c) can then be described as the completed A_*-comodule represented by the corresponding quotient of the $\hat{E}^\infty$-term:

$$\hat{E}^\infty(T(BP)^{tC_2})/(\lambda_k^c \mid k \geq 1) \cong P(u, u^{-1}) \otimes P(\bar{\xi}_1^4, \bar{\xi}_2^4, \ldots) \tag{10.2.6}$$

Proof of theorem 10.1.1. By 8.4.1 and the cyclotomic structure of $T(BP)$, we have maps

$$\begin{array}{ccc} & & H_*(T) \\ & & \big\downarrow \gamma_* \\ R_+(H_*BP) & \xrightarrow{\Psi_*} & H_*^c(T(BP)^{tC_2})\,. \end{array} \tag{10.2.7}$$

The composition

$$R_+H_*BP \xrightarrow{\Psi_*} H_*^cT^{tC_2} \to H_*^cT^{tC_2}/(\lambda_k^c | k \geq 1)$$

is an isomorphism. Indeed, the image of Ψ_* consists of classes represented by the sub algebra $P(u, u^{-1}) \otimes P(\bar{\xi}_k^4 \mid \geq 1) \subset \hat{E}_{*,*}^\infty(T(BP)^{tC_2})$.

We then define a map of filtered A_*-comodules $E(\lambda_k^c) \otimes R_+(H_*BP) \to H_*^c(T(BP)^{tC_2})$ by the formula $\lambda_k^c \otimes x \mapsto \lambda_k^c \Psi(x)$. Since Ψ_* is split, this map is injective. Surjectivity follows since it induces a surjection of associated graded comodules. Since all the generators of $E(\lambda_k^c)$ are A_*-comodule primitives, we have an isomorphism of filtered A_*-comodules

$$E(\lambda_k^c | k \geq 1) \otimes R_+(H_*(BP)) \cong R_+\big[E(\lambda_k | k \geq 1) \otimes H_*(BP)\big] \cong R_+H_*T(BP)\,. \tag{10.2.8}$$

Therefore,

$$H_*^c(T(BP)^{tC_2}) \cong R_+(H_*T(BP)) \tag{10.2.9}$$

as a completed A_*-comodule. Remember that the continuous homology is the inverse limit of bounded below A_*-comodules of finite type. By taking into account that the isomorphism (10.2.9) is an isomorphism of filtered A_*-comodules, we may take the dual of this inverse limit system. We then get that as A-modules,

$$H_c^*(T(BP)^{tC_2}) \cong R_+(H^*T(BP)) \cong E(\lambda_k^c)^* \otimes R_+(H^*(BP))\,. \tag{10.2.10}$$

The map γ_* in homology is nontrivial. This follows from proposition 4.3.3 using the unit $T(S) \to T(BP)$. In addition we know that the class λ_k corresponds to the classes λ_k^c for $k \geq 1$. Dually, γ^* is non-trivial on the dual of the exterior algebra $E(\lambda_k \mid k \geq 1)$.

Moreover, we have that $H^*B \cong A/\!\!/E$ is a cyclic A-module so by corollary 5.2.4 we conclude that $\gamma^* : R_+(H^*(T(BP)) \to H^*(T(BP))$ can be identified with the evaluation map ϵ up to some non-zero scalar. □

Remark: Assuming that we have an analogue of proposition 4.3.3 for $B = MU$, we could prove, using the same proof as above, the Segal conjecture for MU. The problem is that the classes $b_k\sigma b_k$ are not all generated by $b_1\sigma b_1$ over the Dyer-Lashof algebra. We are using that the homology of BP is cyclic over the Dyer-Lashof algebra, which is not the case for H_*MU.

Chapter 11

$BP\langle m-1\rangle$

In this last chapter we prove the last part of theorem 0.0.3. The proof is obtained by noticing that the continuous cohomology $H_c^*(T(BP\langle m-1\rangle)^{tC_2})$ is induced up over A_{m-1}. The main point is that A_{m-1} is a finite sub Hopf algebra of A.

The key tool for this chapter is proposition 5.4.2.

11.1 Overview

Assume that $B = BP\langle m-1\rangle$ has the structure of a commutative S-algebra. Proposition 8.3.4 gave the $\hat{E}^\infty$-term of the spectral sequence converging to the continuous homology of $T(BP\langle m-1\rangle)^{tC_2}$ as the algebra

$$\hat{E}^\infty_{*,*}(m-1) \cong P(u, u^{-1}) \otimes P(m)_* \otimes E(\lambda_k^c | 1 \le k \le m)\,, \tag{11.1.1}$$

where the subalgebra $P(m)_* := P(\bar{\xi}_1^4, \dots, \bar{\xi}_m^4, \bar{\xi}_{m+1}^2, \bar{\xi}_k | k \ge m+2)$ has filtration zero and the exterior class λ_k^c has bidegree $(-2(2^k-1), 2^{k+3}-3)$. Note that $P(m)_*$ has nothing to do with the homotopy of the Brown-Peterson spectra $P(m)$.

