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Orthogonal Spectra

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Contents

Introduction

Some basic motivation

This paper contains a (brief) description of some of the basic constructions and tools used in stable homotopy theory (and other mathematical subjects), more specifically: structured ring spectra. Using a rather informal description, *stable homotopy theory* is a branch of algebraic topology that describes homotopical properties of spaces that are "invariant" under the suspension functor, meaning if a space X has the "stable property" P, then so should the space $\Sigma X \approx X \wedge S^1$. Now let X_0 be the space X and let X_n denote the n-th iterated suspension, $\Sigma^n X$. This leads to several questions: Does this set of spaces $\{X_n\}_{n\geq 0}$ have any structure? Can we "add" spaces together, meaning X_n "+" X_m? Can we "multiply" them? What are the properties of maps between such collections? Can we *smash* two such collections and get a new one? The list of questions goes on, but without any added structure to the space X or even the maps in question, several of these questions are somewhat futile and not really interesting. This is where *structured ring spectra* enter, and in our case: *orthogonal spectra*.

Some history

Orthogonal spectra was first described by Mandell, May, Schwede and Shipley in [7], and then in greater detail in [6] by Mandell and May. Note that this is not the only type of structured ring spectra. Elmendorf, Kriz, Mandell and May introduced *S-modules* in [2], and Smith introduced *symmetric spectra* which were explained in detail in [4] by Hovey, Shipley and Smith. There is also an ongoing book by Schwede on symmetric spectra available at `http://www.math.uni-bonn.de/people/schwede/`

Outline of the paper

Section 1

We will go through some basic linear algebra with real inner product spaces, orthogonal groups and group actions, leading up to the most important concept of this section, *equivariance*.

Section 2

We will introduce based spaces and operations on such spaces before revisiting group actions on based spaces. Here we will prove an important result relating sets of equivariant maps, which will be used frequently in the later sections. We will also introduce homotopy groups.

Section 3

Here we will explain some basic category theory and define the category of orthogonal sequences. We will also give a very important example of an orthogonal sequence, namely the *sphere sequence*.

Section 4

Continuing with the category of orthogonal sequences, we shall construct the tensor product of orthogonal sequences and describe its features.

Section 5

In this section we shall introduce orthogonal spectra and see that the sphere sequence is actually an orthogonal spectrum, the *sphere spectrum*.

Section 6

In this section we will introduce homotopy groups of orthogonal spectra. We will also state some results regarding the homotopy groups of the sphere spectrum.

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1 Inner product spaces and the orthogonal group

1.1 Inner product spaces

Definition 1.1 (Real inner product space). Let V be a vector space over $\mathbb{R}$. An *inner product* on V is a bilinear function $\langle , \rangle : V \times V \to \mathbb{R}$ such that the following properties hold for all $u, v \in V$:

- $\langle v, v \rangle \geq 0$,
- $\langle v, v \rangle = 0 \iff v = 0$, and
- $\langle v, u \rangle = \langle u, v \rangle$.

A real vector space equipped with an inner product, $(V, \langle , \rangle)$ is called a *real inner product space*.

Example 1.2. $\mathbb{R}^n$ with the *Euclidean inner product* or *dot product*, defined by

$$\langle v, w \rangle = \langle (v_1, \ldots, v_n), (w_1, \ldots, w_n) \rangle = v_1 w_1 + \cdots + v_n w_n = v \cdot w \,.$$

We usually write $(\mathbb{R}^n, \cdot)$ for this inner product space.

For the sake of simplicity in several explicit constructions, we shall restrict ourselves to the real inner product spaces $\mathbb{R}^n, n = 0, 1, \ldots$ So unless otherwise specified, *inner product space* shall always mean $(\mathbb{R}^n, \cdot)$, $n = 0, 1, \ldots$, with the standard orthonormal basis for the vector space $\mathbb{R}^n$.

Definition 1.3 (Direct sum). The direct sum of the two real vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ is the real vector space $\mathbb{R}^n \oplus \mathbb{R}^m = \{(v, u) \mid v \in \mathbb{R}^n, u \in \mathbb{R}^m\}$ with vector addition and scalar multiplication defined as follows:

$$\begin{aligned} (v, u) + (v', u') &= (v + v', u + u') \,, \\ c\,(v, u) &= (cv, cu) \text{ for all } c \in \mathbb{R} \,. \end{aligned}$$

Lemma 1.4. *The direct sum $\mathbb{R}^n \oplus \mathbb{R}^m$ of vector spaces and the vector space $\mathbb{R}^{n+m}$ are isomorphic. That is, there exists an isomorphism*

$$f : \mathbb{R}^n \oplus \mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^{n+m} \,.$$

Proof. Let $f : \mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^{n+m}$ be the linear map defined by

$$f(x, y) = f((x_1, \ldots, x_n), (y_1, \ldots, y_m)) = (x_1, \ldots, x_n, y_1, \ldots y_m) \,.$$

This is an obvious isomorphism, with inverse $f^{-1} : \mathbb{R}^{n+m} \to \mathbb{R}^n \oplus \mathbb{R}^m$ defined by

$$f^{-1}(x) = f^{-1}(x_1, \ldots, x_{n+m}) = ((x_1, \ldots, x_n), (x_{n+1}, \ldots, x_{n+m})) \,.$$

□

Definition 1.5 (Orthogonality). Let $(V, \langle , \rangle)$ be an inner product space.

- Two vectors $u, v \in V$ are *orthogonal*, written $u \perp v$, if $\langle u, v \rangle = 0$.
- Two subsets $X, Y \subset V$ are *orthogonal*, written $X \perp Y$, if $x \perp y$ for all $x \in X$ and $y \in Y$.
- The *orthogonal complement* of a subset $X \subset V$ is the set $X^{\perp} = \{v \in V \mid \{v\} \perp X\}$.

Definition 1.6 (Orthogonal direct sum). Let $(V, \langle , \rangle_V)$ and $(U, \langle , \rangle_U)$ be inner product spaces. Let $W = U \oplus V$ and let $\langle , \rangle_W$ be the inner product on W defined by

$$\langle (u_1, v_1), (u_2, v_2) \rangle_W = \langle u_1, u_2 \rangle_U + \langle v_1, v_2 \rangle_V .$$

Note that $\langle (u_1, 0), (u_2, 0) \rangle_W = \langle u_1, u_2 \rangle_U$. So $\langle , \rangle_W = \langle , \rangle_U$ when restricted to U, so we may view U as an inner product subspace of W with respect to $\langle , \rangle_W$. The same holds for V.

Also note that $\langle (u, 0), (0, v) \rangle_W = \langle u, 0 \rangle_U + \langle 0, v \rangle_V = 0 + 0 = 0$. In other words, U and V are orthogonal to one another in W with respect to $\langle , \rangle_W$.

This construction of $(W, \langle , \rangle_W)$ is called the *orthogonal direct sum* of the inner product spaces U and V.

Example 1.7. Considering $(\mathbb{R}^n, \cdot)$ and $(\mathbb{R}^m, \cdot)$ we obtain the inner product space $(\mathbb{R}^n \oplus \mathbb{R}^m, \langle , \rangle_{\oplus})$ such that the isomorphism $f : \mathbb{R}^n \oplus \mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^{n+m}$ preserves the inner product. That is,

$$\begin{aligned} & \langle (x, y), (u, v) \rangle_{\oplus} = x \cdot u + y \cdot v \\ = \ & \sum_{i=1}^{n} x_i u_i + \sum_{j=1}^{m} y_j v_j \\ = \ & (x_1, \ldots, x_n, y_1, \ldots, y_m) \cdot (u_1, \ldots, u_n, v_1, \ldots, v_m) \\ = \ & f(x, y) \cdot f(u, v) . \end{aligned}$$

1.2 The orthogonal group and group actions

Definition 1.8 (The orthogonal group). Let $O(n)$ be the set of orthogonal real $n \times n$-matrices. That is,

$$O(n) = \{A \mid AA^t = A^t A = I_n\}.$$

Since $\left(A^t\right)^t = A$ and $I_n^t = I_n$, this is a topological group with matrix multiplication as group operation and identity I_n, the $n \times n$-identity matrix. This group is called the *orthogonal group* of $n \times n$-matrices. Note that $O(n)$ preserves the Euclidean inner product on $\mathbb{R}^n$:

$$Ax \cdot Ax = (Ax)^t Ax = x^t A^t Ax = x^t x = x \cdot x .$$

Definition 1.9 (Permutation matrices). We let $\chi_{n,m}$ denote the permutation matrix that "switches" the first n coordinates of an $(n+m)$-vector with the last m coordinates. That is,

$$\chi_{n,m} = \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix}$$

such that for any $x = (x_1, \ldots, x_{n+m}) \in \mathbb{R}^{n+m}$ we have

$$\chi_{n,m} x^t = \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix} (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^t = (x_{n+1}, \ldots, x_{n+m}, x_1, \ldots, x_n)^t .$$

Note that $det\,(\chi_{n,m}) = \pm 1$, and $\chi_{n,m}$ is an isomorphism of $\mathbb{R}^{n+m}$ with inverse $\chi_{n,m}^{-1} = \chi_{n,m}^t = \chi_{m,n}$, the transposed matrix.

Definition 1.10 (Conjugation by $\chi_{n,m}$). We define the operator $\mathrm{conj}_{n,m}$ to be the conjugation by $\chi_{n,m}$. That is, for a real $(n+m) \times (n+m)$-matrix A then

$$\mathrm{conj}_{n,m}(A) = \chi_{n,m} A \chi_{n,m}^{-1} .$$

Note that $\mathrm{conj}_{n,m}(AB) = \mathrm{conj}_{n,m}(A)\mathrm{conj}_{n,m}(B)$.

Definition 1.11 (Inclusion homomorphism). Let $\iota : O(n) \times O(m) \to O(n+m)$ denote the injective homomorphism defined by

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} .$$

Definition 1.12 (Twist isomorphism). Let $\gamma : O(n) \times O(m) \to O(m) \times O(n)$ denote the isomorphism defined by

$$(A, B) \mapsto (B, A) .$$

Lemma 1.13. *The following diagram commutes:*

$$\begin{array}{ccc} O(n) \times O(m) & \xrightarrow{\iota} & O(n+m) \\ \gamma \downarrow & & \downarrow \mathrm{conj}_{n,m} \\ O(m) \times O(n) & \xrightarrow{\iota} & O(m+n) \end{array}$$

Proof.

$$\mathrm{conj}_{n,m} \circ \iota(A, B) = \chi_{n,m} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \chi_{n,m}^{-1}$$

$$= \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix}$$

$$= \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} = \iota \circ \gamma(A, B) .$$

□

Definition 1.14 (Group action). Let X be a set and G a group. A *left action* of G *on* X is a map $G \times X \to X$, denoted by $(g, x) \mapsto gx$, such that:

1. $ex = x$ for all $x \in X$, where e is the identity of G,
2. $(g_1 g_2)\, x = g_1 (g_2 x)$ for all $x \in X$ and $g_1, g_2 \in G$.

In the same manner, we define a *right action* of G *on* X as a map $X \times G \to X$, denoted by $(x, g) \mapsto xg$, such that:

1. $xe = x$ for all $x \in X$, where e is the identity of G,
2. $x\,(g_1 g_2) = (x g_1)\, g_2$ for all $x \in X$ and $g_1, g_2 \in G$.

We say that X is a *left (right) G-set* if there is a chosen left (right) action of G on X.

If G is a topological group and X is a topological space, we say that the group action is *continuous* if the map $G \times X \to X$ is continuous, and we say that X is a *G-space*.

