Foreword

First of all, I want to thank my advisor, professor John Rognes. This thesis could never have been written without his ideas and his help. I also want to thank all the other students at C207/C205 for making studying interesting, in more than one way.

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0 Introduction

Bökstedt introduced topological Hochschild homology (THH) in an unpublished paper, [4], and showed that $THH(R)$ accepts an interesting map from $K(R)$, algebraic K-theory. Then Bökstedt, Hsiang and Madsen showed that this map factors through topological cyclic homology (TC), in [3]. Bökstedt calculated $THH(\mathbb{Z}/p)$ and $THH(\mathbb{Z})$ in [5], and there are some other calculations elsewhere, for example in [15], where McClure and Staffeldt calculate $\pi_*(THH(\ell) \wedge S^0/p)$ for odd primes.

In the hope of understanding $K(R)$ for other rings or S-algebra spectra, the first step is to get as much information as possible about $THH(R)$. Often it is not feasible to obtain the homotopy type of $THH(R)$, but this might not be necessary, it might be enough to know the homotopy after smashing with a finite CW-spectrum. We will show that Bökstedt's spectral sequence is an algebra spectral sequence, and under a flatness condition also a coalgebra spectral sequence, and we will use it to calculate the homology of $THH(R)$ for several S-algebra spectra, including ku, ko, ju and j (at odd primes), and we will calculate the homotopy of $THH(ku)$ and $THH(ko)$ after smashing with suitable finite CW-spectra.

For our definition of topological Hochschild homology to work, we need to work in a category of spectra with coherent isomorphisms $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$ and $X \wedge Y \cong Y \wedge X$, a symmetric monoidal category. At the time Bökstedt introduced THH there was no known category of spectra with these properties, but now there are at least two such frameworks. We can either use the symmetric spectra by Hovey, Shipley and Smith in [11], or the category of commutative S-algebras in [10], by Elmendorf, Kriz, Mandell and May.

Here is a short summary of the article.

In chapter 1 we first define the Hochschild homology of an algebra, and then we define $THH(R)$ as the geometric realization of a simplicial spectrum.

In chapter 2 we recall some facts about the Steenrod algebra and define the Adams spectral sequence (ASS). We also introduce Dyer-Lashof operations.

In chapter 3 we recall from [14] that $THH(R) \cong R \otimes S^1$, with a suitable interpretation of $R \otimes S^1$, and show that this gives $THH(R)$ the structure of an R-Hopf algebra, at least in the homotopy category.

In chapter 4 we set up Bökstedt's spectral sequence (BSS), which is our main tool for doing calculations in homology, and show that under a flatness condition there is a Hopf algebra structure on the spectral sequence, which gives very strong conditions on the differentials.

In chapter 5 we calculate $H_*(THH(R); \mathbb{Z}/p)$ for $R = \mathbb{Z}/p$, $\mathbb{Z}$, ku, ko and ℓ, and in chapter 6 we do the same for j at odd primes and ju at $p = 2$. Here j is the fiber $j \longrightarrow \ell \xrightarrow{\psi^r - 1} \Sigma^{2p-2}\ell$ and ju is the fiber $ju \longrightarrow ku \xrightarrow{\psi^3 - 1} bu$.

In chapter 7 we take a closer look at $THH(ku)$ and calculate $\pi_*(THH(ku) \wedge S^0/2)$, and in chapter 8 we calculate $\pi_*(THH(ko) \wedge Y)$, where $Y = S^0/2 \wedge C_\eta$.

1 Definition of THH

The idea of topological Hochschild homology, THH, is to mimic the definition of Hochschild homology by replacing tensor products of rings with smash products of spectra. Let us first review the definition of Hochschild homology. See e.g. [7, IX] for a more complete treatment.

Let k be a commutative ring. The Hochschild complex of a graded k-algebra Λ is the chain complex

$$\ldots \longrightarrow \Lambda^{\otimes(q+1)} \longrightarrow \Lambda^{\otimes q} \longrightarrow \ldots \longrightarrow \Lambda^{\otimes 3} \longrightarrow \Lambda^{\otimes 2} \longrightarrow \Lambda,$$

where $\Lambda^{\otimes n}$ denotes the n-fold tensor product over k, with boundary maps

$$\begin{aligned} d(a_0 \otimes a_1 \otimes \ldots \otimes a_q) = \sum_{i=0}^{q-1} & (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_q + \\ & (-1)^{q+|a_q|(|a_0|+\ldots+|a_{q-1}|)} a_q a_0 \otimes a_1 \otimes \ldots \otimes a_{q-1}. \end{aligned}$$

Hochschild homology of Λ is then defined as the homology of this chain complex, and is denoted by $HH_*^k(\Lambda)$, or simply $HH_*(\Lambda)$.

When Λ is flat over k, which is always the case when k is a field, we can also describe $HH_*(\Lambda)$ as a Tor-group,

$$HH_*(\Lambda) \cong Tor_*^{\Lambda\otimes\Lambda^{op}}(\Lambda, \Lambda),$$

because a natural flat resolution of Λ as a $\Lambda \otimes \Lambda^{op}$-module is $B(\Lambda)_* = B(\Lambda, \Lambda, \Lambda)_*$ with $B(\Lambda)_k = \Lambda \otimes \Lambda^{\otimes k} \otimes \Lambda$, and $\Lambda \otimes_{\Lambda\otimes\Lambda^{op}} B(\Lambda)_*$ is the Hochschild complex.

We will need to calculate the Hochschild homology of some rings, so we recall the following basic results. Let $k = \mathbb{Z}/p$, $P(x) = \mathbb{Z}/p[x]$, $E(x) = \mathbb{Z}/p[x]/(x^2)$, and let $\Gamma(x)$ be the divided power algebra generated by $\{\gamma_k(x)|k \geq 0\}$ as a $\mathbb{Z}/p$-module, with multiplication $\gamma_i(x) \otimes \gamma_j(x) = \binom{i+j}{j}\gamma_{i+j}(x)$. We will write $P(x_1, x_2, \ldots)$ for $P(x_1) \otimes P(x_2) \otimes \ldots$, and so on. If p is odd, we also require $|x|$ to be odd in $E(x)$.

Proposition 1.1. *Let $\sigma x \in HH_1(\Lambda)$ be the class represented by $1 \otimes x$ in the Hochschild complex. Then*

i. $HH_*(P(x)) = P(x) \otimes E(\sigma x)$,

ii. $HH_*(E(x)) = E(x) \otimes \Gamma(\sigma x)$,

iii. $HH_*(\Lambda_1 \otimes \Lambda_2) = HH_*(\Lambda_1) \otimes HH_*(\Lambda_2)$.

Proof. See for example [15, Proposition 2.1]. ☐

Recall the following result about the structure of a Γ-algebra.

Lemma 1.2. *Over $k = \mathbb{Z}/p$ we have*

$$\Gamma(x) \cong P_p(x, \gamma_p(x), \gamma_{p^2}(x), \ldots),$$

where $P_k(x) = P(x)/(x^k)$.

We can also describe the Hochschild complex as a simplicial abelian group, by letting

$$[q] \mapsto \Lambda^{\otimes(q+1)}.$$

If Λ is commutative, the Hochschild complex also has a k- and a Λ-algebra structure, we get maps $HH_p(\Lambda) \otimes HH_q(\Lambda) \longrightarrow HH_{p+q}(\Lambda)$ and $HH_p(\Lambda) \otimes_\Lambda HH_q(\Lambda) \longrightarrow HH_{p+q}(\Lambda)$, see [7, XI.6].

This is also the way we will define topological Hochschild homology of an S-algebra R. Let R be an S-algebra, and define $THH(R)$ as the geometric realization of the simplicial spectrum $THH(R)_\bullet$ with

$$THH(R)_q = R^{\wedge(q+1)},$$

where $R^{\wedge k}$ denotes the k-fold smash product of R. Let μ be the multiplication on R. The face maps $d_i : R^{\wedge(q+1)} \longrightarrow R^{\wedge q}$ are given by $id_R^{\wedge i} \wedge \mu \wedge id_R^{\wedge q-i-1}$ for $0 \leq i < q$ and by a series of twist maps that puts the last factor first followed by $\mu \wedge id_R^{q-1}$ for $i = q$, which we will write as

$$d_i(r_0 \wedge r_1 \wedge \ldots \wedge r_q) = \begin{cases} r_0 \wedge \ldots \wedge r_i r_{i+1} \wedge \ldots \wedge r_q & \text{for } 0 \leq i < q, \\ r_q r_0 \wedge r_1 \wedge \ldots \wedge r_{q-1} & \text{for } i = q. \end{cases}$$

The degeneracy maps are defined as $s_j = id_R^j \wedge i \wedge id_R^{q-j+1}$, where i is the inclusion $i : S \longrightarrow R$, which we will write as

$$s_j(r_0 \wedge r_1 \wedge \ldots \wedge r_q) = r_0 \wedge \ldots \wedge 1 \wedge \ldots \wedge r_q.$$

The idea of describing $THH(R)$ this way is not new, but it requires a symmetric monoidal category of spectra, and that was, until a few years ago, believed not to exist.

Note that if we only require R to be an S-algebra, $THH(R)$ is only an S-module. If R is also commutative, the face and degeneracy maps are S-algebra maps, and in fact also R-algebra maps, and we get an S- and an R-algebra structure on $THH(R)$. The S-algebra multiplication factors through the R-algebra multiplication, $THH(R) \wedge THH(R) \longrightarrow THH(R) \wedge_R THH(R) \longrightarrow THH(R)$.

We also need an inclusion $i : R \longrightarrow THH(R)$ and a map $\sigma : \Sigma R \longrightarrow THH(R)$. We will construct these maps in the same way as in [15, §3].

Let $S(R)_\bullet$ be the simplicial spectrum with $S(R)_q = R^{\vee(q+1)}$, with face and degeneracy maps as suggested by the notation

$$d_i(r_0 \vee r_1 \vee \ldots \vee r_q) = \begin{cases} r_0 \vee \ldots \vee r_i r_{i+1} \vee \ldots \wedge r_q & \text{for } 0 \leq i < q, \\ r_q r_0 \vee r_1 \vee \ldots \vee r_{q-1} & \text{for } i = q, \end{cases}$$

and

$$s_j(r_0 \vee r_1 \vee \ldots \vee r_q) = r_0 \vee \ldots \vee 1 \vee \ldots \vee r_q,$$

with a 1 inserted in summand j.

Then we obviously have a simplicial map $S(R)_\bullet \longrightarrow THH(R)_\bullet$, which gives us a map

$$\omega : |S(R)_\bullet| \longrightarrow THH(R).$$

Using that

$$|S(R)_\bullet| \cong |S(S^0)_\bullet| \wedge R \cong S^1_+ \wedge R$$

we get $i : R \longrightarrow THH(R)$ as the composition $R \longrightarrow S^1_+ \wedge R \longrightarrow THH(R)$. Using that $S^1_+ \wedge R \simeq R \vee \Sigma R$ we get the map $\sigma : \Sigma R \longrightarrow THH(R)$ as the composition $\Sigma R \longrightarrow S^1_+ \wedge R \longrightarrow THH(R)$ in the homotopy category.

2 Hopf algebras and the Steenrod algebra

We will see that the Steenrod algebra is a Hopf algebra, and also that $THH(R)$ is a Hopf algebra spectrum when R is a commutative S-algebra, so let us recall the definition of a Hopf algebra. Let k be a commutative ring, and let H be a graded k-module. We say that H is a k-algebra if we have a multiplication $\phi : H \otimes H \longrightarrow H$ and a unit $\eta : k \longrightarrow H$ such that the following diagrams commute:

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "\phi\otimes 1"] \arrow[d, "1\otimes\phi"] & H \otimes H \arrow[d, "\phi"] \\ H \otimes H \arrow[r, "\phi"] & H \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} H \otimes k \arrow[r, "1\otimes\eta"] \arrow[dr, "\cong"'] & H \otimes H \arrow[d, "\phi"] & k \otimes H \arrow[l, "\eta\otimes 1"'] \arrow[dl, "\cong"] \\ & H & \end{tikzcd} \tag{2}$$

We say that H is a coalgebra if we have a coproduct $\psi : H \longrightarrow H \otimes H$ and a counit $\epsilon : H \longrightarrow k$ such that the following diagrams commute:

$$\begin{tikzcd} H \otimes H \otimes H & H \otimes H \arrow[l, "1\otimes\psi"'] \\ H \otimes H \arrow[u, "\psi\otimes 1"] & H \arrow[l, "\psi"'] \arrow[u, "\psi"'] \end{tikzcd} \tag{3}$$

$$\begin{tikzcd} H \otimes k & H \otimes H \arrow[l, "1\otimes\epsilon"'] \arrow[r, "\epsilon\otimes 1"] & k \otimes H \\ & H \arrow[ul, "\cong"] \arrow[u, "\psi"] \arrow[ur, "\cong"'] & \end{tikzcd} \tag{4}$$

H is a bialgebra if ψ and ϵ are algebra maps, or equivalently, if ϕ and η are coalgebra maps. This gives four extra conditions. We will always let τ be the twist map. For ϕ and ψ we get the commutative diagram

$$\begin{tikzcd} H \otimes H \arrow[r, "\phi"] \arrow[d, "\psi\otimes\psi"] & H \arrow[d, "\psi"] \\ H \otimes H \otimes H \otimes H \arrow[r, "(\phi\otimes\phi)\tau"] & H \otimes H \end{tikzcd} \tag{5}$$

For ϕ and ϵ we get $\epsilon\phi = \epsilon \otimes \epsilon$. For η and ψ we get $\psi\eta = \eta \otimes \eta$, and for η and ϵ we get $\epsilon\eta = 1$.

A Hopf algebra is a bialgebra with a conjugation $\chi : H \longrightarrow H$ such that the following diagram commutes:

$$\begin{tikzcd} & H \arrow[r, "\psi"] \arrow[d, "\psi"] \arrow[ddr, "\epsilon"] & H \otimes H \arrow[ddrr, "1\otimes\chi"] & & \\ & H \otimes H \arrow[dd, "\chi\otimes 1"] & & & \\ & & k \arrow[ddrr, "\eta"] & & H \otimes H \arrow[dd, "\phi"] \\ & & & & \\ & H \otimes H \arrow[rrr, "\phi"] & & & H \end{tikzcd} \tag{6}$$

We say that H is connected if $H_n = 0$ for $n < 0$ and $\epsilon : H \longrightarrow k$ is an isomorphism in degree 0.

