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Commutativity and Bordism Spectra

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Abstract

The real bordism spectra for unoriented, oriented, spin, and string bordism each have an orientation map which is a map of E_∞ ring spectra. In the first three cases the orientations have sections, but these sections are not maps of E_∞ spectra. In this text the author uses Dyer-Lashof operations to place bounds on the existence of E_n sections of the orientation maps, as well as the existence of E_n sections of the topological Hochschild homology of the orientation maps.

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Chapter 1

Introduction

In [Tho54], René Thom showed that the the unoriented and oriented bordism rings $\mathcal{N}_*$ and Ω_* are isomorphic to the homotopy groups of the MO and MSO, the Thom spectra associated with the classifying spaces of the orthogonal and special orthogonal groups. Thom determined the structure of $\mathcal{N}_*$ by producing an equivalence of spectra between MO and a wedge sum of suspensions of the Eilenberg-Maclane spectrum $H\mathbb{F}_2$. Similarly, in [Wal60], C.T.C Wall determined the structure of Ω_* by demonstrating a 2-local equivalence between MSO and wedge sums of suspensions of $H\mathbb{F}_2$ and $H\mathbb{Z}$. Associated to the 4 and 8 connective covers of BO, there are the string and spin bordism spectra $MSpin$ and $MString$. Just as MO has an orientation map $MO \to H\mathbb{F}_2$ and MSO has an orientation $MSO \to H\mathbb{Z}$, there are canonical orientation maps $MSping \to ko$ and $MString \to tmf$. In, [ABP67], D.W. Anderson, E.H. Brown Jr., and F.P. Peterson demonstrated a splitting of the orientation $MSpin \to ko$ analogous to those produced by Thom and Wall, but no such splitting is currently known for $MString$.

The bordism spectra MO, MSO, $MSpin$, and $MString$ are not just spectra, however. They are E_∞ ring spectra, each having an action by an E_∞ operad defining a product which not only commutes up to homotopy, but for which all commuting homotopies are themselves homotopic, all homotopies between commuting homotopies are homotopic, and so on. The spectra $H\mathbb{F}_2$, $H\mathbb{Z}$, ko, and tmf are also E_∞ spectra, and the orientation maps $MO \to H\mathbb{F}_2$, $MSO \to H_{\mathbb{Z}}$, $MSpin \to ko$, and $MString \to tmf$ respect this E_∞ structure, but the sections of these maps do not. By a result of Mark Mahowald in [Mah77], there exists an E_2 section of $MO \to H\mathbb{F}_2$, but this is still a long way from the E_∞ section we might hope for. Thus the question arises: how commutative can a section of these orientation maps be?

One of the best ways to find obstructions to the existence of E_n structures, or to the existence of E_n maps, is through the use of Dyer-Lashof operations. These operations, first defined by Shôrô Araki and Tatsuji Kudo for $p = 2$ in [KA56], and extended to odd primes by Eldon Dyer and Richard K. Lashof in [DL62], are homology operations applying to the mod p homology of any E_n or E_∞ spectrum. Since the first n operations $Q_0, \ldots, Q_{n-1}$ are defined for, and natural with respect to maps of, E_n spectra, these may be used to place constraints on the existence of such structures.

In this text, we will study the mod 2 homology of the real bordism spectra MO, MSO, $MSpin$, and $MString$, together with their Dyer-Lashof operations. We will then use this information to prove the following:

1. The orientation $MO \to H\mathbb{F}_2$ does not admit an E_3 section.

2. The orientation $MSO \to H\mathbb{Z}$ does not admit an E_5 section.

3. The orientation $MSpin \to ko$ does not admit an E_9 section.

4. The orientation $MString \to tmf$ does not admit an E_17 section.

See Proposition 5.1.3 and *Proposition* 5.1.4.

In [BFV07], Morten Brun, Zbigniew Fiedorowicz, and Rainer M. Vogt showed that topological Hochschild homology defines a functor from E_{n+1} spectra to E_n spectra. Thus, if there exists an E_{n+1} section of, say, $MO \to H\mathbb{F}_2$, then there is an induced E_n section of $THH(MO) \to H\mathbb{F}_2$. It is possible, however, that there might exist sections of $THH(MO) \to THH(H\mathbb{F}_2)$ which do not arise in this way, and these sections might preserve more of the E_∞ structure. In order to place bounds on this, we will use the Bökstedt spectral sequence to determine the mod 2 homology of $THH(MO)$, $THH(MSO)$, $THH(MSpin)$, and $THH(MString)$, as well as the relative topological Hochshcild homology spectra $THH(MO, H\mathbb{F}_2)$, $THH(MSO, H\mathbb{Z})$, $THH(MSpin, ko)$, and $THH(MString, tmf)$. We will then use this information, together with Dyer-Lashof operations, to prove the following.

1. $THH(MO) \to THH(H\mathbb{F}_2)$ does not admit an E_3 section.

2. $THH(MSO) \to THH(H\mathbb{Z})$ does not admit an E_5 section.

3. $THH(MSpin) \to THH(ko)$ does not admit an E_9 section.

4. $THH(MString) \to THH(tmf)$ does not admit an E_{17} section.

See

The approximate structure of this text is as follows. In Chapter 2, we define the classifying spaces BO, BSO, $BSpin$, and $BString$, as well as their Thom spectra, and we describe the mod 2 homology of these in terms of the Husemoller-Witt decomposition of bipolynomial Hopf algebras. In Chapter 3 we define operads and E_n operads, then show how the linear isometries operad gives an E_∞ structure to the real bordism spectra. We also discuss Dyer-Lashof operations and Steenrod (co-)operations, and describe how these act on the homology of MO and $H\mathbb{F}_2$. In Chapter 4 we define topological Hochschild homology, and discuss how the tensor product of operads is used to make it a functor from E_{n+1} spectra to E_n spectra. In Chapter 5, we do a number of computations to establish our main original results,

1.1 Notation

Throughout this text, $\mathbb{F}_p$ denotes the field with p elements for p a prime, $\mathbb{Z}_{(p)}$ will denote the integers localized at the prime (p), and all homology and cohomology is taken with coefficients in $\mathbb{F}_2$ unless otherwise stated. In addition, all spaces are assumed to be compactly generated weak Hausdorff, and all spectra lie in a modern category of spectra with a symmetric monoidal smash product.

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Chapter 2

The Bordism Spectra

2.1 Grassmann Manifolds

Let us begin by defining the spectra BO and MO and describing their homology, following the treatment in Milnor and Stasheff [MS74].

Definition 2.1.1. [MS74, p. 56] Let $0 \leq n, m$. As a set, the Grassmann manifold $G_n(\mathbb{R}^{n+m})$ is defined to be the set of n-dimensional linear subspaces of $\mathbb{R}^{n+m}$ and the Stiefel manifold $V_n(\mathbb{R}^{n+m})$ is defined to be the set of n-frames of $\mathbb{R}^{n+m}$. To topologize these, consider $V_n(\mathbb{R}^{n+m})$ to be a subspace of $(\mathbb{R}^{n+m})^n$ and $G_n(\mathbb{R}^{n+m})$ to be a quotient space of $V_n(\mathbb{R}^m)$, where the quotient map sends each n-frame to the subspace that it spans.

The inclusion $\mathbb{R}^{n+m} \hookrightarrow \mathbb{R}^{n+m} \oplus \mathbb{R} \cong \mathbb{R}^{n+m+1}$ induces an inclusion $G_n(\mathbb{R}^{n+m}) \hookrightarrow G_n(\mathbb{R}^{n+m+1})$. The colimit of these inclusions is denoted $G_n(\mathbb{R}^\infty)$. Similarly, there is an inclusion $G_n(\mathbb{R}^{n+m}) \hookrightarrow G_{n+1}(\mathbb{R}^{n+1+m})$ given by sending an n-plane $\ell \subseteq \mathbb{R}^{n+m} \cong \mathbb{R}^n \oplus \mathbb{R}^m$ to the image of the composite $\ell \oplus \mathbb{R} \hookrightarrow \mathbb{R}^n \oplus \mathbb{R}^m \oplus \mathbb{R} \to \mathbb{R}^n \oplus \mathbb{R} \oplus \mathbb{R}^m \cong \mathbb{R}^{n+1+m}$. These then induce inclusions $G_n(\mathbb{R}^\infty) \hookrightarrow G_{n+1}(\mathbb{R}^\infty)$, and the colimit of these is denoted $G_\infty(\mathbb{R}^\infty)$. The spaces $G_n(\mathbb{R}^\infty)$ turn out to be the classifying spaces of the orthogonal groups $O(n)$, and are thus often denoted $BO(n)$, or simply BO in the case $n = \infty$.

The spaces $G_n(\mathbb{R}^{n+m})$ have canonical n-dimensional vector bundles, often called tautological vector bundes, whose total spaces are $E_n(\mathbb{R}^{n+m}) = \{(v, \ell) \in \mathbb{R}^{n+m} \times G_n(\mathbb{R}^{n+m}) \mid v \in \ell\}$. The map $\gamma^n_m : E_n(\mathbb{R}^{n+m}) \to G_n(\mathbb{R}^{n+m})$ is given by projection onto the second factor, and each fiber of this map inherits a vector space structure as a subspace of $\mathbb{R}^{n+m}$. As before, there are inclusions $E_n(\mathbb{R}^{n+m}) \hookrightarrow E_n(\mathbb{R}^{n+m+1})$, and the colimit of these is denoted $E_n(\mathbb{R}^\infty)$ or E_n. One may check that the projections $E_n(\mathbb{R}^{n+m}) \to G_n(\mathbb{R}^{n+m})$ give rise to a map $\gamma^n : E_n \to BO(n)$, and that γ^n inherits the structure of a vector bundle, called the universal bundle [MS74, p. 60].

Now let DE_n and SE_n denote the disk and sphere bundles of γ^n, i.e. the space of vectors v of norm $|v| \leq 1$ and $|v| = 1$, respectively. The Thom space of γ^n is then defined to be $Th(\gamma^n) = DE_n/SE_n$. Now let ϵ_1 denote the trivial line bundle over $BO(n)$ and note that the pullback of γ^{n+1} under the inclusion $BO(n) \hookrightarrow BO(n+1)$ is isomorphic to $\epsilon_1 \oplus \gamma^n$. Thus there is an inclusion $\mathbb{R} \times E_n \hookrightarrow E_{n+1}$, and this induces, after a suitable rescaling, a map $\Sigma Th(\gamma^n) \to Th(\gamma^{n+1})$. The Thom spectrum MO may now be defined by letting $MO(n) = Th(\gamma^n)$, and letting the structure maps $\Sigma MO(n) \to MO(n+1)$ be the maps just defined.

2.2 Cohomology of BO(n) and the Thom Isomorphism

The cohomology of BO is best understood in terms of certain elements called Stiefel-Whitney classes.

Definition-Proposition 2.2.1. [MS74, p. 37-38, Chapter 8] To each vector bundle $\xi : E(\xi) \to B(\xi)$ over a paracompact base space $B(\xi)$ there is associated a sequence of cohomology classes $w_i(\xi) \in H^i(B(\xi); \mathbb{F}_2)$, $i \geq 0$, called Stiefel-Whitney classes. These satisfy and are uniquely characterised by the following axioms.

1) We have $w_0(\xi) = 1 \in H^0(B(\xi); \mathbb{F}_2)$. If ξ is an n-plane bundle, then $w_i(\xi) = 0$ for $i > n$.

2) (Naturality) If a map $f : B(\xi) \to B(\eta)$ is covered by a bundle map between vector bundles ξ and η, then $f^*(w_i(\eta)) = w_i(\xi)$ for each i.

3) (Whitney Product Theorem) If ξ and η are vector bundles over the same base space B, then $w_i(\xi \oplus \eta) = \sum_{j+k=i} w_j(\xi) \cup w_k(\eta)$.

4) For the canonical line bundle γ_1^1 over $\mathbb{R}P^1 = G_1(\mathbb{R}^2)$, the class $w_1(\gamma_1^1)$ is nonzero.

We will also need the following basic result.

Corollary 2.2.2. *If ϵ is a trivial vector bundle over a base space B, and η is any vector bundle over B, then $w_i(\epsilon \oplus \eta) = w_i(\eta)$.*

The mod 2 cohomology of $BO(n)$ may now be described as a polynomial algebra in the Stiefel-Whitney classes associated to the universal bundles γ^n.

Proposition 2.2.3. [MS74, Theorem 7.1]

$$H^*(BO(n)) = \mathbb{F}_2[w_i \mid 1 \leq i \leq n].$$

Let $i_n : BO(n) \hookrightarrow BO(n+1)$ be the inclusion, and note that $i_n^*(\gamma^{n+1}) = \epsilon_1 \oplus \gamma^n$, where ϵ_1 is the trivial line bundle. Thus $i_n^* : H^*(BO(n+1)) \to H^*(BO(n))$ is the quotient map sending w_{n+1} to 0. By the description in [MS74, Chapter 6], $BO(n) \hookrightarrow BO(n+1)$ may be realized as an inclusion of subcomplexes, so that $H^*(\mathrm{colim}_{n\to\infty} BO(n)) \cong \lim_{n\to\infty} H^*(BO(n)) / \lim^1_{n\to\infty} H^{*-1}(BO(n))$. Since $H^*(BO(n+1)) \to H^*(BO(n))$ is surjective, $\lim^1_{n\to\infty} H^{*-1}(BO(n)) = 0$, so that $H^*(BO) = \mathbb{F}_2[w_i \mid i \geq 1]$.

The cohomology of BO may be related to that of MO via the so-called Thom isomorphism theorem.

Theorem 2.2.4. [MS74, Theorem 10.2] *Let $\xi \colon E \to B$ be an n-plane bundle. Let E_0 denote the space of nonzero vectors in E, and, for any fiber F, let F_0 denote the space of nonzero vectors in F. Then there exists a unique class $u \in H^n(E, E_0)$ such that the restriction of u to $H^n(F, F_0) \cong \mathbb{F}_2$ is nonzero for every fiber F. The map $- \cup u \colon H^*(E) \to H^{*+n}(E, E_0)$ is an $H^*(E)$-module isomorphism.*

Since $H^*(E_n, (E_n)_0) \cong H^*(DE_n, SE_n) \cong \tilde{H}^*(MO(n))$ and $H^*(BO(n)) \cong H^*(E_n)$, the Thom isomorphism theorem gives isomorphisms $H^*(BO(n)) \cong \tilde{H}^{*+n}(MO(n))$. Further, it is not difficult to check that the map $\tilde{H}^{*+1}(MO(n+1)) \to \tilde{H}^{*+1}(\Sigma MO(n)) \cong \tilde{H}^*(MO(n))$ sends u to u, so that we get an induced isomorphism $H^*(BO) \cong H^*(MO)$.

2.3 Products and Homology

The Whitney sum of vector bundles induces a product $\mu_{m,n}\colon BO(m) \times BO(n) \to BO(m+n)$ which is covered by a bundle map $\gamma^m \times \gamma^n \to \gamma^{m+n}$. This then induces a homomorphism $\mu^*_{m,n}\colon H^*(BO(m+n)) \to H^*(BO(m)) \otimes H^*(BO(n))$. Since $\mu_{m,n}$ is covered by a bundle map, $\mu^*_{m,n}(w_i) = \sum w_j \otimes w_k$, where the sum is taken over all j, k with $0 \le j \le m$, $0 \le k \le n$, and $j + k = i$.

In Section 3.2 we will see that the Whitney sums induce a product $\mu\colon BO \times BO \to BO$ which is associative, commutative, and unital up to homotopy. The homomorphism $\mu^*\colon H^*(BO) \to H^*(BO) \otimes H^*(BO)$ is then given by $\mu^*(w_i) = \sum_{j+k=i} w_j \otimes w_k$. In the dual case of homology, μ gives $H_*(BO)$ the structure of an $\mathbb{F}_2$ algebra via the Pontryagin product, and the diagonal $\Delta\colon BO \to BO \times BO$ induces a homomorphism $\Delta_*\colon H_*(BO) \to H_*(BO) \otimes H_*(BO)$ dual to the cup product. We then have the following.

Proposition 2.3.1. *[May12, Theorem 21.4.5] Let $b_i \in H_*(BO)$ be the image of the unique nonzero class in $H_i(BO(1)) = H_i(\mathbb{R}P^\infty)$ under the inclusion $BO(1) \hookrightarrow BO$. Then $H_*(BO) = \mathbb{F}_2[b_i \mid i \ge 1]$, and $\Delta^*(b_i) = \sum_{j+k=i} b_j \otimes b_k$.*

Note that since $H*(BO(1)) = \mathbb{F}_2[w_1]$, b_i is dual in the monomial basis (of $H_*(BO)$)) to w_1^i.

In the case of MO, the Whitney sum may be used to define a product $MO \wedge MO \to MO$ such that the Thom isomorphism $H_*(BO) \cong H_*(MO)$ is an isomorphism of $\mathbb{F}_2$-algebras. Under this identification, we will also write $H_*(MO) = \mathbb{F}_2[b_i \mid i \ge 1]$.

2.4 Hopf Algebras

The product on BO induces a product in its homology, and together with the map $H_*(BO) \to H_*(BO) \otimes H_*(BO)$ which is dual to the cup product it makes $H_*(BO)$ into a kind of object known as a Hopf algebra. Thus to understand its structure, it is best to first consider Hopf algebras more generally.

We begin with some definitions, mostly following the treatment of Milnor and Moore in [MM65]. For the rest of this section R is a commutative ring and all tensor products are taken over R. Given graded R-modules A and B, the twisting isomorphism $\tau : A \otimes B \to B \otimes A$ is defined by $\tau(a \otimes b) = b \otimes a$. Note that in most topological applications, τ includes a sign $(-1)^{|a||b|}$, but in our case it will be convenient to work with an unsigned twist. Note also that the signed and unsigned twists are equal in the cases where $R = \mathbb{F}_2$ or A and B have no odd degree elements.

Definition 2.4.1. [MM65, Definition 2.1] A coalgebra over R consists of a non-negatively graded R-module A together with of graded R-module homomorphisms

$$\Delta : A \to A \otimes A$$
$$\epsilon : A \to R$$

such that the following diagrams commute.

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta\otimes \mathrm{id}} \\ A \otimes A & \xrightarrow{\mathrm{id}\otimes\Delta} & A \otimes A \otimes A \end{array}$$

$$
\begin{array}{ccccc}
 & & A & & \\
 & \nearrow^{\cong} & \big\downarrow{\scriptstyle \Delta} & \nwarrow^{\cong} & \\
A \otimes R & \xleftarrow[\mathrm{id}\otimes\epsilon]{} & A \otimes A & \xrightarrow[\epsilon\otimes\mathrm{id}]{} & R \otimes A
\end{array}
$$

Then Δ is called a comultiplication and ϵ is called a counit for Δ.

In addition to coalgebras, we will also need to be able to speak of comodules.

Definition 2.4.2. [MM65, Definition 2.2] Let (A, Δ, ϵ) be a coalgebra over R. A left A-comodule consists of a non-negatively graded R-module M together with a graded R-module homomorphism $\psi : M \to A \otimes M$ such that the following diagrams commute.

$$
\begin{array}{ccc}
M & \xrightarrow{\psi} & A \otimes M \\
\big\downarrow{\scriptstyle \psi} & & \big\downarrow{\scriptstyle \Delta\otimes\mathrm{id}} \\
A \otimes M & \xrightarrow{\mathrm{id}\otimes\psi} & A \otimes A \otimes M
\end{array}
$$

$$
\begin{array}{ccc}
M & & \\
\big\downarrow{\scriptstyle \psi} & \nwarrow^{\cong} & \\
A \otimes M & \xrightarrow[\epsilon\otimes\mathrm{id}]{} & R \otimes M
\end{array}
$$

Just as the isomorphism $R \otimes R \cong R$ can be used to consider R to be an R-algebra, one may also consider R to be an R-coalgebra. Thus we may consider augmentations of both algebras and coalgebras.

Definition 2.4.3. [MM65] Let (A, μ, η) be an R-algebra. An augmentation of A is an algebra homomorphism $\epsilon : A \to R$. Let (A, Δ, ϵ) be an R-coalgebra. A coaugmentation of A is a coalgebra homomorphism $\eta : R \to A$.

We now have what we need to define a bialgebra.