The map $T(BP)^{tC_2} \to T(BP\langle m-1\rangle)^{tC_2}$ induces an isomorphism of $\hat{E}^\infty$-terms of Tate spectral sequences in vertical degrees less than or equal to $2^{m+2}-3$ and gives representatives for the A_*-comodule primitive exterior classes $\lambda_k^c = \gamma_*(\lambda_k)$ for $m \ge k \ge 1$. We have the following commutative diagram

$$\begin{array}{ccc} H_*T(BP) & \longrightarrow & H_*T(BP\langle m-1\rangle) \\ \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} \\ H_*^cT(BP)^{tC_2} & \longrightarrow & H_*^cT(BP\langle m-1\rangle)^{tC_2} \end{array}$$

and $\gamma_*(\lambda_k) = \lambda_k^c$ everywhere.

Let $E(m-1)$ by the subalgebra generated by the exterior classes $\lambda_k \in H_*T(BP\langle m-1\rangle)$ for $1 \le k \le m$. These generators are all A_*-comodule primitives and correspond 1-1 with the classes $[u^{2(2^{k+1}-1)}\nu_k] \in H_*^cT(BP\langle m-1\rangle)^{tC_2}$ via the map γ_*. For shorter notation, let $\lambda_k^c = \gamma_*(\lambda_k)$ and let $E(m-1)^c$ be the exterior algebra generated by the classes λ_k^c for $1 \le k \le m$.

The collection of $\{\lambda_k^c\}_k$ in $H_*^cT(BP\langle m-1\rangle)^{tC_2}$ form a sub A_*-comodule since the classes are all A_*comodule primitives, and thus the ideal generated by these classes forms a sub A_*-comodule. The quotient $H_*^cT(BP\langle m-1\rangle)^{tC_2}/(\lambda_k^c|1 \le k \le m)$ is a filtered A_*-comodule with associated graded isomorphic to $P(u,u^{-1}) \otimes P(m)_*$.

Note that the quotient splits off $H_*T(BP\langle m-1\rangle)^{tC_2}$ as an algebra, but not as an A_*-comodule. There are dual Steenrod operations going from this algebra to the ideal. See e.g. figure 9.2 for the case of $m = 1$. Dually, in cohomology, we have a filtered sub A-module with associated graded isomorphic to $\Sigma P(v, v^{-1}) \otimes P(m)$ inside the continuous cohomology of $T(BP\langle m-1\rangle)^{tC_2}$.

11.2 The extreme cases

Recall that $H_*BP \cong P(\bar\xi_k^2 \mid k \ge 1) \cong (A/\!\!/E)_*$. For brevity, we will denote the polynomial algebra $P(\xi_k^4 \mid k \ge 1)$ by $D(A/\!\!/E_*)$.

The map $BP \to BP\langle -1\rangle = H\mathbb{F}_2$ induces a map of the associated Tate spectra on THH and their continuous homology groups. All the exterior classes λ_k lie in the kernel. The inclusion of zero-simplices and the map down to $T(\mathbb{F}_2)$ compose into an injective map of filtered A_*comodules $R_+H_*BP \hookrightarrow H_*^cT(BP)^{tC_2} \to H_*^cT(\mathbb{F}_2)^{tC_2}$. Injectiveness is checked by inspection on the induced map of associated graded comodules. Indeed, the composite map above is given on associated graded comodules as the map

$$\begin{array}{c} P(u,u^{-1}) \otimes D(A/\!\!/E_*) \\ \downarrow \\ P(u,u^{-1}) \otimes D(A/\!\!/E_*) \otimes E(\lambda_k^c|k \ge 1) \\ \downarrow \\ P(u,u^{-1}) \otimes P(\bar\xi_1^2, \bar\xi_k|k \ge 2) \end{array}$$

sending $D(A/\!\!/E_*) = P(\bar\xi_k^4|k \ge 1)$ injectively to the obvious sub algebra of $P(0)_* = P(\bar\xi_1^2, \bar\xi_k|k \ge 2)$ and all the exterior generators to zero.

We will now determine the structure of $H_*^cT(\mathbb{F}_2)^{tC_2}$ as a completed A_*-comodule. It will turn out that $H_*^cT(\mathbb{F}_2)^{tC_2}$ is coinduced over A_* from a quotient A_{0*} comodule and we start by describing this quotient.

In cohomology the augmentation $A/\!\!/E \to \mathbb{F}_2$ induced an A-linear surjection $R_+(A/\!\!/E) \to R_+(\mathbb{F}_2)$. On the other hand, the inclusion $\mathbb{F}_2 \hookrightarrow A/\!\!/E$ of $\mathbb{F}_2$-vector spaces is an A_0-homomorphism since there are no room for any Sq^1 in $A/\!\!/E$. This gives a splitting of the augmentation map when restricted to A_0-modules. Dual to the inclusion, the quotient map $A/\!\!/E_* \to \mathbb{F}_2$ is a map of A_{0*}-comodules.