The ring of stable operations on mod p cohomology, or the ring of operations on mod p cohomology of spectra, is called the Steenrod algebra and will be denoted by A. See [22] or [16] for a more complete treatment of this. The ring structure comes from composition of cohomology operations, and

A is a $\mathbb{Z}/p$-algebra in the obvious way. The mod p cohomology of any space or spectrum is a module over A. For $p=2$, A is generated by

$$Sq^i : H^*(-;\mathbb{Z}/2) \longrightarrow H^{*+i}(-;\mathbb{Z}/2)$$

for $i \geq 0$, with Sq^0 as the identity and Sq^1 as the Bockstein. For odd primes A is generated by

$$P^i : H^*(-;\mathbb{Z}/p) \longrightarrow H^{*+2i(p-1)}(-;\mathbb{Z}/p)$$

for $i \geq 0$ and a Bockstein

$$\beta : H^*(-;\mathbb{Z}/p) \longrightarrow H^{*+1}(-;\mathbb{Z}/p).$$

Note that Sq^1 for $p=2$ corresponds to β for odd primes, and that P^i for odd primes corresponds to Sq^{2i} for $p=2$. If $x \in H_n(R;\mathbb{Z}/2)$, where R is a ring spectrum, then $Sq^i(x)=0$ for $i>n$ and $Sq^n(x)=x^2$. If $x \in H_n(R;\mathbb{Z}/p)$ for p odd, then $P^i(x)=0$ for $2i>n$ and $P^i(x)=x^p$ for $2i=n$.

Proposition 2.1. *The composition of Steenrod squares or powers satisfies the following relations, known as the Adém relations.*

$$\begin{aligned}
Sq^aSq^b &= \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j}Sq^j \text{ if } a<2b. \\
P^aP^b &= \sum_j (-1)^{a+j}\binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j}P^j \text{ if } a<pb. \\
P^a\beta P^b &= \sum_j (-1)^{a+j}\binom{(p-1)(b-j)}{a-pj} \beta P^{a+b-j}P^j \\
&\quad -\sum_j (-1)^{a+j}\binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j}\beta P^j \text{ if } a\leq pb.
\end{aligned}$$

Using those relations, it is possible to show that A is actually generated by the elements Sq^{2^k} for $p=2$ and by β and P^{p^k} for p odd.

Recall that A is not only an algebra, we also have a comultiplication and a conjugation making it a Hopf algebra. The comultiplication $\psi : A \longrightarrow A \otimes A$ is given by

$$\psi(Sq^k) = \sum_{i+j=k} Sq^i \otimes Sq^j$$

for $p=2$, and a similar formula for odd primes. The conjugation can be calculated inductively, using the formula $\chi(a) = -a - \sum a'\chi(a'')$, where $\psi(a) = a\otimes 1 + 1 \otimes a + \sum a' \otimes a''$.

The dual of A, denoted by A_*, is often easier to work with than A. The main reason is that while the multiplication in A is not commutative, the comultiplication is. This means that the multiplication in A_* is commutative, while the comultiplication is not. While the algebra structure of A is very complicated, A_* is just a polynomial algebra for $p = 2$, and an algebra on some polynomial and some exterior classes for odd primes. To be precise, we have

$$A_* = P(\xi_1, \xi_2, \ldots)$$

for $p = 2$, where $|\xi_k| = 2^k - 1$, and

$$A_* = P(\xi_1, \xi_2, \ldots) \otimes E(\tau_0, \tau_1, \ldots)$$

for p odd, where $|\xi_k| = 2(p^k - 1)$ and $|\tau_k| = 2p^k - 1$.

Of course the comultiplication is more complicated in A_*. We have

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j$$

for $p = 2$ and similar formulas for odd primes. Here $\xi_0 = 1$.

We will use the conjugated classes in A_* in the rest of the paper, because the homology of most of the spectra we will work with involves the conjugate classes. We will denote $\chi(\xi_k)$ by $\bar{\xi}_k$, and $\chi(\tau_k)$ by $\bar{\tau}_k$. We also have

$$A_* = P(\bar{\xi}_1, \bar{\xi}_2, \ldots)$$

and

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{2^i},$$

and similar formulas for odd primes. This can be seen by using that χ is an anti-homomorphism, so that the diagram

$$\begin{tikzcd} A_* \arrow[r, "\psi"] \arrow[d, "\chi"] & A_* \otimes A_* \arrow[d, "(\chi\otimes\chi)\tau"] \\ A_* \arrow[r, "\psi"] & A_* \otimes A_* \end{tikzcd}$$

commutes, and

$$\psi(\bar{\xi}_k) = (\chi \otimes \chi)\tau(\psi(\xi_k)) = (\chi \otimes \chi)\tau \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{2^i}.$$

There is a very useful spectral sequence, the Adams spectral sequence (ASS), which has the A-module structure, or the A_*-comodule structure, of a spectrum X as input and converges to the homotopy of X, see [1]. We will use $\hat{E}$ for the ASS to distinguish it from Bökstedt's spectral sequence; $\hat{E}(X)$ will always be the ASS converging to $\pi_*(X)^\wedge_p$.

Theorem 2.2. *Let X be a spectrum localized at p.*

i. *Then there is a spectral sequence*

$$\hat{E}^2_{**}(X) = Ext_A^{**}(H^*(X;\mathbb{Z}/p),\mathbb{Z}/p) \Longrightarrow \pi_*(X)^\wedge_p.$$

If X has finite type, i.e., there is an n such that $H^i(X;\mathbb{Z}) = 0$ for $i < n$ and finitely generated for $i \geq n$, this spectral sequence converges strongly. An element $x \in Ext_A^{p,q}(H^*(X;\mathbb{Z}/p),\mathbb{Z}/p)$ that survives to $\hat{E}^\infty(X)$ gives an element in $\pi_{q-p}(X)$, and we say that x is in homotopic degree $q-p$.

ii. *A pairing $X' \wedge X'' \longrightarrow X$ induces a pairing between the spectral sequences. In particular, if X is a ring spectrum, there is a multiplication on $\hat{E}^r(X)$ for each r which makes $\hat{E}(X)$ into a $\mathbb{Z}/p$-algebra spectral sequence.*

Proof. The ASS is described several places in the literature, see for example [1]. The second part can be found in for example [20, Theorem 2.3.3]. □

Note that being an algebra spectral sequence implies that the differentials are derivations, i.e., $d_r(xy) = d_r(x)y + (-1)^{|x|}xd_r(y)$.

Calculation of $Ext_A(M,N)$ is rather complicated in general, but sometimes it simplifies greatly.

Lemma 2.3.

i. *Let $B \subset A$ be a subalgebra, and write $A//B$ for $A \otimes_B \mathbb{Z}/p = A/I(B)$, where $I(B)$ is the two-sided ideal in A generated by the elements in B with positive degree. Then*

$$Ext_A(A \otimes_B M, N) \cong Ext_B(M,N).$$

ii. *If $M = \mathbb{Z}/p\{y_1, y_2, \ldots, z_1, z_1', z_2, z_2', \ldots\}$ is an $E(x)$-module, where x acts trivially on the y_i and $xz_i = z_i'$, then*

$$Ext_{E(x)}(M,\mathbb{Z}/p) = \mathbb{Z}/p\{y_1^*, y_2^*, \ldots\} \otimes P(\bar{x}) \oplus \mathbb{Z}/p\{z_1^*, z_2^*, \ldots\},$$

where $\bar{x}$ is in bidegree $(1, |x|)$, or in homotopic degree $|x|-1$.

There are operations similar to the Steenrod operations which operate on the homology of commutative S-algebras, called Dyer-Lashof operations, with

$$Q^k : H_*(-;\mathbb{Z}/2) \longrightarrow H_{*+k}(-;\mathbb{Z}/2)$$

for $p = 2$ and

$$Q^k : H_*(-;\mathbb{Z}/p) \longrightarrow H_{*+2k(p-1)}(-;\mathbb{Z}/p)$$

for odd p. See for example [6, Chapter III] by Mark Steinberger. These operations satisfy $Q^k(x) = 0$ for $k < |x|$ and $Q^k(x) = x^2$ for $|x| = k$ for $p = 2$, and $Q^k(x) = 0$ for $2k < |x|$ and $Q^k(x) = x^p$ for $|x| = 2k$ for p odd. Using composition of homology operations as the ring operation, the Dyer-Lashof operations form a $\mathbb{Z}/p$-algebra, the Dyer-Lashof algebra.

From [6, Chapter III, Remark 1.2] we get that the Dyer-Lashof operations are dual to the Steenrod operations in the following way:

Proposition 2.4. *Let X be a finite CW-spectrum. Then the following diagram commutes:*

$$\begin{tikzcd} H^*(X;\mathbb{Z}/2) \arrow[r, "Sq^i"] \arrow[d, "\cong"] & H^*(X;\mathbb{Z}/2) \arrow[d, "\cong"] \\ H_{-*}(DX;\mathbb{Z}/2) \arrow[r, "Q^{-i}"] & H_{-*}(DX;\mathbb{Z}/2) \end{tikzcd}$$

Here $DX = F(X,S)$ is a commutative S-algebra, so the Q^i are well defined. For odd primes, substitute P^i for Sq^i.

There are also other relations between Steenrod operations and Dyer-Lashof operations, the Nishida relations.

Proposition 2.5. *In the homology of a commutative S-algebra the Nishida relations hold: For $p = 2$ and n sufficiently large,*

$$(Sq^r)_* Q^s = \sum_i \binom{2^n + s - r}{r - 2i} Q^{s-r+i}(Sq^i)_*.$$

For p odd and n sufficiently large,

$$(P^r)_* Q^s = \sum_i (-1)^{r+i} \binom{p^n + (s-r)(p-1)}{r - pi} Q^{s-r+i}(P^i)_*$$

and

$$(P^r)_* \beta Q^s = \sum_i (-1)^{r+i} \binom{p^n + (s-r)(p-1) - 1}{r - pi} \beta Q^{s-r+i}(P^i)_* - \sum_i (-1)^{r+i} \binom{p^n + (s-r)(p-1) - 1}{r - pi - 1} Q^{s-r+i}(P^i)_* \beta.$$

3 Hopf algebra structure on $THH(R)$

Let $X_\bullet$ be a simplicial finite set, and let R be a commutative S-algebra. By a simplicial finite set we mean a simplicial set which is finite in each degree.

Then we define $R \otimes X_\bullet$ as the simplicial spectrum with

$$(R \otimes X_\bullet)_q = R^{\wedge X_q},$$

where $R^{\wedge X_q}$ means R smashed with itself $|X_q|$ times. The face and degeneracy maps in $R \otimes X_\bullet$ come from the corresponding maps in $X_\bullet$ in the following way. For any finite sets X and Y, and any function

$$f : X \longrightarrow Y$$

we define

$$f_* : R^{\wedge X} \longrightarrow R^{\wedge Y}$$

as a series of twists, multiplications and inclusions, as suggested by the notation

$$f_*(r_{x_0} \wedge r_{x_1} \wedge \ldots) = \prod_{f(x_i)=y_0} r_{x_i} \wedge \prod_{f(x_i)=y_1} r_{x_i} \wedge \ldots$$

Note that this construction is functorial in X. If R is not commutative, we can make a similar definition with ordered sets and order-preserving maps. Suppose that X, Y and Z are finite sets with

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

then

$$(gf)_*(r_{x_0} \wedge r_{x_1} \wedge \ldots) = \prod_{gf(x_i)=z_0} r_{x_i} \wedge \prod_{gf(x_i)=z_1} r_{x_i} \wedge \ldots$$

and

$$g_* f_*(r_{x_0} \wedge r_{x_1} \wedge \ldots) = \prod_{g(y_j)=z_0} \prod_{f(x_i)=y_j} r_{x_i} \wedge \prod_{g(y_j)=z_1} \prod_{f(x_i)=y_j} r_{x_i} \wedge \ldots$$

These products have the same factors, although they might come in different order. Applying this definition to the face and degeneracy maps of $X_\bullet$ gives $R \otimes X_\bullet$ the structure of a simplicial spectrum.

For example, let $S^1_\bullet = I_\bullet / \partial I_\bullet$ be the simplicial model for S^1 with $S^1_q = \{x_0, x_1, \ldots, x_q\}$, and face and degeneracy maps

$$d_i(x_j) = \begin{cases} x_j & \text{if } j \le i, \\ x_{j-1} & \text{if } j > i, \end{cases}$$

and

$$s_i(x_j) = \begin{cases} x_j & \text{if } j < i, \\ x_{j+1} & \text{if } j \geq i. \end{cases}$$

We see that $R \otimes S^1_\bullet$ has R smashed with itself $q+1$ times in degree q, and the same face and degeneracy maps as $THH(R)$. Thus we get the following theorem, which is proved in [14].

Theorem 3.1. *Let R be a commutative S-algebra spectrum. Then $THH(R)$ is the geometric realization of the simplicial spectrum $R \otimes S^1_\bullet$.*

Note that for simplicial finite sets $X_\bullet$ and $Y_\bullet$ we have

$$(R \otimes (X_\bullet \amalg Y_\bullet))_q = R^{\wedge(X_q \amalg Y_q)} \cong (R \otimes X_q) \wedge (R \otimes Y_q),$$

so that

$$R \otimes (X_\bullet \amalg Y_\bullet) \cong (R \otimes X_\bullet) \wedge (R \otimes Y_\bullet).$$

In the same way we get

$$R \otimes (X_\bullet \vee Y_\bullet) \cong (R \otimes X_\bullet) \wedge_R (R \otimes Y_\bullet),$$

for pointed simplicial sets, where taking smash product over R compensates for losing one 0-simplex. Note the special role of the first R-factor in $R \otimes X_\bullet$.

Assume for the rest of the chapter that R is commutative. Then simplicial maps of $S^1_\bullet$ immediately give us some interesting maps of spectra. The folding map $S^1_\bullet \amalg S^1_\bullet \longrightarrow S^1_\bullet$ gives us the multiplication $\mu : THH(R) \wedge THH(R) \longrightarrow THH(R)$ making $THH(R)$ an S-algebra spectrum. The folding map factors as $S^1_\bullet \amalg S^1_\bullet \longrightarrow S^1_\bullet \vee S^1_\bullet \longrightarrow S^1_\bullet$, and the last map gives us a map $\phi : THH(R) \wedge_R THH(R) \longrightarrow THH(R)$. We also have a simplicial map $* \longrightarrow S^1_\bullet$, which gives us $\eta : R \longrightarrow THH(R)$ and a map $S^1_\bullet \longrightarrow *$, which gives us an augumentation $\epsilon : THH(R) \longrightarrow R$.

Proposition 3.2. *The maps ϕ, η and ϵ make $THH(R)$ into an augumented commutative R-algebra spectrum.*

Proof. We need to show that the analogous of diagram 1 and 2 commute, plus the two conditions $\epsilon\phi = \epsilon \otimes \epsilon$ and $\epsilon\eta = 1$. Let ϕ also denote the folding map $S^1_\bullet \vee S^1_\bullet \longrightarrow S^1_\bullet$, and write $T = THH(R)$. Then the commutative diagram

$$\begin{tikzcd} S^1_\bullet \vee S^1_\bullet \vee S^1_\bullet \arrow[r, "\phi \vee 1"] \arrow[d, "1 \vee \phi"] & S^1_\bullet \vee S^1_\bullet \arrow[d, "\phi"] \\ S^1_\bullet \vee S^1_\bullet \arrow[r, "\phi"] & S^1_\bullet \end{tikzcd}$$

of simplicial sets implies that the diagram

$$\begin{tikzcd}
T \wedge_R T \wedge_R T \arrow[r, "\phi\wedge_R 1"] \arrow[d, "1\wedge_R\phi"] & T \wedge_R T \arrow[d, "\phi"] \\
T \wedge_R T \arrow[r, "\phi"] & T
\end{tikzcd}$$

commutes.