Unless otherwise stated, all our examples and uses of group actions will be continuous group actions of topological groups on topological spaces.

Example 1.15. An obvious example is the left action $O(n) \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $(A, x) \mapsto Ax$.

Example 1.16. For $n = p+q$, we define a right action $O(n) \times (O(p) \times O(q)) \to O(n)$ by

$$(A, (B, C)) \mapsto A \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}.$$

Example 1.17. As $O(n)$ gives a group action on $\mathbb{R}^n$, products of orthogonal groups give a group action on $\mathbb{R}^n \oplus \mathbb{R}^m$:

$$(O(n) \times O(m)) \times (\mathbb{R}^n \oplus \mathbb{R}^m) \to \mathbb{R}^n \oplus \mathbb{R}^m, \text{ defined by}$$
$$((A, B), (x, y)) \mapsto (Ax, By).$$

Definition 1.18 (Equivariance). Let X, Y be G-spaces. A map $f : X \to Y$ is a G-map if it is G-equivariant. That is, f commutes with the group action of G:

$$f(gx) = gf(x) \text{ for all } g \in G.$$

Example 1.19. Let $\iota : O(n) \times O(m) \to O(n+m)$ be the inclusion homomorphism as previously defined and let $f : \mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^{n+m}$ be the isomorphism defined in lemma 1.4. f is then $O(n) \times O(m)$-equivariant in the following manner:

$$f(Ax, By) = \begin{pmatrix} Ax \\ By \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \iota(A, B)\, f(x, y)$$

where $A \in O(n)$ and $B \in O(m)$.

2 Topology

2.1 Based spaces

Definition 2.1 (Based topological spaces). A *based space*, (X, x_0), consists of a topological space X together with a *basepoint* $x_0 \in X$. A continuous map $f : (X, x_0) \to (Y, y_0)$ is *basepoint preserving* if $f(x_0) = y_0$.

Unless any ambiguity arises, we will simply write X for the based space (X, x_0).

Definition 2.2 (Operations on based spaces). Let X and Y be based spaces.

- We define the product space $(X \times Y, (x_0, y_0))$ with $X \times Y$ being the topological product and (x_0, y_0) serving as basepoint.
- We define the *wedge sum* $X \vee Y \subset X \times Y$ as the quotient space of $X \amalg Y$ where we identify the basepoints x_0 and y_0. That is,

$$X \vee Y = X \amalg Y / (x_0 \sim y_0) ,$$

  where we let the equivalence class $\{*\}$ of x_0 and y_0 serve as the basepoint. Note that this embeds as a subspace of $X \times Y$:

$$X \vee Y \approx X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y .$$

- We define the *smash product* $X \wedge Y$ as the quotient space $X \times Y / X \vee Y$. The equivalence class of (x_0, y_0) serves as the basepoint of this space. We let $x \wedge y$ denote the image of $(x, y) \in X \times Y$.

Definition 2.3. Let X and Y be based spaces. The homeomorphisms $X \times Y \approx Y \times X$ and $X \vee Y \approx Y \vee X$ induce a *twist homeomorphism*

$$\gamma : X \wedge Y \xrightarrow{\approx} Y \wedge X.$$

Example 2.4. Let $S^1 \subset \mathbb{R}^2$ denote the based space $\left(S^1, s_0\right)$ where $s_0 = (1, 0)$.

- $S^1 \times S^1 = T^2$, the familiar "coffee cup".
- $S^1 \vee S^1$ is homeomorphic to a figure eight.
- $S^1 \wedge S^1 \approx S^2$ by collapsing the longitude and meridian circles of the torus T^2 to a point. More generally, $S^n \wedge S^m \approx S^{n+m}$.

Definition 2.5 (Reduced suspension). Let X be a based space. We define the *reduced suspension* of X to be $\Sigma X = X \wedge S^1$, i.e.,

$$\Sigma X = X \times S^1 / \left(\left(\{x_0\} \times S^1\right) \cup (X \times \{s_0\})\right).$$

The equivalence class of (x_0, s_0) is the basepoint of ΣX.

Example 2.6. As seen above, $\Sigma S^1 \approx S^2$. More generally, $\Sigma^n S^0 \approx S^n$.

Definition 2.7 (Loop space). Let (X, x_0) be a based space. We define the *loop space* ΩX as the space of loops in $x_0 \in X$ with the compact-open topology ([3] p. 529). That is, $\Omega X = \{f : S^1 \to X \mid f \text{ is a based map}\} = \operatorname{map}_* \left(S^1, X\right)$.

2.2 Group action revisited

Proposition 2.8. *Let X, Y be based spaces such that X is a G-space and Y is an H-space where the group actions preserve the basepoints of X and Y. Then we have a basepoint preserving group action of $G \times H$ on $X \times Y$, $X \vee Y$ and $X \wedge Y$.*

Proof. We can assume that both group actions are left actions. The proofs for the other scenarios are the same.
1. $(G \times H) \times (X \times Y) \mapsto X \times Y$ is defined by $(g, h)(x, y) \mapsto (gx, hy)$. As G and H preserve the basepoints of X and Y, respectively, $G \times H$ preserve the basepoint of $X \times Y$.
2. Since $X \vee Y$ embeds as a $(G \times H)$-invariant subspace of $X \times Y$, this follows from *1.*
3. If $(x, y) \in X \times Y$, let $x \wedge y$ denote the image in $X \wedge Y$. *1.* and *2.* then induce an $G \times H$-action on $X \wedge Y$ by $(A, B)(x \wedge y) = Ax \wedge By$. □

Definition 2.9 (Balanced smash). Let X be a right G-space and Y a left G-space. We define the *balanced smash product* of X and Y *with respect to* G as the quotient space $X \wedge Y / \sim_G$ where

$$(xg \wedge y) \sim_G \left(x \wedge g^{-1} y\right) \iff (x \wedge y) \sim_G \left(xg \wedge g^{-1} y\right) \text{ for all } g \in G.$$

We let $x \wedge_G y$ denote the equivalence class of $x \wedge y$.

Example 2.10. Let G be a topological group with subgroup H and let X be a based left H-space where the group action preserves the basepoint. Let G_+ denote the union of G and an external basepoint. That is, $G_+ = G \sqcup \{*\}$. G_+ is then a right H-space and we can construct the space $G_+ \wedge_H X$.

Proposition 2.11. *Continuing the previous example, $G_+ \wedge_H X$ is a left G-space with group action: $(f, g \wedge_H x) \mapsto fg \wedge_H x$.*

Proof. We must prove that this is well-defined regarding the equivalence relation $\sim_H$. That is, for all $(g \wedge x) \sim_H \left(gh \wedge h^{-1} x\right)$, $h \in H$, then $(fg \wedge x) \sim_H \left((fg)\, h \wedge h^{-1} x\right)$. Since H is a subgroup of G, h is in G where everything is nice and associative. Hence,

$$f\,(gh) \wedge h^{-1} x = \left((fg)\, h \wedge h^{-1} x\right) \sim_H (fg \wedge x)\ .$$

□

Definition 2.12. Let X and Y be based G-spaces. Let $\mathrm{map}_G(X,Y)$ denote the set of based G-maps. That is,

$$\mathrm{map}_G(X,Y) = \{f : X \to Y \mid f \text{ is basepoint preserving and } G\text{-equivariant}\}.$$

The following result will be used frequently in the later sections.

Proposition 2.13. *Let X be a left H-space and Y a left G-space, where H is a subgroup of G. Note that Y is also an H-space. Then there exists a bijection between the sets*

$$\mathrm{map}_H(X,Y) \longleftrightarrow \mathrm{map}_G(G_+ \wedge_H X, Y).$$

Comment: This bijection is natural in X and Y, so this is an example of an *adjunction*. See chapter IV of [5] for more details on this.

Proof. Let $\psi : G_+ \wedge_H \to Y$ be a G-map. That is,

$$f\psi(g \wedge_H x) = \psi(fg \wedge_H x), \text{ for all } f \in G.$$

Define $\phi : X \to Y$ by $\phi(x) = \psi(e \wedge_H x)$, where e is the identity of G. As ψ is also an H-map, we obtain that ϕ is an H-map as follows:

$$h\phi(x) = h\psi(e \wedge_H x) = \psi(h \wedge_H x) = \psi(e \wedge_H hx) = \phi(hx).$$

The equality $\psi(h \wedge_H x) = \psi(e \wedge_H hx)$ follows from the equivalence $(h \wedge x) \sim_H \left(hh^{-1} \wedge hx\right) = e \wedge hx$.

Let $\phi : X \to Y$ be an H-map. Define $\psi : G_+ \wedge_H X \to Y$ by $\psi(g \wedge_H x) = g\phi(x)$. The fact that ψ is well-defined follows from

$$\psi(g \wedge_H x) = g\phi(x) = ghh^{-1}\phi(x) = gh\phi\left(h^{-1}x\right) = \psi\left(gh \wedge_H h^{-1}x\right)$$

for all $h \in H$.
The G-equivariance follows from

$$f\psi(g \wedge_H x) = f(g\phi(x)) = (fg)\phi(x) = \psi(fg \wedge_H x) \text{ for all } f \in G.$$

□

2.3 Homotopy and homotopy groups

Definition 2.14 (Homotopy). Let X, Y be spaces and $f, g : X \to Y$ maps. A *homotopy* $h : f \simeq g$ between f and g is a map $h : X \times I \to Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Usually we will write $h_t(x)$ instead of $h(x,t)$, so that $h_0 = f$ and $h_1 = g$.

Proposition 2.15. *Let X and Y be spaces. Homotopy, $\simeq$, is an equivalence relation on the set of maps $X \to Y$. Let $[X,Y]$ denote the set of such equivalence classes.*

Proof. *1. Reflexive*: If $f : X \to Y$, then $h : f \simeq f$ by the constant homotopy $h_t = f$ for all $t \in I$.
2. Symmetric: If $f, g : X \to Y$ and $h : f \simeq g$ then $\bar{h} : g \simeq f$ where $\bar{h}(x,t) = h(x, 1-t)$.
3. Transitive: If $f, g, h : X \to Y$ where $i : f \simeq g$ and $j : g \simeq h$, then $k : f \simeq h$ defined by

$$k_t = \begin{cases} i_{2t} & \text{for } t \in [0, 1/2] \\ j_{2t-1} & \text{for } t \in [1/2, 1] . \end{cases}$$

□

In a similar manner, we define homotopy for based spaces, where all our maps are basepoint preserving. Following the notation of [3], we will write $\langle X, Y \rangle$ for the equivalence classes in the case of based homotopy.

Definition 2.16 (Homotopy groups). Let I^n be the n-dimensional unit cube. That is, the product of n copies of the unit interval $I = [0, 1]$. The boundary ∂I^n is the subspace of points with at least one coordinate equal to 0 or 1. For a based space (X, x_0), we define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$, where the homotopies satisfy $h_t(\partial I^n) = x_0$ for all t.

In the case of $n = 0$ where we identify I^0 with a point and $\partial I^0 = \emptyset$, $\pi_0(X, x_0)$ is the set of path-components of X.