Definition 2.4.4. [MM65, Definition 4.1] A bialgebra over R consists of a non-negatively graded R-module A together with graded R-module homomorphisms

$$
\begin{aligned}
&\mu : A \otimes A \to A \\
&\eta : R \to A \\
&\Delta : A \to A \otimes A \\
&\epsilon : A \to R
\end{aligned}
$$

such that

1. The triple (A, μ, η) forms an R-algebra with augmentation ϵ.

2. The triple (A, Δ, ϵ) forms an R-coalgebra with coaugmentation η.

3. The following diagram commutes.

$$
\begin{array}{ccccc}
A \otimes A & \xrightarrow{\mu} & A & \xrightarrow{\Delta} & A \otimes A \\
\big\downarrow{\scriptstyle \Delta\otimes\Delta} & & & & \big\uparrow{\scriptstyle \mu\otimes\mu} \\
A \otimes A \otimes A \otimes A & & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & & A \otimes A \otimes A \otimes A
\end{array}
$$

Just as the tensor product of two algebras can be given a product and unit making it an algebra, the tensor product of two coalgebras can be given a coproduct and counit making it a coalgebra. Combining these, we then get a bialgebra structure on the tensor product of bialgebras. Concretely, if $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ are coalgebras, the coproduct on $A \otimes B$ is given by

$$A \otimes B \xrightarrow{\Delta_A \otimes \Delta_B} A \otimes A \otimes B \otimes B \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} A \otimes B \otimes A \otimes B$$

and the counit is given by

$$A \otimes B \xrightarrow{\epsilon_A \otimes \epsilon_B} R \otimes R \xrightarrow{\cong} R.$$

With this convention, point (3) in Definition 2.4.4 is equivalent to either Δ being an algebra homomorphism or μ being a coalgebra homomorphism. (Unitality follows from η and ϵ being (co)augmentations.)

A Hopf algebra is a bialgebra together with a certain conjugation endomorphism.

Definition 2.4.5. Let $(A, \mu, \eta, \Delta, \epsilon)$ be an R-bialgebra. A conjugation on A is an R-module homomorphism $c \colon A \to A$ such that

$$\mu(\mathrm{id} \otimes c)\Delta = \mu(c \otimes \mathrm{id})\Delta = \eta\epsilon.$$

A bialgebra together with a conjugation is a Hopf algebra.

For many bialgebras, this extra structure is in fact automatic.

Proposition 2.4.6. [MM65, Proposition 8.2] *Let $(A, \mu, \eta, \Delta, \epsilon)$ be an R-bialgebra. If A is connected, i.e., if $\eta \colon R \to A_0$ and $\epsilon \colon A_0 \to R$ are inverse isomorphisms, then there exists a unique conjugation $c \colon A \to A$.*

Proposition 2.4.7. [MM65, Propositions 8.6, 8.7, 8.8] *Let $(A, \mu, \eta, \Delta, \epsilon, c)$ be a connected Hopf algebra. Then the following diagrams commute.*

$$\begin{array}{ccc} A \otimes A & \xrightarrow{c \otimes c} & A \otimes A \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \tau} \\ A & & A \otimes A \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \mu} \\ A & = & A \end{array} \qquad \begin{array}{ccc} A & = & A \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle c} \\ A \otimes A & & A \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \Delta} \\ A \otimes A & \xrightarrow{c \otimes c} & A \otimes A \end{array}$$

Thus c is an antiautomorphism. In addition, if (A, μ, η) is a commutative algebra or (A, Δ, ϵ) is a cocommutative coalgebra, then $c^2 = id$.

One of the most important properties of Hopf algebras is that their duals also inherit Hopf algebra structures, allowing one to work equally well with either A or A^*, or in our case, with homology or cohomology.

Proposition 2.4.8. [MM65, Proposition 4.8] *Let $(A, \mu, \eta, \Delta, \epsilon, c)$ be a Hopf algebra with each A_n projective and finitely generated. Then $(A^*, \Delta^*, \epsilon^*, \mu^*, \eta^*, c^*)$ has the structure of a Hopf algebra.*

In order to understand the structure of Hopf algebras, it is often useful to consider the the quotient module of indecomposables and its dual notion, the submodule of primitives.

Definition 2.4.9. [MM65, Definition 3.7] Let (A,μ,η) be an R-algebra with augmentation $\epsilon : A \to R$. The augmentation ideal of A is the defined to be $I(A) = \ker(\epsilon)$. The module of indecomposables of A is $Q(A) = \operatorname{coker}(I(A) \otimes I(A)) \xrightarrow{\mu} I(A))$.

Let (A,Δ,ϵ) be an R-coalgebra with coaugmentation $\eta : R \to A$. Define $J(A) = \operatorname{coker}(\eta)$. The module of primitive elements of A is $P(A) = \ker(J(A) \xrightarrow{\Delta} J(A) \otimes J(A))$.

Note that a Hopf algebra homomorphism $A \to B$ induces maps $Q(A) \to Q(B)$ and $P(A) \to P(B)$, so that Q and P are functors.

In addition to BO, we will also consider certain n-connective covers of BO, that is, spaces $BO\langle n\rangle$ together with covering maps $BO\langle n\rangle \to BO$ inducing isomorphisms $\pi_i(BO\langle n\rangle) \cong \pi_i(BO)$ for $i \geq n$ and with $\pi_i(BO\langle n\rangle) = 0$ for $i < n$. For $n = 2, 4, 8$, the homology of $BO\langle n\rangle$ is best understood as a sub-Hopf algebra of $H_*(BO)$, and for this purpose it is helpful to view $H_*(BO)$ as a certain tensor product of Hopf algebras known as the Husemoller-Witt decomposition, so let us define this.

Definition 2.4.10. Let p be prime, let R be a $\mathbb{Z}_{(p)}$-algebra and let $d \geq 1$. Define a bipolynomial Hopf algebra $B[d] = B^R[d]$ by letting $B[d] = R[b_i \mid i \geq 1]$ as an algebra, with $|b_i| = di$. Define the coproduct by $\Delta(b_i) = \sum_{j+k=i} b_j \otimes b_k$, with the convention that $b_0 = 1$, and define a counit via $\epsilon : B[d] \to B[d]_0 \cong R$.

Clearly the indecomposable elements of $B[d]$ are given by $QB[d] = R\{b_i \mid i \geq 1\}$. For the primitive elements, the standard basis for $PB[d]$ is given by the following.

Proposition 2.4.11. [Hus71, Proposition 4.2] *For $n \geq 1$, define elements q_n inductively by the Newton relations,*

$$q_n = \sum_{i=1}^{n-1} (-1)^{i+1} b_i q_{n-i} + (-1)^{n+1} n b_n.$$

Then $PB[d] = R\{q_i \mid i \geq 1\}$.

Husemoller constructed sub-Hopf algebras $B_{(p)}[d] \subset B[d]$ as the kernel of a certain homomorphism $B[d] \to \bigotimes_{p\nmid \ell, \ell \text{prime}} B[\ell d]$, see [Hus71, Notation 6.3]. Using a Hopf algebra homomorphism $f_r \colon B[rd] \to B[d]$ satisfying $f_r(q_i) = q_{ri}$, one may also consider $B_{(p)}[rd]$ as sub-Hopf algebras of $B[d]$ for $r \geq 1$ [Hus71, Proposition 5.1]. He then shows that $B_{(p)}[r]$ is a bipolynomial Hopf algebra generated by certain elements $a_{r,j}$, and for our purposes we may consider this to be the definition of $B_{(p)}[rd]$

Definition-Proposition 2.4.12. [Hus71, Propositions 8.2, 8.3] Let $r \geq 1$. Then there exist elements $a_{r,j} \in B[d]$ for $j \geq 0$ with $|a_{r,j}| = rp^j$ that are uniquely defined by $q_{rp^j} = \sum_{i=0}^{j} p^i a_{r,i}^{p^{j-i}}$. Let $B_{(p)}[rd]$ denote $R[a_{r,j} \mid j \geq 0]$. Then $B_{(p)}[rd]$ is a sub-Hopf algebra of $B[d]$.

Note that although the formula $q_{rp^j} = \sum_{i=0}^{j} p^i a_{r,i}^{p^{j-i}}$ is insufficient to define the $a_{r,j}$ over any ring with p-torsion, it does define the $a_{r,j}$ over $\mathbb{Z}_{(p)}$, and taking a tensor product with R gives unique elements in each $B^R[d]$.

The Husemoller Witt decomposition is then given by the following.

Proposition 2.4.13. [Hus71, Theorem 6.5] *The inclusions $B[kd] \hookrightarrow B[d]$ for k coprime to p induce an isomorphism of Hopf algebras*

$$\bigotimes_{k\geq 1, p\nmid k} B_{(p)}[kd] \cong B[d].$$

Table 2.1: The primitive elements q_i and the generators $a_{k,j}$ in $B^{\mathbb{Z}_{(2)}}[1]$ in degrees 0 through 6.

$$
\begin{aligned}
q_1 &= b_1 \\
q_2 &= b_1^2 - 2b_2 \\
q_3 &= b_1^3 - 3b_1b_2 + 3b_3 \\
q_4 &= b_1^4 - 4b_1^2b_2 + 4b_1b_3 + 2b_2^2 - 4b_4 \\
q_5 &= b_1^5 - 5b_1^3b_2 + b_1^2b_3 + 5b_1b_2^2 - 5b_1b_4 - 5b_2b_3 + 5b_5 \\
q_6 &= b_1^6 - 6b_1^4b_2 + 6b_1^3b_3 + 9b_1^2b_2^2 - 6b_1^2b_4 - 12b_1b_2b_3 + 6b_1b_5 - 2b_2^3 + 6b_2b_4 + 3b_3^2 - 6b_6
\end{aligned}
$$

$$
\begin{aligned}
a_{1,0} &= b_1 \\
a_{1,1} &= -b_2 \\
a_{1,2} &= -b_1^2b_2 + b_1b_3 - b_4 \\
a_{3,0} &= b_1^3 - 3b_1b_2 + 3b_3 \\
a_{3,1} &= -b_2^3 + 3b_1b_2b_3 - 3b_1^2b_4 - 3b_3^2 + 3b_2b_4 + 3b_1b_5 - 3b_6 \\
a_{5,0} &= b_1^5 - 5b_1^3b_2 + 5b_1b_2^2 + 5b_1^2b_3 - 5b_2b_3 - 5b_1b_4 + 5b_5
\end{aligned}
$$

Thus, in particular, $B[d] = R[a_{k,j} \mid j \geq 0, k \geq 1, p \nmid k]$ as an algebra. Note also that if R is an $\mathbb{F}_p$ algebra, then $q_{kp^j} = a_{k,0}^{p^j} + \ldots + p^j a_{k,j} = a_{k,0}^{p^j}$, which gives a much simpler description of the primitives q_i than the generators b_i allow.

2.5 Connected Covers of BO

We are now ready to consider the homology of BO and its covers in this new context, using $p = 2$ and $R = \mathbb{F}_2$. To begin with, one sees that $H_*(BO; \mathbb{F}_2) \cong B^{\mathbb{F}_2}[1]$, and that we may thus use this new basis to write $H_*(BO) = \mathbb{F}_2[a_{k,j} \mid j \geq 0, k \geq 1, 2 \nmid k]$ as a Hopf algebra. ($H^*(BO)$ is also isomorphic to $B^{\mathbb{F}_2}[1]$, and Proposition 2.3.1 is a consequence of the more general fact that the Hopf algebras $B^R[d]$ are self dual.)

In addition to BO, we will consider the connected covers $BSO = BO\langle 2\rangle$, $BSpin = BO\langle 4\rangle$ and $BString = BO\langle 8\rangle$. The cohomology of these may be described in terms of the action by the Steenrod algebra, discussed in Section 3.4.

Proposition 2.5.1. *[Koc83, Corollary 2.6] Let* $1 \leq n \leq 3$. *Then the ideal* $(\mathcal{A}w_2^{n-1}) \subseteq H^*(BO\langle 2^{n-1}\rangle)$ *is a Hopf ideal, where* w_i *denotes the i'th Stiefel-Whitney class and* $\mathcal{A}$ *denotes the Steenrod algebra, and the covering map* $BO\langle 2^n\rangle \to BO\langle 2^{n-1}\rangle$ *induces an isomorphism of Hopf algebras*

$$H^*(BO\langle 2^{n-1}\rangle)/(\mathcal{A}w_{2^{n-1}}) \cong H^*(BO\langle 2^n\rangle).$$

Note that this pattern does not continue. In fact, by [Koc83, Theorem 2.9], there is no space X with a map $X \to BO\langle 8\rangle$ identifying $H_*(X)$ with $H_*(BO\langle 8\rangle)/(\mathcal{A}w_8)$. For $1 \leq n \leq 3$, however, we may view $H^*(BO\langle 2^n\rangle)$ as quotient Hopf algebras of $H^*(BO)$. Dualising to homology, we instead get a sequence of sub-Hopf algebras. To describe these, we must first define some functions $\alpha, \rho : \mathbb{N} \to \mathbb{N}$.

Definition 2.5.2. Given $k \geq 0$, write $k = \sum_{i=0}^{m} r_i 2^i$ for $m \geq 0$ and $0 \leq r_i \leq 1$. The bit sum of k is then $\alpha(k) = \sum_{i=0}^{m} r_i$. Given $n \geq 0$, define $\rho_n(k) = \max(n + 1 - \alpha(k), 0)$.

The homology of $BO\langle 2^n \rangle$ is then given by the following.

Proposition 2.5.3. [Bak85, Theorem 1.13] *Let $0 \leq n \leq 3$. Then the map $BO\langle 2^n \rangle \to BO$ induces an isomorphism of Hopf algebras*

$$H_*(BO\langle 2^n \rangle) \cong \bigotimes_{\substack{k \geq 0 \\ 2 \nmid k}} \mathbb{F}_2[a_{k,j}^{2^{\rho_n(k)}} \mid j \geq 0].$$

In particular, this gives the following descriptions of $H_*(BSO)$, $H_*(BSpin)$, and $H_*(BString)$:

$$\begin{aligned}
H_*(BSO) &= \mathbb{F}_2[a_{k,j} \mid 2 \nmid k, k \geq 3, j \geq 0] \otimes \mathbb{F}_2[a_{1,j}^2 \mid j \geq 0] \\
H_*(BSpin) &= \mathbb{F}_2[a_{k,j} \mid 2 \nmid k, k \geq 7, (\nexists m)(k = 2^m + 1), j \geq 0] \\
&\quad \otimes \mathbb{F}_2[a_{2^m+1,j}^2 \mid m \geq 1, j \geq 0] \\
&\quad \otimes \mathbb{F}_2[a_{1,j}^4 \mid j \geq 0] \\
H_*(BString) &= \mathbb{F}_2[a_{k,j} \mid j \geq 0, k \geq 15, \alpha(k) \geq 4] \\
&\quad \otimes \mathbb{F}_2[a_{k,j}^2 \mid j \geq 0, k \geq 7, \alpha(k) = 3] \\
&\quad \otimes \mathbb{F}_2[a_{k,j}^4 \mid j \geq 0, k \geq 3, \alpha(k) = 2] \\
&\quad \otimes \mathbb{F}_2[a_{1,j}^8 \mid j \geq 0]
\end{aligned}$$

Note that rather then considering connected covers of BO directly, one may instead take connected covers of the spaces $BO(n)$ and take a similar colimit. This gives another way of constructing the covers of BO, and from this we may define Thom spectra $MO\langle 2^n \rangle$. Since the Thom isomorphism theorem still applies, we see that $H_*(MO\langle 2^n \rangle) \cong H_*(BO\langle 2^n \rangle)$ just as in the BO case, and we will use this identification to also write $H_*(MO\langle 2^n \rangle) = \mathbb{F}_2[a_{k,j}^{2^{\rho_n(k)}} \mid j \geq 0, k \geq 1, 2 \nmid k]$.

Note 2.5.4. Just as $H_*(BO) \cong B^{\mathbb{F}_2}[1]$, one may show that $H_*(BU) \cong B^{\mathbb{F}_2}[2]$. Many of the results cited throughout this text are written about the homology of BU, not BO. However, there is a non-grade preserving isomorphism of Hopf algebras $B[2] \to B[1]$, and this isomorphism can, in many cases, be used to convert results about BU to results about BO.

Chapter 3

Operads and Operations

3.1 Operads

There is a product $\phi : BO \times BO \to BO$ representing the Whitney sum of vector bundles, and this product is not commutative. It does, however, have a commuting homotopy $H : \phi \simeq \phi\tau$, where $\tau \colon BO \times BO \to BO \times BO$ is the map $(a, b) \mapsto (b, a)$. This is enough to ensure, for instance, that $H_*(BO)$ becomes a commutative ring, but it is not all the information that can be gleaned. We could, for instance, construct a second homotopy $\tilde{H} \colon \phi \simeq \phi\tau$ by precomposing H by $\tau \times f$, where $f \colon I \to I$ is given by $t \mapsto 1 - t$. This new homotopy will in general not be the same as H, but we may ask if there is a homotopy G between them. If such a G does exist, then we may of course construct a $\tilde{G}$ and so on and so forth. In order to keep track of these kinds of commuting homotopies, we will need some additional machinery, and the concept of an operad provides this.

Definition 3.1.1. [May72, Definition 1.1] [Man22, Definition 2.1] An operad $\mathcal{O}$ in the category of topological spaces consists of a sequence of spaces $\mathcal{O}(n)$ for $n \geq 0$ together with composition maps $\Gamma : \mathcal{O}(n) \times \prod_{i=1}^{n} \mathcal{O}(j_i) \to \mathcal{O}(\sum_{i=1}^{n} j_i)$ for each n and $j_1, \dots, j_n$, a unit map $\mathbb{1} : * \to \mathcal{O}(1)$, and a right action of the symmetric group Σ_n on $\mathcal{O}(n)$ for each n such that the following axioms are satisfied:

1. (Associativity) The following diagram commutes for all m, $j_1, \dots, j_m$, and $k_1, \dots, k_j$, where $j = \sum_{i=1}^{m} j_i$, $k = \sum_{i=1}^{j} k_i$, $t_i = \sum_{\ell=1}^{i-1} j_\ell$, and $s_i = \sum_{\ell=t_i+1}^{t_i+j_i} k_\ell$.

$$\begin{array}{ccc}
\left(\mathcal{O}(m) \times \prod_{i=1}^{m} \mathcal{O}(j_i)\right) \times \prod_{i=1}^{j} \mathcal{O}(k_i) & \longrightarrow & \mathcal{O}(m) \times \prod_{i=1}^{m} \left(\mathcal{O}(j_i) \times \prod_{\ell=t_i+1}^{t_i+j_i} \mathcal{O}(k_\ell)\right) \\
\downarrow{\scriptstyle \Gamma \times \mathrm{id} \times \ldots \times \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id} \times \Gamma \times \ldots \times \Gamma} \\
\mathcal{O}(j) \times \prod_{i=1}^{j} \mathcal{O}(k_i) & & \mathcal{O}(m) \times \prod_{i=1}^{m} \mathcal{O}(s_i) \\
\downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma} \\
\mathcal{O}(k) & = & \mathcal{O}(k)
\end{array}$$

2. (Unitality) The following diagrams commute for all m.

$$* \times \mathcal{O}(m) \xrightarrow{\mathbb{1} \times \mathrm{id}} \mathcal{O}(1) \times \mathcal{O}(m) \xrightarrow{\Gamma} \mathcal{O}(m), \qquad * \times \mathcal{O}(m) \xrightarrow{\cong} \mathcal{O}(m)$$

$$\begin{tikzcd} \mathcal{O}(m) \times *^m \arrow[r, "\mathrm{id}\times\mathbb{1}\times\ldots\times\mathbb{1}"] \arrow[rr, bend right, "\cong"'] & \mathcal{O}(m)\otimes\mathcal{O}(1)^m \arrow[r, "\Gamma"] & \mathcal{O}(m) \end{tikzcd}$$

3. The composition map $\Gamma\colon \mathcal{O}(m) \times \prod_{i=1}^{m} \mathcal{O}(j_i) \to \mathcal{O}(j)$ is $(\Sigma_{j_1} \times \ldots \times \Sigma_{j_m})$-equivariant for each $m, j_1, \ldots, j_m$ and $j = \sum_{i=1}^{m} j_i$, where $(\Sigma_{j_1} \times \ldots \times \Sigma_{j_m})$ acts on $\mathcal{O}(m) \times \prod_{i=1}^{m} \mathcal{O}(j_i)$ via its action on $\prod_{i=1}^{m} \mathcal{O}(j_i)$ and acts on $\mathcal{O}(j)$ via the block sum inclusion $(\Sigma_{j_1} \times \ldots \times \Sigma_{j_m}) \hookrightarrow \Sigma_j$.