Similarly, the commutative diagram

$$\begin{tikzcd}
S^1_\bullet \vee * \arrow[r, "1\vee i"] \arrow[dr, "\cong"'] & S^1_\bullet \vee S^1_\bullet \arrow[d, "\phi"] & * \vee S^1_\bullet \arrow[l, "i\vee 1"'] \arrow[dl, "\cong"] \\
& S^1_\bullet &
\end{tikzcd}$$

implies that

$$\begin{tikzcd}
T \wedge_R R \arrow[r, "1\wedge_R\eta"] \arrow[dr, "\cong"'] & T \wedge_R T \arrow[d, "\phi"] & R \wedge_R T \arrow[l, "\eta\wedge_R 1"'] \arrow[dl, "\cong"] \\
& T &
\end{tikzcd}$$

commutes. $\epsilon\phi = \epsilon \otimes \epsilon$ and $\epsilon\eta = 1$ follow from similar diagrams. $\square$

Now a natural question to ask is whether we can do something with the pinch map

$$S^1 \longrightarrow S^1 \vee S^1.$$

This is not a simplicial map with the usual simplicial model for S^1, but we can use the "double" model of S^1 obtained from $I_\bullet \amalg I_\bullet$ by identifying x_0 with y_0 and x_{q+1} with y_{q+1} in $(I_\bullet \amalg I_\bullet)_q$. This model of S^1 has two non-degenerate 0-simplices and two non-degenerate 1-simplices, and will be denoted by $DS^1_\bullet$. We get a simplicial map

$$DS^1_\bullet \longrightarrow S^1_\bullet \vee S^1_\bullet$$

by identifying the two 0-simplices, see figure 1. Let $DTHH(R)_\bullet = R \otimes DS^1_\bullet$ and let $DTHH(R) = |DTHH(R)_\bullet|$. Then the simplicial pinch map gives us a map

$$DTHH(R) \xrightarrow{\psi} THH(R) \wedge_R THH(R).$$

Now let $p : DS^1_\bullet \longrightarrow S^1_\bullet$ be the map that collapses the second 1-simplex.

For the next lemma we need to assume that R is what Elmendorf et. al. calls a q-cofibrant commutative S-algebra. Given a commutative S-algebra R, there is a q-cofibrant commutative S-algebra R' and a weak equivalence $R' \longrightarrow R$. Because the results we obtain from this lemma are only valid in the homotopy category anyway, there is no real loss of generality by assuming that R is q-cofibrant.

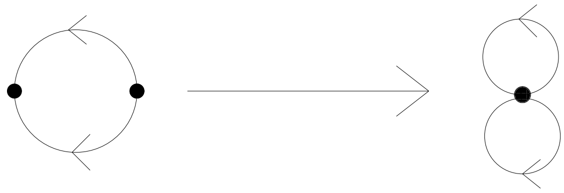

Figure 1: The simplicial map $DS^1_\bullet \longrightarrow S^1_\bullet \vee S^1_\bullet$.

Lemma 3.3. *Let R be a q-cofibrant commutative S-algebra. Then the map $p : DTHH(R) \longrightarrow THH(R)$ induced from $p : DS^1_\bullet \longrightarrow S^1_\bullet$ is a weak equivalence of spectra.*

Proof. Look at the diagram

$$\begin{tikzcd} I_\bullet \arrow[d] & \partial I_\bullet \arrow[l] \arrow[r] \arrow[d] & I_\bullet \arrow[d] \\ * & \partial I_\bullet \arrow[l] \arrow[r] & I_\bullet \end{tikzcd}$$

Pushout along the upper row gives $DS^1_\bullet$ and pushout along the lower row gives $S^1_\bullet$. Tensoring this diagram with R gives us

$$\begin{tikzcd} B(R) \arrow[d] & R \wedge R \arrow[l] \arrow[r] \arrow[d] & B(R) \arrow[d] \\ R & R \wedge R \arrow[l] \arrow[r] & B(R) \end{tikzcd}$$

Here $B(R) = B(R, R, R)$ is the two-sided bar construction, and we know that $B(R) \longrightarrow R$ is a weak equivalence. From [10, Lemma VII.7.1] we know that $R \wedge R \longrightarrow B(R)$ is a q-cofibration of q-cofibrant S-algebras, and we know ([10, Proposition VII.7.4]) that pushout over a q-cofibration of q-cofibrant S-algebras preserves weak equivalences between q-cofibrant S-algebras, so the result follows.

$\square$

Note that we do not have any good map in the opposite direction, $S^1_\bullet \longrightarrow DS^1_\bullet$, because the 0-simplex in $S^1_\bullet$ has to go to one of the two 0-simplices, so the 1-simplex cannot go to any of the non-degenerate 1-simplices. So because we cannot invert this weak equivalence, the map

$$THH(R) \xrightarrow{\psi} THH(R) \wedge_R THH(R)$$

only exists in the homotopy category.

By a Hopf algebra spectrum we will mean a spectrum with maps ϕ, η, ψ, ϵ and χ that satisfy the analogue of the definition of a Hopf algebra.

Proposition 3.4. *$THH(R)$ is an R-bialgebra spectrum, in the homotopy category.*

Proof. We need to show that the analogous of diagram 3 and 4 commute, in the homotopy category, and that ψ is an R-algebra map. Let $TS^1_\bullet$ be the "triple" model of S^1, obtained from $I_\bullet \amalg I_\bullet \amalg I_\bullet$ by identifying x_0 with y_0, x_{q+1} with z_0, and y_{q+1} with z_{q+1}, and let $TTHH(R) = |R \otimes TS^1_\bullet|$. In the same way as we got a weak equivalence between $DTHH(R)$ and $THH(R)$, we get a weak equivalence between $TTHH(R)$ and $THH(R)$ by collapsing the last two 1-simplices. We also get maps $TTHH(R) \longrightarrow DTHH(R) \wedge_R THH(R)$ and $TTHH(R) \longrightarrow THH(R) \wedge_R DTHH(R)$ by identifying x_{q+1} with y_{q+1}, and by identifying x_0 with x_{q+1}, respectively. These are both weakly equivalent to ψ, so the commutative diagram

$$\begin{tikzcd} S^1_\bullet \vee S^1_\bullet \vee S^1_\bullet & S^1_\bullet \vee DS^1_\bullet \arrow[l] \\ DS^1_\bullet \vee S^1_\bullet \arrow[u] & TS^1_\bullet \arrow[l] \arrow[u] \end{tikzcd}$$

gives a commutative diagram

$$\begin{tikzcd} THH(R) \wedge_R THH(R) \wedge_R THH(R) & THH(R) \wedge_R DTHH(R) \arrow[l] \\ DTHH(R) \wedge_R THH(R) \arrow[u] & TTHH(R) \arrow[l] \arrow[u] \end{tikzcd}$$

which gives coassociativity in the homotopy category by inverting $DTHH(R) \xrightarrow{\simeq} THH(R)$ and $TTHH(R) \xrightarrow{\simeq} THH(R)$.

For counitality, let $p' : DS^1_\bullet \longrightarrow S^1_\bullet$ be the map that collapses the first 1-simplex. Then we get a commutative diagram

$$\begin{tikzcd} S^1_\bullet \vee * & S^1_\bullet \vee S^1_\bullet \arrow[l] \arrow[r] & * \vee S^1_\bullet \\ & DS^1_\bullet \arrow[ul, "p"] \arrow[u] \arrow[ur, "p'"'] & \end{tikzcd}$$

which, after tensoring with R, gives us counitality in the homotopy category.

Write T for $THH(R)$. To show that ψ is an R-algebra map, we need to show that the diagram

$$\begin{tikzcd}
T \wedge_R T \arrow[r, "\phi"] \arrow[d, "\psi\wedge\psi"] & T \arrow[d, "\psi"] \\
T \wedge_R T \wedge_R T \wedge_R T \arrow[r, "(\phi\wedge\phi)\tau"] & T \wedge_R T
\end{tikzcd}$$

commutes in the homotopy category. This follows from the commutative diagram

$$\begin{tikzcd}
DS^1_\bullet \vee DS^1_\bullet \arrow[r] \arrow[d] & DS^1_\bullet \arrow[d] \\
S^1_\bullet \vee S^1_\bullet \vee S^1_\bullet \vee S^1_\bullet \arrow[r] & S^1_\bullet \vee S^1_\bullet
\end{tikzcd}$$

The last condition, $\psi\eta = \eta \otimes \eta$, follows from a similar diagram of simplicial sets. $\square$

We also get a weak conjugation map on $THH(R)$ using this construction. There is a conjugation $\chi : DS^1_\bullet \longrightarrow DS^1_\bullet$, which flips the two 1-simplices. This map also induces a map

$$\chi : THH(R) \longrightarrow THH(R)$$

in the homotopy category.

Theorem 3.5. *$THH(R)$ is an R-Hopf algebra, in the homotopy category.*

Proof. All we need to do is to show that the analogue of diagram 6 commutes in the homotopy category.

$$\begin{tikzcd}
T \arrow[r, "\psi"] \arrow[d, "\psi"] \arrow[drr, "\epsilon"] & & T \wedge_R T \arrow[dr, "1\wedge\chi"] & \\
T \wedge_R T \arrow[dr, "\chi\wedge 1"] & & R \arrow[dr, "\eta"] & T \wedge_R T \arrow[d, "\phi"] \\
& T \wedge_R T \arrow[rr, "\phi"] & & T
\end{tikzcd}$$

For the upper part, look at the following diagram of simplicial spaces.

$$\begin{tikzcd}
TS^1_\bullet \arrow[r, "\psi"] \arrow[dr, "\epsilon"'] & S^1_\bullet \vee DS^1_\bullet \arrow[drr, "1\vee\chi"] & & \\
& * \arrow[drr, "\eta"'] & & S^1_\bullet \vee DS^1_\bullet \arrow[d, "\phi"] \\
& & & S^1_\bullet
\end{tikzcd}$$

Here ϕ denotes collapsing the second 1-simplex in $DS^1_\bullet$, followed by a folding map, and ψ is the map that identifies x_0 and x_{q+1}. We see that $\phi(1 \vee \chi)\psi$ factors as $TS^1_\bullet \longrightarrow DS^1_\bullet \longrightarrow I_\bullet \longrightarrow S^1_\bullet$, where $TS^1_\bullet \longrightarrow DS^1_\bullet$ collapses one of the 1-simplices, $DS^1_\bullet \longrightarrow I_\bullet$ identifies the remaining 1-simplices and $I_\bullet \longrightarrow S^1_\bullet$ identifies the 0-simplices. Thus $\phi(1 \vee \chi)\psi$ factors through something simplicially contractible, and is therefore simplicially homotopic to $\eta\epsilon$. Tensoring with R carries simplicial homotopies to homotopies, and the result follows. Commutativity of the lower part of the diagram follows in the same way.

$\square$

4 Bökstedt's spectral sequence

Our primary tool for calculating $THH(R)$ is Bökstedt's spectral sequence (BSS). One way of setting it up is to use the filtration coming from $S^1_\bullet$, or the thick geometric realization of $R \otimes S^1_\bullet$.

Let h_* be a homology theory that satisfies the Künneth formula

$$h_*(X \wedge Y) = h_*X \otimes_{h_*S^0} h_*Y,$$

for example homology with mod p or $\mathbb{Q}$-coefficients, or Morava K-theory.

Then we get a spectral sequence with

$$E^1_{p,*}(R) = h_*R \otimes_{h_*S^0} h_*R \otimes_{h_*S^0} \ldots \otimes_{h_*S^0} h_*R = (h_*R)^{\otimes(p+1)}.$$

The d^1 differential is given by the face maps on $S^1_\bullet$, which we recognize as the Hochschild differential, so the E^2-term is given by $E^2_{**}(R) = HH^{h_*S^0}_*(h_*R)$, and we get a spectral sequence

$$E^2_{**}(R) = HH^{h_*S^0}_*(h_*R) \Longrightarrow h_*(THH(R)).$$

We will always let $E(R)$ be the BSS converging to $h_*(THH(R))$. Note that using the normal geometric realization instead of the thick realization corresponds to using the normalized Hochschild complex instead of the standard complex, see [7, IX.6].

Note that if R is commutative, $HH_0(h_*R) = h_*R$. The composition $R \longrightarrow THH(R) \longrightarrow R$ is induced by the simplicial maps $* \longrightarrow S^1_\bullet \longrightarrow *$, so we get an inclusion of h_*R into $E^r_{0*}(R)$ for every r. Thus there can be no differentials involving elements in filtration 0. Assume for the rest of the chapter that R is commutative, so that we have a Hopf algebra structure on R.

Most of the results in this chapter are valid for any homology theory with a Künneth formula, but let us specialize to mod p homology for simplicity. We get very strong restrictions on the differentials in the BSS, as corollary 4.3, 4.6 and 4.9 will show. Assume for the rest of the chapter that R is connected, i.e., $H_n(R;\mathbb{Z}/p) = 0$ for $n < 0$ and $H_0(R;\mathbb{Z}/p) = \mathbb{Z}/p$.

Proposition 4.1.

i. $H_*(THH(R);\mathbb{Z}/p)$ is a $\mathbb{Z}/p$-algebra.

ii. $E(R)$ is a $\mathbb{Z}/p$-algebra spectral sequence.

Proof. This follows from the map $THH(R) \wedge THH(R) \cong |R \otimes (S^1_\bullet \amalg S^1_\bullet)| \longrightarrow |R \otimes S^1_\bullet| \cong THH(R)$. This map respects the skeleton filtration of $THH(R)$, so we also get a map between the spectral sequences. $\square$

If R is augumented, i.e., if we have a map $\epsilon : R \longrightarrow S$, we obviously get an augumentation $THH(R) \longrightarrow THH(S) = S$ and augumentations on homology and the spectral sequence.

In fact, we can get an even stronger result using the filtration-preserving algebra maps $THH(R) \wedge_R THH(R) \cong |R \otimes (S^1_\bullet \vee S^1_\bullet)| \longrightarrow |R \otimes S^1_\bullet| \cong THH(R)$ and $THH(R) \cong |R \otimes S^1_\bullet| \longrightarrow |R \otimes *| \cong R$.

Proposition 4.2.

i. $H_*(THH(R);\mathbb{Z}/p)$ is an augumented $H_*(R;\mathbb{Z}/p)$ algebra.

ii. $E(R)$ is an augumented $H_*(R;\mathbb{Z}/p)$ algebra spectral sequence.

To determine the differentials we only have to find out what they are on the generators, either the $\mathbb{Z}/p$-algebra generators, or the $H_*(R;\mathbb{Z}/p)$ algebra generators. In most cases, it makes no difference which ones we check, because using $\mathbb{Z}/p$-algebra generators just gives us some extra classes in filtration 0. By a first differential at the E^r-term in a first quadrant spectral sequence we mean a d^r-differential such that there are no other d^r-differentials in lower total degree.