Proposition 2.17. *For $n \geq 1$, $\pi_n(X, x_0)$ is a group and it is abelian for $n \geq 2$.*

Proof. For $n = 1$ let $f \cdot g$ be *composition of loops* defined by

$$(f \cdot g)(s) = \begin{cases} f(2s) & \text{for } s \in [0, 1/2] \\ g(2s-1) & \text{for } s \in [1/2, 1] . \end{cases}$$

This composition preserves homotopy classes: If $h_t : h_0 \simeq h_1$ and $i_t : i_0 \simeq i_1$, then $h_t \cdot i_t : h_0 \cdot i_0 \simeq h_1 \cdot i_1$. This means that the product of equivalence classes in $\pi_1(X, x_0)$ is well-defined. That is, $[f][g] = [f \cdot g]$. Let c_{x_0} be the constant loop at x_0. That is, $c_{x_0}(s) = x_0$ for all s. If $f : I \to X$ is a path, define $\bar{f} : I \to X$ by $\bar{f}(s) = f(1-s)$.

Define a *reparametrization* ([3], p. 27) of a loop $f : I \to X$ in x_0 to be the composition $f \circ \phi$ where $\phi : I \to I$ is a map such that $\phi(0) = 0$ and $\phi(1) = 1$. Then there exists a homotopy $f \circ \phi_t : f \circ \phi \simeq f$ where $\phi_t(s) = (1-t)\phi(s) + ts$.

Let $f, g, h : I \to X$ be loops in x_0. Then $f \cdot (g \cdot h)$ and $(f \cdot g) \cdot h$ are defined. More importantly, $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$ by a reparametrization using a $\phi : I \to I$ defined by

$$\phi(s) = \begin{cases} s/2 & \text{for } s \in [0, 1/2] \\ s - 1/4 & \text{for } s \in [1/2, 3/4] \\ 2s - 1 & \text{for } s \in [3/4, 1]\,. \end{cases}$$

This establishes associativity in $\pi_1(X, x_0)$.

Let $f : I \to X$ be a loop in x_0 and c_{x_0} the constant loop. Then $f \cdot c_{x_0} \simeq f$ by a reparametrization using $\phi : I \to I$ defined as

$$\phi(s) = \begin{cases} 2s & \text{for } s \in [0, 1/2] \\ 1 & \text{for } s \in [1/2, 1]\,. \end{cases}$$

$c_{x_0} \cdot f \simeq f$ is established by $\bar{\phi}$, using the notation as defined above. This establishes $[c_{x_0}]$ as the identity of $\pi_1(X, x_0)$.

Let $f : I \to X$ be a loop at x_0. Then $f \cdot \bar{f} \simeq c_{x_0}$ by the homotopy $h_t = i_t \cdot j_t$ where i_t is defined as

$$i_t(s) = \begin{cases} f(s) & \text{for } s \in [0, 1-t] \\ f(1-t) & \text{otherwise} \end{cases}$$

and $j_t = \bar{i_t}$. This establishes $[f]^{-1} = [\bar{f}]$ as an inverse of $[f]$ in $\pi_1(X, x_0)$. This finalizes the fact that $\pi_1(X, x_0)$ is a group, the *fundamental group* of (X, x_0).

In the case of $n \geq 2$, a sum operation in $\pi_n(X, x_0)$ generalizing the composition in $\pi_1(X, x_0)$ can be defined by

$$(f + g)(s_1, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n)\,, & \text{for } s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \ldots, s_n)\,, & \text{for } s_1 \in [1/2, 1]\,. \end{cases}$$

This sum operation is well-defined on homotopy classes, and since only the first coordinate is involved, the same arguments as for $\pi_1(X, x_0)$ show that $\pi_n(X, x_0)$ is a group. The identity is the constant map that takes I^n to x_0 and the inverses are $-f(s_1, \ldots, s_n) = f(1 - s_1, s_2, \ldots, s_n)$.

The fact that $\pi_n(X, x_0)$ an abelian group for $n \geq 2$, is established by the homotopy $f+g \simeq g+f$ as follows:

$$\begin{aligned}
&(f+g)(s_1,\ldots,s_n)\\
&= \begin{cases} f(2s_1,s_2,\ldots,s_n), & \text{for } s_1 \in [0,1/2] \\ g(2s_1-1,s_2,\ldots,s_n), & \text{for } s_1 \in [1/2,1] \end{cases}\\
&\simeq \begin{cases} f(2s_1,2s_2,\ldots,s_n), & \text{for } s_1,s_2 \in [0,1/2] \\ g(2s_1-1,2s_2-1,\ldots,s_n), & \text{for } s_1,s_2 \in [1/2,1] \end{cases}\\
&\simeq \begin{cases} f(2s_1-1,2s_2,\ldots,s_n), & \text{for } s_1 \in [1/2,1] \text{ and } s_2 \in [0,1/2] \\ g(2s_1,2s_2-1,\ldots,s_n), & \text{for } s_1 \in [0,1/2] \text{ and } s_2 \in [1/2,1] \end{cases}\\
&\simeq \begin{cases} f(2s_1-1,s_2,\ldots,s_n), & \text{for } s_1 \in [1/2,1] \\ g(2s_1,s_2,\ldots,s_n), & \text{for } s_1 \in [0,1/2] \end{cases}\\
&= (g+f)(s_1,\ldots,s_n)
\end{aligned}$$

where we map everything outside our "shrunk" cubes to the basepoint. □

2.4 Compactly generated spaces

Definition 2.18 (Compactly generated spaces). Let K be a compact Hausdorff space. Following chapter 5 of [8], we say that a subset A of a space X is *compactly closed*, if for every map $\phi : K \to X$ then $\phi^{-1}(A)$ is closed in K. A space X is called a *k-space* if every compactly closed subset of X is closed.

A space X is a *weak Hausdorff* space if for every map $\phi : K \to X$, $\phi(K)$ is closed in X.

A space X is *compactly generated* if it is a weak Hausdorff k-space.

Unless otherwise specified, we shall only consider *compactly generated based spaces* and simply write "spaces" for this long term. This restriction is still general enough to include all compact Hausdorff spaces, metric spaces, topological manifolds and CW-complexes. Our main reason for this restriction is the following result:

Proposition 2.19. *Let X, Y and Z be (implicitly: compactly generated based) spaces. Then we have an isomorphism of mapping spaces of basepoint preserving maps*

$$\operatorname{map}_*(X \wedge Y, Z) \cong \operatorname{map}_*(X, \operatorname{map}_*(Y,Z)) .$$

Proof. See [8], chapter 5. □

Example 2.20. Using this, we see that

$$\operatorname{map}_*(\Sigma X, Y) = \operatorname{map}_*\left(X \wedge S^1, Y\right) \cong \operatorname{map}_*\left(X, \operatorname{map}_*\left(S^1, Y\right)\right) = \operatorname{map}_*(X, \Omega Y) .$$

Corollary 2.21. *This adjunction isomorphism passes to homotopy classes. That is,*

$$\langle \Sigma X, Y \rangle \cong \langle X, \Omega Y \rangle.$$

And by taking $X = S^n$, we see that

$$\pi_{n+1}(Y) \cong \pi_n(\Omega Y).$$

3 Category theory

3.1 Categories and functors

Definition 3.1 (Category). A *category* $\mathscr{C}$ consists of a collection of objects, $\mathrm{obj}(\mathscr{C})$, and a set of *morphisms* or *arrows*, $\mathscr{C}(A,B)$, between any two objects A and B such that:

- there is a chosen identity morphism $id_A = id \in \mathscr{C}(A,A)$ for each object A, and
- there is a chosen composition $\circ : \mathscr{C}(B,C) \times \mathscr{C}(A,B) \to \mathscr{C}(A,C)$ for each triple of objects A, B, C, such that the composition of morphisms is associative and unital. That is,

  $$h \circ (g \circ f) = (h \circ g) \circ f,\ id \circ f = f \text{ and } f \circ id = f$$

  whenever the specified composites are defined.

Two objects $A, B \in \mathrm{obj}(\mathscr{C})$ are *isomorphic* if there exist morphisms $f : A \to B$ and $g : B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$. We say that a category is *"small"* if the collection of objects is a set.

Example 3.2. Let **Set** denote the category of all small sets and functions. That is,

- $\mathrm{obj}(\mathbf{Set})$ is the collection of all sets belonging to some universe $\mathscr{U}$, and
- $\mathbf{Set}(A,B)$ is the set of functions $f : A \to B$. For all $A \in \mathrm{obj}(\mathbf{Set})$ we have $id_A \in \mathbf{Set}(A,A)$, where $id_A(a) = a$ for all $a \in A$. Associativity is established by the following proposition.

The isomorphic objects of **Set** are the sets of the same cardinality.

Proposition 3.3 (Associativity of composition). *Let A, B, C and D be arbitrary sets and let $h : A \to B$, $g : B \to C$ and $f : C \to D$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.*

Proof. Let x be any element in A. Computing $(f \circ (g \circ h))(x)$ and $((f \circ g) \circ h)(x)$ we find that

$$(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x)))$$

and

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))),$$

hence composition of functions is associative. □

Example 3.4. Let $\mathscr{O}$ be the category of *finite dimensional real inner product spaces* and *linear isometric isomorphisms*. That is, with our restriction to Euclidean inner product spaces from section 1:

- $\text{obj}(\mathscr{O})$ is the set $\{\mathbb{R}^n \mid n = 0, 1, \ldots\}$, and
- $\mathscr{O}(A, B) = \begin{cases} O(n) & \text{if } A = B = \mathbb{R}^n \\ \emptyset & \text{otherwise.} \end{cases}$

Example 3.5. Let $\mathscr{T}$ be the category of *based topological spaces* and *continuous maps*. That is,

- $\text{obj}(\mathscr{T})$ is the collection of topological spaces with basepoints, and
- $\mathscr{T}(X, Y)$ is the set of basepoint preserving maps $f : X \to Y$.

$X, Y \in \text{obj}(\mathscr{T})$ are isomorphic if there exists a basepoint preserving homeomorphism $f : X \to Y$.

Definition 3.6 (Product categories). If $\mathscr{C}, \mathscr{D}$ are categories, we can construct the product category $\mathscr{C} \times \mathscr{D}$. That is,

- $\text{obj}(\mathscr{C} \times \mathscr{D})$ is the class of pairs of objects (A, B), where $A \in \text{obj}(\mathscr{C})$ and $B \in \text{obj}(\mathscr{D})$, and
- $(\mathscr{C} \times \mathscr{D})((A, B), (C, D))$ is the set of pairs of morphisms (f, g), where $f \in \mathscr{C}(A, C)$ and $g \in \mathscr{D}(B, D)$. The identity morphisms and compositions are defined componentwise in the obvious way.

Example 3.7. We construct the category $\mathscr{O} \times \mathscr{O}$ where

- $\text{obj}(\mathscr{O} \times \mathscr{O}) = \{(\mathbb{R}^n, \mathbb{R}^m) \mid \mathbb{R}^n, \mathbb{R}^m \in \text{obj}(\mathscr{O})\}$, and
- $(\mathscr{O} \times \mathscr{O})((A, B), (C, D))$
  $= \begin{cases} O(n) \times O(m) & \text{if } A = C = \mathbb{R}^n \text{ and } B = D = \mathbb{R}^m \\ \emptyset & \text{otherwise.} \end{cases}$

Example 3.8. In the same manner we construct the category $\mathscr{T} \times \mathscr{T}$ where

- $\text{obj}(\mathscr{T} \times \mathscr{T})$ is the class of pairs of based topological spaces (A, B) where $A, B \in \text{obj}(\mathscr{T})$, and
- $(\mathscr{T} \times \mathscr{T})((A, B), (C, D)) = \{(f, g) \mid f \in \mathscr{T}(A, C) \text{ and } g \in \mathscr{T}(B, D)\}$.