Likewise, in cohomology, there is a map $R_+\mathbb{F}_2 \hookrightarrow H^*_cT(\mathbb{F}_2)^{tC_2}$ which on associated graded modules is the inclusion $\Sigma P(v,v^{-1}) \hookrightarrow \Sigma P(v,v^{-1}) \otimes A/\!\!/A_0$. This map is well defined since there is only one non-trivial class in each degree of maximal filtration. Because of degree reasons, there is also in this case not any room for any vertical Sq^1's, so the inclusion is an A_0-homomorphism. The dual map $H^c_*T(\mathbb{F}_2)^{tC_2} \to R_+\mathbb{F}_2$ of completed A_{0*}-comodules given on the level of associated graded comodules is the map $P(u,u^{-1}) \otimes P(0)_* \to P(u,u^{-1})$ dividing out by the ideal $(\bar\xi_1^2, \bar\xi_k | k \geq 2)$.

The above discussion is summarized in the following diagram:

$$\begin{array}{ccccc} R_+(A/\!\!/E_*) & \hookrightarrow & H^c_*T(BP)^{tC_2} & \longrightarrow & H^c_*T(\mathbb{F}_2)^{tC_2} \\ \uparrow\!\downarrow q & & & & \downarrow q \\ R_+(\mathbb{F}_2) & = & = & = & R_+(\mathbb{F}_2)\,. \end{array}$$

Disregarding the left split, this diagram is commutative. The vertical maps are maps of completed A_{0*}-comodules. The upper composite map is injective, and it is easily checked by considering the map of associated graded comodules that the composite self map of $R_+\mathbb{F}_2$ is the identity.

Proposition 11.2.1. *There is an isomorphism of completed A_*-comodules $H^c_*T(\mathbb{F}_2)^{tC_2} \to A_*\square_{A_{0*}} R_+\mathbb{F}_2$.*

Proof. The isomorphism is defined by the composite

$$H^c_*T(\mathbb{F}_2)^{tC_2} \to A_*\square_{A_{0*}} H^c_*T(\mathbb{F}_2)^{tC_2} \overset{1\square q}{\to} A_*\square_{A_{0*}} R_+\mathbb{F}_2\,. \tag{11.2.1}$$

The first map is the completed A_*-coaction map. Both these maps are maps of filtered A_*-comodules and the induced map on filtration quotients are suspensions of the map

$$P(\bar\xi_1^2, \bar\xi_k | k \geq 2) \overset{\nu}{\to} A_*\square_{A_{0*}} P(\bar\xi_1^2, \bar\xi_k^2 | k \geq 2)_* \to A_*\square_{A_{0*}} \mathbb{F}_2\,. \tag{11.2.2}$$

In cohomology, this composite is the identity on $A/\!\!/A_0$ and in particular an isomorphism. □

We can now describe the map $R_+H_*BP \hookrightarrow H^c_*T(\mathbb{F}_2)^{tC_2}$ explicitly. We have a commutative diagram

$$\begin{array}{ccc} A_*\Box_{A_{0*}}R_+\mathbb{F}_2 & = & A_*\Box_{A_{0*}}R_+\mathbb{F}_2 \\ \uparrow{\scriptstyle 1\Box q} & & \uparrow{\scriptstyle 1\Box q} \\ A_*\Box_{A_{0*}}R_+A/\!\!/E_* & \hookrightarrow & A_*\Box_{A_{0*}}H^c_*T(\mathbb{F}_2)^{tC_2} \\ \uparrow{\scriptstyle \nu} & & \uparrow{\scriptstyle \nu} \\ R_+A/\!\!/E_* & \hookrightarrow & H^c_*T(\mathbb{F}_2)^{tC_2}\,. \end{array} \tag{11.2.3}$$

The right vertical composite is an isomorphism by proposition 11.2.1. Via this identification, $R_+A/\!\!/E_* \to H^c_*T(\mathbb{F}_2)^{tC_2} \cong A_*\Box_{A_{0*}}R_+\mathbb{F}_2$ is the left vertical composite. In cohomology this is the map

$$A \otimes_{A_0} R_+\mathbb{F}_2 \hookrightarrow A \otimes_{A_0} R_+A/\!\!/E \to R_+A/\!\!/E \tag{11.2.4}$$

defined by first including $R_+\mathbb{F}_2 \hookrightarrow R_+A/\!\!/E$ and then using the A-module action map.

11.3 The intermediate cases

As noted, the map $\Psi_* : R_+A/\!\!/E_* \to A_*\Box_{A_{0*}}R_+\mathbb{F}_2$ factors through the quotient $H^c_*T(BP\langle m-1\rangle)^{tC_2}/(\lambda^c_k|1 \le k \le m)$ for all $m \ge 1$.