Corollary 4.3. *If there are differentials $d^r : E^r_{**}(R) \longrightarrow E^r_{**}(R)$, any first differential goes from an indecomposable element.*

Proof. The condition that $E(R)$ is a $\mathbb{Z}/p$-algebra spectral sequence means that the diagram

$$\begin{tikzcd} E^r \otimes E^r \arrow[r, "\mu"] \arrow[d, "d^r \otimes 1 \pm 1 \otimes d^r"'] & E^r \arrow[d, "d^r"] \\ E^r \otimes E^r \arrow[r, "\mu"] & E^r \end{tikzcd}$$

commutes, for a suitable choice (depending on the degree) of plus or minus. Suppose that a first differential goes from a decomposable class ab. Then the diagram says that $d^r(ab) = d^r(a)b \pm ad^r(b)$. a and b have lower total degree than ab, so $d^r(a) = d^r(b) = 0$, and $d^r(ab) = 0$. □

Corollary 4.4. *If $H_*(R;\mathbb{Z}/p)$ is a polynomial algebra, $E(R)$ collapses at $E^2(R) = E^\infty(R)$.*

Proof. If $H_*(R;\mathbb{Z}/p)$ is a polynomial algebra, $E^2(R)$ is generated by classes in filtration 0 and 1 by proposition 1.1, and the differentials on these classes are zero because the BSS is a first quadrant spectral sequence. □

We will see several examples of spectra where $H_*(R;\mathbb{Z}/2)$ is a polynomial algebra. For odd p we usually have some exterior classes in the homology as well, so the situation is a little bit more complicated. The d^{p-1}-differential is usually nontrivial, and in simple cases $E^p_{**}(R)$ is generated by polynomial classes, exterior classes, and polynomial classes truncated in height p, all in filtration 0 and 1, so the BSS collapses at $E^p(R) = E^\infty(R)$.

Let us first include a few more results about the BSS which we will need to get a grip on $THH(R)$ when $H_*(R;\mathbb{Z}/p)$ is a little bit more complicated.

Proposition 4.5. *Let $\Lambda = H_*(R;\mathbb{Z}/p)$.*

i. If $H_*(THH(R);\mathbb{Z}/p)$ is flat over Λ, there is a coproduct

$$H_*(THH(R);\mathbb{Z}/p) \longrightarrow H_*(THH(R);\mathbb{Z}/p) \otimes_\Lambda H_*(THH(R);\mathbb{Z}/p),$$

and $H_*(THH(R);\ \mathbb{Z}/p)$ is a Λ-Hopf algebra.

ii. If $E^r_{**}(R)$ is flat over Λ and $r \geq 2$, there is a coproduct $E^r_{**}(R) \longrightarrow E^r_{**}(R) \otimes_\Lambda E^r_{**}(R)$ that respects the d^r-differential. If $E^{r'}(R)$ is flat over Λ for $2 \leq r' \leq r$, we say that $E(R)$ is a Hopf algebra spectral sequence up to the E^r-term. If $E(R)$ is a Hopf algebra spectral sequence up to the E^r-term for all r, we say that $E(R)$ is a Hopf algebra spectral sequence.

Proof. Let $T = THH(R)$. There is a Künneth spectral sequence with

$$E^2_{**} = Tor^\Lambda_*(H_*(T;\mathbb{Z}/p), H_*(T;\mathbb{Z}/p)) \Longrightarrow H_*(T \wedge_R T;\mathbb{Z}/p),$$

see for example [10, Theorem IV.4.7]. When $H_*(T;\mathbb{Z}/p)$ is flat over Λ, this spectral sequence collapses, and gives the first part.

For the second part, let $E^r_{**}(T \wedge_R T)$ be the spectral sequence obtained from the filtration of $S^1_\bullet \vee S^1_\bullet$, and similar for the other cases. Let $N(\Lambda)$ be the (normalized) Hochschild complex. We know that there is a chain equivalence between $N(\Lambda)\otimes N(\Lambda)$ and $N(\Lambda\otimes\Lambda)$. In fact, [7, XI.§6] gives maps $f : N(\Lambda)\otimes N(\Lambda) \longrightarrow N(\Lambda\otimes\Lambda)$ and $g : N(\Lambda\otimes\Lambda) \longrightarrow N(\Lambda)\otimes N(\Lambda)$ that are chain equivalences. This corresponds to an equivalence $E^1_{**}(R)\otimes E^1_{**}(R) \longrightarrow E^1_{**}(T\wedge T)$. This is an equivalence of $\Lambda\otimes\Lambda$-module spectral sequences.

Identifying the two 0-simplices corresponds to identifying the zero'th factor in each $N(\Lambda)$, or equivalently to apply $\Lambda_{\Lambda\otimes\Lambda}(-)$. The maps f and g do nothing to the zero'th factor, so we obtain chain equivalences $f' : N(\Lambda)\otimes_\Lambda N(\Lambda) \longrightarrow \Lambda\otimes_{\Lambda\otimes\Lambda} N(\Lambda\otimes\Lambda)$ and $g' : \Lambda\otimes_{\Lambda\otimes\Lambda} N(\Lambda\otimes\Lambda) \longrightarrow N(\Lambda)\otimes_\Lambda N(\Lambda)$, and we get an equivalence $E^1_{**}(T\wedge_R T) \longrightarrow E^1_{**}(R)\otimes_\Lambda E^1_{**}(R)$.

In general, there is a Künneth spectral sequence with

$$E^2_{**} = Tor^\Lambda_*(E^r_{**}(R), E^r_{**}(r)) \Longrightarrow E^r_{**}(T\wedge_R T),$$

so the flatness hypothesis makes sure that we get an isomorphism $E^r_{**}(R)\otimes_\Lambda E^r_{**}(R) \longrightarrow E^r_{**}(T\wedge_R T)$.

Let $DE^r_{**}(R)$ be the spectral sequence associated with $DTHH(R)\cdot$ We have a map $DE^r_{**}(R) \longrightarrow E^r_{**}$ which is an isomorphism for $r\geq 2$. Inverting this isomorphism and composing with the pinch map gives us a map $E^r_{**}(R) \longrightarrow E^r_{**}(T\wedge_R T)$ for $r\geq 2$, and using the isomorphism $E^r_{**}(R)\otimes_\Lambda E^r_{**}(R) \longrightarrow E^r_{**}(T\wedge_R T)$ we get the map we want. The conjugation on $E^r_{**}(R)$, which we will not use, comes from the map $DS^1_\bullet \longrightarrow DS^1_\bullet$ that flips the two 1-simplices, so we get a Hopf algebra structure on E^r_{**}, provided that $r\geq 2$. The rest is clear. □

Corollary 4.6. *If $E(R)$ is a Hopf algebra spectral sequence up to $E^r(R)$, and there are differentials $d^r : E^r_{**}(R) \longrightarrow E^r_{**}(R)$, any first differential hits a $H_*(R;\mathbb{Z}/p)$ coalgebra primitive element.*

Proof. Suppose that $d^r(a)\neq 0$, and that d^r is zero on elements of degree less than $|a|$. Look at the commutative diagram

$$\begin{tikzcd} E^r \arrow[r, "\psi"] \arrow[d, "d^r"'] & E^r\otimes E^r \arrow[d, "d^r\otimes 1\pm 1\otimes d^r"] \\ E^r \arrow[r, "\psi"] & E^r\otimes E^r \end{tikzcd}$$

where the tensor products are over $H_*(R;\mathbb{Z}/p)$. Since d^r is zero on elements of degree less than $|a|$, we get that

$$(d^r\otimes 1\pm 1\otimes d^r)(\psi(a)) = d^r(a)\otimes 1 + 1\otimes d^r(a),$$

which shows that

$$\psi(d^r(a)) = d^r(a) \otimes 1 + 1 \otimes d^r(a),$$

so $d^r(a)$ is primitive. $\square$

Lemma 4.7. *Let $x \in H_*(R;\mathbb{Z}/p) = \Lambda$ be an exterior class that gives a Γ-algebra on σx in $HH_*(\Lambda) = E^2_{**}(R)$. Then the coproduct on $\Gamma(\sigma x)$ is given by*

$$\psi(\gamma_k(\sigma x)) = \sum_{i+j=k} \gamma_i(\sigma x) \otimes \gamma_j(\sigma x).$$

Proof. We will need an explicit chain inverse to the map $DE^1 \longrightarrow E^1$ in this case. Write $x_0(x_1,\ldots,x_k)$ for an element in $E^1_{k,*}$, and $(x_1,\ldots,x_k)$ if $x_0 = 1$. Also, write $x_0(x_1,\ldots,x_k) \otimes (y_1,\ldots,y_k)y_{k+1}$ for an element in $DE^1_{p,*}$. Recall the interpretation $E^1_{k,*} = \Lambda \otimes_{\Lambda\otimes\Lambda} B(\Lambda)_k$, where $B(\Lambda)_k = \Lambda \otimes \Lambda^{\otimes k} \otimes \Lambda$. Then $DE^1_{k,*} = \Lambda \otimes_{\Lambda\otimes\Lambda} B(\Lambda)_k \otimes B(\Lambda)_k \otimes_{\Lambda\otimes\Lambda} \Lambda$, where tensoring on the left identifies the first Λ in each $B(\Lambda)_k$, and tensoring on the right identifies the last Λ in each $B(\Lambda)_k$. Recall that $\gamma_k(\sigma x)$ is represented by $(x,\ldots,x)$ in $E^1_{k,*}$.

We claim that $\Sigma \pm (x^{\epsilon_1},\ldots,x^{\epsilon_k}) \otimes (x^{1-\epsilon_1},\ldots,x^{1-\epsilon_k})$, where the sum runs over all $(\epsilon_1,\ldots,\epsilon_k) \in \{0,1\}^k$ is a chain inverse in this case. For $p = 2$ there are no signs to worry about, and for p odd $|x|$ is odd, and the sign conventions are as follows: For $(x,\ldots,x) \otimes (1,\ldots,1)$ we use $+$, and we use $(-1)^{k-\sum\epsilon_i}$ times the sign we would have used for $(x^{\epsilon_1},\ldots,x^{\epsilon_k}) \otimes (x^{1-\epsilon_1},\ldots,x^{1-\epsilon_k})$ following conventions.

To prove that this is a chain inverse, we need to check that it is a cycle and that it maps to something that represents $(x,\ldots,x)$ in Hochschild homology under $p : DE^1 \longrightarrow E^1$. Note that $p((x_1,\ldots,x_k) \otimes (y_1,\ldots,y_k)) = (-1)^{\sum|x_i|\sum|y_i|} y_1 \cdots y_k(x_1,\ldots,x_k)$. Using this, we see that $p(\sum \pm(x^{\epsilon_1},\ldots,x^{\epsilon_k}) \otimes(x^{1-\epsilon_1},\ldots,x^{1-\epsilon_k})) = (x,\ldots,x) + \sum \pm x(\ldots)$, and that only $(x,\ldots,x)$ represents a nontrivial cycle in $E^1_{k,*}$, the rest of the terms are boundaries.

To see that it is a cycle, let $d = \sum(-1)^i d_i$ be the differential on $B(\Lambda)_k$, and use that $d((x_1,\ldots,x_k)\otimes(y_1,\ldots,y_k)) = \sum(-1)^i d_i(x_1,\ldots,x_k)\otimes d_i(y_1,\ldots,y_k)$, suitably interpreted.

Then we observe that for $1 \leq i \leq k-1$,

$$d_i(x^{\epsilon_1},\ldots,x^{\epsilon_k}) \otimes d_i(x^{1-\epsilon_1},\ldots,x^{1-\epsilon_k}) = \begin{cases} 0 \text{ if } \epsilon_i = \epsilon_{i+1}, \\ (x^{\epsilon_1},\ldots,x,\ldots,x^{\epsilon_k}) \otimes (x^{1-\epsilon_1},\ldots,x,\ldots,x^{1-\epsilon_k}) \text{ if } \epsilon_i = 1-\epsilon_{i+1}. \end{cases}$$

Because the $(x^{\epsilon_1},\ldots,x^{\epsilon_i},x^{\epsilon_{i+1}},\ldots,x^{\epsilon_k}) \otimes (x^{1-\epsilon_1},\ldots,x^{1-\epsilon_i},x^{1-\epsilon_{i+1}},\ldots,x^{\epsilon_k})$-term and the $(x^{\epsilon_1},\ldots,x^{1-\epsilon_i},x^{1-\epsilon_{i+1}},\ldots,x^{\epsilon_k}) \otimes (x^{1-\epsilon_1},\ldots,x^{\epsilon_i},x^{\epsilon_{i+1}},\ldots,x^{\epsilon_k})$-term have opposite signs, d_i of the first term cancels with d_i of the second term.

If $i = 0$, $d_0(x, \ldots, x^{\epsilon_k}) \otimes d_0(1, \ldots, x^{1-\epsilon_k})$ cancels with $d_0(1, \ldots, x^{\epsilon_k}) \otimes d_0(x, \ldots, x^{1-\epsilon_k})$, and similarly with $i = k$.

Identifying the 0-simplices does nothing much, and when we apply the shuffle map $g : N(\Lambda)_k \otimes_\Lambda N(\Lambda)_k \longrightarrow \oplus_{i+j=k} N(\Lambda)_i \otimes_\Lambda N(\Lambda)_j$, which in this case looks like

$$g\big((x_1, \ldots, x_k) \otimes_\Lambda (y_1, \ldots, y_k)\big) = \\ \sum_i (-1)^{k-i}(x_1, \ldots, x_i)x_{i+1} \cdots x_k \otimes_\Lambda y_1 \cdots y_i(y_{i+1}, \ldots, y_k),$$

we kill all elements except $(\underbrace{x, \ldots, x}_{i}, 1, \ldots, 1) \otimes_\Lambda (1, \ldots, 1, \underbrace{x, \ldots, x}_{j})$, which maps to $(x, \ldots, x) \otimes_\Lambda (x, \ldots, x)$, and represents $\gamma_i(\sigma x) \otimes_\Lambda \gamma_j(\sigma x)$.

$\square$

Proposition 4.8.

i. *$H_*(THH(R); \mathbb{Z}/p)$ is an A_*-comodule.*

ii. *$E^r_{**}(R)$ is an A_*-comodule spectral sequence, i.e., there is a filtration-preserving map*

$$\nu : E^r_{**}(R) \longrightarrow A_* \otimes E^r_{**}(R).$$

Corollary 4.9. *Any first differential hits an A_*-comodule primitive element.*

5 Some examples

As we have seen, if $H_*(R; \mathbb{Z}/p)$ is a polynomial algebra, the BSS collapses at the E^2-term, and this happens for several interesting spectra for $p = 2$.

Proposition 5.1. *Let ku and ko be complex and real connective topological K-theory, respectively. Then $H_*(R; \mathbb{Z}/2)$ is a polynomial algebra for $R = H\mathbb{Z}/2$, $H\mathbb{Z}$, ku and ko. More precisely we have*

i. $H_*(H\mathbb{Z}/2; \mathbb{Z}/2) = A_* = P(\bar{\xi}_k | k \geq 1)$.

ii. $H_*(H\mathbb{Z}; \mathbb{Z}/2) = P(\bar{\xi}_1^2, \bar{\xi}_k | k \geq 2)$.

iii. $H_*(ku; \mathbb{Z}/2) = P(\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_k | k \geq 3)$.

iv. $H_*(ko; \mathbb{Z}/2) = P(\bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_k | k \geq 3)$.