Definition 3.9 (Functor). A *functor* is a morphism of categories, $F : \mathscr{C} \to \mathscr{D}$. It assigns to each object $C \in \text{obj}(\mathscr{C})$ an object $F(C) \in \text{obj}(\mathscr{D})$ and to each morphism $f \in \mathscr{C}(A, B)$ a morphism $F(f) \in \mathscr{D}(F(A), F(B))$ such that:

$$F(id_A) = id_{F(A)} \text{ for all } A \in \text{obj}(\mathscr{C}), \text{ and}$$
$$F(f \circ g) = F(f) \circ F(g) \text{ wherever } f \text{ and } g \text{ are composable.}$$

More precisely, this is a *covariant* functor. A *contravariant* functor F reverses the direction of the morphisms. That is, F sends the morphism $f : A \to B$ to $F(f) : F(B) \to F(A)$ such that $F(g \circ f) = F(f) \circ F(g)$.

Note that functors preserve isomorphic objects: Let $F : \mathscr{C} \to \mathscr{D}$ be a covariant functor. If $A, B \in \mathrm{obj}(\mathscr{C})$ with morphisms $f : A \to B$ and $g : B \to A$ such that $f \circ g = id_B$ and $g \circ f = id_A$, we see that $F(g \circ f) = F(id_A) = id_{F(A)} = F(g) \circ F(f)$ and $F(f \circ g) = F(id_B) = id_{F(B)} = F(f) \circ F(g)$. Thus $F(f)$ and $F(g)$ are mutually inverse isomorphisms between $F(A)$ and $F(B)$. The contravariant case is similar.

Example 3.10. Let $\mathbb{S} : \mathscr{O} \to \mathscr{T}$ be the one-point compactification functor:

$$\mathbb{S}(\mathbb{R}^n) = S^n = \mathbb{R}^n \cup \{\infty\}.$$

where $\{\infty\}$ denotes the basepoint of S^n. Since any orthogonal matrix A induces a proper map $A : \mathbb{R}^n \to \mathbb{R}^n$, A extends continuously to a basepoint preserving map $\mathbb{S}(A) : S^n \to S^n$.

Example 3.11. The operation of orthogonal sum, $\oplus$, is a functor $\oplus : \mathscr{O} \times \mathscr{O} \to \mathscr{O}$ where

- $\oplus(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \oplus \mathbb{R}^m \cong \mathbb{R}^{n+m}$
- Let $(A, B) \in (\mathscr{O} \times \mathscr{O})((\mathbb{R}^n, \mathbb{R}^m), (\mathbb{R}^n, \mathbb{R}^m))$. Then

  $$\oplus(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in O(n+m) .$$

  It is easily seen that $(I_n, I_m) \mapsto I_{n+m}$ and preservation of composition follows from

  $$(A, B) \circ (C, D) = (AC, BD) \mapsto \begin{pmatrix} AC & 0 \\ 0 & BD \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} .$$

Proposition 3.12. *The smash product $\wedge$ is a functor $\wedge : \mathscr{T} \times \mathscr{T} \to \mathscr{T}$.*

Proof. We have already seen that $\wedge(A, B) = A \wedge B \in \mathrm{obj}(\mathscr{T})$ for all $(A, B) \in \mathrm{obj}(\mathscr{T} \times \mathscr{T})$.

$(f, g) \in \mathrm{obj}(\mathscr{T} \times \mathscr{T})((A, B), (C, D))$ induces maps $(f \times g) : A \times C \to B \times D$ defined by

$$(f \times g)(a, c) = (f(a), g(c)) \in C \times D$$

and $(f \vee g) : A \vee C \to B \vee D$ defined by

$$(f \vee g)(x) = \begin{cases} f(x) & \text{for all}\, x \in A \\ g(x) & \text{for all}\, x \in C \end{cases},$$

hence we have an induced basepoint preserving map

$$\wedge(f,g) = (f \wedge g) : A \wedge C \to B \wedge D.$$

If $(E,F) \in \mathrm{obj}(\mathscr{T} \times \mathscr{T})$ and $(h,i) \in (\mathscr{T} \times \mathscr{T})((C,D),(E,F))$, the composition

$$(h,i) \circ (f,g) = (h \circ f, i \circ g) \in (\mathscr{T} \times \mathscr{T})((A,B),(E,F))$$

induces compositions

$$(h \times i) \circ (f \times g) = (h \circ f) \times (i \circ g) : A \times B \to E \times F$$

and

$$(h \vee i) \circ (f \vee g) = (h \circ f) \vee (i \circ g) : A \vee B \to E \vee F,$$

hence we have an induced basepoint preserving map

$$(h \wedge i) \circ (f \wedge g) = (h \circ f) \wedge (i \circ g) : A \wedge B \to E \wedge F.$$

We have $(id_A, id_B) = id_{(A,B)}$, $(id_A \times id_B) = id_{A \times B} : A \times B \xrightarrow{\equiv} A \times B$ and $id_{A \vee B} : A \vee B \xrightarrow{\equiv} A \vee B$, hence $\wedge(id_A, id_B) = id_{A \wedge B} : A \wedge B \xrightarrow{\equiv} A \wedge B$. This concludes the proof that $\wedge : \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ is a functor. □

3.2 Natural transformations and functor categories

Definition 3.13 (Natural transformations). Let $\mathscr{C}, \mathscr{D}$ be categories and $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A *natural transformation* $\alpha : F \dot{\to} G$ consists of morphisms $\alpha_A : F(A) \to G(A)$ for each $A \in \mathrm{obj}(\mathscr{C})$ such that for all $B \in \mathrm{obj}(\mathscr{C})$ and $f \in \mathscr{C}(A,B)$ the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array}$$

Composition of natural transformations is done componentwise and it is associative. That is, if $H, I : \mathscr{C} \to \mathscr{D}$ are two other functors with natural transformations $\beta : G \dot{\to} H$ and $\gamma : H \dot{\to} I$, we have that

$$\begin{aligned} & (\gamma \circ (\beta \circ \alpha))_X = \gamma_X \circ (\beta \circ \alpha)_X = \gamma_X \circ \beta_X \circ \alpha_X \\ = \; & (\gamma \circ \beta)_X \circ \alpha_X = ((\gamma \circ \beta) \circ \alpha)_X . \end{aligned}$$

The identity natural transformation taking each object and morphism to themselves is a unit for this composition. For a functor F we denote this identity natural transformation as $id_F : F \dot{\to} F$.

Two functors $F, G : \mathscr{C} \to \mathscr{D}$ are *naturally isomorphic* if there exist natural transformations $\tau : F \dot{\to} G$ and $\mu : G \dot{\to} F$ such that $\mu \circ \tau = id_F : F \dot{\to} F$ and $\tau \circ \mu = id_G : G \dot{\to} G$

Definition 3.14 (Functor category). Let $\mathscr{C}, \mathscr{D}$ be categories where $\mathscr{C}$ is small. We define the *functor category* $\mathscr{D}^{\mathscr{C}}$ as:

- $\operatorname{obj}\left(\mathscr{D}^{\mathscr{C}}\right) = \{F : \mathscr{C} \to \mathscr{D} \mid F \text{ is a functor}\}$.
- $\mathscr{D}^{\mathscr{C}}(F, G) = \{\alpha : F \dot{\to} G \mid \alpha \text{ is a natural transformation}\}$. The smallness of $\mathscr{C}$ makes this a set of natural transformations.

Example 3.15 (Orthogonal sequences). Let $\mathscr{O}, \mathscr{T}$ be as defined above. We define the category of *orthogonal sequences* as the functor category $\mathscr{T}^{\mathscr{O}}$:

- $\operatorname{obj}\left(\mathscr{T}^{\mathscr{O}}\right)$ are functors $X : \mathscr{O} \to \mathscr{T}$ that take each inner product space $\mathbb{R}^n$ to a based topological space $X(\mathbb{R}^n) = X_n$ such that we have a continuous basepoint preserving left-action of the orthogonal group $O(n)$ on X_n for each $n = 0, 1, \ldots$
- $\mathscr{T}^{\mathscr{O}}(X, Y) = \{\phi : X \dot{\to} Y \mid \phi \text{ is a natural transformation}\}$. Such a natural transformation consists of a set of basepoint preserving maps $\phi_n : X_n \to Y_n$ that are $O(n)$-*equivariant* for all n. This means that ϕ_n commutes with the group action of $O(n)$ on X_n and Y_n.

The functor $\mathbb{S}$ in example 3.10 was an important example of an orthogonal sequence. For now we shall call it the *sphere sequence*.

Example 3.16 (Free functor). Following the notation of [4], we want to construct orthogonal sequences starting with an $O(n)$-space at level n and then fill in the remaining as freely as possible. We will do this using the *free functor* G_n, defined as follows:
Let K be any topological space with basepoint. We define the orthogonal sequence G_pK as

$$(G_pK)_n = \begin{cases} O(n)_+ \wedge K & \text{if } n = p, \\ \{*\} & \text{otherwise}. \end{cases}$$

The *unit sequence* $G_0S^0 = \{S^0, *, *, \ldots\}$ is an important example of such a sequence.

Example 3.17 (Biorthogonal sequences). We define the functor category of *biorthogonal sequences* $\mathscr{T}^{\mathscr{O} \times \mathscr{O}}$ as follows:

- $\operatorname{obj}\left(\mathscr{T}^{\mathscr{O} \times \mathscr{O}}\right) = \{X : \mathscr{O} \times \mathscr{O} \to \mathscr{T} \mid X \text{ is a functor}\}$ such that for all pairs of inner product spaces $(\mathbb{R}^n, \mathbb{R}^m)$ we have a continuous basepoint preserving left-action of $O(n) \times O(m)$ on $X(\mathbb{R}^n, \mathbb{R}^m) = X_{n,m}$.
- $\mathscr{T}^{\mathscr{O} \times \mathscr{O}}(X, Y) = \{\tau : X \dot{\to} Y \mid \tau \text{ is a natural transformation}\}$. Such a natural transformation consists of basepoint preserving maps $\tau_{n,m} : X_{n,m} \to Y_{n,m}$ that are $O(n) \times O(m)$-equivariant for all $n, m = 0, 1, \ldots$

To avoid any confusion with morphisms of orthogonal sequences, we will refer to morphisms of biorthogonal sequences as *bimorphisms*.

Example 3.18 (External smash product). Let X, Y be orthogonal sequences. We define the *external smash product* $\bar{\wedge}$ of X and Y, $X \bar{\wedge} Y \in \mathscr{T}^{\mathscr{O} \times \mathscr{O}}$ to be the composition of the functors

$$\mathscr{O} \times \mathscr{O} \xrightarrow{(X \times Y)} \mathscr{T} \times \mathscr{T} \xrightarrow{\wedge} \mathscr{T}$$

so that $(X \bar{\wedge} Y)_{n,m} = (X \bar{\wedge} Y)(\mathbb{R}^n, \mathbb{R}^m) = X(\mathbb{R}^n) \wedge Y(\mathbb{R}^m) = X_n \wedge Y_m$. Each space $X_n \wedge Y_m$ has an $O(n) \times O(m)$-action as described in proposition 2.8.

Example 3.19. Let Z be any orthogonal sequence. Then the composite $Z \circ \oplus$ is a biorthogonal sequence. That is,

$$(Z \circ \oplus)_{n,m} = Z \circ \oplus (\mathbb{R}^n, \mathbb{R}^m) = Z\left(\mathbb{R}^{n+m}\right) = Z_{n+m}\,,$$

where the $O(n) \times O(m)$-action is via the inclusion $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

4 Tensor product of orthogonal sequences

As a motivating example for what is to come, let A be a commutative ring with unity and let M, N and P be A-modules. We can then construct the *tensor product* of M and N, denoted $M \otimes_A N$, such that the set of A-bilinear maps $M \times N \to P$ is in a natural bijective correspondence with the set of A-linear maps $M \otimes_A N \to P$. For more details in this, see chapter 2 of [1]. We will do an analogous construction for orthogonal sequences.