Theorem 11.3.1. *The continuous cohomology of $T(BP\langle m-1\rangle)^{tC_2}$ sits in a short exact sequence of filtered A-modules*

$$0 \to A \otimes_{A_m} R_+(A_{m-1}/\!\!/E_{m-1}) \to H^*_cT(BP\langle m-1\rangle)^{tC_2} \to (\lambda^c_k|k \ge 1)^* \to 0 \tag{11.3.1}$$

where the quotient is the dual of the ideal generated by the exterior classes λ^c_k.

The map $H^*T(BP\langle m-1\rangle)^{tC_2} \to H^*_cT(BP) \to R_+(H^*BP) \cong R_+(A/\!\!/E)$ restricted to the submodule in the sequence above is given by

$$A \otimes_{A_m} R_+(A_{m-1}/\!\!/E_{m-1}) \hookrightarrow A \otimes_{A_m} R_+(A/\!\!/E) \to R_+(A/\!\!/E)\,.$$

The first map is induced by the inclusion of A_{m-1}-modules $A_{m-1}/\!\!/E_{m-1} \hookrightarrow A/\!\!/E$ and the last map is the A-action map on $R_+(A/\!\!/E)$.

Proof. In cohomology, we have the following diagram where the upper horizontal composite is the map (11.2.4).

$$\begin{array}{ccccc} A\otimes_{A_0} R_+(\mathbb{F}_2) & \longrightarrow & A\otimes_{A_0} R_+(A/\!\!/E) & \longrightarrow & R_+A/\!\!/E \\ \uparrow & & \uparrow & & \uparrow \\ A_m\otimes_{A_0} R_+(\mathbb{F}_2) & \longrightarrow & A_m\otimes_{A_0} R_+(A_{m-1}/\!\!/E_{m-1}) & \longrightarrow & R_+(A_{m-1}/\!\!/E_{m-1})\,. \end{array} \tag{11.3.2}$$

The vertical maps are A_m-homomorphisms. We claim that the lower composite is surjective. Indeed, the diagram is a diagram of filtered A_m-modules, and the lower horizontal map is given on filtration quotients as suspensions of the map

$$A_m/\!\!/A_0 \cong A_m\otimes_{A_0}\mathbb{F}_2 \to A_m\otimes_{A_0} D(A_{m-1}/\!\!/E_{m-1}) \to D(A_{m-1}/\!\!/E_{m-1})\,. \tag{11.3.3}$$

Here, $D(A_{m-1}/\!\!/E_{m-1})$ is the double module of $A_{m-1}/\!\!/E_{m-1}$.

The target is cyclic over A_m on one generator in degree zero and this element is hit by $1\in A_m/\!\!/A_0$. It then follows that the composite (11.3.3) is a surjection of filtered A_m-modules.

Look now at the injective composite

$$A\otimes_{A_0} R_+(\mathbb{F}_2) \to H_c^*T(BP\langle m-1\rangle)^{tC_2} \to H_c^*T(BP)^{tC_2} \to R_+(A/\!\!/E)\,.$$

We claim that the image of the submodule $A_m\otimes_{A_0} R_+(A_{m-1}/\!\!/E_{m-1})$ in $H_c^*T(BP)^{tC_2}$ is also isomorphic to $R_+(A_{m-1}/\!\!/E_{m-1})$.

To see this, we dualize and work in homology. The inclusion $A_m\otimes_{A_0} R_+(\mathbb{F}_2)\hookrightarrow A\otimes_{A_0} R_+(\mathbb{F}_2)$ is dual to a surjection of A_{m*}-comodules. Denote its kernel $I_0(m)$. Similarly, we have a surjection of A_{m*}-comodule $R_+(A/\!\!/E_*)\to R_+(A_{m-1}/\!\!/E_{m-1_*})$ with kernel $I_\infty(m)$. We have the following diagram of filtered completed A_{m*}-comodules

$$\begin{array}{ccccc} I_\infty(m) & \hookrightarrow & R_+(A/\!\!/E_*) & \xrightarrow{q_\infty(m)} & R_+(A_{m-1}/\!\!/E_{m-1_*}) \\ \downarrow & & \downarrow{\scriptstyle i_m} & & \downarrow \\ I_m(m) & \hookrightarrow & H_*^cT(BP\langle m-1\rangle)^{tC_2}/(\lambda_k^c \mid k\le m) & \xrightarrow{q_m(m)} & Q_m \\ \downarrow & & \downarrow{\scriptstyle j_m} & & \downarrow \\ I_0(m) & \hookrightarrow & A_*\square_{A_{0*}} R_+(\mathbb{F}_2) & \xrightarrow{q_0(m)} & A_{m*}\square_{A_{0*}} R_+(\mathbb{F}_2)\,. \end{array} \tag{11.3.4}$$

The outer square is the dual of the outer square in diagram (11.3.2). The horizontal maps are inclusions, hence the upper outer square is a pullback square.