There are maps $ko \longrightarrow ku \longrightarrow H\mathbb{Z} \longrightarrow H\mathbb{Z}/2$ that induces the inclusions in homology suggested by the notation.

Proof. i and ii are well known. For iii and iv, see for example Stong's article [23]. $\square$

According to corollary 4.3, the BSS collapses at the E^2-term in all these cases, and we only need to find the right A_*-comodule algebra extension to determine the $\mathbb{Z}/2$-homology of THH of these spectra.

Corollary 5.2. *The BSS for these spectra collapses at the E^2-term, and we get*

i. $E^{\infty}_{**}(\mathbb{Z}/2) = H_*(H\mathbb{Z}/2;\mathbb{Z}/2)\otimes E(\sigma\bar{\xi}_k|k\geq 1)$.

ii. $E^{\infty}_{**}(\mathbb{Z}) = H_*(H\mathbb{Z};\mathbb{Z}/2)\otimes E(\sigma\bar{\xi}_1^2,\sigma\bar{\xi}_k|k\geq 2)$.

iii. $E^{\infty}_{**}(ku) = H_*(ku;\mathbb{Z}/2)\otimes E(\sigma\bar{\xi}_1^2,\sigma\bar{\xi}_2^2,\sigma\bar{\xi}_k|k\geq 3)$.

iv. $E^{\infty}_{**}(ko) = H_*(ko;\mathbb{Z}/2)\otimes E(\sigma\bar{\xi}_1^4,\sigma\bar{\xi}_2^2,\sigma\bar{\xi}_k|k\geq 3)$.

Because the homology is a free $\mathbb{Z}/2$-module, to determine the algebra structure the only question is whether a product that looks like it is zero in E^∞ is nonzero and lies in lower filtration. To determine the extensions we use Dyer-Lashof operations. The key result we need is this, see e.g. theorem 2.2 and theorem 2.3 in chapter III of [6].

Proposition 5.3. *In $A_* = H_*(H\mathbb{Z}/p;\mathbb{Z}/p)$ the following formula holds for all k.*

i. $Q^{p^k}(\bar{\xi}_k) = \bar{\xi}_{k+1}$ *for all primes p.*

ii. $Q^{p^k}(\bar{\tau}_k) = \bar{\tau}_{k+1}$ *for p odd.*

An immediate consequence of this is that the same formulas hold in the homology of $H\mathbb{Z}$, ku, ko and ℓ whenever $\bar{\xi}_k$ and $\bar{\xi}_{k+1}$, or $\bar{\tau}_k$ and $\bar{\tau}_{k+1}$, are defined, because we have maps from these spectra to $H\mathbb{Z}/p$ which induce injections in homology. The formulas also hold in THH of these spectra, because $THH(R)$ is an augumented R-algebra.

We also need to know that the Dyer-Lashof operations commute with σ.

Lemma 5.4. *Let $a\in H_*(R;\mathbb{Z}/p)$. Then in $H_*(THH(R);\mathbb{Z}/p)$ we have*

$$Q^k(\sigma a) = \sigma Q^k(a).$$

Proof. (John Rognes) We use the map $\omega : S^1_+\wedge R \longrightarrow THH(R)$. Adjoint to this we have a map

$$\tilde{\omega} : R \longrightarrow F(S^1_+, THH(R))$$

Now we use that $F(S^1_+, X) \simeq D(S^1_+) \wedge X$ for any spectrum X and that $D(S^1_+) \simeq S^0 \vee S^{-1}$, so we get a weak map

$$\tilde{\omega} : R \longrightarrow (S^0 \vee S^{-1}) \wedge THH(R).$$

Let σa denote the image of Σa under $\Sigma R \xrightarrow{\sigma} THH(R)$, and let a also denote the image of a under $R \xrightarrow{i} THH(R)$. Then we have $\tilde{\omega}(a) = 1 \otimes a + x \otimes \sigma a \in H_*(S^0 \vee S^{-1}; \mathbb{Z}/p) \otimes H_*(THH(R); \mathbb{Z}/p)$, where $x \in H_{-1}(S^0 \vee S^{-1}; \mathbb{Z}/p)$ is dual to a generator in $H_1(S^1_+; \mathbb{Z}/p)$.

There is a Künneth formula for the Dyer-Lashof operations,

$$Q^k(x \otimes y) = \sum_{i+j=k} Q^i(x) \otimes Q^j(y).$$

According to proposition 2.4 and because all Steenrod operations act trivially on S^1_+, the Dyer-Lashof operations act trivially on $D(S^1_+) \simeq S^0 \vee S^{-1}$. Thus we get

$$Q^k(\tilde{\omega}(a)) = Q^k(1 \otimes a + x \otimes \sigma a) = 1 \otimes Q^k(a) + x \otimes Q^k(\sigma a).$$

But we also have

$$Q^k(\tilde{\omega}(a)) = \tilde{\omega}(Q^k(a)) = 1 \otimes Q^k(a) + x \otimes \sigma Q^k(a),$$

so we see that $Q^k(\sigma a) = \sigma Q^k(a)$. □

Now we can proceed with determining the extensions in the BSS. Let us look at $H_*(THH(\mathbb{Z}/2); \mathbb{Z}/2)$ first. In $E^\infty(\mathbb{Z}/2)$ we have exterior classes $\sigma\bar{\xi}_k$ for $k \geq 1$. But according to the above results, we have

$$(\sigma\bar{\xi}_k)^2 = Q^{2^k}(\sigma\bar{\xi}_k) = \sigma Q^{2^k}(\bar{\xi}_k) = \sigma\bar{\xi}_{k+1}.$$

This shows that what looks like an exterior algebra in $E^\infty(\mathbb{Z}/2)$ is actually a polynomial algebra on $\sigma\bar{\xi}_1$ in $H_*(THH(\mathbb{Z}/2); \mathbb{Z}/2)$. There is no room for any other identifications, so we get that

$$H_*(THH(\mathbb{Z}/2); \mathbb{Z}/2) = A_* \otimes P(\sigma\bar{\xi}_1).$$

The other cases are similar. For $THH(\mathbb{Z})$ we get a polynomial algebra on $\sigma\bar{\xi}_2$, while we get a polynomial algebra on $\sigma\bar{\xi}_3$ for $THH(ku)$ and $THH(ko)$. Let us summarize these results.

Theorem 5.5. *As a $H_*(R; \mathbb{Z}/2)$-algebra, $H_*(THH(R); \mathbb{Z}/2)$ is given by the following, for $R = \mathbb{Z}/2$, $\mathbb{Z}$, ku and ko.*

i. $H_*(THH(\mathbb{Z}/2);\mathbb{Z}/2) = A_* \otimes P(\sigma\bar{\xi}_1)$.

ii. $H_*(THH(\mathbb{Z});\mathbb{Z}/2) = H_*(H\mathbb{Z};\mathbb{Z}/2) \otimes E(\sigma\bar{\xi}_1^2) \otimes P(\sigma\bar{\xi}_2)$.

iii. $H_*(THH(ku);\mathbb{Z}/2) = H_*(ku;\mathbb{Z}/2) \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3)$.

iv. $H_*(THH(ko);\mathbb{Z}/2) = H_*(ko;\mathbb{Z}/2) \otimes E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3)$.

The A_*-comodule structure is determined by the A_*-comodule structure in $H_*(R;\mathbb{Z}/2)$ and the action on the extra polynomial and exterior classes. For $R = \mathbb{Z}/2$, $\sigma\bar{\xi}_1$ is primitive, for $R = \mathbb{Z}$, $\sigma\bar{\xi}_1^2$ is primitive while $\nu(\sigma\bar{\xi}_2) = 1 \otimes \sigma\bar{\xi}_2 + \bar{\xi}_1 \otimes \sigma\bar{\xi}_1^2$. For $R = ku$, $\sigma\bar{\xi}_1^2$ and $\sigma\bar{\xi}_2^2$ are primitive, while $\nu(\sigma\bar{\xi}_3) = 1 \otimes \sigma\bar{\xi}_3 + \bar{\xi}_1^2 \otimes \sigma\bar{\xi}_2^2$. For $R = ko$, $\sigma\bar{\xi}_1^4$ is primitive, $\nu(\sigma\bar{\xi}_2^2) = 1 \otimes \sigma\bar{\xi}_2^2 + \bar{\xi}_1^2 \otimes \sigma\bar{\xi}_1^4$ and $\nu(\sigma\bar{\xi}_3) = 1 \otimes \sigma\bar{\xi}_3 + \bar{\xi}_1^2 \otimes \sigma\bar{\xi}_2^2 + \bar{\xi}_2 \otimes \sigma\bar{\xi}_1^4$.

The cases $R = \mathbb{Z}/p$ and $R = \mathbb{Z}$ are due to Bökstedt, calculated first in [5].

Proof. We have proved most of this already, the only things that remain are to see that there can be no other algebra identifications in the E^∞-term and that the A_*-comodule structure is as claimed. The A_*-comodule structure on the elements in filtration 1 follows from the formula $\nu(\sigma x) = (1 \otimes \sigma)(\nu x)$, and we already know the comodule structure on the elements in filtration 0. The only thing that can happen to the algebra structure when going from E^∞ to homology is that something that looks like it is zero in E^∞ is nonzero and lies in a lower filtration. So we only need to check that the squares of the exterior powers are zero.

For $\mathbb{Z}/2$ there can be no other identifications. For $\mathbb{Z}$ the only thing that can happen is $(\sigma\bar{\xi}_1^2)^2 \neq 0$, which is impossible, because $\sigma\bar{\xi}_1^2$ is A_*-comodule primitive, so $(\sigma\bar{\xi}_1^2)^2$ has to be A_*-comodule primitive too. Note that if x is in positive filtration, x^2 cannot be in filtration 0, because that is inconsistent with the R-algebra map $THH(R) \longrightarrow R$. There are no primitives in filtration 1 and degree 6, so $(\sigma\bar{\xi}_1^2)^2 = 0$. Similarly, there are no primitives in filtration 1 and degree 6 or 14 in $H_*(THH(ku);\mathbb{Z}/2)$, so $(\sigma\bar{\xi}_1^2)^2 = 0$ and $(\sigma\bar{\xi}_2^2)^2 = 0$. For $THH(ko)$ we find that $(\sigma\bar{\xi}_1^4)^2 = 0$ because there are no primitives in filtration 1 and degree 10, and because $\nu(\sigma\bar{\xi}_2^2) = 1 \otimes \sigma\bar{\xi}_2^2 + \bar{\xi}_1^2 \otimes \sigma\bar{\xi}_1^4$, we get $\nu((\sigma\bar{\xi}_2^2)^2) = 1 \otimes (\sigma\bar{\xi}_2^2)^2 + \bar{\xi}_1^4 \otimes (\sigma\bar{\xi}_1^4)^2 = 1 \otimes (\sigma\bar{\xi}_2^2)^2$, so $(\sigma\bar{\xi}_2^2)^2$ is also primitive, but there are no primitives in filtration 1 and degree 14, so we can conclude that $(\sigma\bar{\xi}_2^2)^2 = 0$ and there are no more extensions. Because the homology is generated by elements from filtration 0 and 1, this also determines the A_*-comodule structure. ∎

There are unpublished works about another spectrum that fits into this pattern. This spectrum, which is called eo_2, is connected to elliptic modular forms. eo_2 is a commutative S-algebra spectrum, and we have $H^*(eo_2;\mathbb{Z}/2) = A//A_2$, where A_2 is generated by Sq^1, Sq^2 and Sq^4, or dually

$$H_*(eo_2;\mathbb{Z}/2) = P(\bar{\xi}_1^8, \bar{\xi}_2^4, \bar{\xi}_3^2, \bar{\xi}_k | k \geq 4).$$

Using that $H_*(eo_2;\mathbb{Z}/2)$ is a polynomial algebra, we can easily calculate $H_*(THH(eo_2);\mathbb{Z}/2)$ in the same way as for the last examples. We get

$$E^2_{**}(eo_2) = E^\infty_{**}(eo_2) = H_*(eo_2;\mathbb{Z}/2) \otimes E(\sigma\bar{\xi}_1^8, \sigma\bar{\xi}_2^4, \sigma\bar{\xi}_3^2, \sigma\bar{\xi}_k | k \geq 4),$$

and making the usual identifications going from E^∞ to homology we get

Proposition 5.6. *The $\mathbb{Z}/2$-homology of $THH(eo_2)$ is given by*

$$H_*(THH(eo_2);\mathbb{Z}/2) = H_*(eo_2;\mathbb{Z}/2) \otimes E(\sigma\bar{\xi}_1^8, \sigma\bar{\xi}_2^4, \sigma\bar{\xi}_3^2) \otimes P(\sigma\bar{\xi}_4)$$

as an $H_*(eo_2;\mathbb{Z}/2)$-algebra. The A_*-comodule action is determined by $\nu(\sigma\bar{\xi}_1^8) = 1 \otimes \sigma\bar{\xi}_1^8$, $\nu(\sigma\bar{\xi}_2^4) = 1 \otimes \sigma\bar{\xi}_2^4 + \bar{\xi}_1^4 \otimes \sigma\bar{\xi}_1^8$, $\nu(\sigma\bar{\xi}_3^2) = 1 \otimes \sigma\bar{\xi}_3^2 + \bar{\xi}_1^2 \otimes \sigma\bar{\xi}_2^4 + \bar{\xi}_2^2 \otimes \sigma\bar{\xi}_1^8$ *and* $\nu(\sigma\bar{\xi}_4) = 1 \otimes \sigma\bar{\xi}_4 + \bar{\xi}_1 \otimes \sigma\bar{\xi}_3^2 + \bar{\xi}_2 \otimes \sigma\bar{\xi}_2^4 + \bar{\xi}_3 \otimes \sigma\bar{\xi}_1^8$.

Now let us look at some cases for p odd.

Proposition 5.7. *Let ℓ be the p-completed Adams summand of p-local complex connected K-theory. Then we have*

i. $H_*(H\mathbb{Z}/p;\mathbb{Z}/p) = A_* = P(\bar{\xi}_k | k \geq 1) \otimes E(\bar{\tau}_k | k \geq 0)$.

ii. $H_*(H\mathbb{Z};\mathbb{Z}/p) = P(\bar{\xi}_k | k \geq 1) \otimes E(\bar{\tau}_k | k \geq 1)$.

iii. $H_*(\ell;\mathbb{Z}/p) = P(\bar{\xi}_k | k \geq 1) \otimes E(\bar{\tau}_k | k \geq 2)$.

Now we get that

$$E^2_{**}(\mathbb{Z}/p) = A_* \otimes E(\sigma\bar{\xi}_k | k \geq 1) \otimes \Gamma(\sigma\bar{\tau}_k | k \geq 0)$$

as a $\mathbb{Z}/p$-algebra spectral sequence, and similar expressions in the other cases.