Notation 4.1. Since all our balanced smash products will be over orthogonal groups, we shall simplify "$\wedge_{O(p)\times O(q)}$" to the more aesthetic combination of symbols "$\wedge_{p\times q}$".

Let X, Y and Z be orthogonal sequences, and consider a bimorphism $b : X \bar{\wedge} Y \to Z \circ \oplus$. Using proposition 2.13 we see that each $O(p) \times O(q)$-equivariant map $b_{p,q} : X_p \wedge Y_q \to Z_{p+q}$ corresponds to an $O(n)$-equivariant map $\bar{b}_{p,q} : O(n)_+ \wedge_{p\times q} X_p \wedge Y_q \to Z_n$, where $n = p + q$. By fixing this n and varying p and q, we obtain an $O(n)$-equivariant map $\bar{b}_n : \bigvee_{p+q=n} O(n)_+ \wedge_{p\times q} X_p \wedge Y_q \to Z_n$, where $\bar{b}_n = \bigvee_{p+q=n} \bar{b}_{p,q}$.

Definition 4.2 (Tensor product of orthogonal sequences). Let X and Y be orthogonal sequences. We define the *tensor product* of X and Y, denoted $X \otimes Y$, as the orthogonal sequence

$$(X \otimes Y)_n = \bigvee_{p+q=n} O(n)_+ \wedge_{p\times q} X_p \wedge Y_q \,,$$

where the $O(n)$-action on $(X \otimes Y)_n$ is defined by acting on each wedge summand.

Proposition 4.3. *Let X, Y and Z be orthogonal sequences. There is a natural bijection,*

$$\mathscr{T}^{\mathscr{O}\times\mathscr{O}}(X\bar{\wedge}Y, Z \circ \oplus) \xleftrightarrow{\cong} \mathscr{T}^{\mathscr{O}}(X \otimes Y, Z).$$

This isomorphism is natural, hence it is an adjunction ([5], chapter IV).

Proof. As explained above, each bimorphism $b \in \mathscr{T}^{\mathscr{O}\times\mathscr{O}}(X\bar{\wedge}Y, Z \circ \oplus)$ give rise to a morphism $\bar{b} : \mathscr{T}^{\mathscr{O}}(X \otimes Y, Z)$. Going the other way, we see that a morphism $f \in \mathscr{T}^{\mathscr{O}}(X \otimes Y, Z)$ is a collection of $O(n)$-equivariant maps

$$f_n : \bigvee_{p+q=n} O(n)_+ \wedge_{p\times q} X_p \wedge Y_q \to Z_n \text{ for } n = 0, 1, \ldots$$

Each such f_n can be written as a wedge sum $f_n = \bigvee_{p+q=n} f_{p,q}$, where each $f_{p,q} : O(n)_+ \wedge_{p\times q} X_p \wedge Y_q \to Z_n$ is $O(n)$-equivariant. And by using proposition 2.13 again, we obtain a bijection of the sets

$$\mathscr{T}^{\mathscr{O}\times\mathscr{O}}(X\bar{\wedge}Y, Z \circ \oplus) \xleftrightarrow{\cong} \mathscr{T}^{\mathscr{O}}(X \otimes Y, Z).$$

□

4.1 The symmetric monoidal category $\mathscr{T}^{\mathscr{O}}$

Following the notation of Mac Lane we will define and give examples of symmetric monoidal categories. For more details on this, se chapter VII and XI of [5].

Definition 4.4 (Symmetric monoidal categories). A *monoidal category* $(\mathscr{C}, \Box, e)$ is a category $\mathscr{C}$ together with a functor $\Box : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, a unit $e \in \mathrm{obj}(\mathscr{C})$ and natural isomorphisms

$$\alpha : A\Box(B\Box C) \xrightarrow{\cong} (A\Box B)\Box C,$$
$$\lambda : e\Box A \xrightarrow{\cong} A, \text{ and } \rho : A\Box e \xrightarrow{\cong} A,$$

such that $\lambda = \rho : e\Box e \xrightarrow{\cong} e$ and the following two diagrams commute for all $A, B, C, D \in \mathrm{obj}(\mathscr{C})$:

$$\begin{array}{ccccc} A\Box(B\Box(C\Box D)) & \xrightarrow{\alpha} & (A\Box B)\Box(C\Box D) & \xrightarrow{\alpha} & ((A\Box B)\Box C)\Box D \\ \downarrow{\scriptstyle id_A\Box\alpha} & & & & \uparrow{\scriptstyle \alpha\Box id_D} \\ A\Box((B\Box C)\Box D) & & \xrightarrow[\alpha]{} & & (A\Box(B\Box C))\Box D \end{array}$$

and

$$\begin{array}{ccc} A\Box(e\Box B) & \xrightarrow{\alpha} & (A\Box e)\Box B \\ & \searrow{\scriptstyle id_A\Box\lambda} & \downarrow{\scriptstyle \rho\Box id_B} \\ & & A\Box B \end{array}$$

A *braiding* for a monoidal category $\mathscr{C}$ consists of a family of isomorphisms

$$\gamma = \gamma_{A,B} : A\Box B \xrightarrow{\cong} B\Box A$$

natural in $A, B \in \mathscr{C}$, which satisfy for the unit e the commutativity

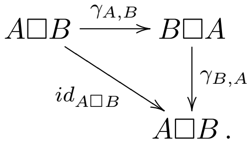
$$\begin{array}{ccc} A\Box e & \xrightarrow{\gamma} & e\Box A \\ \downarrow{\scriptstyle \rho} & \swarrow{\scriptstyle \lambda} & \\ A & & \end{array}$$

and which, with the associativity isomorphism α, make both the following diagrams commute:

$$\begin{array}{ccc} (A\Box B)\Box C & \xrightarrow{\gamma} & C\Box(A\Box B) \\ \downarrow{\scriptstyle \alpha^{-1}} & & \downarrow{\scriptstyle \alpha} \\ A\Box(B\Box C) & & (C\Box A)\Box B \\ \downarrow{\scriptstyle id_A\Box\gamma} & & \downarrow{\scriptstyle \gamma\Box id_B} \\ A\Box(C\Box B) & \xrightarrow[\alpha]{} & (A\Box C)\Box B \end{array}, \qquad \begin{array}{ccc} A\Box(B\Box C) & \xrightarrow{\gamma} & (B\Box C)\Box A \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha^{-1}} \\ (A\Box B)\Box C & & B\Box(C\Box A) \\ \downarrow{\scriptstyle \gamma\Box id_C} & & \downarrow{\scriptstyle id_B\Box\gamma} \\ (B\Box A)\Box C & \xrightarrow[\alpha^{-1}]{} & B\Box(A\Box C) \end{array}.$$

Note that these two diagrams imply that if γ is a braiding, so is γ^{-1}.

A *symmetric monoidal category*, is a monoidal category $\mathscr{C}$ with a braiding γ such that for all $A, B \in \text{obj}(\mathscr{C})$ the following diagram commutes:



If we have a symmetric monoidal category, the two hexagonal diagrams above involving γ and α will imply each other.

Proposition 4.5 (Coherence for symmetric monoidal categories). *For a symmetric monoidal category $(\mathscr{C}, \square, e)$, the associativity α and commutativity γ are coherent isomorphisms. That is, all the formal diagrams involving just α and γ that have a chance to commute, actually do commute.*

Proof. See [5], theorem 1, p. 253. □

Example 4.6 (Abelian groups). The category of abelian groups $\mathcal{A}b$ is a symmetric monoidal category $(\mathcal{A}b, \otimes, \mathbb{Z})$, where $\otimes$ is the tensor product of abelian groups. See chapter 2 of [1] for more details on this.

Proposition 4.7 (The symmetric monoidal category $\mathscr{O}$). *$\mathscr{O}$ is a symmetric monoidal category, $(\mathscr{O}, \oplus, \mathbb{R}^0)$, where $\oplus$ is the operation of orthogonal sum. That is, there exist isomorphisms*

1. $\gamma : \mathbb{R}^n \oplus \mathbb{R}^m \cong \mathbb{R}^m \oplus \mathbb{R}^n$,
2. $\alpha : \mathbb{R}^k \oplus (\mathbb{R}^l \oplus \mathbb{R}^m) \cong (\mathbb{R}^k \oplus \mathbb{R}^l) \oplus \mathbb{R}^m$,
3. $\rho : \mathbb{R}^k \oplus \{0\} \cong \mathbb{R}^k$ *and* $\lambda : \{0\} \oplus \mathbb{R}^k \cong \mathbb{R}^k$.

These isomorphisms preserve the inner products and the necessary diagrams commute.

Proof. First of all, due to the construction of orthogonal direct sum and our restriction to the Euclidean inner product, it is obvious that the inner product is preserved.

1. Let $f : \mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^{n+m}$ and $g : \mathbb{R}^m \oplus \mathbb{R}^n \to \mathbb{R}^{m+n}$ be linear isomorphisms constructed the same way as in lemma 1.4. Let $\gamma_{n,m} = g^{-1} \circ \chi_{n,m} \circ f$ and $\gamma_{m,n} = \gamma_{n,m}^{-1} = f^{-1} \circ \chi_{n,m}^t \circ g$, where $\chi_{n,m}$ is the permutation matrix defined in definition 1.9. These constructions then give direct expressions for the isomorphisms:

$$\gamma_{n,m} : \mathbb{R}^n \oplus \mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^m \oplus \mathbb{R}^n,$$
$$\gamma_{m,n} : \mathbb{R}^m \oplus \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n \oplus \mathbb{R}^m.$$

With some abuse of notation we will usually just let γ denote this twist isomorphism, $\gamma : \mathbb{R}^n \oplus \mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^m \oplus \mathbb{R}^n$.

2. From lemma 1.4 we have the following isomorphisms:

$$\mathbb{R}^k \oplus (\mathbb{R}^l \oplus \mathbb{R}^m) \cong \mathbb{R}^k \oplus \mathbb{R}^{l+m} \cong \mathbb{R}^{k+l+m}, \text{ and}$$
$$(\mathbb{R}^k \oplus \mathbb{R}^l) \oplus \mathbb{R}^m \cong \mathbb{R}^{k+l} \oplus \mathbb{R}^m \cong \mathbb{R}^{k+l+m},$$

hence the operation of direct sum is associative. We will write $\alpha : \mathbb{R}^k \oplus (\mathbb{R}^l \oplus \mathbb{R}^m) \xrightarrow{\cong} (\mathbb{R}^k \oplus \mathbb{R}^l) \oplus \mathbb{R}^m$ for this isomorphism.

3. This follows from lemma 1.4, *1.* and the fact that the zero dimensional vector space $\mathbb{R}^0 \cong \{0\}$. Hence we have isomorphisms

$$\rho : \mathbb{R}^n \oplus \{0\} \xrightarrow{\cong} \mathbb{R}^n, \text{ and}$$
$$\lambda : \{0\} \oplus \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n .$$

We have to verify that the required diagrams do commute. This is straight forward task, but it is very time- and spaceconsuming. And since it is similar to that of proposition 4.8, we will skip this part. □

Proposition 4.8 (The symmetric monoidal category of orthogonal sequences). $\langle \mathscr{T}^{\mathscr{O}}, \otimes, G_0S^0 \rangle$ *is a symmetric monoidal category where* $\otimes$ *is the tensor product of orthogonal sequences defined above.*

The proof of this proposition will be in several steps:

1. Establishing associativity α.
2. Establishing commutativity γ.
3. Proving that the coherence diagrams for α and γ commute.
4. Proving that G_0S^0 is a unit for the tensor product.