The left hand squares are also pullback squares, i.e. $I_\infty(m) = i_m(R_+(A/\!\!/E_*)) \cap I_m(m)$ and similarly for $I_m(m)$. This implies that the vertical maps into and out of Q_m are injections. We will show that $R_+(A_{m-1}/\!\!/E_{m-1_*}) \hookrightarrow Q_m$ is surjective.

To do this we analyze the maps in the above diagram on associated graded comodules. The lower horizontal surjection in the diagram is given on filtered comodules by suspensions of the map $A/\!\!/A_{0*} \to A_{m*}$ by dividing out by the ideal $I(m) = (\bar\xi_1^{2^{m+1}}, \bar\xi_2^{2^m}, \ldots, \bar\xi_{m+1}^2, \bar\xi_k | k \geq m+2) \subset P(\bar\xi_1^2, \bar\xi_k | k \geq 2) \cong A/\!\!/A_{0*}$. Hence, the filtration kernels of $q_0(m)$ are given by suspensions $I(m)$. The middle vertical map induces the inclusion map $D(A/\!\!/E_*) \hookrightarrow P(m)_* \hookrightarrow A/\!\!/A_{0*}$ on filtration kernels. Hence, we get a diagram like above of maps of filtration kernels

$$\begin{array}{ccc}
(\bar\xi_k^4 | k \geq 1) & \hookrightarrow & P(\bar\xi_k^4 | k \geq 1) \\
\downarrow & & \downarrow \\
(\bar\xi_1^{2^{m+1}}, \ldots, \bar\xi_m^4, \bar\xi_{m+1}^2, \bar\xi_k | k \geq m+2) & \hookrightarrow & P(\bar\xi_1^4, \ldots, \bar\xi_m^4, \bar\xi_{m+1}^2, \bar\xi_k | k \geq m+2) \\
\downarrow & & \downarrow \\
(\bar\xi_1^{2^{m+1}}, \ldots, \bar\xi_m^4, \bar\xi_{m+1}^2, \bar\xi_k | k \geq m+2) & \hookrightarrow & P(\bar\xi_1^2, \bar\xi_k | k \geq 2)\,.
\end{array}$$

It is now clear that the quotients of the two upper horizontal maps are isomorphic. Hence, the map $R_+(A_{m-1}/\!\!/E_{m-1*}) \hookrightarrow Q_m$ in diagram (11.3.4) is an isomorphism of A_{m*}-comodules. We now have a commutative diagram analogous the case for $m = 0$:

$$\begin{array}{ccc}
A_* \Box_{A_{m*}} R_+(A_{m-1}/\!\!/E_{m-1*}) & = & A_* \Box_{A_{m*}} R_+(A_{m-1}/\!\!/E_{m-1*}) \\
\uparrow{\scriptstyle 1\Box q_\infty(m)} & & \uparrow{\scriptstyle 1\Box q_m(m)} \\
A_* \Box_{A_{m*}} R_+(A/\!\!/E_*) & \hookrightarrow & A_* \Box_{A_{m*}} H_*^c T(BP\langle m-1\rangle)^{tC_2}/(\lambda_k^c) \\
\uparrow{\scriptstyle \nu} & & \uparrow{\scriptstyle \nu} \\
R_+(A/\!\!/E_*) & \hookrightarrow & H_*^c T(BP\langle m-1\rangle)^{tC_2}/(\lambda_k^c)\,.
\end{array} \tag{11.3.5}$$

In order to show the theorem, we need to show that the vertical composite map on the right is an isomorphism of completed A_*-comodules. Again, this is a map of filtered A_*-comodules, which on filtration kernels are given by the upper row in diagram (11.3.6):

$$\begin{array}{ccccc} P(m)_* & \longrightarrow & A_*\square_{A_{m*}}P(m)_* & \longrightarrow & A_*\square_{A_{m*}}P(m)_*/I(m) \\ \downarrow & & \downarrow & & \downarrow \\ A_* & \xrightarrow{\psi} & A_*\square_{A_{m*}}A_* & \longrightarrow & A_*\square_{A_{m*}}A_*/I(m)\,. \end{array} \tag{11.3.6}$$

The lower row in this diagram is the dual of the maps $A\otimes_{A_m}A_m \hookrightarrow A\otimes_{A_m} A \to A$ whose composite is the isomorphism $A\otimes_{A_m}A_m\cong A$. Hence, both the horizontal composite maps are injections. Both source and target are comodules of finite type in each degree so to show surjectivity, it is enough to show that they are of the same dimension degreewise.

It is clear that as a vector space, $P(m)_*$ is isomorphic to the tensor product $P(\bar\xi_1^{2^{m+1}},\dots,\bar\xi_m^4,\bar\xi_{m+1}^2,\bar\xi_{m+2},\dots)\otimes P(\bar\xi_k^4|k\geq 1)/I(m)$ which in turn is isomorphic to $(A/\!\!/A_m)_*\otimes P(\bar\xi_k^4|k\geq 1)/I(m)$. This vector space has the same degreewise dimension over $\mathbb{F}_2$ as $A_*\square_{A_{m*}}P(\bar\xi_k^4|k\geq 1)/I(m)$.