In each case, the E^2-term is flat over $H_*(R;\mathbb{Z}/p)$, so we have a coproduct on the BSS. For the coalgebra structure, note that the classes in filtration 0 are treated as constants, and that the classes in filtration 1 are primitive by default.

A closer examination of the E^2-term of the spectral sequences for $\mathbb{Z}/p$, $\mathbb{Z}$ and ℓ shows that there are no coalgebra primitives in filtration greater than 1. This is because all the generators except the Γ-classes are in filtration 0 and 1, and $\psi(\gamma_q(\sigma\bar\tau_k)) = \sum_{i+j=q} \gamma_i(\sigma\bar\tau_k) \otimes \gamma_j(\sigma\bar\tau_k)$. According to corollary 4.3 and 4.6, a first differential has to go from $\gamma_{p^k}(\sigma x)$ for some exterior class x to something in filtration 1. This means that the first possible differential is d^{p-1}, so we have

$$\begin{aligned}
&E^2_{**}(\mathbb{Z}/p) = E^{p-1}_{**}(\mathbb{Z}/p) = A_* \otimes E(\sigma\bar\xi_k | k \geq 1) \otimes \Gamma(\sigma\bar\tau_k | k \geq 0);\\
&E^2_{**}(\mathbb{Z}) = E^{p-1}_{**}(\mathbb{Z}) = H_*(H\mathbb{Z}; \mathbb{Z}/p) \otimes E(\sigma\bar\xi_k | k \geq 1) \otimes \Gamma(\sigma\bar\tau_k | k \geq 1);\\
&E^2_{**}(\ell) = E^{p-1}_{**}(\ell) = H_*(\ell; \mathbb{Z}/p) \otimes E(\sigma\bar\xi_k | k \geq 1) \otimes \Gamma(\sigma\bar\tau_k | k \geq 2).
\end{aligned}$$

Bökstedt [5] showed that there are differentials in $E^{p-1}_{**}(\mathbb{Z}/p)$. He showed that

$$d^{p-1}(\gamma_p(\sigma\bar\tau_k)) = \sigma\bar\xi_{k+1}$$

for all $k \geq 0$, and also that there is a d^{p-1}-differential on $\gamma_k(\sigma\bar\tau_k)$ for $k > p$. For a published reference, see for example [12]. As we will see, the rest of the d^{p-1}-differentials are a consequence of the differentials on $\gamma_p(\sigma\bar\tau_k)$, using the coalgebra structure on the spectral sequence.

We have maps $\ell \longrightarrow H\mathbb{Z} \longrightarrow \mathbb{Z}/p$, which give maps between THH of these spectra and also between the spectral sequences. This shows that we have the same d^{p-1}-differentials for $k \geq 1$ in $E^{p-1}(\mathbb{Z})$ and for $k \geq 2$ in $E^{p-1}(\ell)$.

Now we can use the coalgebra map on the spectral sequences to determine the rest of the d^{p-1}-differentials.

Lemma 5.8. *Let $R = \mathbb{Z}/p$, $\mathbb{Z}$ or ℓ, and let $x = \sigma\bar\tau_k$ and $y = \sigma\bar\xi_{k+1}$, where $k \geq 1$ for $R = \mathbb{Z}$ and $k \geq 2$ for $R = \ell$. If $d^{p-1}(\gamma_p(x)) = y$ then $d^{p-1}(\gamma_q(x)) = \gamma_{q-p}(x)y$ for all $q \geq p$.*

Proof. In each case we have a coalgebra map on the E^{p-1}-term of the BSS, so we can use the commutative diagram

$$\begin{tikzcd}
E^{p-1} \arrow[r, "\psi"] \arrow[d, "d^{p-1}"'] & E^{p-1} \otimes_\Lambda E^{p-1} \arrow[d, "d^{p-1}\otimes 1 \pm 1 \otimes d^{p-1}"] \\
E^{p-1} \arrow[r, "\psi"] & E^{p-1} \otimes_\Lambda E^{p-1}
\end{tikzcd}$$

of spectral sequences and induction on q.

$$d^{p-1}(\psi(\gamma_q(x))) = d^{p-1}\big(\sum_{i+j=q} \gamma_i(x) \otimes \gamma_j(x)\big) = \sum_{i+j=q-p, j\neq 0} \gamma_i(x)y \otimes \gamma_j(x) + \sum_{i+j=q-p, i\neq 0} \gamma_i(x) \otimes \gamma_j(x)y + d^{p-1}(\gamma_q(x)) \otimes 1 + 1 \otimes d^{p-1}(\gamma_q(x)).$$

This shows that $\psi(d^{p-1}(\gamma_q(x)))$ is nonzero for $q \geq p$. Because of the structure of the coproduct, we have to have $d^{p-1}(\gamma_q(x)) = \gamma_{q-p}(x)y + z$, where z is coalgebra primitive. Because there are no coalgebra primitives in filtration $q-(p-1)$ for $q > p$, the result follows. □

Proposition 5.9. *The BSS for $\mathbb{Z}/p$, $\mathbb{Z}$ and ℓ collapses at the E^p-term, and we get the following.*

i. $E^p_{**}(\mathbb{Z}/p) = E^\infty_{**}(\mathbb{Z}/p) = A_* \otimes P_p(\sigma\bar{\tau}_k | k \geq 0)$.

ii. $E^p_{**}(\mathbb{Z}) = E^\infty_{**}(\mathbb{Z}) = H_*(H\mathbb{Z}; \mathbb{Z}/p) \otimes E(\sigma\bar{\xi}_1) \otimes P_p(\sigma\bar{\tau}_k | k \geq 1)$.

iii. $E^p_{**}(\ell) = E^\infty_{**}(\ell) = H_*(\ell; \mathbb{Z}/p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P_p(\sigma\bar{\tau}_k | k \geq 2)$.

Proof. The d^{p-1}-differentials already accounted for leaves us with these E^p-terms, when we use that $\mathbb{Z}/p\{1, \gamma_1(x), \ldots, \gamma_{p-1}(x)\} \cong P_p(x)$. There is no room for any more differentials after this, because all the algebra generators are in filtration 0 and 1, so the spectral sequences collapse. □

Now it is straightforward to determine the right algebra extensions using what we know about Dyer-Lashof operations. We have that

$$(\sigma\bar{\tau}_k)^p = Q^{p^k}(\sigma\bar{\tau}_k) = \sigma Q^{p^k}(\bar{\tau}_k) = \sigma\bar{\tau}_{k+1},$$

so what looks like a truncated polynomial algebra on the $\sigma\bar{\tau}_k$ is a polynomial algebra on $\sigma\bar{\tau}_0$ in $H_*(THH(\mathbb{Z}/p); \mathbb{Z}/p)$, on $\sigma\bar{\tau}_1$ in $H_*(THH(\mathbb{Z}); \mathbb{Z}/p)$ and on $\sigma\bar{\tau}_2$ in $H_*(THH(\ell); \mathbb{Z}/p)$. The exterior classes still have square zero, because they are in odd degree. Thus we get

Proposition 5.10. *As $H_*(R; \mathbb{Z}/2)$-algebras, $H_*(THH(R); \mathbb{Z}/p)$ is given by the following, for $R = H\mathbb{Z}/p$, $H\mathbb{Z}$ and ℓ.*

i. $H_*(THH(\mathbb{Z}/p); \mathbb{Z}/p) = A_* \otimes P(\sigma\bar{\tau}_0)$.

ii. $H_*(THH(\mathbb{Z}); \mathbb{Z}/p) = H_*(H\mathbb{Z}; \mathbb{Z}/p) \otimes E(\sigma\bar{\xi}_1) \otimes P(\sigma\bar{\tau}_1)$.

iii. $H_*(THH(\ell); \mathbb{Z}/p) = H_*(\ell; \mathbb{Z}/p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P(\sigma\bar{\tau}_2)$.

The A_*-comodule structure is determined by the A_*-comodule structure in $H_*(R; \mathbb{Z}/p)$ and the action on the extra polynomial and exterior classes. For $R = \mathbb{Z}/p$, $\sigma\bar{\tau}_0$ is primitive. For $R = \mathbb{Z}$, $\sigma\bar{\xi}_1$ is primitive, while $\nu(\sigma\bar{\tau}_1) = 1 \otimes \sigma\bar{\tau}_1 + \bar{\tau}_0 \otimes \sigma\bar{\xi}_1$. For $R = \ell$, $\sigma\bar{\xi}_1$ and $\sigma\bar{\xi}_2$ are primitive, while $\nu(\sigma\bar{\tau}_2) = 1 \otimes \sigma\bar{\tau}_2 + \bar{\tau}_0 \otimes \sigma\bar{\xi}_2$.

Proof. The only thing that we have not proved yet is the A_*-comodule structure. It follows from the formula $\nu(\sigma x) = (1 \otimes \sigma)\nu(x)$ and that σ is a derivation. For example we get $\nu(\sigma\bar{\tau}_2) = (1 \otimes \sigma)(\bar{\tau}_2 \otimes 1 + 1 \otimes \bar{\tau}_2 + \bar{\tau}_1 \otimes \bar{\xi}_1^p + \bar{\tau}_0 \otimes \bar{\xi}_2) = 1 \otimes \sigma\bar{\tau}_2 + \bar{\tau}_0 \otimes \sigma\bar{\xi}_2$, using that $\sigma(x^p) = 0$ for all x, because σ is a derivation. $\square$

The first two parts are due to Bökstedt, calculated in [5], while the last part can be found in McClure and Staffeldt's article [15].

6 $THH(ju)$ and $THH(j)$

For $p = 2$ the spectrum j comes in two different variations, one complex and one real, which we will denote by ju and j, respectively. j is the most interesting case, but unfortunately also the most difficult to calculate.

For $p = 2$ there are maps $\psi^3 : ku \longrightarrow ku$ and $\psi^3 : ko \longrightarrow ko$ that induce multiplication by 3^k on $\pi_{2k}(ku)$ and $\pi_{2k}(ko)$. In the complex case, ψ^3 is the identity on $\pi_0(ku)$, so $\psi^3 - 1$ lifts to the 1-connected cover of ku, which we will denote by bu. ju is defined as the fiber of this map,

$$ju \longrightarrow ku \xrightarrow{\psi^3 - 1} bu.$$

As an alternative definition, we can use $ju = K(\mathbb{F}_3)$, which ensures that we have an E_∞-structure on ju, see May's book [13]. Theorem VIII.3.2 says that the two definitions give equivalent spectra. K-theory of a field has an E_∞-structure, and [10, corollary II.3.6] assures us that given an E_∞ structure on a ring spectrum, it is equivalent to a commutative S-algebra, so $THH(ju)$ is well defined.

In a similar way we can define j at odd primes as K-theory of a field. Let r be any prime power such that the image of r under $\mathbb{Z} \longrightarrow \mathbb{Z}/p^2$ generates $(\mathbb{Z}/p^2)^*$, the group of units in $\mathbb{Z}/p^2$. Then there is a field $\mathbb{F}_r$ with r elements, and we can define j as $K(\mathbb{F}_r)$, as in [13, definition VIII.3.1]. This ensures that j is equivalent to a commutative S-algebra, so $THH(j)$ is well defined.

As an alternative, which makes calculations in homology easier, we can define j as a fiber. Let $q = 2p - 2$. We have a map $\psi^r : \ell \longrightarrow \ell$, which induces multiplication with r^k on $\pi_{2k}(\ell)$, i.e., multiplication with $r^{(p-1)k}$ on π_{qk}. ψ^r induces the identity on $\pi_n(\ell)$ for $n < q$, so $\psi^r - 1$ factors through the $(q-1)$-connected cover of ℓ, which is equivalent to $\Sigma^q \ell$ by Bott periodicity. We can define j as the fiber

$$j \longrightarrow \ell \xrightarrow{\psi^r - 1} \Sigma^q \ell.$$

[13, Theorem VIII.3.2] says that the two definitions give equivalent spectra. By using the fiber definitions of ju and j we get the following well-known result about the homotopy groups.

Theorem 6.1.

i.

$$\pi_i(ju) = \begin{cases} \mathbb{Z} & \textit{for } i = 0; \\ \mathbb{Z}/2 & \textit{for } i = 4k+1,\ k \geq 0; \\ \mathbb{Z}/2^{m+3} & \textit{for } i = 4k-1,\ k > 0,\ k = s2^m,\ 2 \nmid s; \\ 0 & \textit{otherwise.} \end{cases}$$

ii.

$$\pi_i(j) = \begin{cases} \mathbb{Z} & \textit{for } i = 0; \\ \mathbb{Z}/p^{m+1} & \textit{for } i = qk-1,\ k > 0,\ k = sp^m,\ p \nmid s; \\ 0 & \textit{otherwise.} \end{cases}$$

The spectra ju at $p = 2$ and j at odd p have very similar homology, and we can use the same techniques for determining $H_*(THH(ju); \mathbb{Z}/2)$ and $H_*(THH(j); \mathbb{Z}/p)$ for odd p. Let us first determine $H_*(ju; \mathbb{Z}/2)$ and $H_*(j; \mathbb{Z}/p)$. From the fiber sequence

$$ju \longrightarrow ku \xrightarrow{\psi^3 - 1} bu \longrightarrow \Sigma ju$$

we get a long exact sequence in homology, which gives the following short exact sequence.

$$0 \longrightarrow \Sigma coker(\psi^3 - 1)_* \longrightarrow H_*(ju; \mathbb{Z}/2) \longrightarrow ker(\psi^3 - 1)_* \longrightarrow 0.$$

In the same way, we get a short exact sequence for $H_*(j; \mathbb{Z}/p)$.

$$0 \longrightarrow \Sigma coker(\psi^r - 1)_* \longrightarrow H_*(j; \mathbb{Z}/p) \longrightarrow ker(\psi^r - 1)_* \longrightarrow 0.$$

Calculating these kernels and cokernels is easier in cohomology, so we will use the short exact sequences

$$\begin{aligned} 0 \longleftarrow \Sigma ker(\psi^3 - 1)^* \longleftarrow H^*(ju; \mathbb{Z}/2) \longleftarrow coker(\psi^3 - 1)^* \longleftarrow 0 \\ 0 \longleftarrow \Sigma ker(\psi^r - 1)^* \longleftarrow H^*(j; \mathbb{Z}/p) \longleftarrow coker(\psi^r - 1)^* \longleftarrow 0 \end{aligned}$$

instead. We will need the cohomology of ku and ℓ. The following proposition is just a dualization of proposition 5.1*iii* and 5.7*iii*.

Proposition 6.2.

i. *$H^*(ku; \mathbb{Z}/2) \cong A//E_1$, where E_1 is the subalgebra generated by the Milnor primitives $Q_0 = Sq^1$ and $Q_1 = Sq^1 Sq^2 + Sq^2 Sq^1$.*

ii. *$H^*(\ell; \mathbb{Z}/p) \cong A//E_1$, where E_1 is the subalgebra generated by the Milnor primitives $Q_0 = \beta$ and $Q_1 = P^1\beta - \beta P^1$.*

$\psi^3 - 1$ and $\psi^r - 1$ give left A-module homomorphisms in cohomology,

$$(\psi^3 - 1)^* : \Sigma^2 A//E_1 \longrightarrow A//E_1 \text{ for } p = 2.$$
$$(\psi^r - 1)^* : \Sigma^q A//E_1 \longrightarrow A//E_1 \text{ for } p \text{ odd}.$$

These maps are determined by where they send the identity, because $A//E_1$ is generated by 1 as an A-module.