These steps will be done with a fairly high level of detail and since it's not crucial to understand all these details, the reader may perfectly take lightly on the rest of this section.

4.1.1 Associativity

Let X, Y and Z be orthogonal sequences and note that

$$\begin{aligned} & (X \otimes (Y \otimes Z))_n \\ = & \bigvee_{p+s=n} O(n)_+ \wedge_{p\times s} X_p \wedge (Y \otimes Z)_s \\ = & \bigvee_{p+s=n} O(n)_+ \wedge_{p\times s} X_p \wedge (\bigvee_{q+r=s} O(s)_+ \wedge_{q\times r} Y_q \wedge Z_r) \\ = & \bigvee_{p+q+r=n} O(n)_+ \wedge_{p\times q+r} X_p \wedge (O(q+r)_+ \wedge_{q\times r} Y_q \wedge Z_r) \end{aligned}$$

where "$\wedge_{p\times q+r}$" is shorthand for "$\wedge_{O(p)\times O(q+r)}$". And similarly we have that,

$$\begin{aligned} & ((X \otimes Y) \otimes Z)_n \\ = & \bigvee_{p+q+r=n} O(n)_+ \wedge_{p+q\times r} (O(p+q)_+ \wedge_{p\times q} X_p \wedge Y_q) \wedge Z_r \,, \end{aligned}$$

where the wedge sums run over all non negative p, q and r that sum up to n.

Fix a triple $p+q+r = n$, and consider the maps $u = u_{p,q,r}$ and $d = d_{p,q,r}$ between the two wedge summands

$$\begin{array}{c} O(n)_+ \wedge_{p\times q+r} X_p \wedge (O(q+r)_+ \wedge_{q\times r} Y_q \wedge Z_r) \\ d \big\downarrow \; \big\uparrow u \\ O(n)_+ \wedge_{p+q\times r} (O(p+q)_+ \wedge_{p\times q} X_p \wedge Y_q) \wedge Z_r \end{array}$$

defined by

$$\begin{array}{ccc} A \wedge_{p\times q+r} x \wedge (B \wedge_{q\times r} y \wedge z) & & A(C, I_r)^t \wedge_{p\times q+r} x \wedge (I_{q+r} \wedge_{q\times r} y \wedge z) \\ \big\downarrow d & & \big\uparrow u \\ A(I_p, B)^t \wedge_{p+q\times r} (I_{p+q} \wedge_{p\times q} x \wedge y) \wedge z & & A \wedge_{p+q\times r} (C \wedge_{p\times q} x \wedge y) \wedge z \,, \end{array}$$

where $A \in O(n)$, $B \in O(q+r)$, $C \in O(p+q)$, $A(C, I_r)^t = A \begin{pmatrix} C & 0 \\ 0 & I_r \end{pmatrix}^t$ and

$A(I_p, B)^t = A \begin{pmatrix} I_p & 0 \\ 0 & B \end{pmatrix}^t$.

Choosing the lesser of two typographic evils, we will stick to this paranthesisnotation, meaning that

$$(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

$$((A, B), C) = \begin{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} & 0 \\ 0 & C \end{pmatrix}$$

and so on.

Lemma 4.9. *The maps d and u are well-defined and are inverse homeomorphisms.*

Proof. Since the proofs are basicly the same, we will only show that d is well-defined. Let A, B, x, y and z be as above. Then

$$\begin{array}{ll} & A \wedge_{p \times q+r} x \wedge (B \wedge_{q \times r} y \wedge z) \\ \sim & A(I_p, B)^t \wedge_{p \times q+r} x \wedge (I_{q+r} \wedge_{q \times r} y \wedge z) \\ & \xmapsto{\quad d \quad} \\ & A(I_p, B)^t (I_p, I_{q+r})^t \wedge_{p+q \times r} (I_{p+q} \wedge_{p+q} x \wedge y) \wedge z \\ = & A(I_p, B)^t \wedge_{p+q \times r} (I_{p+q} \wedge_{p+q} x \wedge y) \wedge z \,. \end{array}$$

The fact that they are inverse homeomorphisms follows easily. □

And sincethis holds for all triples $p + q + r = n$, we can wedge over all these maps and obtain the inverse homeomorphisms

$$\begin{array}{c} \bigvee_{p+q+r=n} O(n)_+ \wedge_{p \times q+r} X_p \wedge (O(q+r)_+ \wedge_{q \times r} Y_q \wedge Z_r) \\ D_n \big\downarrow \big\uparrow U_n \\ \bigvee_{p+q+r=n} O(n)_+ \wedge_{p+q \times r} (O(p+q)_+ \wedge_{p \times q} X_p \wedge Y_q) \wedge Z_r \end{array}$$

where

$$U_n = \bigvee_{p+q+r=n} u_{p,q,r} \text{ and } D_n = \bigvee_{p+q+r=n} d_{p,q,r} \,.$$

Hence we have an associativity isomorphism between the orthogonal sequences

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$$

which is the set of basepoint preserving homeomorphisms

$$\{((X \otimes Y) \otimes Z)_n \xrightarrow{U_n} (X \otimes (Y \otimes Z))_n \mid U_n \text{ is } O(n)\text{-equivariant}, n = 0, 1, \ldots\}$$

with inverse

$$\alpha^{-1}_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z$$

being the set of the D_n-maps.

Comment: Note that the author has switched the notation of α and α^{-1} compared to Mac Lane. This is due to some unknown "mental defect".

4.1.2 Commutativity

Let X and Y be orthogonal sequences. Let $p+q=n$ and consider the map $t_{p,q}$

$$O(n)_+ \wedge_{p\times q} X_p \wedge Y_q \xrightarrow{t_{p,q}} O(n)_+ \wedge_{q\times p} Y_q \wedge X_p$$

defined by

$$A \wedge_{p\times q} x \wedge y \xmapsto{t_{p,q}} \mathrm{conj}_{p,q}(A) \wedge_{q\times p} y \wedge x$$

where $\mathrm{conj}_{p,q}(A) = \chi_{p,q} A \chi_{q,p}$.

Lemma 4.10. *$t_{p,q}$ is a well-defined homeomorphism with inverse $t_{q,p}$*

Proof. Let X and Y be orthogonal sequences and fix an $n = p+q$, $p,q \geq 0$. Let $A \in O(n)$, $x \in X_p$, $y \in Y_q$ such that $A \wedge_{p\times q} x \wedge y$ is an element of $(X \otimes Y)_n$. Then we have that,

$$\begin{aligned}
& A \wedge_{p\times q} x \wedge y \\
\sim\ & A(B,C)^t \wedge_{p\times q} Bx \wedge Cy \\
& \xmapsto{\quad t_{p,q} \quad} \\
& \mathrm{conj}_{p,q}(A(B,C)^t) \wedge_{q\times p} Cy \wedge Bx \\
=\ & \mathrm{conj}_{p,q}(A)\mathrm{conj}_{p,q}((B,C)^t) \wedge_{q\times p} Cy \wedge Bx \\
=\ & \mathrm{conj}_{p,q}(A)(C,B)^t \wedge_{q\times p} Cy \wedge Bx \\
\sim\ & \mathrm{conj}_{p,q}(A) \wedge_{q\times p} y \wedge x
\end{aligned}$$

The fact that $t_{p,q}$ are inverse homeomorphisms is obvious. □

And as before, by letting $T_n = \bigvee_{p+q=n} t_{p,q}$ and $T_n^{-1} = \bigvee_{p+q=} t_{q,p}$ (summed in the same manner as T_n) we obtain a natural isomorphism

$$\gamma_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$

which is the set of basepoint preserving homeomorphisms

$$\{(X\otimes Y)_n \xrightarrow{T_n} (Y \otimes X)_n \mid T_n \text{ is } O(n)\text{-equivariant, } n = 0,1,\dots\}$$

with inverse isomorphism

$$\gamma_{X,Y}^{-1} = \gamma_{Y,X} : Y \otimes X \xrightarrow{\cong} X \otimes Y$$

being the set

$$\{(Y\otimes X)_n \xrightarrow{T_n^{-1}} (X \otimes Y)_n \mid T_n^{-1} \text{ is } O(n)\text{-equivariant, } n = 0,1,\dots\}$$

Proposition 4.11. *Let X and Y be orthogonal sequences. The diagram*

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\gamma_{X,Y}} & Y \otimes B \\ & \searrow^{id_{X\otimes Y}} & \downarrow{\scriptstyle \gamma_{Y,X}} \\ & & X \otimes Y \end{array}$$

commutes.

Proof. Fix an $n = p + q$, $p, q \geq 0$, and let $A \in O(n))$, $x \in X_p$ and $y \in Y_p$. $A \wedge_{p\times q} x_p \wedge y_q$ is then an element of the space $(X \otimes Y)_n$. Going clockwise, we see that

$$\begin{aligned} & A \wedge_{p\times q} x_p \wedge y_q \\ & \xmapsto{t_{p,q}} \\ & \mathrm{conj}_{p,q}(A) \wedge_{q\times p} y \wedge x \\ & \xmapsto{t_{q,p}} \\ & \mathrm{conj}_{q,p}(\mathrm{conj}_{p,q}(A)) \wedge_{p\times q} x_p \wedge y_q \\ = \; & A \wedge_{p\times q} x_p \wedge y_q \,, \end{aligned}$$

hence $\gamma_{Y,X} \circ \gamma_{X,Y} = id_{X\otimes Y}$. □

4.1.3 Coherence diagrams

Lemma 4.12. *Let W, X, Y and Z be orthogonal sequences. Then the following pentagonal diagram commutes*

$$\begin{array}{ccccc} & & ((W \otimes X) \otimes Y) \otimes Z & & \\ & \swarrow^{\alpha_{X,Y,Z} \otimes id_Z} & & \searrow^{\alpha_{W\otimes X,Y,Z}} & \\ (W \otimes (X \otimes Y)) \otimes Z & & & & (W \otimes X) \otimes (Y \otimes Z) \\ \downarrow{\scriptstyle \alpha_{W,X\otimes Y,Z}} & & & & \downarrow{\scriptstyle \alpha_{W,X,Y\otimes Z}} \\ W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow[id_W \otimes \alpha_{X,Y,Z}]{} & & W \otimes (X \otimes (Y \otimes Z)) \end{array}$$

Proof. Fix an $n = o + p + q + r$, $o, p, q, r \geq 0$, and let $A \in O(n)$, $B \in O(o + p + q)$, $C \in O(o + p)$, $w \in W_o$, $x \in X_p$, $y \in Y_q$ and $z \in Z_r$. Then $(A \wedge_{o+p+q\times r} (B \wedge_{o+p\times q} (C \wedge_{o\times p} w \wedge x) \wedge y) \wedge z)$ is an element of $(((W \otimes X) \otimes Y) \otimes Z)_n$. We must show that this element maps to the same image in