4. The following diagram commutes for all $m, j_1, \ldots, j_m$, and $\sigma \in \Sigma_m$, where $j = \sum_{i=1}^{m} j_i$, c_σ permutes the $\mathcal{O}(j_i)$ factors according to σ, and $\sigma_{j_1,\ldots j_m} \in \Sigma_j$ permutes blocks of size $j_1, \ldots, j_m$ by σ.

$$\begin{tikzcd} \mathcal{O}(m) \times \mathcal{O}(j_1) \times \ldots \times \mathcal{O}(j_m) \arrow[r, "\sigma\times\mathrm{id}\times\ldots\times\mathrm{id}"] \arrow[dd, "\Gamma"] & \mathcal{O}(m) \times \mathcal{O}(j_1) \times \ldots \times \mathcal{O}(j_m) \arrow[d, "\mathrm{id}\times c_\sigma"] \\ & \mathcal{O}(m) \times \mathcal{O}(j_{\sigma^{-1}(1)}) \times \ldots \times \mathcal{O}(j_{\sigma^{-1}(m)}) \arrow[d, "\Gamma"] \\ \mathcal{O}(j) \arrow[r, "\sigma_{j_1,\ldots,j_m}"] & \mathcal{O}(j) \end{tikzcd}$$

The archetypal example of an operad is the endomorphism operad $\mathcal{E}_X$. Given a space X, the endomorphism operad is defined by $\mathcal{E}_X(n) = \mathrm{Map}(X^n, X)$. The composition maps Γ are then defined by composition of maps, the identity $\mathbb{1}$ is the inclusion of the identity on X, and the symmetric group Σ_n acts on $\mathcal{E}_X$ via the left action on X^n, i.e., by permuting arguments. With this example in mind, the axioms of the previous definition simply express that:

1. Composition of functions is associative.

2. $f \circ \mathrm{id} = f = \mathrm{id} \circ f$ for any f.

3. When composing operations as in $f = g \circ (h_1 \times \ldots \times h_m)$, permuting the arguments of each h_i is equivalent to permuting the arguments of f in blocks.

4. When composing functions as in $f = g \circ (h_1 \times \ldots \times h_m)$, permuting the h_i and the arguments of g is equivalent to permuting blocks of the arguments of f.

In addition to the composition and identity maps, the endomorphism operad comes with action maps $\xi_m\colon \mathcal{E}_X(m) \times X^m \to X$ defined by $\xi_m(f, x_1, \ldots, x_m) = f(x_1, \ldots, x_m)$. Given a morphism of operads $\mathcal{O} \to \mathcal{E}_X$, i.e., a sequence of Σ_n-equivariant maps $O(n) \to \mathcal{E}_X(n)$ commuting with composition and identity maps, we get induced action maps $\mathcal{O}(m) \times X^m \to X$. In this case we say that X is an $\mathcal{O}$-algebra or an $\mathcal{O}$-space. Alternatively, we may define $\mathcal{O}$-spaces without making reference to $\mathcal{E}_X$ as follows.

Definition 3.1.2. [Man22, Definition 4.1] Let $\mathcal{O}$ be an operad in the category of topological spaces. An $\mathcal{O}$-space consists of a space X together with action maps $\xi_m\colon \mathcal{O}(m) \times X^m \to X$ for $m \geq 0$ such that the following axioms are satisfied.

1. For each $m \geq 0$, the map $\xi_m \colon \mathcal{O}(m) \times X^m \to X$ is Σ_m equivariant, where Σ_m acts diagonally on $\mathcal{O}(m) \times X^m$ via the right action on $\mathcal{O}(m)$ and the right action by inverses on X^m (i.e. the action given by $(x_1, \ldots, x_m)\sigma = \sigma^{-1}(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$), and Σ_m acts trivially on X.

2. The following diagram commutes for each $m, j_1, \ldots j_m$, where $j = \sum_{i=1}^{m} j_i$.

$$\begin{array}{ccc}
(\mathcal{O}(m) \times \prod_{i=1}^{m} \mathcal{O}(j_i)) \times X^j & \longrightarrow & \mathcal{O}(m) \times \prod_{i=1}^{m} (\mathcal{O}(j_i) \times X^{j_i}) \\
\downarrow{\scriptstyle \Gamma \times \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id} \times \xi_{j_1} \times \ldots \times \xi_{j_m}} \\
\mathcal{O}(j) \times X^j & & \mathcal{O}(m) \times X^m \\
\downarrow{\scriptstyle \xi_j} & & \downarrow{\scriptstyle \xi_m} \\
X & = & X
\end{array}$$

3. The following diagram commutes.

$$* \times X \xrightarrow{\mathbb{1} \times \mathrm{id}} \mathcal{O}(1) \times X \xrightarrow{\xi_1} X, \qquad * \times X \xrightarrow{\cong} X$$

Note that although the definitions here are written in terms of spaces, they apply just as well in categories of spectra that have a symmetric monoidal smash product, or more generally to any symmetric monoidal category. Simply replace the cartesian product of spaces with the smash product (or the monoidal product) and replace the one point space $*$ with the sphere spectrum S (or the unit of the monoidal product).

Apart from the endomorphism operad, perhaps the simplest operads are the commutative operad $\mathcal{C}om$ and the associative operad $\mathcal{A}ss$. The commutative operad is defined simply by $\mathcal{C}om(m) = *$ with the only possible compositions, identity, and Σ_m actions. The triviality of the action by Σ_2 implies that any pairing defined by it must be commutative, and in fact it is not difficult to show that giving a space the structure of a $\mathcal{C}om$-algebra is equivalent to giving it the structure of a commutative monoid. If we allow for noncommutativity, we get the associative operad $\mathcal{A}ss$, defined by letting $\mathcal{A}ss(m) = \Sigma_m$. The action by Σ_m is then given by composition, and the composition $\Gamma \colon \Sigma_m \times \prod_{i=1}^{m} \Sigma_{j_i} \to \Sigma_j$ is given by $(\sigma, \sigma_1, \ldots \sigma_m) \mapsto \sigma_{j_1, \ldots, j_m}(\sigma_1, \ldots, \sigma_m)$. As in the commutative case, one may show that an $\mathcal{A}ss$-algebra structure on a space is equivalent to the structure of a monoid.

3.2 E_n Operads and Commutativity

Where the associative and commutative operads parametrize operations that are strictly associative and commutative, the E_n operads are designed to parametrize operations with some degree of associativity and commutativity up to homotopy.

Definition 3.2.1. [BV68, Example 2.5][Man22, Construction 3.5] Let $n \geq 1$. The Boardman Vogt little n-cubes operad $\mathcal{C}_n$ is defined as follows. The space $\mathcal{C}_n(m)$ consists of ordered m-tuples $(f_1, \ldots, f_m)$ of embeddings $f_i \colon [0,1]^n \to [0,1]^n$ of the form $f_i(x_1, \ldots, x_n) = (y_1 + a_1 x_1, \ldots, y_n + a_n x_n)$ for $0 \leq y_i < 1$ and $0 < a_i \leq 1 - y_i$ such that the interiors $f_i((0,1)^n)$ are parwise disjoint. This set is topologized as a subspace of $\mathrm{Map}([0,1]^n,[0,1]^n)^m$. The identity is given by $\mathrm{id}\colon [0,1]^n \to [0,1]^n$. The group Σ_m acts by reordering the m embeddings. Composition is given by

$$\Gamma((f_1,\dots,f_m),(g_{1,1}\dots,g_{1,j_1}),\dots,(g_{m,1},\dots,g_{m,j_m})) = \\ (f_1g_{1,1},\dots,f_1g_{1,j_1},\dots,f_mg_{m,1},\dots,f_mg_{m,j_m})$$

Given an embedding $f\colon [0,1]^n \to [0,1]^n$, one constructs $f\times \mathrm{id}\colon [0,1]^{n+1} \to [0,1]^{n+1}$, and this induces a morphism of operads $\mathcal{C}_n \to \mathcal{C}_{n+1}$. Taking colimits of the relevant spaces and maps, one obtains the little ∞-cubes operad $\mathcal{C}_\infty$

The natural setting of the little n-cubes operads is in describing the products in an n-fold loop space. Given a pointed space (X,x_0), one may define an action by $\mathcal{C}_n$ on $\Omega^n X$ in the following way. Viewing points of $\Omega^n X$ as maps $\gamma\colon [0,1]^n \to X$ sending the boundary to x_0, define $\xi_m\colon \mathcal{C}_n(m)\times(\Omega^n X)^m \to \Omega^n X$ by

$$\xi_m((f_1,\dots,f_m),\gamma_1,\dots,\gamma_m)(u) = \begin{cases} \gamma_i(v) & f_i(v)=u \\ x_0 & u\notin \bigcup_i \mathrm{im}(f_i). \end{cases}$$

The point in $\mathcal{C}_1(2)$ given by $t\mapsto t/2$ and $t\mapsto 1/2+t/2$ then represents the usual product on a loopspace, and the fact that the product in $\Omega^2 X$ is commutative up to homotopy is simply a consequence of $\mathcal{C}_2(2)$ being path connected. In general, the spaces of $\mathcal{C}_n$ become more and more highly connected as n increases, culminating in $\mathcal{C}_\infty$ consisting of contractible spaces.

Proposition 3.2.2. [May72, Theorem 4.8][Man22, p. 12] *For $1\le n\le\infty$, let $C(m,\mathbb{R}^n)$ denote the space of m ordered parwise distinct points in $\mathbb{R}^n$. Then $\mathcal{C}_n(m)$ is Σ_m-equivariantly homotopy equivalent to $C(m,\mathbb{R}^n)$.*

In particular, we have that $\mathcal{C}_1(m)\simeq \mathcal{A}ss(m)$ while $\mathcal{C}_\infty(m)\simeq \mathcal{C}om(m)$, corresponding to the least and greatest degrees of commutativity we might expect. We see also that $\mathcal{C}_n(2)\simeq S^{n-1}$, so that a $\mathcal{C}_n$-algebra in some sense extends the older notion of an H_n-space, see f.ex., [Bro60].

While the little n-cubes operads are a useful model for describing commutativity of operations, there are many contexts, such as that of BO and MO, in which other operads are more practical to work with. Hence we extend our view somewhat to the notion of E_n operads.

Definition 3.2.3. [Man22, Definition 3.6] An operad $\mathcal{O}$ in the category of topological spaces is an E_n operad if there exists a zigzag of morphisms of operads relating $\mathcal{O}$ to the little n-cubes operad $\mathcal{C}_n$ such that the m'th component map of each morphism is a Σ_m equivariant homotopy equivalence for each m. An E_n space is then an $\mathcal{O}$-space for $\mathcal{O}$ any E_n operad.

An operad $\mathcal{O}$ in the category of spectra is an E_n operad if there exists a zigzag of morphisms of operads relating $\mathcal{O}$ to $\mathcal{C}_n$ such that the m'th component map of each morphism is a weak equivalence for each m. An E_n spectrum is then an $\mathcal{O}$-spectrum for any E_n operad $\mathcal{O}$.

Let $f\colon \mathcal{A}\to\mathcal{B}$ be a morphism of operads. An action ξ of $\mathcal{B}$ on a space X may then be pulled back along f to define an action $f^*(\xi)$ of $\mathcal{A}$ on X by letting $f^*(\xi)_m = \xi_m(f_m\times\mathrm{id})\colon \mathcal{A}\times X^m\to X$. A map of E_n spaces is then a morphism of operads $f\colon\mathcal{A}\to\mathcal{B}$ between E_n operads together with a map of spaces $g\colon X\to Y$ from an $\mathcal{A}$ space to a $\mathcal{B}$ space, such that g is a map of $\mathcal{A}$ spaces, where the $\mathcal{A}$ space structure on Y is the pullback of the $\mathcal{B}$ space structure along f. Maps of E_n spectra are defined analagously.

In the case of E_∞ operads in spaces there is a somewhat simpler characterization then the definition in therms of little ∞-cubes.

Proposition 3.2.4. [Man22, Proposition 3.7] *An operad $\mathcal{O}$ in spaces is an E_∞ operad if and only if $\mathcal{O}(m)$ is contractible and has the Σ_m-equivariant homotopy type of a free Σ_m-cell complex, i.e. a space built of cells of the form $(\Sigma_m \times D^n, \Sigma_m \times S^{n-1})$.*

In our case, we will make use of the Boardman-Vogt linear isometries operad. See [BV68] and [May77, Chapter I.1]. Let $\mathcal{I}((\mathbb{R}^m)^k, \mathbb{R}^n)$ be the space of linear isometries $(\mathbb{R}^m)^k \to \mathbb{R}^n$ and let $\mathcal{L}(k) = \lim_{m\to\infty} \operatorname{colim}_{n\to\infty} \mathcal{I}((\mathbb{R}^m)^k, \mathbb{R}^n)$ denote the space of linear isometries $(\mathbb{R}^\infty)^k \to \mathbb{R}^\infty$. Then defining compositions, identity, and Σ_k actions as in the case of the endomorphism operad gives $\mathcal{L}$ the structure of an E_∞ operad.

Now, the quotient of the orthogonal group $O(\mathbb{R}^n \oplus \mathbb{R}^m)$ by $O(\mathbb{R}^n) \oplus O(\mathbb{R}^m)$ may be identified with $G_n(\mathbb{R}^{n+m})$ by identifying $f \in O(\mathbb{R}^n \oplus \mathbb{R}^m)$ with the subspace $f(\mathbb{R}^n \oplus \{0\}) \subseteq \mathbb{R}^n \oplus \mathbb{R}^m \cong \mathbb{R}^{n+m}$. Note that under this identification, the maps $G_n(\mathbb{R}^{n+m}) \to G_n(\mathbb{R}^{n+m+1}) \to G_{n+1}(\mathbb{R}^{n+1+m+1})$ are induced by the inclusions $O(\mathbb{R}^n \oplus \mathbb{R}^m) \to O(\mathbb{R}^n \oplus \mathbb{R}^{m+1}) \to O(\mathbb{R}^{n+1} \oplus \mathbb{R}^{m+1})$. For $m, n, k \geq 1$, define a map $\tilde{\theta}_{m,n,k} \colon \mathcal{I}((\mathbb{R}^m)^k, \mathbb{R}^n) \times O(\mathbb{R}^m \oplus \mathbb{R}^m)^k \to O(\mathbb{R}^n \oplus \mathbb{R}^n)$ as follows. Let $f \in \mathcal{I}((\mathbb{R}^m)^k, \mathbb{R}^n)$, and let $g_1, \ldots, g_k \in O(\mathbb{R}^m \oplus \mathbb{R}^m)$. Let $V = \operatorname{im}(f) \subseteq \mathbb{R}^n \oplus \mathbb{R}^n$, and let $V^\perp$ denote its orthogonal complement. The map $\tilde{\theta}_{m,n,k}(f, g_1, \ldots, g_k)$ is then given by letting the following diagram commute.

$$\begin{array}{ccc}
\mathbb{R}^n \oplus \mathbb{R}^n & \xrightarrow{\tilde{\theta}_{m,n,k}(f,g_1,\ldots,g_k)} & \mathbb{R}^n \oplus \mathbb{R}^n \\
\big\downarrow \cong & & \cong \big\uparrow \\
V \oplus V^\perp \oplus V \oplus V^\perp & & V \oplus V^\perp \oplus V \oplus V^\perp \\
f\oplus \mathrm{id} \oplus f \oplus \mathrm{id} \big\uparrow \cong & & \cong \big\uparrow f \\
(\mathbb{R}^m)^k \oplus V^\perp \oplus (\mathbb{R}^m)^k \oplus V^\perp & & (\mathbb{R}^m)^k \oplus V^\perp \oplus (\mathbb{R}^m)^k \oplus V^\perp \\
\big\downarrow \cong & & \cong \big\uparrow \\
(\mathbb{R}^m \oplus \mathbb{R}^m)^k \oplus V^\perp \oplus V^\perp & \xrightarrow{g_1 \oplus \ldots \oplus g_k \oplus \mathrm{id} \oplus \mathrm{id}} & (\mathbb{R}^m \oplus \mathbb{R}^m)^k \oplus V^\perp \oplus V^\perp
\end{array}$$

The maps $\tilde{\theta}_{m,n,k}$ then induce maps $\theta_{m,n,k} \colon \mathcal{I}((\mathbb{R}^m))^k, \mathbb{R}^n) \times G_m(\mathbb{R}^{m+m})^k \to G_n(\mathbb{R}^{n+n})$. These now fit together to define $\theta_{m,k} \colon \mathcal{I}((\mathbb{R}^m))^k, \mathbb{R}^\infty) \times G_m(\mathbb{R}^{m+m})^k \to G_\infty(\mathbb{R}^\infty)$, and these in turn fit together to define $\theta_k \colon \mathcal{L}(k) \times G_\infty(\mathbb{R}^\infty) \to G_\infty(\mathbb{R}^\infty)$. This defines an operad action of $\mathcal{L}$ on $BO = G_\infty(\mathbb{R}^\infty)$

Thus BO is an E_∞ space. By [LMS86, Theorem IX.7.1], MO inherits an E_∞ structure from BO, and is thus an E_∞ spectrum. The E_∞ structure on BSO, $BSpin$, $BString$, and their associated Thom spectra are defined completely analogously.

3.3 Dyer-Lashof Operations

Associated to the homology of any E_n spectrum are certain operations, called Dyer-Lashof or Araki-Kudo operations, arising from the geometry of the E_n operad. Since these operations are natural with respect to maps of E_n spectra, they may be used to place bounds on the existence of such maps.

Definition-Proposition 3.3.1. [Law20, Theorem 5.2] For X any E_n-spectrum, for $n \geq 1$, there exist specific functions $Q_i : H_j(X) \to H_{2j+i}(X)$, called Dyer-Lashof operations, for each $0 \leq i \leq n-1$ for which the following hold:

1. Q_i is natural with respect to maps of E_n spectra.
2. Q_i is an $\mathbb{F}_2$-module homomorphism for $1 \leq i \leq n-2$.
3. $Q_0(x) = x^2$, for $x \in H_*(X)$.
4. $Q_i(1) = 0$ for $1 \leq i \leq n-1$.
5. (Cartan Formula) For $x, y \in H_*(X)$ and $1 \leq i \leq n-2$, $Q_i(xy) = \sum_{j+k=i} Q_j(x)Q_k(y)$
6. (Adem Relations) For $x \in H_*(x)$ and $0 \leq i < j \leq n-1$, $Q_j(Q_i(x)) = \sum \binom{k-i-1}{2k-j-i} Q_{j+2i-2k}(Q_k(x))$
7. (Stability) $\sigma Q_0 = 0$ and $\sigma Q_i = Q_{i-1}\sigma$ for $1 \leq i \leq n-1$, where $\sigma \colon H_*(\Omega X) \to H_{*+1}(X)$ is the homology suspension.
8. If the E_n structure on X extends to an E_{n+1} structure on X, then the Dyer-Lashof operations associated to the E_n structure agree with those associated to the E_{n+1} structure.

Note that linearity and the Cartan formula do not in general hold for the topmost operation Q_{n-1}. In this case there exist similar formulas involving a bilinear map $[-,-] \colon H_i(X) \otimes H_j(X) \to H_{i+j+(n-1)}(X)$ called the Browder Bracket. If, however, the E_n structure on X extends to an E_{n+1} structure, then both linearity and the Cartan formula hold for Q_{n-1} as well. Note also that if X is an E_∞-spectrum, then $H_*(X)$ has Dyer-Lashof operations Q_i for all $i \geq 0$. In this case one often writes $Q^{i+|x|}(x) = Q_i(x)$, so that Q^j is the operation that raises degrees by j.

The Dyer-Lashof operations on BO were first determined by Kochman in terms of a recursive algorith, but the following closed formula is due to Priddy.

Proposition 3.3.2. [Law20, Theorem 5.15][Pri75, Theorem 2.4] *In* $H_*(BO) = \mathbb{F}_2[b_i \mid i \geq 1]$ *we have, for each* $n \geq 1$,

$$\sum_{i \geq 0} Q_i(b_n) = \left(\sum_{k=n}^{\infty} \sum_{j=0}^{n} \binom{k-n+j-1}{j} b_{k+j} b_{n-j} \right) \left(\sum_{j=0}^{\infty} b_j \right)^{-1}.$$

By [LMS86, Proposition IX.7.4], the Thom isomorphism $H_*(BO) \to H_*(MO)$ preserves Dyer-Lashof operations, so that this also gives a description for MO.

Although this is somewhat unhelpful when working with the $a_{k,j}$ generators, it does have the useful consequence that $Q_i(b_n) \equiv \binom{n+i-1}{n} b_{2n+i}$ modulo decomposables.

3.4 Steenrod (Co)operations

Just as the Dyer-Lashof operations act on the homology of any E_n spectrum, there exist operations, called Steenrod squares, which act on the mod 2 cohomology of any spectrum. These squares generate an associative, but not commutative, algebra known as the Steenrod algebra $\mathcal{A}$, over which all cohomology (with $\mathbb{F}_2$ coefficients) is a module. For our purposes however, we will be more interested in the dual Steenrod algebra $\mathcal{A}_*$, under which homology becomes a comodule.

Definition-Proposition 3.4.1. [SE62, pp. 1–2] For $i \geq 0$, there exist natural transformations of functors $Sq^i : H^*(-) \to H^{*+i}(-)$, where $H^*(-)$ is viewed as a functor from topological spaces (or spectra) into the category of graded $\mathbb{F}_2$-modules. The homomorphisms Sq^i satisfy the following properties.

1. (Cartan Formula) For $x, y \in H^*(X)$, $Sq^i(x \cup y) = \sum_{j+k=i} Sq^j(x) \cup Sq^k(y)$.