Proposition 6.3.

i. For $p = 2$ the map $\psi^3 - 1$ induces right multiplication with Sq^2.

ii. For p odd the map $\psi^r - 1$ induces right multiplication with P^1.

Proof. Note that $H^2(ju; \mathbb{Z}/2) = \mathbb{Z}/2$ generated by Sq^2, so there are only two possibilities for $(\psi^3 - 1)^*$. We know that $S \longrightarrow ju$ is 2-connected, so $H^*(ju; \mathbb{Z}/2) \longrightarrow H^*(S; \mathbb{Z}/2) = \mathbb{Z}/2$ is an isomorphism for $* \leq 2$. This means that $Sq^2 \notin H^2(ju; \mathbb{Z}/2)$, so $Sq^2 \in H^2(ku; \mathbb{Z}/2)$ is not in $coker(\psi^3 - 1)^*$, and is therefore killed by $(\psi^3 - 1)^*(\Sigma^2 1)$.

In the same way for odd p we know, see e.g. [20], that $S \longrightarrow j$ is $(pq-2)$-connected, so $H^*(j; \mathbb{Z}/p) \longrightarrow H^*(S; \mathbb{Z}/p) = \mathbb{Z}/p$ is an isomorphism for $* \leq pq - 2$. $H^q(\ell; \mathbb{Z}/p) = \mathbb{Z}/p$ generated by P^1, and again we get that because $P^1 \in H^q(j; \mathbb{Z}/p)$ is trivial, $P^1 \in H^q(\ell; \mathbb{Z}/p)$ has to be hit by $\Sigma^q 1$. (To be precise, this proves that $\Sigma^q 1 \mapsto aP^1$, where $a \in \mathbb{Z}/p$ is nonzero, but that is irrelevant.) $\square$

To determine the appropriate kernels and cokernels, let $A_1 \subset A$ be the subalgebra generated by Sq^1 and Sq^2 for $p = 2$ and β and P^1 for odd p. A_1 is a $4p$-dimensional $\mathbb{Z}/p$-algebra. E_1 is 4-dimensional, and we can express $A//E_1$ in terms of $A//A_1$ in the following way.

$$A//E_1 = A \otimes_{A_1} \mathbb{Z}/2\{1, Sq^2\} \text{ for } p = 2.$$
$$A//E_1 = A \otimes_{A_1} \mathbb{Z}/p\{1, P^1, \ldots, P^{p-1}\} \text{ for odd } p.$$

Now we can describe the multiplication with Sq^2 or P^1 in a very simple way, in terms of A_1-modules. For $p = 2$ we get

$$(\psi^3 - 1)^* : A \otimes_{A_1} \mathbb{Z}/2\{\Sigma^2 1, \Sigma^2 Sq^2\} \longrightarrow A \otimes_{A_1} \mathbb{Z}/2\{1, Sq^2\},$$

where $\Sigma^2 1 \mapsto Sq^2$ and $\Sigma^2 Sq^2 \mapsto 0$. For odd p we get

$$(\psi^r - 1)^* : A \otimes_{A_1} \mathbb{Z}/p\{\Sigma^q 1, \Sigma^q P^1, \ldots, \Sigma^q P^{p-1}\} \longrightarrow A \otimes_{A_1} \mathbb{Z}/p\{1, P^1, \ldots, P^{p-1}\},$$

where $\Sigma^q P^i \mapsto P^i P^1 = (-1)^i \binom{p-2}{i} P^{i+1}$ for $0 \leq i \leq p-2$ and $\Sigma^q P^{p-1} \mapsto 0$.

We see that $ker(\psi^3 - 1)^* = A \otimes_{A_1} \mathbb{Z}/2 = A//A_1$ and $coker(\psi^3 - 1)^* = \Sigma^4 A//A_1$ for $p = 2$, and $ker(\psi^r - 1)^* = A//A_1$ and $coker(\psi^r - 1)^* = \Sigma^{pq} A//A_1$ for p odd.

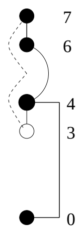

Figure 2: A_1-module extensions of $A//A_1$ and $\Sigma^3\mathbb{Z}/2$.

Proposition 6.4.

i. $0 \longrightarrow A//A_1 \longrightarrow H^*(ju;\mathbb{Z}/2) \longrightarrow \Sigma^3 A//A_1 \longrightarrow 0$ *is a short exact sequence of A-modules. This short exact sequence splits.*

ii. $0 \longrightarrow A//A_1 \longrightarrow H^*(j;\mathbb{Z}/p) \longrightarrow \Sigma^{pq-1}A//A_1 \longrightarrow 0$ *is a short exact sequence of A-modules. This short exact sequence does not split.*

Proof. To show that the first sequence splits, we calculate $Ext_A^{1,0}(\Sigma^3 A//A_1, A//A_1)$. By lemma 2.3 this is the same as $Ext_{A_1}^{1,0}(\Sigma^3\mathbb{Z}/2, A//A_1)$. Now look at any short exact sequence

$$0 \longrightarrow A//A_1 \longrightarrow E \longrightarrow \Sigma^3\mathbb{Z}/2 \longrightarrow 0$$

of A_1-modules. We have that $A//A_1 = \{1, Sq^4, Sq^2Sq^4, Sq^1Sq^2Sq^4, \ldots\}$. Let $b \in E$ be the homogeneous element in degree 3 mapping to the generator of $\Sigma^3\mathbb{Z}/2$. An extension of the above short exact sequence is determined by the action of Sq^1 and Sq^2 on b. Now $Sq^2(b) = 0$ because there are no elements in degree 5, and $Sq^1(b) = 0$ because if $Sq^1(b) \neq 0$ we would get that $Sq^1Sq^2Sq^1(b) \neq 0$, which is impossible, because of the Adém relation $Sq^1Sq^2Sq^1 = Sq^2Sq^2$. Thus the Ext^1-group is zero and the extension splits.

For the other short exact sequence, we get $Ext_A^{1,0}(\Sigma^{pq-1}A//A_1, A//A_1) = Ext_{A_1}^{1,0}(\Sigma^{pq-1}\mathbb{Z}/p, A//A_1)$, so to see how many such extensions there are, we can look at

$$0 \longrightarrow A//A_1 \longrightarrow E \longrightarrow \Sigma^{pq-1}\mathbb{Z}/p \longrightarrow 0.$$

Let $b \in E$ be the homogeneous element in degree $pq-1$ mapping to $1 \in \Sigma^{pq-1}\mathbb{Z}/p$. An extension is determined by how β and P^1 acts on b. $P^1(b) = 0$ for dimensional reasons, but there is no Adém relation that makes $\beta(b)$ trivial. We get that $Ext_A^{1,0}(\Sigma^{pq-1}A//A_1, A//A_1) = \mathbb{Z}/p$, and to see that we actually have a nontrivial extension in this case, we look at the ASS converging to $\pi_*(j)$. Suppose the sequence splits. Then the E^2-term of the ASS would consist of two copies of $Ext_A^{**}(A//A_1, \mathbb{Z}/p)$, one starting in degree 0 and one

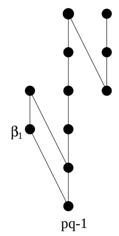

Figure 3: The ASS for j if the sequence splits

in degree $pq-1$, and we would get an infinite a_0-tower in degree $pq-1$. Thus we need to calculate $Ext_A(A//A_1;\mathbb{Z}/p) = Ext_{A_1}(\mathbb{Z}/p,\mathbb{Z}/p)$, at least in a certain range. This can be done by writing down an explicit minimal A_1-resolution of $\mathbb{Z}/p$, which starts as

$$\mathbb{Z}/p \longleftarrow A_1 \longleftarrow A_1\{\beta, P_1\} \longleftarrow$$
$$A_1\{\beta|\beta, P^2|\beta - P^1\beta|P^1 + \frac{p+1}{2}\beta P^1|P^1, P^{p-1}|P^1\} \longleftarrow \dots$$

Thus we see that there is no element in homotopic degree pq and filtration 0, 1 or 2, which is all we need.

The ASS looks the same for j and for S in degrees less than $pq-1$, and from [13, theorem 1.1.14] we know that the only class that is in $\pi_*(S)$ but not in $\pi_*(j)$ for $0 \le * \le pq-2$ is β_1 in degree $pq-2$, which has order p^2 and has to be killed by a d_2-differential, because it is in filtration 2 and 3 in the ASS. We also have that $\pi_{pq-1}(j) = \mathbb{Z}/p^2$, but with an infinite a_0-tower in degree $pq-1$, and possibly an infinite a_0-tower in degree pq starting in filtration 3 or higher, we find that $|\pi_{pq-1}(j)| \ge p^3$ because the classes in filtration t in the a_0-tower survive for $t = 2$, 3 and 4, see figure 3. Because $\pi_{pq-1}(j) = \mathbb{Z}/p^2$, this is a contradiction. $\square$

By duality there are similar short exact sequences in homology. Let $\bar{\bar{A}}_*$ be the dual of $A//A_1$, i.e., $\bar{\bar{A}}_* = P(\bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3|k \ge 3)$ for $p = 2$ and $\bar{\bar{A}}_* = P(\bar{\xi}_1^p, \bar{\xi}_k|k \ge 2) \otimes E(\bar{\tau}_k|k \ge 2)$ for p odd. Then we have the following short exact sequences of A_*-comodules in homology, where the first one is split.

$$0 \longrightarrow \Sigma^3\bar{\bar{A}}_* \longrightarrow H_*(ju;\mathbb{Z}/2) \longrightarrow \bar{\bar{A}}_* \longrightarrow 0.$$
$$0 \longrightarrow \Sigma^{pq-1}\bar{\bar{A}}_* \longrightarrow H_*(j;\mathbb{Z}/p) \longrightarrow \bar{\bar{A}}_* \longrightarrow 0.$$

We also need the algebra structure to get started with the BSS.

Let b be the homogeneous element that generates $H_3(ju;\mathbb{Z}/2)$ for $p=2$ and a homogeneous element that generates $H_{pq-1}(j;\mathbb{Z}/p)$ for p odd.

Lemma 6.5. *For $p = 2$ we have the following.*

i. $b^2 = Q^3(b) = 0$.

ii. $Q^4(b) = 0$.

iii. $Q^{2^k}(\bar{\xi}_k) = \bar{\xi}_{k+1}$ *for* $k \geq 3$.

Proof. Since $|b|$=3, $|b^2| = 6$, and $H_6(ju; \mathbb{Z}/2) = \mathbb{Z}/2$ generated by $\bar{\xi}_2^2$. Because of the above splitting, we know the A_*-comodule structure in $H_*(ju; \mathbb{Z}/2)$. For example we have that $(Sq^2)_* \bar{\xi}_2^2 = \bar{\xi}_1^4$ in $H_*(ju; \mathbb{Z}/2)$, because the same holds in $\bar{\bar{A}}_*$. If $b^2 = \bar{\xi}_2^2$ we would get $(Sq^2)_*(b^2) = \bar{\xi}_1^4 \neq 0$. But this is impossible, because $b^2 = Q^3(b)$ and according to the Nishida relations we have $(Sq^2)_* Q^3 = Q^2 (Sq^1)_*$, and we already know that $(Sq^1)_*(b) = 0$.

To calculate $Q^4(b)$, note that $H_7(ju; \mathbb{Z}/2) = (\mathbb{Z}/2)^2$, generated by $\bar{\xi}_1^4 b$ and $\bar{\xi}_3$. We get that $(Sq^1)_* Q^4(b) = Q^3(b) = 0$ and $(Sq^4)_* Q^4(b) = Q^2 (Sq^2)_*(b) = 0$ from the Nishida relations, so because $(Sq^1)_*$ acts nontrivially on $\bar{\xi}_3$ and $(Sq^4)_*$ acts nontrivially on $\bar{\xi}_1^4 b$, $Q^4(b) = 0$.

The last part follows in the same way as in A_*, using the A_*-comodule structure. To calculate $Q^{2^k} \bar{\xi}_k$ it is enough to calculate $(Sq^{2^m})_* Q^{2^k} \bar{\xi}_k$ for $0 \leq m \leq k$, because the only A_*-comodule primitives in $H_*(ju; \mathbb{Z}/2)$ are 1 and b, so $Q^{2^k} \bar{\xi}_k$ cannot be primitive. Using the Nishida relations and that $Q^n x = 0$ whenever $|x| > n$, we find that $(Sq^1)_* Q^{2^k} \bar{\xi}_k = Q^{2^k - 1} \bar{\xi}_k = \bar{\xi}_k^2$, $(Sq^2)_* Q^{2^k} \bar{\xi}_k = Q^{2^k - 1} (Sq^1)_* \bar{\xi}_k = Q^{2^k - 1}(\bar{\xi}_{k-1}^2) = 0$ because of a Künneth formula and $(Sq^{2^m})_* Q^{2^k} \bar{\xi}_k = Q^{2^k - 2^{m-1}} (Sq^{2^{m-1}})_* \bar{\xi}_k = Q^{2^k - 2^{m-1}}(0) = 0$ for $m \geq 2$, so $Q^{2^k} \bar{\xi}_k = \bar{\xi}_{k+1}$ is the only possibility. □

Proposition 6.6. *As an A_*-comodule algebra,*

$$H_*(ju; \mathbb{Z}/2) \cong \bar{\bar{A}}_* \otimes E(b),$$

where b is A_*-comodule primitive and $|b| = 3$.

Proof. We have the split short exact sequence $\Sigma^3 \bar{\bar{A}}_* \xrightarrow{i} H_*(ju; \mathbb{Z}/2) \xrightarrow{\pi} \bar{\bar{A}}_*$, where π is an algebra map because it is induced by the S-algebra map $ju \longrightarrow \ell$. Let $s : \bar{\bar{A}}_* \longrightarrow H_*(ju; \mathbb{Z}/2)$ be the splitting. Then we can make a map

$$\bar{\bar{A}}_* \otimes E(b) \longrightarrow H_*(ju; \mathbb{Z}/2)$$

by sending $x \otimes 1$ to $s(x)$ and $x \otimes b$ to $bs(x)$. We will show that this is an algebra map and that it is injective, the proposition will then follow, because $\bar{\bar{A}}_* \otimes E(b)$ and $H_*(ju; \mathbb{Z}/p)$ have the same dimension in each degree.