$(W \otimes (X \otimes (Y \otimes Z)))_n$, going through the two paths of the diagram. Going in the counter-clockwise direction, we have that

$$
\begin{aligned}
& A \wedge_{o+p+q\times r} (B \wedge_{o+p\times q} (C \wedge_{o\times p} w \wedge x) \wedge y) \wedge z \\
& \xmapsto{\alpha_{X,Y,Z}\otimes id_Z} \\
& A \wedge_{o+p+q\times r} (B(C, I_q)^t \wedge_{o\times p+q} w \wedge (I_{p+q} \wedge_{p\times q} x \wedge y)) \wedge z \\
& \xmapsto{\alpha_{W,X\otimes Y,Z}} \\
& A(B(C, I_q)^t, I_r)^t \wedge_{o\times p+r+q} w \wedge (I_{p+q+r} \wedge_{p+q\times r} (I_{p+q} \wedge_{p\times q} x \wedge y) \wedge z) \\
& \xmapsto{id_W\otimes\alpha_{X,Y,Z}} \\
& A(B(C, I_q)^t, I_r)^t \wedge_{o\times p+r+q} w \wedge (I_{p+q+r}(I_{p,q}, I_r) \wedge_{p\times q+r} x \wedge (I_{q+r} \wedge_{q\times r} y \wedge z)) \\
= \; & A(B(C, I_q)^t, I_r)^t \wedge_{o\times p+r+q} w \wedge (I_{p+q+r} \wedge_{p\times q+r} x \wedge (I_{q+r} \wedge_{q\times r} y \wedge z)).
\end{aligned}
$$

The other way around,

$$
\begin{aligned}
& A \wedge_{o+p+q\times r} (B \wedge_{o+p\times q} (C \wedge_{o\times p} w \wedge x) \wedge y) \wedge z \\
\sim \; & A \wedge_{o+p+q\times r} (B(C, I_q)^t \wedge_{o+p\times q} (I_{o+p} \wedge_{o\times p} w \wedge x) \wedge y) \wedge z \\
& \xmapsto{\alpha_{W\otimes X,Y,Z}} \\
& A(B(C, I_q)^t, I_r)^t \wedge_{o+p\times q+r} (I_{o+p} \wedge_{o\times p} w \wedge x) \wedge (I_{q+r} \wedge_{q\times r} y \wedge z) \\
& \xmapsto{\alpha_{W,X,Y\otimes Z}} \\
& A(B(C, I_q)^t, I_r)^t \wedge_{o\times p+q+r} w \wedge (I_{p+q+r} \wedge_{p\times q+r} x \wedge (I_{q+r} \wedge_{q\times r} y \wedge z)),
\end{aligned}
$$

hence the image is the same in $(W \otimes (X \otimes (Y \otimes Z)))_n$. Since this holds for all elements and all $n = o + p + q + r$, the diagram commutes. $\square$

Proposition 4.13. *The following diagram commutes:*

$$
\begin{array}{ccccc}
(X \otimes Y) \otimes Z & \xrightarrow{\alpha_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{id_X\otimes\gamma_{Y,Z}} & X \otimes (Z \otimes Y) \\
\downarrow{\scriptstyle \gamma_{X\otimes Y,Z}} & & & & \downarrow{\scriptstyle \alpha^{-1}_{X,Z,Y}} \\
Z \otimes (X \otimes Y) & \xrightarrow[\alpha^{-1}_{Z,X,Y}]{} & (Z \otimes X) \otimes Y & \xrightarrow[\gamma_{Z,X}\otimes id_Y]{} & (X \otimes Z) \otimes Y
\end{array}
$$

Proof. Fix an $n = p + q + r$, $p, q, r \geq 0$ and let $A \in O(n)$, $B \in O(p + q)$, $x \in X_p$, $y \in Y_q$ and $z \in Z_r$. $A \wedge_{p+q\times r} (B \wedge_{p\times q} x \wedge y) \wedge z$ is then an element of the orthogonal sequence $((X \otimes Y) \otimes Z)_n$. Going counter-clockwise, we see that

$$
\begin{aligned}
& A \wedge_{p+q\times r} (B \wedge_{p\times q} x \wedge y) \wedge z \\
& \xmapsto{\gamma_{X\otimes Y,Z}} \\
& \mathrm{conj}_{p+q,r}(A) \wedge_{r\times p+q} z \wedge (B \wedge_{p\times q} x \wedge y) \\
& \xmapsto{\alpha^{-1}_{Z,X,Y}} \\
& \mathrm{conj}_{p+q,r}(A)(I_r, B)^t \wedge_{r+p\times q} (I_{r+p} \wedge_{r\times p} z \wedge x) \wedge y \\
= \; & \mathrm{conj}_{p+q,r}(A(B, I_r)^t) \wedge_{r+p\times q} (I_{r+p} \wedge_{r\times p} z \wedge x) \wedge y \\
& \xmapsto{\gamma_{Z,X}\otimes id_Y} \\
& \mathrm{conj}_{r,p+q}(\mathrm{conj}_{p+q,r}(A(B, I_r)^t)) \wedge_{p+r\times q} (\mathrm{conj}_{r,p}(I_{r+p}) \wedge_{p\times r} x \wedge z) \wedge y \\
= \; & A(B, I_r)^t \wedge_{p+r\times q} (I_{p+r} \wedge_{p\times r} x \wedge z) \wedge y\,.
\end{aligned}
$$

Going clockwise,

$$
\begin{aligned}
& A \wedge_{p+q\times r} (B \wedge_{p\times q} x \wedge y) \wedge z \\
& \xmapsto{\alpha_{X,Y,Z}} \\
& A(B, I_r)^t \wedge_{p\times q+r} x \wedge (I_{q+r} \wedge_{q\times r} y \wedge z) \\
& \xmapsto{id_X\otimes\gamma_{Y,Z}} \\
& A(B, I_r)^t \wedge_{p\times r+q} x \wedge (I_{r+q} \wedge_{r\times q} z \wedge y) \\
& \xmapsto{\alpha^{-1}_{X,Z,Y}} \\
& A(B, I_r)^t \wedge_{p+r\times q} (I_{p+r} \wedge_{p\times r} x \wedge z) \wedge y\,,
\end{aligned}
$$

hence the image in $((X \otimes Z) \otimes Y)_n$ is the same. And as before, since this holds for all elements and $n = p + q + r$, the diagram commutes. □

This concludes coherence between γ and α.

4.1.4 Unit

Definition 4.14. Let X be any orthogonal sequence and G_0S^0 the unit sequence. Fix an $n = p+q$, $p, q \geq 0$. Let $A \in O(n)$, $x \in X_p$ and $g \in (G_0S^0)_q$ such that $A \wedge_{p\times q} x \wedge g$ is an element of $(X \otimes G_0S^0)_n$. We let $r_{p,q}$ denote the map

$$(X \otimes G_0S^0)_n \xrightarrow{r_{p,q}} X_n$$

defined by

$$r_{p,q}(A \wedge_{p\times q} x \wedge g) = \begin{cases} x & \text{if } q = 0, \text{ and} \\ x_n & \text{otherwise,} \end{cases}$$

where x_n denotes the basepoint of X_n. We denote $l_{q,p}$ as the composite $r_{p,q} \circ t_{q,p}$. That is,

$$(G_0S^0 \otimes X)_n \xrightarrow{l_{q,p}} X_n$$

defined by

$$r_{p,q} \circ t_{q,p}(A \wedge_{q\times p} g \wedge x) = r_{p,q}(\mathrm{conj}_{q,p}(A) \wedge_{p\times q} x \wedge g) = \begin{cases} x & \text{if } q = 0\text{, and} \\ x_n & \text{otherwise.} \end{cases}$$

Using this, and the fact that $O(n)_+ \wedge_{n\times 0} X_n \wedge S^0 \approx O(n)_+ \wedge_n X_n \approx X_n$, we see that $r_{p,0} : (X \otimes G_0S^0)_n \xrightarrow{\approx} X_n$ and $l_{0,p} : (G_0S^0 \otimes X)_n \xrightarrow{\approx} X_n$ are homeomorphisms. And by varying p we obtain isomorphisms

$$r_{\bullet,0} = \rho_X : X \otimes G_0S^0 \xrightarrow{\cong} X\text{, and}$$
$$l_{0,\bullet} = \lambda_X : G_0S^0 \otimes X \xrightarrow{\cong} X.$$

It is easily seen that $\lambda_X = \rho_X \circ \gamma_{G_0S^0,X}$.

Proposition 4.15. *The unit sequence G_0S^0 is a unit for the tensor product. That is, for all orthogonal sequences X and Y, there exist natural isomorphisms*

$$\lambda = \lambda_X : G_0S^0 \otimes X \xrightarrow{\cong} X\text{, and}$$
$$\rho = \rho_X : X \otimes G_0S^0 \xrightarrow{\cong} X$$

such that the diagram

$$\begin{array}{ccc} X \otimes (G_0S^0 \otimes Y) & \xrightarrow{\alpha^{-1}_{X,G_0S^0,Y}} & (X \otimes G_0S^0) \otimes Y \\ & {\scriptstyle id_X \otimes \lambda_Y} \searrow \quad \swarrow {\scriptstyle \rho_X \otimes id_Y} & \\ & X \otimes Y & \end{array}$$

commutes.

Proof. Existence of ρ and λ has already been shown. Fix an $n = p + r$, $p, r \geq 0$. Let $A \in O(n)$, $B \in O(r)$, $x \in X_p$, $y \in Y_r$ and $g \in (G_0S^0)_0$. Then $A \wedge_{p\times 0+r} x \wedge (B \wedge_{0\times r} g \wedge y)$ is an element of $(X \otimes (G_0S^0 \otimes Y))_n$. As always, we must show that this element maps to the same image through the two paths of the diagram. Going clockwise, we see that

$$A \wedge_{p\times 0+r} x \wedge (B \wedge_{0\times r} g \wedge y)$$
$$\xmapsto{\alpha^{-1}_{X,G_0S^0,Y}}$$
$$A(I_p, B)^t \wedge_{p+0\times r} (I_{p,0} \wedge_{p\times r} x \wedge g) \wedge y$$
$$\xmapsto{\rho_X \otimes id_Y}$$
$$A(I_p, B)^t \wedge_{p\times r} x \wedge y\,.$$

The other way,

$$
\begin{array}{rl}
& A \wedge_{p\times 0+r} x \wedge (B \wedge_{0\times r} g \wedge y) \\
& \xmapsto{id_X \otimes \gamma_{G_0 S^0, y}} \\
& A \wedge_{p\times r+0} x \wedge (\mathrm{conj}_{0,r}(B) \wedge_{r\times 0} y \wedge g) \\
\sim & A(I_p, \mathrm{conj}_{0,r}(B))^t \wedge_{p\times r+0} x \wedge (I_{r+0} \wedge_{r\times 0} y \wedge g) \\
& \xmapsto{id_X \otimes \rho_Y} \\
& A(I_p, \mathrm{conj}_{0,r}(B))^t \wedge_{p\times r} x \wedge y \\
= & A(I_p, B)^t \wedge_{p\times r} x \wedge y\,.
\end{array}
$$

□

Now only one diagram remains, namely

$$
\begin{array}{ccc}
X \otimes G_0 S^0 & \xrightarrow{\gamma_{X, G_0 S^0}} & G_0 S^0 \otimes X \\
\downarrow{\scriptstyle \rho_X} & \swarrow{\scriptstyle \lambda_X} & \\
X. & &
\end{array}
$$

But this follows easily from the construction of λ. This concludes the proof of proposition 4.8.

5 $\mathbb{S}$-modules and orthogonal spectra

In this section we will construct orthogonal spectra, which are orthogonal sequences with some extra structure. We will also see that $\mathbb{S}$ in some sense plays the same role as the integers $\mathbb{Z}$ in the category of abelian groups, which is why we denote it by a double stroke character.