The last claim follows from the more general fact that $A\otimes_B M \cong_{\mathbb{F}_2} A/\!\!/B\otimes M$ for all bounded above A-modules M of finite degreewise rank over $\mathbb{F}_2$ and all $B\subset A$ such that $A\otimes_B(-)$ is flat.

□

Dividing out by the ideals generated by the exterior classes λ_k and λ_k^c the map γ_* induces a map

$$\begin{array}{ccc} H_*T(BP\langle m-1\rangle)/(\lambda_k|k\leq m) & \xrightarrow{\bar\gamma_*} & H_*^cT(BP\langle m-1\rangle)^{tC_2}/(\lambda_k^c|k\leq m) \\ \cong\Big| & & \cong\Big| \\ P(\mu_m)\otimes A/\!\!/E_{m-1*} & \longrightarrow & A_*\square_{A_{m*}}R_+(A_{m-1}/\!\!/E_{m-1*}) \end{array}$$

in homology. The class μ_m is A_*-comodule primitive since we have killed λ_m, so the dual of the lower horizontal map translates to a map of A-modules

$$\bar\gamma^* : A\otimes_{A_m}R_+(A_{m-1}/\!\!/E_{m-1})\to\bigoplus_{0\leq i}\Sigma^{i2^{m+1}}A/\!\!/E_{m-1}\,. \tag{11.3.7}$$

Proposition 11.3.2. *The map $\bar\gamma^*$ is surjective. The kernel of γ^* satisfies*

$$\mathrm{Ext}_A^{s,t}(\ker(\gamma^*),\mathbb{F}_2)\cong\mathrm{Ext}_A^{s,t}(\bigoplus_{j<0}\Sigma^{j2^{m+1}}A/\!\!/E_{m-1},\mathbb{F}_2)$$

for all s,t.

Proof. We will first show that

$$A \otimes_{A_m} R_+(A_{m-1}/\!\!/E_{m-1}) \xrightarrow{\bar{\gamma}^*} \bigoplus_{i\geq 0} \Sigma^{i2^{m+1}} A/\!\!/E_{m-1}$$

is surjective for all $k \geq 0$. This follows since the A-module generators on the right are dual to μ_m^i for $i \geq 0$ and since $\gamma_*(\mu_m^i)$ is represented by $u^{-i\cdot 2^{m+1}}$ in the homological Tate spectral sequence $\hat{E}^\infty(T(BP\langle m-1\rangle)^{tC_2})$.

Using lemma 5.4.2 in the case $M = A_{m-1}/\!\!/E_{m-1}$ we get the lower horizontal sequence in the following diagram:

$$\begin{array}{ccccc} & & \bigoplus_{i\geq 0}\Sigma^{i2^{m+1}} A/\!\!/E_{m-1} & & \\ & \overset{0}{\nearrow} & \uparrow \bar{\gamma}^* & \nwarrow & \\ A\otimes_{A_m} K & \hookrightarrow & A \otimes_{A_m} R_+(A_{m-1}/\!\!/E_{m-1}) & \twoheadrightarrow & \bigoplus_{j\geq 0}\Sigma^{j2^{m+1}} A/\!\!/E_{m-1}\,. \end{array} \tag{11.3.8}$$

Since $M = A_{m-1}/\!\!/E_{m-1}$ generated over A_{m-1} by classes of degree less than or equal to zero, then, by the same lemma, the kernel $A \otimes_{A_m} K$ is generated by A from classes in degree less than or equal to -2^{m+1}. Thus, $A \otimes_{A_m} K$ is contained in the kernel of the A-module homomorphism $\bar{\gamma}^*$ since the target of this map is concentrated in positive degrees.

The quotient direct sum on the right and the target of $\bar{\gamma}^*$ are abstractly isomorphic and of degreewise finite dimensions as vector spaces over $\mathbb{F}_2$, so surjectivity of $\bar{\gamma}^*$ implies injectivity as well. This implies that $A \otimes_{A_m} K$ is indeed the whole kernel of $\bar{\gamma}^*$. □

In order to estimate the co-connectivity of γ after smashing with some finite complex $F(m)$, we will use the above results to show vanishing results of the E^2-term of the Caruso-May-Priddy spectral sequence for the cofiber of γ. By the fundamental Norm-Restriction square (4.2.1) the same coconnectivity result holds for the map of fixed points $T(B)^{C_2} \to T(B)^{hC_2}$.

The following is an elementary fact about comodule algebras.

Lemma 11.3.3. *Let R be a graded commutative A_*-comodule algebra and assume that $R \cong S \otimes E(e)$ as algebras, where the exterior class e is primitive with respect to the A_*-comodule coaction on R.*

Then the ideal $(e) \subset R$ is a sub A_*-comodule and the map $\Sigma^{|e|}R/(e) \to (e)$ defined by $\Sigma^{|e|}[x] \mapsto e \cdot x$ is an A_*-comodule isomorphism.