2. For $s : H^*(X) \to H^{*+1}(X)$ the usual suspension isomorphism and $x \in H^n(X)$, $Sq^i(s(x)) = s(Sq^i(x))$.

3. For $x \in H^n(X)$, $Sq^0(x) = x$, and, if X is a space, $Sq^n(x) = x^2$ and $Sq^i(x) = 0$ for $i > n$.

4. Sq^1 is the Bockstein homomorphism associated to the short exact sequence
$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

5. (Adem Relations) For $0 < i < 2j$, we have
$$Sq^i Sq^j = \sum_{k=0}^{[i/2]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k.$$

The Steenrod operations interact with Dyer-Lashof operations via the so-called Nishida relations. Note that here too the top Dyer-Lashof operation Q_{n-1} is a special case.

Proposition 3.4.2. [Law20, Theorem 5.18][NIS68] *For X an E_n spectrum, and for $r \geq 0$ and $0 \leq s \leq n-2$,*
$$Sq^r_* Q^s = \sum_i \binom{s-r}{r-2i} Q^{s-r+i} Sq^i_* \colon H_*(X) \to H_{*-r+s}(X),$$
where $Sq^r_* \colon H_*(X) \to H_{*-r}(X)$ is dual to Sq^r.

Since mod 2 cohomology is represented by the Eilenberg-Mac Lane spectrum $H\mathbb{F}_2$, such operations are represented by classes in $H^*(H\mathbb{F}_2)$, by the Yoneda lemma . After identifying an operation with the cohomology class representing it, one has the following by a result of Serre in [Ser52].

Proposition 3.4.3. [Swi02, Theorem 18.15] *Define a finite sequence of integers $I = (i_1, \dots i_n)$ to be admissible if $I = (0)$ or if each i is positive and $i_m \geq 2i_{m+1}$ for $1 \leq m \leq n-1$, and let Sq^I denote the product $Sq^{i_1} \dots Sq^{i_n}$. With this notation, one has*
$$H^*(H\mathbb{F}_2) = \mathbb{F}_2\{Sq^I \mid I \text{ is an admissible sequence}\}.$$

The Steenrod algebra is then by definition $\mathcal{A} = H^*(H\mathbb{F}_2)$, and the fact that the Sq^i give operations on cohomology can be rephrased as saying that for any space (or spectrum) X, $H^*(X)$ has the canonical structure of an $\mathcal{A}$-module.

Dualising, we write $\mathcal{A}_* = H_*(H\mathbb{F}_2)$ for the dual Steenrod algebra. It turns out that $\mathcal{A}_*$ may be given the structure of a Hopf algebra, and this structure is described by the following.

Proposition 3.4.4. [Mil58, Theorems 2, 3] *As an algebra, $\mathcal{A}_* = H_*(H\mathbb{F}_2) = \mathbb{F}_2[\xi_i \mid i \geq 1]$, with $\xi_i \in H_{2^i-1}(H\mathbb{F}_2)$. The coproduct on $\mathcal{A}_*$ is given by $\Delta(\xi_i) = \sum_{j+k=i} \xi_j^{2^k} \otimes \xi_k$, where as usual we use the convention that $\xi_0 = 1$.*

Let $c : \mathcal{A}_* \to \mathcal{A}_*$ be the conjugation and set $\zeta_i = c(\xi_i)$. We then have $\mathcal{A}_* = \mathbb{F}_2[\zeta_i \mid i \geq 1]$ and $\Delta(\zeta_i) = \sum_{j+k=i} \zeta_j \otimes \zeta_k^{2^j}$.

Note that there is some inconsistency between sources on which generators are denoted ξ_i and which are denoted ζ_i. In particular, the notation used here is not the same as is used in [Mil58] for $p = 2$.

Given a space (or spectrum) X, we have an action by the Steenrod algebra that takes the form of a map $\mathcal{A} \otimes H^*(X) \to H^*(X)$. In the homology case, there is a map $\psi : H_*(X) \to \mathcal{A}_* \otimes H_*(X)$ making $H_*(X)$ into a $\mathcal{A}_*$-comodule. If $H_*(X)$ happens to be bounded below and finitely generated in each degree then this is simply the dual of the $\mathcal{A}$-module structure on $H^*(X)$. In the case of $H\mathbb{F}_2$, the $\mathcal{A}$-module structure on $H^*(H\mathbb{F}_2) = \mathcal{A}$ is the obvious one: the action $\mathcal{A} \otimes H^*(H\mathbb{F}_2) \to H^*(H\mathbb{F}_2)$ defining the module structure is simply the product in $\mathcal{A}$. Thus the coaction $H_*(H\mathbb{F}_2) \to A_* \otimes H_*(H\mathbb{F}_2)$ is simply the coproduct in $\mathcal{A}_*$. In the case of BO and MO, the comodule structures are given on the b_i by the following.

Proposition 3.4.5. [Swi73, Theorem 2] *Let X denote the formal sum $\sum_{i=0}^{\infty} \xi_i$. Then the coaction $\psi_{BO} : H_*(BO) \to \mathcal{A}_* \otimes H_*(BO)$ on $H_*(BO) = \mathbb{F}_2[b_i \mid i \geq 1]$ is given by*

$$\psi_{BO}(b_i) = \sum_{j=0}^{i} (X^j)_{i-j} \otimes b_j,$$

where $(X^j)_{i-j}$ denotes the degree $i - j$ component of X^j.

The coaction $\psi_{MO} : H_*(MO) \to \mathcal{A}_* \otimes H_*(MO)$ on $H_*(MO) = \mathbb{F}_2[b_i \mid i \geq 1]$ is given by

$$\psi_{MO}(b_i) = \sum_{j=0}^{i} (X^{j+1})_{i-j} \otimes b_j.$$

Note that although the Thom isomorphism gives $H_*(BO) \cong H_*(MO)$ as algebras, this map does not respect the $\mathcal{A}_*$-coaction. Instead, these can be related by the inclusion $BO(1) \to MO(1)$. Since the 0-sphere bundle $SE_1 \to BO(1)$ has $SE_1 = S^\infty \simeq *$, this map induces an isomorphism on homology given by $b_i \mapsto b_{i-1}$, where the degree shift is due to $MO(1)$ being the first level of the spectrum MO, and this isomorphism respects the $\mathcal{A}_*$-coaction by naturality.

One may show that the coactions on $H^*(BO)$ and $H^*(MO)$ are map of algebras, so that the above proposition is in principle a complete description. In addition, since the inclusions $H_*(BO\langle 2^n \rangle) \subseteq H_*(BO)$ are induced by the covering maps $BO\langle 2^n \rangle \to BO$, naturality also allows us to apply the the above proposition to BSO, $BSpin$, and $BString$, along with their associated Thom spectra.

As an Eilenberg-Mac Lane spectrum over a commutative ring, $H\mathbb{F}_2$ has the structure of an E_∞ spectrum, and the dual Steenrod algebra thus has Dyer-Lashof operations of its own.

Proposition 3.4.6. [Bru+86, Theorems III.2.2, III.2.4] *The Dyer-Lashof operations on $\mathcal{A}_*$ are given by the following formulas.*

1.

$$1 + \xi_1 + \sum_{i=0}^{\infty} Q_i(\xi_1) = \left(\sum_{j=0}^{\infty} \xi_j \right)^{-1}$$

2. *For all $i \geq 0$, $j \geq 1$,*

$$Q_i(\zeta_j) = \begin{cases} Q_{2^{j+1}+i-4}(\xi_1) & i \equiv 0, 1 \mod 2^j \\ 0 & \textit{otherwise.} \end{cases}$$

3. $Q_{2^i-3}(\xi_1) = \zeta_i$ *for* $i \geq 2$.

Inherent in this result are the following useful useful facts, which we will have significant use for later. Note that point (2) below follows from point (1) because the conjugation in $\mathcal{A}_*$ is its own inverse.

Corollary 3.4.7. *1.*

$$\zeta_i = \left(\left(\sum_{j=0}^{\infty} \xi_j\right)^{-1}\right)_{2^i-1}$$

2.

$$\xi_i = \left(\left(\sum_{j=0}^{\infty} \zeta_j\right)^{-1}\right)_{2^i-1}$$

3.

$$Q_1(\zeta_i) = \zeta_{i+1}$$

3.5 Integral Liftings

3.5.1 Lifts of Dyer-Lashof Operations

The formulas given so far for the Dyer-Lashof operations and Steenrod cooperations on $H_*(BO)$ and $H_*(MO)$ were given in terms of the generators b_i, but in order to work in the homology of BSO, $BSpin$, $BString$, or any of their Thom spectra, we will need to understand these operations in terms of the elements $a_{k,j}$. For this purpose, the primitive elements q_i offer a useful middle ground, with one small caveat: the formula $q_{k2^j} = \sum_{i=0}^{j} 2^i a_{k,i}^{2^{j-i}}$, which can be used for many calculations when taken over $\mathbb{Z}_{(2)}$ or $\mathbb{Q}$, reduces simply to $q_{k2^j} = a_{k,0}^{2^j}$ when taken over $\mathbb{F}_2$, leaving most generators uninvolved. To remedy this, one may first do calculations in $B^{\mathbb{Z}_{(2)}}[1]$, then map down to the case of $\mathbb{F}_2$ coefficients at the end. Of course, Dyer-Lashof and Steenrod (co-)operations are not a priori defined in this context, but by the work of Lance in [Lan83] they may be lifted to it anyway.

Lance writes his results in terms of the mod p homology of BU for odd primes p, but the proof adapts quite readily to the case of BO with $p = 2$. The main tools are the following lemma together with Kochman's description of the Dyer-Lashof operations in BO and BU.

Lemma 3.5.1. [Lan83, Lemma 2.1] *Let p be prime and let T_j be the j'th Witt polynomial, given by $T_j(t_0, \ldots, t_j) = \sum_{i=0}^{j} p^i t_i^{p^{j-i}}$. Let $g_0, g_1, \ldots$ be polynomials or formal power series in indeterminates $t_0, t_1, \ldots$ with integral coefficients such that $g_j(t) \equiv g_{j-1}(t^p) \mod p^j$ for $j \geq 1$, where $t = (t_0, t_1, \ldots)$ and $t^p = (t_0^p, t_1^p, \ldots)$. Then the equations*

$$g_j(t) = T_j(\phi_0(t), \ldots, \phi_j(t))$$

inductively define polynomials or formal power series ϕ_j with integral coefficients for $j \geq 0$.

Recall that $H_*(BU;\mathbb{F}_p) = B^{\mathbb{F}_p}[2]$ for p any prime, so we may write $\mathbb{F}_2[\tilde{b}_i \mid i \geq 1]$ with $|\tilde{b}_i| = 2i$ and denote the standard basis for the primitive elements by $\{\tilde{q}_i\}_{i\geq 1}$.

Proposition 3.5.2. [Koc73, Theorem 97] *There is an algorithm for computing $Q^r(\tilde{b}_n)$ in $H_*(BU;\mathbb{F}_p)$ for p an odd prime or $Q^{2r}(\tilde{b}_n)$ in $H_*(BU;\mathbb{F}_2)$ using the following properties.*

1. *The maps $Q^r\colon H_*(BU;\mathbb{F}_p) \to H_{*+2r(p-1)}(BU;\mathbb{F}_p)$ and $Q^{2r}\colon H_*(BU;\mathbb{F}_2) \to H_{*+2r}(BU;\mathbb{F}_2)$ are linear for all $r \geq 0$.*
2. *$Q^r(\tilde{b}_n) = 0$ over $\mathbb{F}_p$ and $Q^{2r}(\tilde{b}_n) = 0$ over $\mathbb{F}_2$ for $n > r \geq 0$.*
3. *(Cartan Formula) For $x, y \in H_*(BU;\mathbb{F}_p)$ or $x, y \in H_*(BU;\mathbb{F}_2)$, $Q^r(xy) = \sum_{i+j=r} Q^i(x)Q^j(y)$.*
4. *(CoCartan Formula) For $x \in H_*(BU;\mathbb{F}_p)$ or $x \in H_*(BU;\mathbb{F}_2)$, if $\Delta(x) = \sum x' \otimes x''$, then $\Delta(Q^r(x)) = \sum_{i+j=r} Q^i(x') \otimes Q^j(x'')$.*
5. *$Q^n(\tilde{b}_n) = \tilde{b}_n^p$ in $H_*(BU;\mathbb{F}_p)$ and $Q^{2n}(\tilde{b}_n) = \tilde{b}_n^2$ in $H_*(BU;\mathbb{F}_2)$.*
6. *(Nishida Relations) $P^s_*Q^r = \sum_i \binom{r(p-1)-(p-1)s}{s-pi} Q^{r-s+i}P^i_*$ as operations on $H_*(BU;\mathbb{F}_p)$ and $Sq^s_*Q^r = \sum_i \binom{r-s}{s-2i} Q^{r-s+i}Sq^i_*$ as operations on $H_*(BU;\mathbb{F}_2)$*
7. *$Q^r(\tilde{q}_n) = (-1)^{r+n}\binom{r-1}{n-1}\tilde{q}_{n+r(p-1)}$ in $H_*(BU;\mathbb{F}_p)$ and $Q^{2r}(\tilde{q}_n) = \binom{r-1}{n-1}\tilde{q}_{n+r}$ in $H_*(BU;\mathbb{F}_2)$.*
8. *$Q^r(b_n) \equiv (-1)^{r+n+1}\binom{r-1}{n}\tilde{b}_{n+r(p-1)}$ modulo decomposables in $H_*(BU;\mathbb{F}_p)$ and $Q^{2r}(\tilde{b}_n) \equiv \binom{r-1}{n}\tilde{b}_{n+r}$ modulo decomposables in $H_*(BU;\mathbb{F}_2)$.*

This theorem can also be used to describe Dyer-Lashof operations in $H_*(BO;\mathbb{F}_2)$. One way to see this is to use that there is a non-grade preserving homomorphism of Hopf algebras $f\colon H_*(BO;\mathbb{F}_2) \to H_*(BU;\mathbb{F}_2)$ sending b_i to $\tilde{b}_i$ and q_i to $\tilde{q}_i$. The homomorphism f respects Dyer-Lashof operations in the sense that $Q^{2r}(f(x)) = f(Q^r(x))$, as can be seen most easily by comparing the descriptions in [Law20, Theorem 5.15], and it respects Steenrod operations in a similar manner, by the Wu formula . Thus the version that applies to $H_*(BO;\mathbb{F}_2)$ is the following.

Corollary 3.5.3. *There is an algorithm for computing $Q^r(b_n)$ in $H_*(BO;\mathbb{F}_2)$ using the following properties.*

1. *The maps $Q^r\colon H_*(BO;\mathbb{F}_2) \to H_{*+r}(BO;\mathbb{F}_2)$ are linear for $r \geq 0$.*
2. *$Q^r(b_n) = 0$ for $n > r \geq 0$.*
3. *(Cartan Formula) For $x, y \in H_*(BO;\mathbb{F}_2)$, and $r \geq 0$, $Q^r(xy) = \sum_{i+j=r} Q^i(x)Q^j(y)$.*
4. *(CoCartan Formula) For $x \in H_*(BO;\mathbb{F}_2)$ and $r \geq 0$, if $\Delta(x) = \sum x' \otimes x''$ then $\Delta(Q^r(x)) = \sum_{i+j=r} Q^i(x') \otimes Q^j(x'')$.*
5. *$Q^n(b_n) = b_n^2$ for $n \geq 1$.*
6. *(Nishida Relations) $Sq^s_*Q^r = \sum_i \binom{r-s}{s-2i} Q^{r-s+i}Sq^i_*$.*
7. *$Q^r(q_n) = \binom{r-1}{n-1}q_{n+r}$.*
8. *$Q^r(b_n) \equiv \binom{r-1}{n}b_{n+r}$ modulo decomposables.*

We now consider the construction and verification of the lift itself in the case of $H_*(BO;\mathbb{F}_2)$, following the proof of [Lan83, Theorem 4.2]. To avoid confusion, denote the generators of $B^{\mathbb{Z}_{(2)}}[1]$ by $\hat{b}_i$ and $\hat{a}_{k,j}$, and denote the standard primitive elements by $\hat{q}_i$, so that the quotient map $B^{\mathbb{Z}_{(2)}}[1] \to B^{\mathbb{F}_2}[1] \cong H_*(BO;\mathbb{F}_2)$ sends $\hat{b}_i$, $\hat{a}_{k,j}$, and $\hat{q}_i$ to b_i, $a_{k,j}$, and q_i. Now note that in $B^{\mathbb{Z}_{(2)}}[1]$, the primitive elements are given by $\hat{q}_{k2^j} = T_j(\hat{a}_{k,0},\dots,\hat{a}_{k,j})$ for $j \geq 0$, $k \geq 1$, and k odd. Let $a = \{\hat{a}_{k,j} \mid j \geq 0, k \geq 1, 2 \nmid k\}$ and define $g_{k,j}(a) = \sum_{r=0}^{\infty} \binom{r-1}{k2^j-1}\hat{q}_{k2^j+r} \in \mathbb{Z}_{(2)}[[\hat{a}_{k,j}]]$. Note that $T_j(\hat{a}_{k,0},\dots,\hat{a}_{k,j}) \equiv T_{j-1}(\hat{a}_{k,0}^2,\dots,\hat{a}_{k,j-1}^2)$ modulo 2^j. This, together with the the identities

$$r\binom{r-1}{k2^j-1} \equiv 0 \mod 2^j$$

and

$$\binom{2r-1}{k2^j-1} - \binom{r-1}{k2^{j-1}-1} \equiv 0 \mod 2^j,$$

ensures that $g_{k,j}(a) \equiv g_{k,j-1}(a^2) \bmod 2^j$. Thus we may inductively define power series $\hat{Q}(\hat{a}_{k,j})$, and thus an algebra homomorphism $\hat{Q}\colon \mathbb{Z}_{(2)}[\hat{a}_{k,j}] \to \mathbb{Z}_{(2)}[[\hat{a}_{k,j}]]$, by requiring that

$$T_j(\hat{Q}(\hat{a}_{k,0}),\dots,\hat{Q}(\hat{a}_{k,j})) = g_{k,j}(a).$$

Setting $\hat{Q}^r(x)$ to be the degree $|x|+r$ term of $\hat{Q}(x)$, we get $\mathbb{Z}_{(2)}$-module homomorphisms $\hat{Q}^r\colon B^{\mathbb{Z}_{(2)}}[1] \to B^{\mathbb{Z}_{(2)}}[1]$ raising degrees by r. These will be our lifted Dyer-Lashof operations.

Thus it remains to show that the homomorphisms $\hat{Q}^r$ reduce mod 2 to the Dyer-Lashof operations. We do this be checking that $\hat{Q}^r$ satisfies each of the requirements for Kochman's algorithm in Corollary 3.5.3 modulo 2. To begin with, (1), (3), and (7) are satisfied by construction. Tensoring with $\mathbb{Q}$, we see that the coCartan formula (4) follows in $B^{\mathbb{Q}}[1]$ since $B^{\mathbb{Q}}[1]$ is primitively generated and the $\hat{Q}^r$ satisfy the Cartan forumula and send primitives to primitives. Since $B^{\mathbb{Z}_{(2)}}[1]$ has no torsion, the coCartan formula follows in $B^{\mathbb{Z}_{(2)}}[1]$ as well. Since $\hat{Q}^r(q_n) = 0$ for $n > r \geq 0$ by construction, it follows by the Cartan formula and $B^{\mathbb{Q}}[1]$ being primitively generated that $Q^r(x) = 0$ for any x with $|x| > r \geq 0$, so (2) holds. Point (8) follows from (7), the Cartan formula, the fact that $\hat{q}_n \equiv (-1)^{n+1}n\hat{b}_n$ modulo decomposables, and the following identity:

$$\frac{n+r}{n}\binom{r-1}{n-1} + \binom{r-1}{n} = 2\binom{r}{n} \equiv 0 \mod 2.$$

It remains to show (5) and the Nishida relations (6). Lance's proofs of these are significantly more involved, but the arguments apply just as well to the $H_*(BO;\mathbb{F}_2)$ case here too.

Thus the homomorphisms $\hat{Q}^r$ reduce modulo 2 to Dyer-Lashof operations. In other words, we have the following.

Proposition 3.5.4. *There exist $\mathbb{Z}_{(2)}$-module homomorphisms $\hat{Q}_i\colon B^{\mathbb{Z}_{(2)}}[1]_j \to B^{\mathbb{Z}_{(2)}}[1]_{2j+i}$ for $i, j \geq 0$ which satisfy and are uniquely defined by the following.*

1. *For any $i \geq 0$ and $x, y \in B^{\mathbb{Z}_{(2)}}[1]$, $\hat{Q}_i(xy) = \sum_{j+k=i}\hat{Q}_j(x)\hat{Q}_k(y)$.*

2. *For any $i \geq 0$ and $j \geq 1$, $\hat{Q}_i(\hat{q}_j) = \binom{i+j-1}{j-1}\hat{q}_{2j+i}$*

The maps $\hat{Q}_i\colon B^{\mathbb{Z}_{(2)}}[1]_j \to B^{\mathbb{Z}_{(2)}}[1]_{2j+i}$ *reduce modulo 2 to the Dyer-Lashof operations* $Q_i\colon H_j(BO) \to H_{2j+i}(BO)$.