Definition 5.1 (Orthogonal spectra). An *orthogonal spectrum* X consists of the following:

- a sequence of based topological spaces X_n, $n = 0, 1, \ldots$,
- a basepoint preserving continuous left action of the orthogonal group $O(n)$ on X_n for each n, and
- based maps $\sigma = \sigma_n : X_n \wedge S^1 \to X_{n+1}$, called *structure maps*,

such that for each $n, m \geq 0$, the iterated structure map

$$\sigma_n^m : X_n \wedge S^m \to X_{n+m}$$

is $O(n) \times O(m)$-equivariant.

A morphism $f : X \to Y$ of orthogonal spectra consists of $O(n)$-equivariant maps $f_n : X_n \to Y_n$, $n \geq 0$, which are compatible with the structure maps in the following sense

$$f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge id_{S^1}) .$$

We will denote the category of orthogonal spectra by $\mathcal{S}p^{\mathscr{O}}$.

Example 5.2 (The sphere spectrum). From this definition, we see that the sphere sequence $\mathbb{S}$ is an orthogonal spectrum, and we shall from now on refer to it as the *sphere spectrum*. The structure maps are the canonical homeomorphisms $\sigma_p^q = m_{p,q} : S^p \wedge S^q \xrightarrow{\approx} S^{p+q}$. And since we regard the spheres S^n as the one-point compactification of $\mathbb{R}^n$, these maps are $O(p) \times O(q)$-equivariant as follows:

$$\begin{aligned} & m_{p,q}(Ax \wedge By) = m_{p,q}((A, B)(x \wedge y)) \\ = \quad & \iota(A, B)\, m_{p,q}(x \wedge y) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} m_{p,q}(x \wedge y) \ , \end{aligned}$$

where $A \in O(p)$ and $B \in O(q)$.

Definition 5.3 (Monoids). A *monoid* (M, μ, η) in a monoidal category $(\mathscr{C}, \square, e)$ is an object $M \in \text{obj}(\mathscr{C})$ together with arrows $\mu : M \square M \to M$ and $\eta : e \to M$ such that the diagrams

$$\begin{array}{ccccc} M\square(M\square M) & \xrightarrow{\alpha} & (M\square M)\square M & \xrightarrow{\mu\square id_M} & M\square M \\ \downarrow{\scriptstyle id_M\square\mu} & & & & \downarrow{\scriptstyle \mu} \\ M\square M & & \xrightarrow{\quad\mu\quad} & & M \end{array}$$

and

$$\begin{array}{ccccc} e\Box M & \xrightarrow{\eta\Box id_M} & M\Box M & \xleftarrow{id_M\Box\eta} & M\Box e \\ & \searrow^{\lambda} & \downarrow{\scriptstyle\mu} & {}^{\rho}\swarrow & \\ & & M & & \end{array}$$

are commutative.

A morphism $f : (M, \mu, \eta) \to (M', \mu', \eta')$ monoids is a morphism $f : M \to M'$ in $\mathscr{C}$ such that

$$f \circ \mu = \mu' \circ (f\Box f) : M\Box M \to M', \text{ and } f \circ \eta = \eta' : e \to M' .$$

With these arrows, the monoids of $\mathscr{C}$ form a category $\mathrm{Mon}_{\mathscr{C}}$.

If the category is symmetric with a braiding γ, we say that M is a *commutative monoid* if the following diagram commutes:

$$\begin{array}{ccc} M\Box M & \xrightarrow{\gamma_{M,M}} & M\Box M \\ & {}_{\mu}\searrow \quad \swarrow_{\mu} & \\ & M & \end{array}$$

Proposition 5.4. *The sphere sequence $\mathbb{S}$ is a commutative monoid in the symmetric monoidal category $\left(\mathscr{T}^{\mathscr{O}}, \otimes, G_0S^0\right)$ of orthogonal sequences.*

Proof. Using proposition 4.3 we see that the canonical homeomorphisms $\{m_{p,q} : S^p \wedge S^q \xrightarrow{\approx} S^{p+q} \mid p, q \geq 0\}$ give a morphism $\mu : \mathbb{S} \otimes \mathbb{S} \to \mathbb{S}$. The commutative diagram

$$\begin{array}{ccc} S^p \wedge S^q \wedge S^r & \xrightarrow{id_{S^p}\wedge m_{q,r}} & S^p \wedge S^{q+r} \\ {\scriptstyle m_{p,q}\wedge id_{S^r}}\downarrow & & \downarrow{\scriptstyle m_{p,q+r}} \\ S^{p+q} \wedge S^r & \xrightarrow[m_{p+q,r}]{} & S^{p+q+r} \end{array}$$

shows the associativity of μ. The basepoint preserving maps $n_p : \left(G_0S^0\right)_p \to S^p$, where n_0 is the unit homeomorphism, give a morphism $\eta : G_0S^0 \to \mathbb{S}$. The composites

$$\left(G_0S^0\right)_p \wedge S^q \xrightarrow{n_p\wedge id_{S^q}} S^p \wedge S^q \xrightarrow[m_{p,q}]{\approx} S^{p+q}$$

$$S^p \wedge \left(G_0S^0\right)_q \xrightarrow{id_{S^p}\wedge n_q} S^p \wedge S^q \xrightarrow[m_{p,q}]{\approx} S^{p+q}$$

shows the second commutative diagram. And finally, to see that $\mathbb{S}$ being a commutative monoid let $A\wedge_{p\times q} x\wedge y$ be an element of $O(p+q)_+\wedge_{p\times q} S^p\wedge S^q$.

Going clockwise, we see that

$$\begin{aligned} A \wedge_{p\times q} x \wedge y &\xmapsto{\gamma_{\mathbb{S},\mathbb{S}}} conj_{p,q}(A) \wedge_{q\times p} y \wedge x \\ &\xmapsto{\mu} \operatorname{conj}_{p,q}(A)(y \wedge x) \\ &= A(x \wedge y) \end{aligned}$$

And counter-clockwise we have

$$A \wedge_{p\times q} x \wedge y \xmapsto{\mu} A(x \wedge y),$$

hence the image is the same in S^{p+q}, and $\mathbb{S}$ is a commutative monoid. $\square$

Definition 5.5 (Modules). Let $(\mathscr{C}, \square, e)$ be a monoidal category with a monoid (M, μ, η). A *right M-module* is an object A with an associative "multiplication" morphism $\nu : A\square M \to A$ such that the following diagram commutes:

$$\begin{array}{ccccccc} (A\square M)\square M & \xrightarrow{\alpha^{-1}} & A\square(M\square M) & \xrightarrow{id_A\square\mu} & A\square M & \xleftarrow{id_A\square\eta} & A\square e \\ \downarrow{\scriptstyle \nu\square id_M} & & & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \rho} \\ A\square M & & \xrightarrow{\quad\nu\quad} & & A & \xleftrightarrow[=]{} & A \end{array}$$

Note that a monoid M is always a module over itself with the morphism $\mu : M\square M \to M$.

A *morphism* $(A, \nu_A) \to (B, \nu_B)$ of right modules of M is a morphism $f : A \to B$ such that $\nu_B \circ (f\square id_M) = f \circ \nu_A : A\square M \to B$. With these morphisms, the collection of right M-modules form a category, denoted **mod**-M.

Example 5.6 (Abelian groups and $\mathbb{Z}$). $\mathbb{Z}$ is a commutative monoid in $(\mathcal{A}b, \otimes, \mathbb{Z})$ and the category of right $\mathbb{Z}$-modules is the category $\mathcal{A}b$. More generally, the category of monoids in $(\mathcal{A}b, \otimes, \mathbb{Z})$ is the category of rings. For more details on this, see [1].

Proposition 5.7. *The category of right* $\mathbb{S}$*-modules,* **mod**-$\mathbb{S}$*, is naturally equivalent to the category of orthogonal spectra,* $\mathcal{S}p^{\mathscr{O}}$.

Proof. Let X be a right $\mathbb{S}$-module with a multiplication map $\nu : X \otimes \mathbb{S} \to X$. Using proposition 4.3, this morphism corresponds to a set of $O(n) \times O(m)$-equivariant maps

$$\{\nu_n^m : X_n \wedge S^m \to X_{n+m} \mid n, m \geq 0\}$$

where ν_n^0 is the unit homomorphism. Due to associativity of action, this set now functions as structure maps. That is, for any n, we have that

$$(\nu_n \circ \nu_n) \circ \nu_n = \nu_n \circ \nu_n \circ \nu_n = \nu_n \circ (\nu_n \circ \nu_n),$$

so every ν_n^m is determined by ν_n.

Conversely, for an orthogonal spectrum X the set of structure maps $\{\sigma_n^p : X_n \wedge S^p \to X_{n+p} \mid p, n \geq 0\}$, where σ_n^0 is the unit homeomorphism of X_n, corresponds to a multiplication morphism $\nu : X \otimes \mathbb{S} \to X$, hence X is a right $\mathbb{S}$-module. These are inverse constructions and give a natural equivalence of the two categories. $\square$

6 Homotopy groups of orthogonal spectra

Definition 6.1. Let $(A_i, f_{i,j})_I$ denote a directed system of abelian groups indexed over some set I. That is, each A_i is an abelian group and $f_{i,j} : A_i \to A_j$, $i \geq j$, are groups homomorphisms such that $f_{i,i}$ is the identity of A_i and $f_{j,k} \circ f_{i,j} = f_{i,k}$. The *direct limit* A of the directed system $(A_i, f_{i,j})_I$ is defined as

$$\varinjlim (A_i) = (\amalg_i A_i) \,/ \sim,$$

where $x_i \sim x_j \iff$ there exists a $k \in I$ such that $f_{i,k}(x_i) = f_{j,k}(x_j)$. This means that two elements are equivalent if they "eventually become equal".

Definition 6.2. We define the *k-th homotopy group* of an orthogonal spectrum X as the direct limit

$$\pi_k (X) = \varinjlim (\pi_{n+k} (X_n))$$

indexed over $n = 0, 1, \ldots$, and using the composites

$$\pi_{n+k} (X_n) \xrightarrow{\quad -\wedge S^1 \quad} \pi_{n+k+1} \left(X_n \wedge S^1\right) \xrightarrow{(\sigma_n)_*} \pi_{n+k+1} (X_{n+1}) \; .$$

as the directed maps.

Example 6.3 (Homotopy groups of $\mathbb{S}$). Homotopy groups for spheres are notoriously hard to compute in general, so it would seem like that finding the homotopy groups of $\mathbb{S}$ is a futile task. But there are some results that are quite useful and make this task a lot easier. In [10] Serre showed that for $m > n \geq 1$

$$\pi_m (S^n) = \begin{cases} (\text{finite group}) \oplus \mathbb{Z} & m = 2n-1, n \text{ even}, \\ (\text{finite group}) & \text{otherwise}. \end{cases}$$

And *Freudenthal's Suspension Theorem* ([3], p. 360) implies that for $n \geq k+2$ the groups $\pi_{n+k} (S^n)$ only depend on k. These groups are the *stable homotopy groups of spheres*, $\pi_k (\mathbb{S})$. These two results combined, then imply that

$$\pi_n (\mathbb{S}) = \begin{cases} 0 & \text{if}\, n < 0, \\ \mathbb{Z} & \text{if}\, n = 0, \\ (\text{finite group}) & \text{otherwise}. \end{cases}$$

This result simplifies things, but as n varies the "pattern" of the homotopy groups is still elusive. For example, the first 9 groups are:

n	0	1	2	3	4	5	6	7	8
$\pi_n (\mathbb{S})$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$

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