Proof. Since e is a primitive element, the degree $|e|$ self map on R defined by multiplication by e is an A_*homomorphism. The image of this map is the ideal $(e) \subset R$ which is then a sub A_*-comodule. Moreover, the kernel of the multiplication map is also equal to the ideal (e), so we have a short exact sequence

$$0 \to \Sigma^{|e|}(e) \subset \Sigma^{|e|}R \xrightarrow{e\cdot} (e) \to 0$$

of A_*-comodules. Hence, $\Sigma^{|e|}R/(e) \to (e)$ given by $\Sigma^{|e|}[x] \mapsto e \cdot x$ is an isomorphism of A_*-comodules. □

Theorem 11.3.4. *Assume that there exists a finite complex $F(m)$ of chromatic type m such that its cohomology with $\mathbb{F}_2$-coefficients is isomorphic as a left A-module to the finite sub-Hopf algebra $A_{m-1} \subset A$.*

After smashing with $F(m)$, the map $\Gamma_1 : T(BP\langle m-1\rangle)^{C_2} \to T(BP\langle m-1\rangle)^{hC_2}$ becomes an isomorphism on homotopy groups in degree strictly greater than $2^{m+1}(m-2)+m+2$.

Proof. After smashing with $F(m)$, the kernel of $\bar{\gamma}^*$ can be estimated on Ext-groups as follows:

$$\begin{aligned}\operatorname{Ext}_A^{s,t}(\ker(\bar{\gamma}^*), \mathbb{F}_2) &\cong \operatorname{Ext}_A^{s,t}((\textstyle\bigoplus_{j<0} \Sigma^{j2^{m+1}} A/\!\!/E_{m-1}) \otimes A_{m-1}, \mathbb{F}_2) \\ &\cong \operatorname{Ext}_A^{s,t}(A \otimes \textstyle\bigoplus_{j<0} \Sigma^{j2^{m+1}} A_{m-1}/\!\!/E_{m-1}), \mathbb{F}_2) \\ &\cong \operatorname{Ext}_{\mathbb{F}_2}^{s,t}(\textstyle\bigoplus_{j<0} \Sigma^{j2^{m+1}} A_{m-1}/\!\!/E_{m-1}), \mathbb{F}_2)\,.\end{aligned} \tag{11.3.9}$$

Here we are using a shearing isomorphism $(A \otimes_{A_{m-1}} A_{m-1}/\!\!/E_{m-1}) \otimes A_{m-1} \cong A \otimes (A_{m-1}/E_{m-1})$. This isomorphism is dependent on the fact that the module A_{m-1} is the cohomology of a spectrum, and has a natural A-action compatible with the self action of the inclusion $A_{m-1} \subset A$.

Since we are taking Ext over a field, the Ext-groups are concentrated in homological degree zero where $\operatorname{Ext}_{\mathbb{F}_2}^{0,t}(-,-) = \operatorname{Hom}_{\mathbb{F}_2}^{t}(-,-)$. We have that $\operatorname{Hom}_{\mathbb{F}_2}^{t}(-,\mathbb{F}_2) = \operatorname{Hom}_{\mathbb{F}_2}(\Sigma^{-t}-,\mathbb{F}_2)$, so it follows that $\operatorname{Ext}_A^{s,t}(\ker\bar{\gamma}^*,\mathbb{F}_2) = 0$ if $s > 0$ or $t > \delta(m)$. Where $\delta(m)$ is the top dimension of $\ker\bar{\gamma}^*$.

We claim that $\delta(m) = 2^{m+1}(m-4)+2m+6$. Indeed,

$$\begin{aligned}\delta(m) &= ||\ker\bar{\gamma}^* \otimes A_{m-1}|| \\ &= ||(\textstyle\bigoplus_{j<0} \Sigma^{j2^{m+1}} A_{m-1}/\!\!/E_{m-1}|| \\ &= ||\Sigma^{-2^{m+1}} A_{m-1}/\!\!/E_{m-1}|| \\ &= -2^{m+1} + ||A_{m-1}/\!\!/E_{m-1}|| = -2^{m+1} + ||D(A_{m-2})|| \\ &= -2^{m+1} + 2(2^m(m-3)+m+3) \\ &= 2^{m+1}(m-4)+2m+6\,.\end{aligned} \tag{11.3.10}$$

Here we are using that $||A_m|| = 2^{m+2}(m-1)+m+5$.

Fix m and let $T = T(BP\langle m-1\rangle)$ and $T^t = T(BP\langle m-1\rangle)^{tC_2}$. The proof of the theorem now follows from an inductive procedure.