When making use of this lifting, we will often take the additional step of tensoring with $\mathbb{Q}$ and working in $B^{\mathbb{Q}}[1]$ for convenience. Thus, if we wished to calculate $Q_1(a_{1,1})$, we would use that in $B^{\mathbb{Q}}[1]$ we have

$$\hat{Q}_1(\hat{a}_{1,1}) = \hat{Q}_1\left(\frac{1}{2}\hat{q}_2 - \frac{1}{2}\hat{q}_1^2\right) = \frac{1}{2}\hat{Q}_1(\hat{q}_2) - \hat{Q}_0(\hat{q}_1)\hat{Q}_1(\hat{q}_1) = \hat{q}_5 - \hat{q}_2\hat{q}_3,$$

so that in $H_*(BO)$ we must have $Q_1(a_{1,1}) = q_5 - q_2q_3 = a_{5,0} + a_{1,1}^2 a_{3,0}$.

3.5.2 Lifts of Steenrod Co-operations

The case of Steenrod co-operations is similar in spirit, as done by Baker in [Bak85, Sections 2, 3]. First, let $\hat{\mathcal{A}}_* = \mathbb{Z}_{(2)}[\hat{\xi}_i \mid i \geq 1]$ with $|\hat{\xi}_i| = 2^i - 1$. Define a $\mathbb{Z}_{(2)}$-algebra homomorphism $\Delta\colon \hat{\mathcal{A}}_* \to \hat{\mathcal{A}}_* \otimes_{\mathbb{Z}_{(2)}} \hat{\mathcal{A}}_*$ by $\Delta(\hat{\xi}_i) = \sum_{j+k=i} \hat{\xi}_j^{2^k} \otimes \hat{\xi}_k$. We may now define homomorphisms $\hat{\psi}_{BO}, \hat{\psi}_{MO}\colon B^{\mathbb{Z}_{(2)}}[1] \to \hat{\mathcal{A}}_* \otimes_{\mathbb{Z}_{(2)}} B^{\mathbb{Z}_{(2)}}[1]$ by simply interpreting the formulas in Proposition 3.4.5 as using integral coefficients. Clearly these reduce modulo 2 to the usual coaction, so it only remains to determine the value of $\hat{\psi}_{BO}(\hat{q}_i)$ and $\hat{\psi}_{MO}(\hat{q}_i)$, for which Baker makes use of a description of $\hat{\psi}_{BO}$ and $\hat{\psi}_{MO}$ in terms of power series.

Proposition 3.5.5. [Propositions 2.6,3.5][Bak85] *Define formal power series*

$$\xi(T) = \sum_{i\geq 0} \hat{\xi}_i T^{2^i} \in \hat{\mathcal{A}}_*[[T]],$$

and

$$b(T) = \sum_{i\geq 0} \hat{b}_i T^i \in B^{\mathbb{Z}_{(2)}}[1][[T]].$$

Then the homomorphisms $\hat{\psi}_{BO}, \hat{\psi}_{MO}\colon B^{\mathbb{Z}_{(2)}}[1][[T]] \to (\hat{\mathcal{A}}_* \otimes_{\mathbb{Z}_{(2)}} B^{\mathbb{Z}_{(2)}}[1])[[T]]$ *satisfy the following.*

$$\hat{\psi}_{BO}\left(\sum_{i\geq 1}(-1)^i \hat{q}_i T^i\right) = T\frac{(\xi\otimes 1)'(T)}{(\xi\otimes 1)(T)} \sum_{j\geq 1} (-1)^j \xi(T)^j \otimes \hat{q}_j$$

$$\hat{\psi}_{MO}\left(\sum_{i\geq 1}(-1)^i \hat{q}_i T^i\right) = T\frac{(\xi\otimes 1)'(T)}{(\xi\otimes 1)(T)} \left(\sum_{j\geq 1} (-1)^j \xi(T)^j \otimes \hat{q}_j - 1\otimes 1\right) + 1\otimes 1$$

Here $(\xi\otimes 1)(T) = \sum_{i\geq 1}(\hat{\xi}_i\otimes 1)T^{2^i} \in (\hat{\mathcal{A}}_* \otimes_{\mathbb{Z}_{(2)}} B^{\mathbb{Z}_{(2)}}[1])[[T]]$, *and* $(\xi\otimes 1)'(T)$ *is the formal derivative of* $(\xi\otimes 1)(T)$ *with respect to* T.

Proof. Baker initially gives a proof in the case of BU which directly translates to BO, then indicates how the proof may be adapted to MU, so we here consider the details of the case of MO. First note that the Newton polynomials in Proposition 2.4.11 describing the relationship between the $\hat{b}_i$ and the $\hat{q}_i$ may be rewritten as

$$\sum_{i\geq 1}(-1)^i \hat{q}_i T^i = -T\frac{b'(T)}{b(T)}.$$

Now let $\bar{b}(T) = Tb(T)$. We then have from Proposition 3.4.5 that

$$\hat{\psi}_{MO}(\hat{b}_i) = \sum_{0\leq j}(\xi(T)^{j+1})_{T^{i+1}} \otimes \hat{b}_j,$$

where $(\xi(T)^{j+1})_{T^{i+1}}$ denotes the term of $(\xi(T)^{j+1})$ of T-degree $i+1$. Thus

$$\hat{\psi}_{MO}(\bar{b})(T) = (1 \otimes \bar{b}) \circ (\xi \otimes 1)(T),$$

where $\circ$ denotes the usual functional composition of power series. Since

$$\sum_{i\geq 1}(-1)^i \hat{q}_i T^i = -T\frac{b'(T)}{b(T)} = -T\frac{\bar{b}'(T)}{\bar{b}(T)} + 1$$

we may then calculate

$$\begin{aligned}
&\hat{\psi}_{MO}\left(\sum_{i\geq 1}(-1)^i \hat{q}_i T^i\right) \\
&= \hat{\psi}_{MO}\left(-T\frac{\bar{b}'(T)}{\bar{b}(T)} + 1\right) \\
&= -T\frac{((1\otimes \bar{b}) \circ (\xi \otimes 1))'(T)}{(1\otimes \bar{b}) \circ (\xi \otimes 1)(T)} + 1 \otimes 1 \\
&= -T\frac{((1\otimes \bar{b})' \circ (\xi \otimes 1)(T)) \cdot (\xi \otimes 1)'(T)}{(1\otimes \bar{b}) \circ (\xi \otimes 1)(T)} + 1 \otimes 1 \\
&= T\frac{(\xi \otimes 1)'(T)}{(\xi \otimes 1)(T)}\left(-(\xi \otimes 1)(T)\frac{(1\otimes \bar{b})' \circ (\xi \otimes 1)(T)}{(1\otimes \bar{b}) \circ (\xi \otimes 1)(T)}\right) + 1 \otimes 1 \\
&= T\frac{(\xi \otimes 1)'(T)}{(\xi \otimes 1)(T)}\left(\sum_{j\geq 1}(-1)^j (1\otimes \hat{q}_j)((\xi \otimes 1)(T))^j - 1 \otimes 1\right) + 1 \otimes 1 \\
&= T\frac{(\xi \otimes 1)'(T)}{(\xi \otimes 1)(T)}\left(\sum_{j\geq 1}(-1)^j \xi(T)^j \otimes \hat{q}_j - 1 \otimes 1\right) + 1 \otimes 1
\end{aligned}$$

□

Chapter 4

Topological Hochschild Homology

4.1 Simplicial Objects

The topological Hochschild homology of an E_1 spectrum and the delooping of an E_1 space may be constructed by glueing together spaces of the form $X_n \times \Delta^n$ along the faces of Δ^n. The structure that allows for such a construction is that of a simplicial space, so let us begin by defining this.

Let Δ be the category whose objects are the linearly ordered sets $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$ and whose morphisms are the weakly increasing functions between these sets. Define a functor $F\colon \Delta \to \mathcal{T}op$ by letting $F([n]) = \Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, \quad t_0, \dots, t_n \geq 0\}$, the standard n-simplex, and letting $F(\mu\colon [q] \to [p])$ be the affine linear map sending the vertex v_i to v_{μ_i}. Among these maps, two kinds are of particular interest. For $0 \leq i \leq n$, define the i'th face map $\delta^i\colon [n-1] \to [n]$ by

$$\delta(j) = \begin{cases} j & j < i \\ j+1 & j \geq i, \end{cases}$$

and define the i'th degeneracy map $\sigma^i\colon [n+1] \to [n]$ by

$$\sigma^i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i. \end{cases}$$

In terms of simplices, $F(\delta^i)$ then defines the inclusion of an $n-1$-dimensional face into Δ^n and $F(\sigma^i)$ defines a map $\Delta^{n+1} \to \Delta^n$ that collapses one of the edges. In fact, these are the only maps one needs to consider in light of the following lemma.

Lemma 4.1.1. *[Mac94, Lemma V.III.5.1] Let $\mu\colon [q] \to [p]$ be any morphism in Δ. Then μ has a unique factorization $\mu = \delta^{i_1} \dots \delta^{i_s} \sigma^{j_1} \dots \sigma^{j_t}$ with $0 \leq i_s < \dots < i_1 \leq p$, $0 \leq j_1 < \dots < j_t < q$ and $q - t + s = p$.*

With this in mind, there are now two equivalent definitions of a simplicial object.

Definition 4.1.2. A simplicial object in a category $\mathcal{C}$ is a (contravariant) functor $S\colon \Delta^{\mathrm{op}} \to \mathcal{C}$. A map of simplicial objects is then a natural transformation of functors.

Alternatively, a simplicial object in a category $\mathcal{C}$ consists of a family of objects $\{S_n\}_{n \geq 0}$ together with face maps $d_i\colon S_n \to S_{n-1}$ and degeneracy maps $s_i\colon S_n \to S_{n+1}$ for $0 \leq i \leq n$ such that the following are satisfied.

1. $d_i d_j = d_{j-1} d_i$ for $i < j$.

2. $s_is_j = s_{j+1}s_i$ for $i \leq j$.

3. $d_is_j = \begin{cases} s_{j-1}d_i & i < j \\ \mathrm{id} & i = j, i = j+1 \\ s_jd_{i-1} & i > j+1 \end{cases}$

A map of simplicial objects is then a family of maps commuting with the face and degeneracy maps.

Definition 4.1.3. Given a simplicial space $X_\bullet$, i.e., a simplicial object in the category of topological spaces, the geometric realization $|X_\bullet|$ is the quotient of $\coprod_{n=0}^\infty X_n \times \Delta^n$ by the relations $(d_i(x), y) = (x, \delta^i(y))$ and $(s_i(x), y) = (x, \sigma^i y)$. This defines a functor from the category of simplicial spaces to topological spaces.

Alternatively, one may define $|X_\bullet|$ as a colimit of spaces $|X_\bullet|_n$, where $|X_\bullet|_0 = X_0$ and $|X_\bullet|_n$ is defined by a pushout

$$\begin{array}{ccc} \coprod_{i=0}^{n-1}(X_{n-1} \times \Delta^n) \sqcup \coprod_{j=0}^{n}(X_n \times \Delta^{n-1}) & \longrightarrow & |X_\bullet|_{n-1} \\ \downarrow & & \downarrow \\ X_n \times \Delta^n & \longrightarrow & |X_\bullet|_n. \end{array}$$

Here the lefthand vertical arrow is given by $s_i \times \mathrm{id}$ on the i'th $X_{n-1} \times \Delta^n$ component and $\mathrm{id} \times \delta^j$ on the j'th $X_n \times \Delta^{n-1}$ component. The upper horizontal arrow is given $\mathrm{id} \times \sigma^i$ on the i'th $X_{n-1} \times \Delta^n$ component and $d_j \times \mathrm{id}$ on the j'th $X_n \times \Delta^{n-1}$ component, followed by the inclusion $X_{n-1} \times \Delta^{n-1} \to |X_\bullet|_{n-1}$. This definition then immediately generalizes to the geometric realization of a simplicial spectrum by replacing products with smash products and Δ^n by Δ^n_+.

In addition to spaces and spectra, one may also consider geometric realization as a functor from simplicial $\mathcal{O}$-algebras to $\mathcal{O}$-algebras in light of the following.

Proposition 4.1.4. [Man22, Theorem 7.5] *Let $\mathcal{O}$ be an operad in the category of topological spaces or spectra and let $X_\bullet$ be a simplicial $\mathcal{O}$-algebra. Then $|X_\bullet|$, the geometric realization of the underlying space or spectrum, has the canonical structure of an $\mathcal{O}$-algebra.*

On the algebraic side of things, a simplicial R-module may be used to construct a chain complex whose homology has a number of nice properties.

Definition 4.1.5. Let R be a commutative ring and let $M_\bullet$ be a simplicial R-module. Define a chain complex (M_*, d) whose component modules are those of $M_\bullet$ by letting $d\colon M_n \to M_{n-1}$ be given by

$$d = \sum_{i=0}^{n} (-1)^i d_i.$$

Then M_* is the chain complex associated to $M_\bullet$, and $H_*(M_*)$ is the homology of $M_\bullet$.

Given simplicial R-modules $A_\bullet$ and $B_\bullet$, there is a product simplicial R-module $(A \times B)_\bullet$ with $(A \times B)_n = (A_n) \otimes_R (B_n)$. The associated chain complex $(A \times B)_*$ is then not the same as the tensor product $A_* \otimes_R B_*$, but they are closely related by the Eilenberg-Zilber theorem.

Proposition 4.1.6. [Mac94, Theorems 8.1, 8.5, 8.8] *Let $A_\bullet$ and $B_\bullet$ be simplicial R-modules for R a commutative ring. There exists a natural chain equivalence*

$$(A \times B)_* \underset{g}{\overset{f}{\rightleftarrows}} (A_*) \otimes (B_*).$$

The component maps $f\colon A_n \otimes B_n \to \bigoplus_{i+j=n} A_i \otimes B_j$ are given by

$$f(a \otimes b) = \sum_{i+j=n} \tilde{d}^j(a) \otimes d_0^i(b).$$

Here $\tilde{d}$ denotes the "last" face operator, i.e., $d_m\colon A_m \to A_{m-1}$ for any m.

The component maps $g\colon A_m \otimes B_n \to A_{m+n} \otimes B_{m+n}$ are given by

$$g(a \otimes b) = \sum_{(\mu,\nu)} (-1)^{sgn(\mu,\nu)} (s_{\nu_n} \dots s_{\nu_1} a) \otimes (s_{\mu_m} \dots s_{\mu_1} b).$$

Here the sum is taken over all (m, n) shuffles (μ, ν), i.e. permutations of $\{1, \dots, m+n\}$ whose restrictions $\mu\colon \{1, \dots, m\} \to \{1, \dots, m+n\}$ and $\nu\colon \{m+1, \dots, m+n\} \to \{1, \dots, m+n\}$ are order preserving.

Given a product $(A \times A)_\bullet \to A_\bullet$, we can then precompose its induced map in homology with the shuffle map g and the homology product $H_*(A) \otimes H_*(A) \to H_*(A_* \otimes A_*)$ to define a product in homology $H_*(A) \otimes H_*(A) \to H_*(A)$.

4.2 Topological Hochschild Homology

Definition 4.2.1. [Lod98, p. 9] Let R be a commutative ring, let A be a (not necessarily commutative) R-algebra, and let M be an A-bimodule. Define a simplicial R-module $C_\bullet(A, M)$ by letting $C_n(A, M) = M \otimes_R A^{\otimes n}$ with degeneracy and face operators given by

$$d_i(m \otimes a_1 \otimes \dots \otimes a_n) = \begin{cases} ma_1 \otimes a_2 \otimes \dots \otimes a_n & i = 0 \\ m \otimes a_1 \otimes \dots \otimes a_i a_{i+1} \otimes a_n & 1 \le i \le n-1 \\ a_n m \otimes a_1 \otimes \dots \otimes a_{n-1} & i = n \end{cases}$$

$$s_i(m \otimes a_1 \otimes \dots \otimes a_n) = m \otimes a_1 \otimes \dots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \dots \otimes a_n$$

The Hochschild homology of (A, M) is then $H_*(A, M) = H_*(C_*(A, M))$. In the case $M = A$ we write $HH_*(A) = H_*(A, A)$.

If A happens to be commutative, then there is a product $C_\bullet(A, A) \otimes C_\bullet(A, A) \to C_\bullet(A, A)$ given by $(a_0 \otimes \dots \otimes a_n)(a'_0 \otimes \dots \otimes a'_n) = a_0 a'_1 \otimes \dots \otimes a_n a'_n$, so that $HH_*(A)$ gains the structure of a commutative R-algebra.

In the case that A is projective, there is another useful description of $HH_*(A)$.

Definition 4.2.2. Let A be an R-algebra M be a right A-module, and N be a left A-module. Let $C_\ell^{bar}(A) = M \otimes_R A^{\otimes \ell} \otimes_R N$ and denote an element $m \otimes a_1 \otimes \dots \otimes a_\ell \otimes n \in C_\ell^{bar}(M, A, N)$ by $m[a_1|\dots|a_n]m$. Define face and degeneracy maps by

$$d_i(m[a_1|\dots|a_\ell]n) = \begin{cases} ma_1[a_1|\dots|a_\ell]n & i = 0 \\ m[a_1|\dots|a_i a_{i+1}|\dots a_\ell]n & 1 \le i \le \ell \\ m[a_1|\dots|a_{n-1}]a_\ell n & i = \ell \end{cases}$$

$$s_i(m[a_1|\dots|a_\ell]n) = m[a_1|\dots|a_i|1|a_{i+1}|\dots a_\ell]n$$

Then $C_\bullet^{bar}(M, A, N)$ is a simplicial R-module, called the two sided bar construction.

The bar construction has a number of useful versions in various categories, but in the case of Hochschild homology, the its main use is this. In the case $M = N = A$, $C_\bullet^{bar}(A) = C_\bullet^{bar}(A, A, A)$ has an "extra degeneracy" given by

$$s(a_0[a_1|\dots|a_\ell]a_{\ell+1}) = 1[a_0|\dots|a_\ell]a_{\ell+1}.$$

The property that $sd_i = d_is$ makes s a contracting homotopy for $C_*^{bar}(A)$ such that, if A happens to be projective, $C_*^{bar}(A)$ provides a projective resolution of A. If A is projective as an R-module, then it is also projective as a module over the enveloping algebra $A^e = A \otimes_R A^{op}$, where the (left) module action is given by

$$(a_1' \otimes a_2')(a_0[a_1|\dots|a_\ell]a_{\ell+1}) = a_1'a_0[a_1|\dots|a_\ell]a_{\ell+1}a_2'$$

Since $M \otimes_{A^e} C_*^{bar}(A) \cong C_*(A, M)$ for any A-bimodule M, this gives the following.

Proposition 4.2.3. *[Lod98] Let R be a commutative ring, A be an R-algebra, and M be an A-bimodule. Then*

$$H_*(A, M) \cong Tor_*^{A^e}(M, A)$$

The geometric analogue of Hochschild homology is the geometric realization of a corresponding simplicial spectrum, called topological Hochschild homology, or THH.

Definition 4.2.4. [EKMM, Definition IX.2.1] Let A be an S-algebra and let M be an A-bimodule. Let $\mu\colon A \wedge A \to A$ and $\eta\colon S \to A$ denote the product and unit of A and let $\phi_r\colon A \wedge M \to M$ and $\phi_l\colon M \wedge A \to M$ denote the left and right actions by A on M. Define a simplicial spectrum $THH_\bullet(A, M)$ by letting $THH_n(A, M) = M \wedge A^{\wedge n}$ and letting the face and degeneracy operators be given by

$$d_i = \begin{cases} \phi_r \wedge \mathrm{id}^{n-1} & i = 0 \\ \mathrm{id} \wedge \mathrm{id}^{i-1} \wedge \mu \wedge \mathrm{id}^{n-i-1} & 1 \le i \le n-1 \\ (\phi_l \wedge \mathrm{id}^{n-1})\tau & i = n \end{cases}$$

$$s_i = \mathrm{id} \wedge \mathrm{id}^i \wedge \eta \wedge \mathrm{id}^{n-i},$$

where $\tau\colon M \wedge A^{\wedge n-1} \wedge A \to A \wedge M \wedge A^{\wedge n-1}$ is the commuting isomorphism. The topological Hochschild homology of A relative to M is then given by $THH(A, M) = |THH_\bullet(A, M)|$. In the case $M = A$ we write $THH(A) = THH(A, A)$.