We know that H_*T and $H_*^cT^t$ are both commutative completed A_*-comodule algebras of the form considered in lemma 11.3.3. Hence, we have short exact sequences

$$\begin{array}{ccccccc}
\Sigma^{|\lambda_m|}H_*T/(\lambda_m) & \xrightarrow{\cong} & (\lambda_m) & \hookrightarrow & H_*T & \twoheadrightarrow & H_*T/(\lambda_m) \\
\downarrow{\scriptstyle \Sigma^{|\lambda_m|}\bar{\gamma}_*} & & \downarrow & & \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \bar{\gamma}_*} \\
\Sigma^{|\lambda_m|}H_*^cT^t/(\lambda_m^c) & \xrightarrow{\cong} & (\lambda_m^c) & \hookrightarrow & H_*^cT^t & \twoheadrightarrow & H_*^cT^t/(\lambda_m^c)
\end{array} \qquad (11.3.11)$$

with maps between them induced by γ_*. We are using that γ_* is a map of completed A_*comodule algebras.

We will now repeat the procedure with H_*T and $H_*^cT^t$ replaced by $H_*T/(\lambda_m)$ and $H_*^cT^t/(\lambda_m^c)$ respectively. Thus, by induction, we get

$$\begin{array}{ccccc}
\Sigma^{|\lambda_r|}H_*T/(\lambda_m,\dots,\lambda_r) & \hookrightarrow & H_*T/(\lambda_m,\dots,\lambda_{r+1}) & \twoheadrightarrow & H_*T/(\lambda_m,\dots,\lambda_r) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{|\lambda_r|}H_*^cT^t/(\lambda_m^c,\dots,\lambda_r^c) & \hookrightarrow & H_*^cT^t/(\lambda_m^c,\dots,\lambda_{r+1}^c) & \twoheadrightarrow & H_*^cT^t/(\lambda_m^c,\dots,\lambda_r^c)
\end{array} \qquad (11.3.12)$$

for each $1 \le r \le m$. The vertical maps are induced by γ_*.

We are now in position to estimate the coconnectivity of γ. We start when $r = 1$. From the long exact sequence of Ext-groups and the first part of this proof, we see that $\mathrm{Ext}_A^{s,t}(\ker(\gamma^*/(\lambda_m,\dots,\lambda_2),\mathbb{F}_2)$ vanishes when $s > 0$ and $t > \max\{|\lambda_1| + \delta(m), \delta(m)\} = |\lambda_1| + \delta(m)$. By induction on r, we then get that $\mathrm{Ext}_A^{s,t}(\ker(\gamma^*),\mathbb{F}_2) = 0$ when $s > 0$ and $t > \sum_{1\le k\le m}|\lambda_k| + \delta(m)$. I.e.

$$\begin{aligned}
t &> \delta(m) + \sum_{1\le k\le m}(2^{k+1}-1) \\
&= \delta(m) - m + 4\cdot\sum_{0\le j\le m-1}2^j \\
&= \delta(m) - m + 4(2^m - 1) \\
&= 2^{m+1}(m-4) + 2m + 6 - m + 2(2^{m+1}-2) \\
&= 2^{m+1}(m-2) + m + 2\,.
\end{aligned}$$

□

Remark: In the presence of a finite spectrum $V(m-1)$ realizing the exterior sub-Hopf algebra $E_{m-1} \subset A$, we might improve on our estimates. In this case, $\mathrm{Ext}_A^{s,t}(\ker\bar\gamma^*,\mathbb{F}_2)\cong\mathrm{Ext}_A^{s,t}(\bigoplus_{j<0}\Sigma^{j2^{m+1}}A,\mathbb{F}_2)$. We then get that $\delta(m)=||\bigoplus_{j<0}\Sigma^{j2^{m+1}}\mathbb{F}_2||=-2^{m+1}$, thus $\mathrm{Ext}_A^{s,t}(\ker\bar\gamma^*,\mathbb{F}_2)$ vanishes for $s>0$ or $t>-2^{m+1}$.

Theorem 11.3.5. *Assume that there exists a Smith-Toda complex $V(m-1)$. After smashing with $V(m-1)$, the map $\Gamma_1 : T(BP\langle m-1\rangle)^{C_2}\to T(BP\langle m-1\rangle)^{hC_2}$ becomes an isomorphism in degrees strictly greater than $2^{m+1}-m-4$.*

Proof. By the induction in the last part of the proof of theorem 11.3.4, we get that $\mathrm{Ext}_A^{s,t}(\ker\gamma^*\otimes E_{m-1},\mathbb{F}_2)$ vanishes for $s>0$ or $t>\delta(m)-m+4(2^m-1)=-2^{m+1}-m+2\cdot 2^{m+1}-4=2^{m+1}-m-4$. □

Note that the improvement of the latter theorem occurs only for $m\geq 2$. For the algebraic cases $T(\mathbb{Z})$ and $T(\mathbb{F}_2)$ the two theorems state the same result, and they are in agreement with earlier calculations by Bökstedt-Madsen for $T(\mathbb{Z})$ [7] and Hesselholt for $T(\mathbb{F}_2)$.

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