Just as in the algebra case, if the product on A is strictly commutative, then $THH(A)$ also inherits the structure of a commutative S-algebra.

One might hope for a close relation between the homology of $THH(A)$ and the Hochschild homology of $H_*(A)$, and indeed this relation appears in the form of the Bökstedt spectral sequence.

Proposition 4.2.5. [EKMM, Theoremm IX.2.9] *Let E be a commutative ring spectrum, let A be an S-algebra, and let M be a cell A^e-module. If $E_*(A)$ is E_*-flat then there is a spectral sequence of the form*

$$E^2_{p,q} = H^{E_*}_{p,q}(E_*(A), E_*(M)) \Rightarrow E_{p+q}(THH(A, M)),$$

where $H^{E_*}_{p,q}$ denotes Hochschild homology over the ring E_* with homological degree p and internal degree q. The composite map

$$E_*(M) \to E^2_{0,*} \to E^\infty_{0,*} \to E_*(THH(A; M))$$

is the E_*-module homomorphism i_* induced by the inclusion $M = |THH_\bullet(A; M)|_0 \to THH(A, M)$. If A is a commutative S-algebra then the spectral sequence

$$E^2_{p,q} = HH^{E_*}_{p,q}(E_*(A)) \Rightarrow E_{p+q}(THH(A))$$

is a spectral sequence of differential $E_*(A)$-algebras, and the composition

$$E_*(A) \to E^2_{1,*} \to E^\infty_{1,*} \to E_{*+1}(THH(A))/im(i_*)$$

is the E_*-module homomorphism σ induced by the composition

$$\Sigma A \cong \Sigma(S \wedge A) \to \Sigma(A \wedge A) \to A \wedge A \wedge \Delta^1_+ \to |THH_\bullet(A)|_1 \to THH(A).$$

Here the map $E_*(A) \to E^2_{1,*} = HH_1(E_*(A))$ is given by $a \mapsto [1 \otimes a] \in HH_1(E_*(A))$.

4.3 THH of E_n Spectra

If we are not so lucky as to be working with a strictly associative and/or commutative S-algebra, there is still a way to make use of THH by replacing an E_n spectrum with an E_{n-1}-algebra in S-algebras. The key ingredient in this procedure is the tensor product of operads.

Definition 4.3.1. [BV79, p. 120] Let $\mathcal{A}$ and $\mathcal{B}$ be operads in the category of topological spaces. The tensor product of $\mathcal{A}$ and $\mathcal{B}$ is then an operad $\mathcal{A}\otimes\mathcal{B}$ together with morphisms $f\colon \mathcal{A} \to \mathcal{A}\otimes\mathcal{B}$ and $g\colon \mathcal{B} \to \mathcal{A}\otimes\mathcal{B}$ such that the following interchange diagram commutes, and such that $\mathcal{A}\otimes\mathcal{B}$ is the initial object among such operads.

$$\begin{array}{ccc}
\mathcal{A}(n) \times \mathcal{B}(m) & \longrightarrow & \mathcal{B}(m) \times \mathcal{A}(n) \\
\big\downarrow{\scriptstyle f_n \times g_m} & & \big\downarrow{\scriptstyle g_m \times f_n} \\
(\mathcal{A}\otimes\mathcal{B})(n) \times (\mathcal{A}\otimes\mathcal{B})(m) & & (\mathcal{A}\otimes\mathcal{B})(m) \times (\mathcal{A}\otimes\mathcal{B})(n) \\
\big\downarrow{\scriptstyle \mathrm{id}\times\Delta} & & \big\downarrow{\scriptstyle \mathrm{id}\times\Delta} \\
(\mathcal{A}\otimes\mathcal{B})(n) \times ((\mathcal{A}\otimes\mathcal{B})(m))^n & & (\mathcal{A}\otimes\mathcal{B})(m) \times ((\mathcal{A}\otimes\mathcal{B})(n))^m \\
\big\downarrow{\scriptstyle \Gamma} & & \big\downarrow{\scriptstyle \Gamma} \\
(\mathcal{A}\otimes\mathcal{B})(nm) & \xrightarrow{\ \sigma\ } & (\mathcal{A}\otimes\mathcal{B})(mn)
\end{array}$$

Here Δ denotes the diagonal map in each case, and $\sigma \in \Sigma_{mn}$ is the permutation sending $im + j + 1$ to $jn + i + 1$ for $0 \le i \le n-1$ and $0 \le j \le m-1$. Thus an operad action by $\mathcal{A}\otimes\mathcal{B}$ on a space X determines and is uniquely determined by an $\mathcal{A}$-action and a $\mathcal{B}$-actions such that, for any $a \in \mathcal{A}(n)$ and $b \in \mathcal{B}(m)$, the following interchange diagram commutes.

$$\begin{array}{ccc}
(X^n)^m = (X^m)^n & \xrightarrow{\ b^n\ } & X^n \\
\big\downarrow{\scriptstyle a^m} & & \big\downarrow{\scriptstyle a} \\
X^m & \xrightarrow{\ b\ } & X
\end{array}$$

In other words, an $\mathcal{A}\otimes\mathcal{B}$-algebra is equivalent to an $\mathcal{A}$-algebra in the category of $\mathcal{B}$-algebras.

Note that in the case $\mathcal{A}$ is the associative operad $\mathcal{A}ss$, the structure of an $\mathcal{A}ss \otimes \mathcal{B}$-algebra is equivalent to the structure of a monoid in the category of $\mathcal{B}$-algebras. In [BFV07], the authors show that $\mathcal{A}ss \otimes \mathcal{C}_n$ is an E_{n+1} operad, and they then use that to prove the following.

Proposition 4.3.2. [BFV07, Theorem 3.4] *Let $f\colon A \to M$ be a map of E_{n+1} spectra. Then there exists a commutative diagram of E_{n+1} spectra and maps of E_{n+1} spectra*

$$\begin{array}{ccccc} Y_A & \longleftarrow & X_A & \longrightarrow & A \\ \downarrow{\scriptstyle f_Y} & & \downarrow{\scriptstyle f_X} & & \downarrow{\scriptstyle f} \\ Y_M & \longleftarrow & X_M & \longrightarrow & M \end{array}$$

such that the horizontal arrows are homotopy equivalences, Y_A and Y_M are $\mathcal{A}ss \otimes \mathcal{C}_n$-algebras, and f_Y is a map of $\mathcal{A}ss \otimes \mathcal{C}_n$-algebras.

Thus if we are given E_{n+1} spectra A and M and an E_{n+1} map $A \to M$, we can first replace these with $\mathcal{A}ss \otimes \mathcal{C}_n$ spectra Y_A and Y_M. These are then monoids in the category of $\mathcal{C}_n$-algebras, so they may be used to define a simplicial $\mathcal{C}_n$-algebra $THH_\bullet(Y_A, Y_M)$, whose geometric realization is then a $\mathcal{C}_n$-algebra $THH(Y_A, Y_M)$, which one may also call $THH(A; M)$.

Given an E_{n+1}-spectrum A, we then get an E_n spectrum $THH(A)$. One would hope that the Bökstedt spectral sequence then also becomes an algebra spectral sequence as in the strictly commutative case, and indeed it is.

Proposition 4.3.3. [AR05, Theorem 4.3] *Let A be an E_2-spectrum. The Böckstedt spectral sequence*

$$E^2_{p,q} = HH^{\mathbb{F}_2}_{p,q}(H_*(A)) \Rightarrow H_{p+q}(THH(A))$$

is a spectral sequence of $\mathcal{A}_*$-comodule $\mathbb{F}_2$ algebras. If A is an E_3-spectrum then $E^r_{*,*}$ is a spectral sequence of commutative $H_*(A)$-algebras in $\mathcal{A}_*$-comodules.

Chapter 5

Computations

5.1 Orientation Maps

Let $u\colon MO \to H\mathbb{F}_2$ denote the orientation map, i.e., the map represented by the cohomology class $1 \in H^*(MO)$. This map may be realized as a map of E_∞-ring spectra [Law20, Proposition 5.29]. Thom showed in [Tho54] that MO splits as a wedge sum of suspensions $\Sigma^i H\mathbb{F}_2$, and this result then gave an essentially complete description of the of the unoriented bordism ring $\pi_*(MO)$. This splitting does not necessarily preserve the E_∞ structure of MO, but Mahowald showed in [Mah77] that $H\mathbb{F}_2$ is the Thom spectrum of a map $\Omega^2 S^3 \to BO$, from which one gets an E_2 section $H\mathbb{F}_2 \to MO$.

For the connected covers MSO, $MSpin$, and $MString$ there is a similar story. Let $\mathcal{A}_n$ denote the subalgebra of $\mathcal{A}$ generated by $Sq^1, \ldots, Sq^{2^n}$. Then the quotient algebras $\mathcal{A}//\mathcal{A}_0$, $\mathcal{A}//\mathcal{A}_1$ and $\mathcal{A}//\mathcal{A}_2$ may be realized as the mod 2 cohomology of E_∞ spectra $H\mathbb{Z}$, ko, and tmf [AR05, p. 1257]. The map u then lifts to orientation maps $MSO \to H\mathbb{Z}$, $MSpin \to ko$, and $MString \to tmf$, which are also E_∞ maps, and which we will also denote by u[AHR10, Theorems 6.1, 12.3]. Wall showed in [Wal60] that MSO splits 2-locally as a wedge sum of suspensions of $H\mathbb{Z}$ and $H\mathbb{F}_2$, and Anderson, Brown, and Shapiro produced a similar splitting for $MSpin$ in [ABP67] . For $MString$ no such splitting is known. See Fig. 5.1. We aim here to find upper bounds on the possible commutativity of such splittings by determining which Dyer-Lashof operations it is possible for a section of u_* to respect.

For $-1 \le n \le 2$ let

$$eo(n) = \begin{cases} H\mathbb{F}_2 & n = -1 \\ H\mathbb{Z} & n = 0 \\ ko & n = 1 \\ tmf & n = 2. \end{cases}$$

The homology of $eo(n)$ is then given in each case by the following.

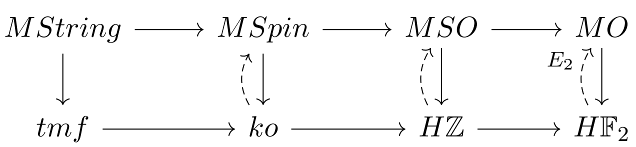

Figure 5.1: The orientations $u\colon MO\langle 2^n\rangle \to eo(n-1)$ and sections of these.

Proposition 5.1.1. [AR05, Propsosition 6.1] *Let $-1 \leq n \leq 3$. Then there is a map of commutative ring spectra $eo(n) \to H\mathbb{F}_2$ inducing the following identification in homology.*

$$H_*(eo(n)) = \mathbb{F}_2[\zeta_i^{2^{n+2-i}} \mid i \leq n+1] \otimes \mathbb{F}_2[\zeta_i \mid i \geq n+2]$$

Thus we have

$$\begin{aligned} H_*(H\mathbb{Z}) &= \mathbb{F}_2[\zeta_1^2, \zeta_2, \ldots] \\ H_*(ko) &= \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \ldots] \\ H_*(tmf) &= \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \ldots]. \end{aligned}$$

In order to understand how the orientation maps look in homology, we will make use of the Steenrod cooperations. Let ϵ denote the counit of the Hopf algebra structures on $H_*(MO)$ and $\mathcal{A}_*$, and let $\psi : H_*(-) \to \mathcal{A}_* \otimes H_*(-)$ denote the coaction on homology. We then have the following commutative diagram.

$$\begin{array}{ccccc} H_*(MO) & \xrightarrow{\psi} & \mathcal{A}_* \otimes H_*(MO) & \xrightarrow{\mathrm{id}\otimes\epsilon} & \mathcal{A}_* \otimes \mathbb{F}_2 \\ \downarrow{\scriptstyle u_*} & & \downarrow{\scriptstyle \mathrm{id}\otimes u_*} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathcal{A}_* & \xrightarrow{\psi} & \mathcal{A}_* \otimes \mathcal{A}_* & \xrightarrow{\mathrm{id}\otimes\epsilon} & \mathcal{A}_* \otimes \mathbb{F}_2 \\ & \searrow^{\mathrm{id}} & & & \downarrow{\scriptstyle \cong} \\ & & & & \mathcal{A}_* \end{array}$$

Here the lefthand square commutes becuase ψ is natural, and the bottom triangle commutes since the coaction $\psi : H_*(H\mathbb{F}_2) \to \mathcal{A}_* \otimes H_*(H\mathbb{F}_2)$ is equal to the coproduct $\Delta : \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$. To see that the righthand square commutes, we need only that $u_*(1) = 1$, since ϵ is in both cases the map sending all positive degree terms to 0. This is simply a consequence of u representing the nontrivial element of $H^0(MO) \cong \mathbb{F}_2$.

Thus to find $u_*(x)$ it suffices to find the term of the form $- \otimes 1$ in $\psi(x)$. For the generators b_i this is reasonably straightforward. We have from Proposition 3.4.5 that $\psi(b_i) = \sum_{j=0}^{i} (X^{j+1})_{i-j} \otimes b_j$, where $X = \sum_{i=0}^{\infty} \xi_i$, so that

$$u_*(b_i) = (X)_i = \begin{cases} \xi_m & 2^m - 1 = i \\ 0 & (\nexists m)(2^m - 1 = i). \end{cases}$$

In terms of the generators $a_{k,j}$ and ζ_i there is not such a nice formula, although the previous description can be used to do calculations in low degrees. To do this, use the formulas in Proposition 2.4.11 and Definition-Proposition 2.4.12 with integral coefficients to write the elements $a_{k,j}$ as polynomials in the b_i, then use point (2) in Corollary 3.4.7 to write each ξ_i as a polynomial in the elements ζ_j. See Table 5.1 for the results of such a calculation. For the indecomposable elements q_i, however, Proposition 3.5.5 gives that $u_*(q_i) = (X^{-1})_i$, so that in particular, $u_*(q_{2^i-1}) = \zeta_i$ by Corollary 3.4.7.

Note that although these calculations are done in the case $MO \to H\mathbb{F}_2$, they also give descriptions of the MSO, $MSpin$, and $MString$ cases by Proposition 2.5.3 and Proposition 5.1.1.

Before we go on, we recall some important formulas. By Proposition 2.5.3, the mod 2 homology of $M(\langle 2^n \rangle)$ is given by

$$H_*(M\langle 2^n \rangle) = \mathbb{F}_2[a_{k,j}^{2^{\rho_n(k)}} \mid j \geq 0, k \geq 1, 2 \nmid k],$$

Table 5.1: The homomorphism $u_* : H_*(MO) \to H_*(H\mathbb{F}_2)$ on generators in degrees 0 through 19.

$$
\begin{aligned}
u_*(a_{1,0}) &= \zeta_1 \\
u_*(a_{1,1}) &= 0 \\
u_*(a_{1,2}) &= \zeta_1^4 + \zeta_1\zeta_2 \\
u_*(a_{1,3}) &= \zeta_1^8 + \zeta_1\zeta_3 \\
u_*(a_{1,4}) &= \zeta_1^{16} + \zeta_1^7\zeta_2^3 + \zeta_1^6\zeta_2\zeta_3 + \zeta_1\zeta_2^5 + \zeta_1\zeta_4 + \zeta_2^3\zeta_3 \\
u_*(a_{3,0}) &= \zeta_2 \\
u_*(a_{3,1}) &= \zeta_1^6 + \zeta_2^2 \\
u_*(a_{3,2}) &= \zeta_1^6\zeta_2^2 + \zeta_1^2\zeta_2\zeta_3 \\
u_*(a_{5,0}) &= \zeta_1^2\zeta_2 \\
u_*(a_{5,1}) &= \zeta_1^{10} + \zeta_2\zeta_3 \\
u_*(a_{7,0}) &= \zeta_3 \\
u_*(a_{7,1}) &= \zeta_1^8\zeta_2^2 + \zeta_3^2 \\
u_*(a_{9,0}) &= \zeta_1^6\zeta_2 + \zeta_1^2\zeta_3 + \zeta_2^3 \\
u_*(a_{9,1}) &= \zeta_1^{18} + \zeta_1^4\zeta_3^2 + \zeta_1^2\zeta_2^3\zeta_3 + \zeta_2\zeta_4 \\
u_*(a_{11,0}) &= \zeta_1^4\zeta_3 \\
u_*(a_{13,0}) &= \zeta_2^2\zeta_3 \\
u_*(a_{15,0}) &= \zeta_4 \\
u_*(a_{17,0}) &= \zeta_1^{14}\zeta_2 + \zeta_1^{10}\zeta_3 + \zeta_1^8\zeta_2^3 + \zeta_1^4\zeta_2^2\zeta_3 + \zeta_1^2\zeta_2^5 + \zeta_1^2\zeta_4 + \zeta_2\zeta_3^2 \\
u_*(a_{19,0}) &= \zeta_1^{12}\zeta_3 + \zeta_1^4\zeta_4 + \zeta_2^4\zeta_3
\end{aligned}
$$

Table 5.2: The homomorphism $u_*\colon H_*(MSO) \to H_*(H\mathbb{Z})$ on monomials in degrees 0 through 4.

$$
\begin{aligned}
u_*(1) &= 1 \\
u_*(a_{1,0}^2) &= \zeta_1^2 \\
u_*(a_{3,0}) &= \zeta_2 \\
u_*(a_{1,1}^2) &= 0 \\
u_*(a_{1,0}^4) &= \zeta_1^4
\end{aligned}
$$

Table 5.3: The homomorphism $u_*\colon H_*(MSpin) \to H_*(ko)$ on monomials in degrees 0 through 8.

$$
\begin{aligned}
u_*(1) &= 1 \\
u_*(a_{1,0}^4) &= \zeta_1^4 \\
u_*(a_{3,0}^2) &= \zeta_2^2 \\
u_*(a_{7,0}) &= \zeta_3 \\
u_*(a_{1,1}^4) &= 0 \\
u_*(a_{1,0}^8) &= \zeta_1^8
\end{aligned}
$$

Table 5.4: The homomorphism $u_*\colon H_*(MString) \to H_*(tmf)$ on monomials in degrees 0 through 16.

$$
\begin{aligned}
u_*(1) &= 1 \\
u_*(a_{1,0}^8) &= \zeta_1^8 \\
u_*(a_{3,0}^4) &= \zeta_2^4 \\
u_*(a_{7,0}^2) &= \zeta_3^2 \\
u_*(a_{15,0}) &= \zeta_4 \\
u_*(a_{1,1}^8) &= 0 \\
u_*(a_{1,0}^{16}) &= \zeta_1^{16}
\end{aligned}
$$

Table 5.5: Dyer-Lashof Operations in $H_*(H\mathbb{F}_2)$.

$Q_2(\zeta_1) = \zeta_1^4$	
$Q_3(\zeta_1) = \zeta_1^2\zeta_2$	Obstruction
$Q_4(\zeta_2) = \zeta_1^4\zeta_2^2$	Obstruction
$Q_5(\zeta_2) = \zeta_1^4\zeta_3$	Obstruction
$Q_8(\zeta_3) = \zeta_1^8\zeta_3^2$	Obstruction
$Q_9(\zeta_3) = \zeta_1^8\zeta_4$	Obstruction
$Q_{16}(\zeta_4) = \zeta_1^{16}\zeta_4^2$	Obstruction
$Q_{17}(\zeta_4) = \zeta_1^{16}\zeta_5$	Obstruction

for $0 \leq n \leq 3$, where $|a_{k,j}| = k2^j$, $\rho_n(k) = \max(n+1-\alpha(k), 0)$, and α denotes the bit sum. By Proposition 3.5.4 Dyer-Lashof operations in $H_*(MO)$ lift to integral operations $\hat{Q}_r \colon \mathbb{Z}_{(2)}[\hat{a}_{k,j}] \to \mathbb{Z}_{(2)}[\hat{a}_{k,j}]$. The lifts $\hat{Q}_r$ are uniquely defined by the Cartan formula and

$$\hat{Q}_r(\hat{q}_i) = \binom{i+r-1}{i-1} \hat{q}_{2i+r}.$$

The primitive elements are given by

$$\hat{q}_{k2^j} = \hat{a}_{k,0}^{2^j} + \ldots + 2^j \hat{a}_{k,j},$$

in the integral case and

$$q_{k2^j} = a_{k,0}^{2^j}$$

in the mod 2 case.

On the other side of things, by Proposition 5.1.1, the homology of $eo(n-1)$ is given by

$$H_*(eo(n-1)) = \mathbb{F}_2[\zeta_i^{2^{n+1-1}} \mid 1 \leq i \leq n] \otimes \mathbb{F}_2[\zeta_i \mid i \geq n+1]$$

for $0 \leq n \leq 3$, where $|\zeta_i| = 2^i - 1$. By Proposition 3.4.6, the Dyer-Lashof operations are given in $H_*(H\mathbb{F}_2)$ by

$$Q_r(\zeta_i) = \begin{cases} Q_{2^{i+1}+r-4}\zeta_1 & r \equiv 0, 1 \mod 2^i \\ 0 & \text{otherwise} \end{cases}$$

and

$$Q_r(\zeta_1) = \left(\left(\sum_{i=0}^{\infty} \xi_i \right)^{-1} \right)_{r+2},$$

where, by Corollary 3.4.7,

$$\xi_i = \left(\left(\sum_{j=0}^{\infty} \zeta_j \right)^{-1} \right)_{2^i-1}.$$

We will also have significant use throughout the rest of this chapter of the $p = 2$ case of the following result from elementary number theory, known as Lucas's theorem.

Proposition 5.1.2. [Fin47, Theorem 1] *Let $\alpha = \sum_{i=0}^{n} \alpha_i p^i$ and $\beta = \sum_{i=0}^{n} \beta_i p^i$, where $0 \leq \alpha_i, \beta_i \leq p-1$ and $n \geq 0$. Then*

$$\binom{\alpha}{\beta} \equiv \prod_{i=0}^{n} \binom{\alpha_i}{\beta_i} \mod p.$$

We begin with the simplest case.

Proposition 5.1.3. *The $\mathbb{F}_2$-algebra homomorphism $u_* : H_*(MO) \to H_*(H\mathbb{F}_2)$ admits a unique algebra section s that commutes with the Dyer-Lashof operation Q_1. s also commutes Q_2, but not Q_3. Thus the orientation map $u : MO \to H\mathbb{F}_2$ does not have an E_4 section.*

Proof. In degree 1, the map u_* is an isomorphism given by $a_{1,0} = q_1 \mapsto \zeta_1$. Thus a section s would have to satisfy $s(\zeta_1) = q_1$. By Corollary 3.5.3, we have for all $i \geq 1$ that $Q_1(q_{2^i-1}) = \binom{2^i-1}{2^i-2} q_{2^{i+1}-1} = q_{2^{i+1}-1}$, while $Q_1(\zeta_i) = \zeta_{i+1}$ by Corollary 3.4.7. Thus by induction, if s commutes with Q_1, then it must be given by $s(\zeta_i) = q_{2^i-1}$. To see that this does define a section of u_*, one could appeal to the existence of an E_2 section of $u : MO \to H\mathbb{F}_2$, but we have also seen directly that $u_*(q_{2^i-1}) = \zeta_i$.

By construction, we have that s commutes with Q_1. For Q_0, $Q_0 s = s Q_0$ because $Q_0(x) = x^2$ for any x, and s is an algebra homomorphism. For Q_2, we have that $Q_2(\zeta_i) = 0$ for $i \geq 2$ by Proposition 3.4.6, while $Q_2(q_{2^i-1}) = \binom{2^i}{2^i-2} q_{2^{i+1}} = 0$ for $i \geq 2$. In the $i = 1$ case, we have $Q_2(\zeta_1) = \zeta_1^4$, while $Q_2(q_1) = q_4 = q_1^4$, so this commutes as well. For Q_3, however, we have that $Q_3(\zeta_1) = \xi_1^5 + \xi_1^2 \xi_2 = \zeta_1^2 \zeta_2$, whereas $Q_3(q_1) = q_5$. Thus,

$$s(Q_3(\zeta_1)) = a_{1,0}^2 a_{3,0} \neq a_{5,0} = Q_3(s(\zeta_1)).$$

$\square$

The cases of MSO, $MSpin$, and $MString$ are all quite similar, and they benefit from a somewhat more systematic approach.

Proposition 5.1.4. *Let $1 \leq n \leq 3$. Then the $\mathbb{F}_2$-algebra homomorphism $u_* : H_*(MO\langle 2^n \rangle) \to H_*(eo(n-1))$ admits a unique algebra section s_n that commutes with the Dyer-Lashof operation Q_1. The section s_n commutes with Q_r for $0 \leq r \leq 2^{n+1} - 1$, but it does not commute with $Q_{2^{n+1}}$. Thus the orientation map $u : MO\langle 2^n \rangle \to eo(n-1)$ does not have an $E_{2^{n+1}+1}$ section.*

Proof. We begin by showing that if s_n commutes with Q_1, then it must be given by $s_n(\zeta_i^{2^{n+1-i}}) = q_{2^i-1}^{2^{n+1-i}}$ for $1 \leq i \leq n$ and $s_n(\zeta_i) = q_{2^i-1}$ for $i \geq n+1$. We see from Table 5.2, Table 5.3, and Table 5.4 that u_* is an isomorphism in degrees 0 through $2^{n+1} - 1$, and that in these degrees s_n must be given by $s_n(\zeta_i^{2^{n+1-i}}) = a_{2^i-1,0}^{2^{n+1-i}} = q_{2^{n+i-1}}$ for $1 \leq i \leq n$ and $s_n(\zeta_n) = q_{2^n-1}$. As in Proposition 5.1.3 we have that $Q_1(\zeta_i) = \zeta_{i+1}$ and $Q_1(q_{2^i-1}) = q_{2^{i+1}-1}$, so the claim follows by induction.

Let s_0 denote the section of $u_* \colon H_*(MO) \to H_*(H\mathbb{F}_2)$ constructed in Proposition 5.1.3, and note that each s_n is merely a restriction of s_0. In particular, s_n is in fact a section of u_*.

To see which Dyer-Lashof operations s_n respects, we use that, by Proposition 3.4.6, Corollary 3.5.3, and Lucas's theorem, we have

$$Q_i(\zeta_j) = \begin{cases} Q_{i+2^{j+1}-4}\zeta_1 & i \equiv 0, 1 \mod 2^j \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

$$Q_i(q_{2^j-1}) = \binom{i + 2^j - 2}{2^j - 2} q_{i+2^{j+1}-2} = \begin{cases} q_{i+2^{j+1}-2} & i \equiv 0, 1 \mod 2^j \\ 0 & \text{otherwise} \end{cases}. \tag{5.2}$$

From these we see that $s_n(Q_r(\zeta_m)) = 0 = Q_r(s_n(\zeta_m))$ for $2 \leq r \leq 2^m - 1$. Note that s_0 commutes with Q_0 and Q_1, and thus s_n does as well. Now, let X, Y be E_∞ spectra and let $x \in H_*(X)$. We then have by the Cartan formula that, for all $r \geq 0$,

$$Q_r(x^2) = \begin{cases} Q_{r/2}(x)^2 & 2 \mid r \\ 0 & 2 \nmid r. \end{cases} \tag{5.3}$$

Let $f \colon H_*(X) \to H_*(Y)$ be an algebra homomorphism and let $i \geq 0$ be such that $f(Q_r(x)) = Q_r(f(x))$ for all $0 \leq r \leq i$. We then have by Eq. (5.3) that

$f(Q_r(x^2)) = Q_r(f(x^2))$ for all $0 \leq r \leq 2i+1$. Thus, for any $j \geq 0$, we have that $f(Q_r(x^{2^j})) = Q_r(f(x^{2^j}))$ for all $0 \leq r \leq 2^j i + 2^j - 1$ by induction.

In our case, letting $f = s_0$ and $i = 2^m - 1$, this gives $s_0(Q_r(\zeta_m^{2^{n+1-m}})) = Q_r(s_0(\zeta_m^{2^{n+1-m}}))$ for all $1 \leq m \leq n+1$ and $0 \leq r \leq 2^{n+1-m}(2^m - 1) + 2^{n+1-m} - 1 = 2^{n+1} - 1$. Since s_n is a restriction of s_0, it follows that s_n commutes with Q_r for all $0 \leq r \leq 2^{n+1} - 1$.

To see that s_n does not commute with $Q_{2^{n+1}}$, note that $Q_{2^{n+1}}(\zeta_{n+1}) = \zeta_1^{2^{n+1}} \zeta_{n+1}^2$, while $Q_{2^{n+1}}(q_{2^{n+1}-1}) = q_{2^{n+2}+2^{n+1}-2}$. Thus

$$\begin{aligned} s_n(Q_{2^{n+1}}) = q_1^{2^{n+1}} q_{2^{n+1}-1}^2 = a_{1,0}^{2^{n+1}} a_{2^{n+1}-1,0}^2 \neq a_{2^{n+1}+2^n-1,0}^2 = q_{2^{n+2}+2^{n+1}-2} \\ = Q_{2^{n+1}}(s_n(\zeta_{n+1})). \end{aligned}$$

□

5.2 THH of Bordism Spectra

As Topological Hochschild homology is a functor from E_{n+1} spectra to E_n spectra, the orientations $u\colon MO\langle 2^n \rangle \to eo(n-1)$ for $0 \leq n \leq 3$ induce E_∞ maps $THH(u)\colon THH(MO\langle 2^n \rangle) \to THH(eo(n-1))$. An E_{n+1} section of u then induces an E_n section of $THH(u)$, but it is possible that $THH(u)$ could have sections with higher degrees of commutativity that are not induced by sections of u. In order to place bounds on this, we will determine the mod 2 homology of $THH(MO\langle 2^n \rangle)$ and $THH(MO\langle 2^n \rangle, eo(n-1))$, and use this to prove analogues of Proposition 5.1.3 and Proposition 5.1.4.

We begin with a simple calculation.

Lemma 5.2.1. $HH_*^{\mathbb{F}_2}(\mathbb{F}_2[t]) \cong \mathbb{F}_2[u] \otimes \bigwedge(v)$, *where* u *lies in degree 0 and is represented by* t, *and* v *lies in degree 1 and is represented by* $1 \otimes t$.

Proof. Since $\mathbb{F}_2[t]$ is projective as an $\mathbb{F}_2$-module, we may make use of the description of Hochschild homology as $HH_*^R(A) = Tor_*^{A^e}(A, A)$ and choose a a simpler resolution, as in Fig. 5.2. From the resolution in the bottom row we see after tensoring with $\mathbb{F}_2[t]$ over $\mathbb{F}_2[t]^e$ that, as an $\mathbb{F}_2$-module,

$$HH_i(\mathbb{F}_2[t]) \cong \begin{cases} \mathbb{F}_2[t] & i \in \{0,1\} \\ 0 & \text{otherwise,} \end{cases}$$

To see how these are represented in the top row, note that α may be taken to be the identity, so that $HH_0[\mathbb{F}_2[t]]$ is generated as a module by the classes of t^n for various n, as expected. For HH_1, we have that $d'(\beta(t^n \otimes t \otimes 1)) = \alpha(d(t^n \otimes t \otimes 1)) = t^{n+1} \otimes 1 + t^n \otimes t$, so that $\beta(t^n \otimes t \otimes 1) = t^n \otimes 1$. Tensoring with $\mathbb{F}_2[t]$, we then have that $HH_1(\mathbb{F}_2[t])$ is generated as a module by the classes of $t^n \otimes t$ for various n.

The product structure in Hochschild homology is defined by composing the homology product with the shuffle map in Proposition 4.1.6 and the homomorphism induced by the product the in simplicial $\mathbb{F}_2$-module $C_\bullet(A, A)$. Unpacking these definitions, we get that the products in $HH_*(\mathbb{F}_2[t])$ are given by

$$[t^n] \otimes [t^m] \mapsto [t^n \otimes t^m] \mapsto [t^n \otimes t^m] \mapsto [t^{n+m}]$$

$$[t^n] \otimes [t^m \otimes t] \mapsto [(t^n) \otimes (t^m \otimes t)] \mapsto [(t^n \otimes 1) \otimes (t^m \otimes t)] \mapsto [t^{n+m} \otimes t]$$

The lemma then follows. □

$$x \otimes y \otimes z \longmapsto xy \otimes z + x \otimes yz$$

$$x \otimes y \longmapsto xy$$

$$\begin{array}{ccccccccc}
\dots \longrightarrow \mathbb{F}_2[t]^{\otimes 4} & \xrightarrow{d} & \mathbb{F}_2[t] \otimes \mathbb{F}_2[t] \otimes \mathbb{F}_2[t] & \xrightarrow{d} & \mathbb{F}_2[t] \otimes \mathbb{F}_2[t] & \xrightarrow{d} & \mathbb{F}_2[t] \\
\downarrow & & \downarrow \beta & & \downarrow \alpha & & \| \\
\dots \longrightarrow 0 & \xrightarrow{d'} & \mathbb{F}_2[t] \otimes \mathbb{F}_2[t] & \xrightarrow{d'} & \mathbb{F}_2[t] \otimes \mathbb{F}_2[t] & \xrightarrow{d'} & \mathbb{F}_2[t]
\end{array}$$

$$x \otimes y \longmapsto (t \otimes 1 + 1 \otimes t)(x \otimes y)$$

$$x \otimes y \longmapsto xy$$

Figure 5.2: The projective resolution used in the definition of $HH_*(\mathbb{F}_2[t])$ compared to a shorter resolution.

Recall that for an S-algebra A, the $\mathbb{F}_2$ homomorphism $\sigma\colon H_*(A) \to H_{*+1}(THH(A))$ is induced by the composition

$$\Sigma A \cong \Sigma(S \wedge A) \to \Sigma(A \wedge A) \to A \wedge A \wedge \Delta^1_+ \to |THH_\bullet(A)|_1 \to THH(A)$$

. We will have significant use of the following facts about σ.

Proposition 5.2.2. *[AR05, Proposition 5.10] For A any E_2 spectrum, the $\mathbb{F}_2$-module homomorphism σ follows a Leibniz rule. In other words, for $x, y \in H_*(A)$, $\sigma(xy) = x\sigma(y) + \sigma(x)y$.*

Proposition 5.2.3. *Let A be an E_{n+1} spectrum, and let $0 \leq r \leq n-2$. Then $Q_r\sigma = \sigma Q_{r+1}$.*

We can now use the Bökstedt spectral sequence described in Proposition 4.2.5 to calculate the homology of $THH(MO)$. The proof given here is based on the proof of [AR05, Theorem 6.2].

Proposition 5.2.4.

$$H_*(THH(MO)) \cong \mathbb{F}_2[a_{k,j} \mid j \geq 0, k \geq 1, 2 \nmid k] \otimes \mathbb{F}_2[\sigma a_{k,j} \mid j \geq 1, k \geq 1, 2 \nmid k] \otimes \mathbb{F}_2[\sigma a_{1,0}]$$

Here we are identifying elements of $H_*(MO)$ with their images under the inclusion $MO = |THH_\bullet(MO)|_0 \to THH(MO)$.

Proof. The second page of the Bökstedt spectral sequence in this case is given by $E^2_{**} \cong HH_*(H_*(MO)) = HH_*(\mathbb{F}_2[a_{k,j} \mid j \geq 0, k \geq 1, 2 \nmid k])$. Since Hochschild homology commutes with tensor products (when everything is flat) , we have

$$E^2_{*,*} \cong \bigotimes_{\substack{j \geq 0 \\ k \geq 1 \\ 2 \nmid k}} HH_{*,*}(\mathbb{F}_2[a_{k,j}]) \cong \bigotimes_{\substack{j \geq 0 \\ k \geq 1 \\ 2 \nmid k}} (\mathbb{F}_2[u_{k,j}] \otimes \bigwedge(v_{k,j}))$$

$$\cong \mathbb{F}_2[u_{k,j} \mid j \geq 0, k \geq 1, 2 \nmid k] \otimes \bigwedge(v_{k,j} \mid j \geq 0, k \geq 1, 2 \nmid k)$$

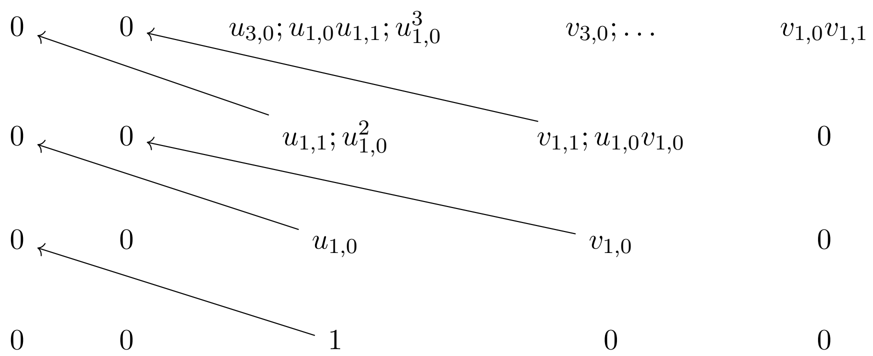

Figure 5.3: The E^2_{**} term of the Bökstedt spectral sequence.

where $u_{k,j}$ lies in bidegree $(0, k2^j)$ and $v_{k,j}$ lies in bidegree $(1, k2^j)$.

We now claim that the spectral sequence collapses at the E^2_{**} term. Since the differential in the Bökstedt spectral sequence follows a Leibniz rule, it suffices to check that d is zero on generators. This may be seen by checking that $d(u_{j,k})$ and $d(v_{j,k})$ lie in degrees where E^2_{pq} is trivial. See Fig. 5.3 for the case of the differentials in $E^2_{*,*}$.

Thus we may write $E^\infty_{**} \cong E^2_{**} \cong \mathbb{F}_2[u_{k,j} \mid k,j] \otimes \bigwedge(v_{k,j} \mid k,j)$. The elements $u_{k,j}$ and $v_{k,j}$ are represented in $HH_*(H_*(MO))$ by $a_{k,j}$ and $1 \otimes a_{k,j}$, so by Proposition 4.2.5 $u_{k,j} = i_*(a_{k,j}) \in H_*(THH(MO))$, where $i\colon MO \to THH(MO)$ is the inclusion of the 0-skeleton, and $v_{k,j} = \sigma(a_{k,j}) \in H_*(THH(MO))/\mathrm{im}(i_*)$. Thus the associated graded of $H_*(THH(MO))$ is given by $\mathbb{F}_2[a_{k,j} \mid j \geq 0, k \geq 1, 2 \nmid k] \otimes \bigwedge(\sigma a_{k,j} \mid j \geq 0, k \geq 1, 2 \nmid k)$. As an $\mathbb{F}_2$-module, this is free, so the additive structure of $H_*(THH(MO))$ is given, and the multiplicative structure is almost given. We have a list of generators, and we see that there are no relations between products of these generators, with one exception: the squares of the generators $\sigma a_{k,j}$ remain unkown.

In order to determine squares, we use Dyer-Lashof operations. By Proposition 5.2.3, we have that $(\sigma a_{k,j})^2 = Q_0(\sigma a_{k,j}) = \sigma(Q_1 a_{k,j})$. To calculate these, we will use Lance's integral lifting, modulo decomposables, as defined in Proposition 3.5.4. In $B^{\mathbb{Q}}[1]$ we have $\hat{Q}_1(\hat{a}_{k,j}) \equiv \hat{Q}_1(2^{-j}\hat{q}_{k2^j}) = k\hat{q}_{k2^{j+1}+1} \equiv k\hat{a}_{k2^{j+1}+1,0}$ modulo decomposables, so that in $H_*(MO)$ we have $Q_1(a_{k,j}) \equiv a_{k2^{j+1}+1,0}$ modulo decomposables. Since σ follows a Leibniz rule, it takes decomposable elements to decomposable elements. Thus we get $(\sigma a_{k,j})^2 \equiv a_{k2^{j+1}+1,0}$ mod decomposables. The proposition immediately follows. □

Making similar identifications in the MSO, $MSpin$, and $MString$ cases, we get the following.

Proposition 5.2.5. *For $0 \leq n \leq 3$, we have*

$$\begin{aligned} H_*(THH(MO\langle 2^n\rangle) \cong \mathbb{F}_2[a_{k,j}^{2^{\rho_n(k)}} &\mid j \geq 0, k \geq 1, 2 \nmid k] \\ &\otimes \bigwedge(\sigma(a_{k,j}^{2^{\rho_n(k)}}) \mid j \geq 0, k \geq 1, 2 \nmid k, \alpha(k) < n+1) \\ &\otimes \mathbb{F}_2[\sigma(a_{k,j}) \mid j \geq 0, k \geq 2^{n+1}-1, 2 \nmid k, \alpha(k) = n+1] \\ &\otimes \mathbb{F}_2[\sigma(a_{k,j}) \mid j \geq 1, k \geq 2^{n+2}-1, 2 \nmid k, \alpha(k) > n+1]. \end{aligned}$$

Proof. The majority of the argument in Proposition 5.2.4 generalizes immediately. The one major point of difference is in determining the squares of $\sigma(a_{k,j}^{2^{\rho_n(k)}})$. For k such that $\alpha(k) \leq n$, $a_{k,j}^{2^{\rho_n(k)}}$ is a square, so that $Q_1(a_{k,j}^{2^{\rho_n(k)}}) = 0$ and $\sigma(a_{k,j}^{2^{\rho_n(k)}})^2 = 0$. For k such

Table 5.6: The homomorphism $THH(u)_*\colon H_*(THH(MO)) \to H_*(THH(H\mathbb{F}_2))$ on monomials in degrees 0 through 2.

$$\begin{aligned} THH(u)_*(1) &= 1 \\ THH(u)_*(a_{1,0}) &= \zeta_1 \\ THH(u)_*(\sigma a_{1,0}) &= \sigma(\zeta_1) \\ THH(u)_*(a_{1,1}) &= 0 \\ THH(u)_*(a_{1,0}^2) &= \zeta_1^2 \end{aligned}$$

that $\alpha(k) \geq n+1$, $a_{k,j}^{2^{\rho_n(k)}} = a_{k,j}$, and we claim that $\sigma(a_{k,j})^2 \equiv \sigma(a_{k2^{j+1}+1,0})$ modulo decomposables.

Note, however, that we must be slightly more careful about which elements are truly decomposable here. Let $B(n)$ denote the preimage of $H_*(MO\langle 2^n\rangle)$ under the quotient map $B^{\mathbb{Z}_{(2)}}[1] \to H_*(MO)$. Thus $B(n)$ is generated by the elements $a_{k,j}^{2^{\rho_n(k)}}$ together with the elements of $2B^{\mathbb{Z}_{(2)}}[1]$. In addition, the quotient map $B(n) \to H_*(MO\langle 2^n\rangle)$ still takes decomposable elements to decomposable elements, and because $H_*(MO\langle 2^n\rangle)$ is closed under the Dyer-Lashof operations, $B(n)$ is closed under the lifted Dyer-Lashof operations $\hat{Q}_i$. Since the $\hat{Q}_i$ follow a Cartan formula, they take decomposable elements in $B(n)$ to decomposable elements in $B(n)$. Now let $k \geq 1$ be such that k is odd and $\alpha(k) \geq n+1$. Then we have that, for $j \geq 0$, $\hat{q}_{k2^j} = \sum_{i=0}^{j} 2^i \hat{a}_{k,i}^{2^{j-i}} \equiv 2^j \hat{a}_{k,j}$ modulo decomposables in $B(n)$. Thus $2^j \hat{Q}_1(\hat{a}_{k,j}) = k2^j \hat{a}_{k2^{j+1}+1,0} + x$, where x is some decomposable element of degree $k2^{j+1}+1$. Now the indecomposable elements of $B(n)$ have an additive basis given by $\{2\hat{a}_{i,\ell}\}_{i,\ell,\rho_n(i)\geq 1} \cup \{\hat{a}_{i,\ell}^{2^{\rho_n(i)}}\}_{i,\ell}$. The elements of this basis which become decomposable when multiplied by 2^j are precisely those elements of the form $\hat{a}_{i,\ell}^{2^{\rho_n(i)}}$ for $\rho_n(i) \geq 1$, but these all lie in even degrees. Since x lies in an odd degree, $2^{-j}x$ must then also be decomposable in $B(n)$. Thus $\hat{Q}_1(\hat{a}_{k,j}) \equiv \hat{a}_{k2^{j+1}+1,0}$ modulo decomposables in $B(n)$, so that $Q_1(a_{k,j}) \equiv a_{k2^{j+1}+1,0}$ modulo decomposables in $H_*(MO\langle 2^n\rangle)$, and $\sigma(a_{k,j})^2 \equiv \sigma(a_{k2^{j+1}+1,0})$ modulo decomposables in $H_*(THH(MO\langle(2^n\rangle))$.

Finally, let $i \geq 3$ be odd. Then there exist unique $j \geq 0$ and $k \geq 1$ with k odd such that $i = k2^{j+1}+1$, and $\alpha(i) = \alpha(k)+1$. Thus $\sigma(a_{i,\ell})) \equiv \sigma(a_{k,j})^2$ modulo decomposables for some $j \geq 0$ and $k \geq 1$ with $\alpha(k) \geq n+1$ if and only if $\ell = 0$ and $\alpha(i) \geq n+2$. The result then follows. □

For $THH(eo(n-1))$, the cases $n=2$ and $n=3$ are done in [AR05], and the cases $n=0$ and $n=1$ are no different.

Proposition 5.2.6. [AR05, Theorom 6.2] *Let $0 \leq n \leq 3$. Then we have*

$$\begin{aligned} H_*(THH(eo(n-1))) \cong & \mathbb{F}_2[\zeta_m^{2^{n+1-m}} \mid 1 \leq m \leq n] \otimes \mathbb{F}_2[\zeta_m \mid m \geq n+1] \\ & \otimes \bigwedge(\sigma(\zeta_m^{2^{n+1-m}}) \mid 1 \leq m \leq n) \otimes \mathbb{F}_2[\sigma(\zeta_{n+1})] \end{aligned}$$

5.3 Sections of THH of the Orientation

Proposition 5.3.1. *Let $0 \leq n \leq 3$. The $\mathbb{F}_2$-algebra homomorphism $THH(u)_*$: $H_*(THH(MO\langle 2^n\rangle) \to H_*(THH(eo(n-1)))$ admits a unique algebra section s_n that*

Table 5.7: The homomorphism $THH(u)_*\colon H_*(THH(MSO)) \to H_*(THH(H\mathbb{Z}))$ on monomials in degrees 0 through 4.

$$\begin{aligned}
THH(u)_*(1) &= 1\\
THH(u)_*(a_{1,0}^2) &= \zeta_1^2\\
THH(u)_*(\sigma(a_{1,0}^2)) &= \sigma(\zeta_1^2)\\
THH(u)_*(a_{3,0}) &= \zeta_2\\
THH(u)_*(\sigma(a_{3,0})) &= \sigma(\zeta_2)\\
THH(u)_*(a_{1,1}^2) &= 0\\
THH(u)_*(a_{1,0}^4) &= \zeta_1^4
\end{aligned}$$

Table 5.8: The homomorphism $THH(u)_*\colon H_*(THH(MSpin)) \to H_*(THH(ko))$ on monomials in degrees 0 through 8.

$$\begin{aligned}
THH(u)_*(1) &= 1\\
THH(u)_*(a_{1,0}^4) &= \zeta_1^4\\
THH(u)_*(\sigma(a_{1,0}^4)) &= \sigma(\zeta_1^4)\\
THH(u)_*(a_{3,0}^2) &= \zeta_2^2\\
THH(u)_*(\sigma(a_{3,0}^2)) &= \sigma(\zeta_2^2)\\
THH(u)_*(a_{7,0}) &= \zeta_3\\
THH(u)_*(\sigma(a_{7,0})) &= \sigma(\zeta_3)\\
THH(u)_*(a_{1,1}^4) &= 0\\
THH(u)_*(a_{1,0}^8) &= \zeta_1^8
\end{aligned}$$

Table 5.9: The homomorphism $THH(u)_*\colon H_*(THH(MString)) \to H_*(THH(tmf))$ on monomials in degrees 0 through 16.

$$THH(u)_*(1) = 1$$
$$THH(u)_*(a_{1,0}^8) = \zeta_1^8$$
$$THH(u)_*(\sigma(a_{1,0}^8)) = \sigma(\zeta_1^8)$$
$$THH(u)_*(a_{3,0}^4) = \zeta_2^4$$
$$THH(u)_*(\sigma(a_{3,0}^4)) = \sigma(\zeta_2^4)$$
$$THH(u)_*(a_{7,0}^2) = \zeta_3^2$$
$$THH(u)_*(\sigma(a_{7,0}^2)) = \sigma(\zeta_3^2)$$
$$THH(u)_*(a_{15,0}) = \zeta_4$$
$$THH(u)_*(\sigma(a_{15,0})) = \sigma(\zeta_4)$$
$$THH(u)_*(a_{1,1}^8) = 0$$
$$THH(u)_*(a_{1,0}^{16}) = \zeta_1^{16}$$

commutes with Q_1 and Q_{2^n}. The section s_n commutes with Q_r for all $0 \le r \le 2^{n+1} - 1$, but it does not commute with $Q_{2^{n+1}}$. Thus the map of spectra $u : THH(MO\langle 2^n \rangle) \to THH(eo(n-1))$ does not admit an $E_{2^{n+1}+1}$ section.

Proof. We begin by showing that if s_n commutes with Q_1 and Q_{2^n}, then it must be given by $s_n(\xi_m^{2^{n+1-m}}) = a_{2^m-1,0}^{2^{n+1-m}}$ and $s_n(\sigma(\zeta_m^{2^{n+1-m}})) = \sigma(a_{2^m-1,0}^{2^{n+1-m}})$ for $1 \le m \le n$, $s_n(\zeta_m) = q_{2^m-1} = a_{2^m-1,0}$ for $m \ge n+1$, and $s_n(\sigma(\zeta_{n+1})) = \sigma(a_{2^{n+1}-1}$. First, we see from Table 5.6, Table 5.7, Table 5.8, and Table 5.9 that $THH(u)_*$ is an isomorphism in degrees 0 through $2^{n+1}-1$, and that a section s_n must satisfy $s_n(\zeta_m^{2^{n+1-m}}) = a_{2^m-1,0}^{2^{n+1-m}}$ for $1 \le m \le n+1$ and $s_n(\sigma(\zeta_m)) = \sigma(a_{2^m-1,0}^{2^{n+1-m}})$ for $1 \le m \le n$. Since $s_n(\zeta_{n+1}) = a_{2^{n+1}-1,0} = q_{2^{n+1}-1}$, we must have $s_n(\zeta_m) = q_{2^m-1} = a_{2^m-1,0}$ for all $m \ge n+1$ by the same argument as in Proposition 5.1.3. Thus it remains to determine $s_n(\sigma(\zeta_{n+1})$.

We see from Table 5.6, Table 5.7, Table 5.8, and Table 5.9 that $s_n(\sigma(\zeta_{n+1}))$ must be equal to either $\sigma(a_{2^{n+1}-1,0})$ or $\sigma(a_{2^{n+1}-1,0}) + a_{1,1}^{2^n}$. We claim that if s_n commutes with Q_{2^n}, then we must have $s_n(\sigma(\zeta_{n+1}) = \sigma(a_{2^{n+1}-1,0})$. First, by Proposition 3.4.6 we have

$$Q_{2^n}(\sigma(\zeta_{n+1}) = \sigma(Q_{2^n+1}(\zeta_{n+1}) = 0$$

in the case that $n \ge 1$ and

$$Q_1(\sigma(\zeta_1)) = \sigma(Q_2(\zeta_1)) = \sigma(\zeta_1^4) = 0$$

in the case that $n = 1$, since σ follows a Leibniz rule and is thus zero on squares. Similarly, by Corollary 3.5.3 we have that

$$Q_{2^n}(\sigma(a_{2^{n+1}-1,0}) = \sigma(Q_{2^n+1}(q_{2^{n+1}-1}) = \sigma\left(\binom{2^{n+1}+2^n-1}{2^{n+1}-1} q_{2^{n+2}+2^n-1}\right).$$

If $n \ge 1$, then $\binom{2^{n+1}+2^n-1}{2^{n+1}-1} \equiv 0$ modulo 2, so that $Q_{2^n}(\sigma(a_{2^{n+1}-1,0})) = 0$. In the case $n = 0$, $q_{4+1-1} = q_4 = q_1^4$, so that $Q_1(\sigma(a_{1,0})) = 0$. To calculate $Q_{2^n}(a_{1,1}^{2^n})$ we make use of the integral lifts again. In $B^{\mathbb{Q}}[1]$ we have

$$\hat{q}_2 = \hat{a}_{1,0}^2 + 2\hat{a}_{1,1} = \hat{q}_1^2 + 2\hat{a}_{1,1}$$

by Definition-Proposition 2.4.12, so that

$$\hat{Q}_1(\hat{a}_{1,1}) = \hat{Q}_1\left(\frac{1}{2}\hat{q}_2 - \frac{1}{2}q_1^2\right) = \frac{1}{2}\hat{Q}_1(\hat{q}_2) + \hat{Q}_0(\hat{(}q)_1)\hat{Q}_1(\hat{q}_1) = \hat{q}_5 - \hat{q}_2\hat{q}_3.$$

Thus we get $Q_1(a_{1,1}) = q_5 + q_2q_3 = a_{5,0} + a_{1,0}^2a_{3,0} \neq 0$, and by the Cartan formula $Q^{2^n}(a_{1,1}^{2^n}) = Q_1(a_{1,1})^{2^n} \neq 0$. Thus $s_n(\sigma(\zeta_{n+1}))$ must be $\sigma(a_{2^{n+1}-1,0})$.

It now remains to check that s_n actually is a section of $THH(u)_*$ and to check which Dyer-Lashof Operations s_n respects. To begin with, note that $THH(u)_*$ factors as a tensor product of algebra homomorphisms

$$u'_*\colon H_*(MO\langle 2^n\rangle) \to H_*(eo(n-1))$$

and

$$\begin{aligned} u''_*\colon & \bigwedge(\sigma(a_{k,j}^{2^{\rho_n(k)}}) \mid j \geq 0, k \geq 1, 2 \nmid k, \alpha(k) < n+1) \\ & \otimes \mathbb{F}_2[\sigma(a_{k,j}) \mid j \geq 0, k \geq 2^{n+1}-1, 2 \nmid k, \alpha(k) = n+1] \\ & \otimes \mathbb{F}_2[\sigma(a_{k,j}) \mid j \geq 1, k \geq 2^{n+2}-1, 2 \nmid k, \alpha(k) > n+1] \\ \to & \bigwedge(\sigma(\zeta_m) \mid 1 \leq m \leq n) \otimes \mathbb{F}_2[\sigma(\zeta_{n+1})]. \end{aligned}$$

Here we are identifying $H_*(MO\langle 2^n\rangle)$ and $H_*(eo(n-1))$ with their images in $H_*(THH(MO\langle 2^n\rangle)$ and $H_*(THH(eo(n-1)))$, and under this identification $u'_* = u_*$. Similarly, the s_n we have just defined factors as a tensor product of

$$s'_n\colon H_*(eo(n-1)) \to H_*(MO\langle 2^n\rangle)$$

with

$$\begin{aligned} s''_n\colon & \bigwedge(\sigma(\zeta_m) \mid 1 \leq m \leq n) \otimes \mathbb{F}_2[\sigma(\zeta_{n+1})] \\ \to & \bigwedge(\sigma(a_{k,j}^{2^{\rho_n(k)}}) \mid j \geq 0, k \geq 1, 2 \nmid k, \alpha(k) < n+1) \\ & \otimes \mathbb{F}_2[\sigma(a_{k,j}) \mid j \geq 0, k \geq 2^{n+1}-1, 2 \nmid k, \alpha(k) = n+1] \\ & \otimes \mathbb{F}_2[\sigma(a_{k,j}) \mid j \geq 1, k \geq 2^{n+2}-1, 2 \nmid k, \alpha(k) > n+1], \end{aligned}$$

where s'_n is just the section of u_* defined inProposition 5.1.4.

We already know from Proposition 5.1.4 that s'_n is a section of u'_* and that s'_n commutes with Q_r for $0 \leq r \leq 2^{n+1}-1$. For s''_n, we have that $u''_*(s''_n(\sigma(\zeta_{n+1}))) = \sigma(\zeta_{n+1})$ and $u''_*(s''_n(\sigma(\zeta_m^{2^{n+1-m}}))) = \sigma(\zeta_m^{2^{n+1-m}})$ for $1 \leq m \leq n$ by construction. Thus s_n is an algebra section of $THH(u)_*$, and it remains to check for which r, $Q_r(s_n(\sigma(\zeta_m^{2^{n+1-m}}))) = s_n(Q_r(\sigma(\zeta_m^{2^{n+1-m}})))$ for $1 \leq m \leq n+1$.

First, let $1 \leq m \leq n$. Then as usual $Q_r(\sigma(\zeta_m^{2^{n+1-m}})) = \sigma(Q_{r+1}(\zeta_m^{2^{n+1-m}}))$. If r is even, then $Q_{r+1}(\zeta_m^{2^{n+1-m}}) = 0$, and if r is odd, then $Q_{r+1}(\zeta_m^{2^{n+1-m}}) = Q_{(r+1)/2}(\zeta_m^{2^{n-m}})^2$. In either case, $s_n(Q_{r+1}(\sigma(\zeta_m^{2^{n+1-m}}))) = 0$, and $Q_{r+1}(s_n(\sigma(\zeta_m^{2^{n+1-m}}))) = Q_{r+1}(\sigma(a_{2^m-1,0}^{n+1-m})) = 0$ by the same argument. Finally, for $1 \leq r \leq 2^{n+1}-2$, we have

$$Q_r(\sigma(\zeta_{n+1})) = \sigma(Q_{r+1}(\zeta_{n+1})) = 0$$

and

$$Q_r(\sigma(a_{2^{n+1}-1,0})) = \sigma(Q_{r+1}(q_{2^{n+1}-1})) = \sigma\left(\binom{r+2^{n+1}-1}{2^{n+1}-1} q_{2^{n+2}-1+r}\right) = 0.$$

For $r = 2^{n+1}-1$, we have

$$Q_{2^{n+1}-1}(\sigma(\zeta_{n+1})) = \sigma(Q_{2^{n+1}}(\zeta_{n+1})) = \sigma(\zeta_1^{2^{n+1}}\zeta_{n+1}^2) = 0,$$

and

$$Q_{2^{n+1}-1}(\sigma(a_{2^{n+1}-1,0})) = \sigma(Q_{2^{n+1}}(q_{2^{n+1}-1})) = \sigma\left(\binom{2^{n+2}-2}{2^{n+1}-2} q_{2^{n+2}+2^{n+1}-2}\right)$$
$$= \sigma(q^2_{2^{n+1}+2^n-1}) = 0.$$

For $r = 2^{n+1}$, however, we have

$$Q_{2^{n+1}}(\sigma(\zeta_{n+1})) = \sigma(Q_{2^{n+1}+1}(\zeta_{n+1})) = \sigma(\zeta_1^{2^{n+1}}\zeta_{n+2}) = \zeta_1^{n+1}\sigma(\zeta_{n+2}).$$

Since $\sigma(\zeta_{n+1})^2 = \sigma(Q_1(\zeta_{n+1})) = \sigma(\zeta_2)$, we then have $s_n(Q_{2^{n+1}}(\sigma(\zeta_{n+1}))) = s_n(\zeta_1^{n+1}\sigma(\zeta_{n+1})^2) = a_{1,0}^{n+1}\sigma(a_{2^{n+1}-1,0})^2$. On the other hand,

$$Q_{2^{n+1}}(\sigma(a_{2^{n+1}-1,0})) = \sigma(Q_{2^{n+1}-1}(q_{2^{n+1}-1})) = \sigma\left(\binom{2^{n+2}-1}{2^{n+1}-2} q_{2^{n+2}+2^{n+1}-1}\right)$$
$$= \sigma(a_{2^{n+2}+2^{n+1}-1,0}).$$

Now $\alpha(2^{n+2}+2^{n+1}-1) = n+2$, so $\sigma(a_{2^{n+2}+2^{n+1}-1,0})$ is decomposable in $H_*(THH(MO\langle(2^n\rangle)))$. To determine how it decomposes, we split into two cases.

First, assume that $n \geq 1$. Then $\sigma(a_{2^{n+1}+2^n-1,0})$ is indecomposable, and we have that

$$\sigma(a_{2^{n+1}+2^n-1,0})^2 = \sigma(Q_1(q_{2^{n+1}+2^n-1})) = \sigma\left(\binom{2^{n+1}+2^n-1}{2^{n+1}+2^n-2} q_{2^{n+2}+2^{n+1}-1}\right)$$
$$= a_{2^{n+2}+2^{n+1}-1,0}.$$

Thus we have

$$Q_{2^{n+1}}(s_n(\sigma(\zeta_{n+1}))) = \sigma(a_{2^{n+1}+2^n-1,0})^2 \neq a_{1,0}^{n+1}\sigma(a_{2^{n+1}-1,0})^2 = s_n(Q_{2^{n+1}}(\sigma(\zeta_{n+1}))).$$

Now assume $n = 0$. Then $\sigma(a_{1,1})$, $\sigma(a_{1,0})$, and $a_{1,0}$ are indecomposable. As we have previously seen, $Q_1(a_{1,1}) = a_{5,0} + a_{1,0}^2 a_{3,0}$, so that $\sigma(a_{1,1})^2 = \sigma(a_{5,0} + a_{1,0}^2 a_{3,0}) = \sigma(a_{5,0}) + a_{1,0}^2\sigma(a_{3,0})$. In addition, we have

$$\sigma(a_{1,0})^2 = \sigma(Q_1(q_1)) = \sigma(q_3) = \sigma(a_{3,0}).$$

Putting these together, we have that $\sigma(a_{2^2+2^1-1,0}) = \sigma(a_{5,0}) = \sigma(a_{1,1})^2 + a_{1,0}^2\sigma(a_{3,0})$. Thus,

$$Q_2(s_0(\sigma(\zeta_1))) = \sigma(a_{1,1})^2 + a_{1,0}^2\sigma(a_{3,0}) \neq a_{1,0}^2\sigma(a_{1,0})^2 = s_0(Q_2(\sigma(\zeta))).$$

□

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