First we check that $s(x)s(y) = s(xy)$. First note that because 1 and b are the only A_*-comodule primitives, we can check whether $x = y$ by comparing the $(-)\otimes 1$ and $(-)\otimes b$ terms in $\nu(x)$ and $\nu(y)$. Because s is an A_*-comodule homomorphism, the diagram

$$\begin{tikzcd} \bar{\bar{A}}_* \arrow[r, "s"] \arrow[d, "\nu"] & H_*(ju;\mathbb{Z}/2) \arrow[d, "\nu"] \\ A_* \otimes \bar{\bar{A}}_* \arrow[r, "1\otimes s"] & A_* \otimes H_*(ju;\mathbb{Z}/2) \end{tikzcd}$$

commutes. This means that $\nu(s(xy)) = (1\otimes s)(\nu(xy)) = xy\otimes 1+\ldots$, without any $(-)\otimes b$-term. Also $\nu(s(x)s(y)) = \nu(s(x))\nu(s(y)) = (x\otimes 1+\ldots)(y\otimes 1+\ldots) = xy\otimes 1+\ldots$, without any $(-)\otimes b$-term by induction on $|xy|$, so $s(x)s(y) = s(xy)$.

Now take two elements $x\otimes b^{\epsilon_1}$ and $y\otimes b^{\epsilon_2}$ in $\bar{\bar{A}}_*\otimes E(b)$, where $\epsilon_i = 0$ or 1 for $i = 1, 2$. We need to show that

$$(b^{\epsilon_1}s(x))(b^{\epsilon_2}s(y)) = b^{\epsilon_1+\epsilon_2}s(xy).$$

To do this we use the coproduct again. Because b is primitive we get

$$\begin{aligned} \nu((b^{\epsilon_1}s(x))(b^{\epsilon_2}s(y))) = (1\otimes b^{\epsilon_1})(x\otimes 1+\ldots)(1\otimes b^{\epsilon_2})(y\otimes 1+\ldots) = \\ (1\otimes b^{\epsilon_1+\epsilon_2})(xy\otimes 1+\ldots) = xy\otimes b^{\epsilon_1+\epsilon_2}+\ldots, \end{aligned}$$

where $xy\otimes b^{\epsilon_1+\epsilon_2}$ is the only $(-)\otimes 1$ or $(-)\otimes b$-term, again by induction on $|xy|$. We also get

$$\nu(b^{\epsilon_1+\epsilon_2}s(xy)) = (1\otimes b^{\epsilon_1+\epsilon_2})(xy\otimes 1+\ldots) = xy\otimes b^{\epsilon_1+\epsilon_2}+\ldots,$$

again with $xy\otimes b^{\epsilon_1+\epsilon_2}$ as the only $(-)\otimes 1$ or $(-)\otimes b$-term, so it follows that $\bar{\bar{A}}_*\otimes E(b) \longrightarrow H_*(ju;\mathbb{Z}/2)$ is multiplicative. To see that it is injective, note that $\nu(b^\epsilon s(x))$ contains an $x\otimes b^\epsilon$, and for $\epsilon = 0, 1$, this is nonzero, so $b^\epsilon s(x)$ is nonzero, and the result follows.

$\square$

For odd primes, b is a generator of $H_{pq-1}(j;\mathbb{Z}/p)$. We automatically get $b^2 = 0$, because $|b|$ is odd, and we get a similar result.

Proposition 6.7. *As an algebra,*

$$H_*(j;\mathbb{Z}/p) \cong \bar{\bar{A}}_*\otimes E(b),$$

where $|b| = pq-1$. The A_*-coalgebra structure is determined by $\nu(\bar{\xi}_1^p\otimes 1) = 1\otimes(\bar{\xi}_1^p\otimes 1)+\bar{\xi}_1^p\otimes(1\otimes 1)+\bar{\tau}_0\otimes(1\otimes b)$ and that b is primitive.

Proof. We cannot do the same as we did for ju at $p=2$, because we have no splitting of $\Sigma^{pq-1}\bar{\bar{A}}_* \longrightarrow H_*(j;\mathbb{Z}/p) \longrightarrow \bar{\bar{A}}_*$. Thus we have to make a map the other way instead, $f : H_*(j;\mathbb{Z}/p) \longrightarrow \bar{\bar{A}}_* \otimes E(b)$. We make the map f in the following way. Consider A_* as a $\mathbb{Z}/p$-vector space with monomials in the conjugate classes as a basis. Then we have a projection map $A_* \longrightarrow \bar{\bar{A}}$ using this basis. We also have an obvious map $H_*(j;\mathbb{Z}/p) \longrightarrow E(b)$, and we define f as the composition

$$H_*(j;\mathbb{Z}/p) \xrightarrow{\nu} A_* \otimes H_*(\mathbb{Z}/p) \xrightarrow{1\otimes\pi} A_* \otimes E(b) \xrightarrow{\pi\otimes 1} \bar{\bar{A}}_* \otimes E(b).$$

We need to show that f is an isomorphism and that it is multiplicative. To see that it is an isomorphism, look at the commutative diagram

$$\begin{tikzcd}
\Sigma^{pq-1}\bar{\bar{A}}_* \arrow[r] \arrow[d, "\nu"] & H_*(j;\mathbb{Z}/p) \arrow[r] \arrow[d, "\nu"] & \bar{\bar{A}}_* \arrow[d, "\nu"] \\
A_* \otimes \Sigma^{pq-1}\bar{\bar{A}}_* \arrow[r] \arrow[d, "1\otimes\pi"] & A_* \otimes H_*(j;\mathbb{Z}/p) \arrow[r] \arrow[d, "1\otimes\pi"] & A_* \otimes \bar{\bar{A}}_* \arrow[d, "1\otimes\pi"] \\
A_* \otimes \Sigma^{pq-1}\mathbb{Z}/p \arrow[r] \arrow[d, "\pi\otimes 1"] & A_* \otimes E(b) \arrow[r] \arrow[d, "\pi\otimes 1"] & A_* \arrow[d, "\pi\otimes 1"] \\
\Sigma^{pq-1}\bar{\bar{A}}_* \arrow[r] & \bar{\bar{A}}_* \otimes E(b) \arrow[r] & \bar{\bar{A}}_*
\end{tikzcd}$$

Here each row is a short exact sequence, not necessarily of A_*-comodules. The compositions along the left and right columns are clearly isomorphisms, and it follows from the 5-lemma that f is an isomorphism.

To show that f is multiplicative, observe that because we know what the A_*-comodule extension of the short exact sequence $\Sigma^{pq-1}\bar{\bar{A}}_* \longrightarrow H_*(j;\mathbb{Z}/p) \longrightarrow \bar{\bar{A}}_*$ looks like, we know that for $x \in H_*(j;\mathbb{Z}/p)$, we have $\nu(x) = x_1 \otimes 1 + \bar{\tau}_0 x_2 \otimes b + x_3 \otimes b + \sum x_i' \otimes x_i''$, where $x_i \in \bar{\bar{A}}_*$ for $i = 1, 2, 3$ and each x_i'' is different from 1 and b. Obviously we have $f(x) = x_1 \otimes 1 + x_3 \otimes b$. Using a similar expression for $y \in H_*(j;\mathbb{Z}/p)$ we need to show that $f(xy) = x_1y_1 \otimes 1 + (x_1y_3 + x_3y_1) \otimes b$, and this follows from calculating $\nu(xy) = \nu(x)\nu(y) = x_1y_1 \otimes 1 + \bar{\tau}_0(x_1y_2 + x_2y_1) \otimes b + (x_1y_3 + x_3y_1) \otimes b + \ldots$. □

Now we are ready to look at the BSS. The E^2-terms are

$$E^2_{**}(ju) = H_*(ju;\mathbb{Z}/2) \otimes E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_k | k \geq 3) \otimes \Gamma(\sigma b)$$
$$E^2_{**}(j) = H_*(j;\mathbb{Z}/p) \otimes E(\sigma\bar{\xi}_1^p, \sigma\bar{\xi}_k | k \geq 2) \otimes \Gamma(\sigma\bar{\tau}_k | k \geq 2) \otimes \Gamma(\sigma b).$$

Let us look at $THH(ju)$ first.

Theorem 6.8. *The BSS converging to $H_*(THH(ju);\mathbb{Z}/2)$ collapses at the E^2-term, and*

$$H_*(THH(ju);\mathbb{Z}/2) = H_*(ju;\mathbb{Z}/2) \otimes E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3) \otimes \Gamma(\sigma b),$$

as an algebra.

Proof. $\Gamma(\sigma b)$ has algebra-generators $\gamma_{2^k}(\sigma b)$ for $k \geq 1$, where $\gamma_{2^k}(\sigma b)$ is in filtration 2^k, or in bidegree $(2^k, 3\cdot 2^k)$. $\gamma_{2^k}(\sigma b)$ for $k \geq 2$ are the only algebra-generators in $E^2_{**}(ju)$ that can support a differential.

Because the E^2-term is flat over $H_*(ju;\mathbb{Z}/2)$, we have a coproduct, and any first differential also has to hit a $H_*(ju;\mathbb{Z}/2)$-coalgebra primitive. The elements in $H_*(ju;\mathbb{Z}/2)$ are treated as constants, and the elements in filtration 1 are primitive by default. Using that $\psi(\gamma_k(\sigma b)) = \sum_{i+j=k} \gamma_i(\sigma b) \otimes \gamma_j(\sigma b)$ we see that all the primitives are in filtration 1. Thus the first differential has to be $d^{2^k-1}(\gamma_{2^k}(\sigma b))$ for some $k \geq 2$.

This differential also hits an A_*-comodule primitive, so if we can show that there are no such elements in bidegree $(1, 2^{k+2}-2)$, the spectral sequence collapses. By using that $\nu(\sigma x) = (1 \otimes \sigma)(\nu(x))$ and that $\gamma_n(\sigma b)$ has to be primitive for dimensional reasons, there are no elements in filtration n and lower degree than $\gamma_n(\sigma b)$, we can determine the comodule structure. Observe that if $x \in H_*(ju;\mathbb{Z}/2) \otimes E(\sigma\bar{\xi}_k | k \geq 4) \otimes \Gamma(\sigma b)$ is not primitive, then xy is not primitive if $xy \neq 0$, and that $E(b) \otimes E(\sigma\bar{\xi}_k | k \geq 4) \otimes \Gamma(\sigma b)$ are all the primitives in $H_*(ju;\mathbb{Z}/2) \otimes E(\sigma\bar{\xi}_k | k \geq 4) \otimes \Gamma(\sigma b)$. By checking the A_*-comodule structure on $E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_3)$, we find that the A_*-comodule primitives in the E^{2^k-1}-term are precisely

$$E(b) \otimes \mathbb{Z}/2\{1, \sigma\bar{\xi}_1^4, \sigma\bar{\xi}_1^4\sigma\bar{\xi}_2^2, \sigma\bar{\xi}_1^4\sigma\bar{\xi}_2^2\sigma\bar{\xi}_3\} \otimes E(\sigma\bar{\xi}_k | k \geq 4) \otimes \Gamma(\sigma b).$$

Thus the only possible targets of the first differential are σb, $b\sigma b$, $\sigma\bar{\xi}_1^4$, $b\sigma\bar{\xi}_1^4$, $\sigma\bar{\xi}_m$ and $b\sigma\bar{\xi}_m$ for $m \geq 4$. The first differential lands in degree $2^{k+2}-1$ for some $k \geq 2$, and these elements are in degree 4, 7, 5, 8, 2^m and 2^m+3, so we can conclude that the spectral sequence collapses.

We can also conclude that $(\sigma\bar{\xi}_k)^2 = \sigma\bar{\xi}_{k+1}$ for $k \geq 3$ by using lemma 6.5. The only thing we need to prove to get the above result as an algebra, is that $\Gamma(\sigma b)$ is still a Γ-algebra in homology, i.e., that $\gamma_{2^k}(\sigma b)^2 = 0$ for all k. To see this we use the map $THH(ju) \longrightarrow THH(ku)$, induced by $ju \longrightarrow ku$. Write γ_n for $\gamma_n(\sigma b)$. Assume, per induction, that $\gamma_{2^i}^2 = 0$ for $i < k$, i.e., $\gamma_n^2 = 0$ for $n < 2^k$, and look at the commutative diagram

$$\begin{tikzcd} H_*(THH(ju);\mathbb{Z}/2) \arrow[r] \arrow[d, "\nu"] & H_*(THH(ku);\mathbb{Z}/2) \arrow[d, "\nu"] \\ A_* \otimes H_*(THH(ju);\mathbb{Z}/2) \arrow[r] & A_* \otimes H_*(THH(ku);\mathbb{Z}/2) \end{tikzcd}$$

Because $\gamma_{2^k} \mapsto 0$ in $H_*(THH(ku);\mathbb{Z}/2)$, $\nu(\gamma_{2^k})$ cannot survive to $A_* \otimes H_*(THH(ku);\mathbb{Z}/2)$. The monomials that are killed have to have a b, $\sigma\bar{\xi}_1^4$ or γ_n factor for some n.

Thus we can write

$$\nu(\gamma_{2^k}) = \sum x_i' \otimes bx_i'' + \sum y_i' \otimes \sigma\bar{\xi}_1^4 y_i'' + \sum z_i' \otimes \gamma_{n_i} z_i'',$$

and we get that $\nu(\gamma_{2^k}^2) = 1 \otimes \gamma_{2^k}^2$ because all the other terms are zero by the induction hypothesis.

Because $\gamma_{2^k}^2$ is primitive, we return to the A_*-comodule primitives in $H_*(THH(ju);\mathbb{Z}/2)$. The A_*-comodule primitives not involving $\Gamma(\sigma b)$ are

$$E(b) \otimes \mathbb{Z}/2\{1, \sigma\bar{\xi}_1^4, \sigma\bar{\xi}_1^4\sigma\bar{\xi}_2^2, \sigma\bar{\xi}_1^4\sigma\bar{\xi}_2^2\sigma\bar{\xi}_3\} \otimes P((\sigma\bar{\xi}_3)^2).$$

Now assume that $\gamma_{2^k}^2 \neq 0$. $|\gamma_{2^k}^2| = 2^{k+3}$, so the only possibility is $(\sigma\bar{\xi}_3)^{2^k}$, but we can assume that $\gamma_{2^k}^2$ does not contain $(\sigma\bar{\xi}_3)^{2^k}$. If it does, we can use $\gamma_{2^k} + (\sigma\bar{\xi}_3)^{2^{k-1}}$ instead, because an element in the BSS is only determined modulo elements in lower filtration. Thus we have $\gamma_{2^k}^2 = \sum x_i$, where $x_i = x_i' x_i''$ and $x_i' \in \Gamma(\sigma b)$.

Let $\psi(\gamma_{2^k}) = \sum_{i+j=2^k} \gamma_i \otimes \gamma_j + \sum y_i \otimes z_i$. Then $\psi(\gamma_{2^k}^2) = \gamma_{2^k}^2 \otimes 1 + 1 \otimes \gamma_{2^k}^2 + \sum y_i^2 \otimes z_i^2$, but we also have $\psi(\gamma_{2^k}^2) = x_1' \otimes x_1'' + \ldots$, with $x_1' \in \Gamma(\sigma b)$, which is impossible. □

Theorem 6.9. *The BSS converging to $H_*(THH(j);\mathbb{Z}/p)$, for p odd, collapses at the E^p-term, and*

$$H_*(THH(j);\mathbb{Z}/p) = H_*(j;\mathbb{Z}/p) \otimes E(\sigma\bar{\xi}_1^p, \sigma\bar{\xi}_2) \otimes P(\sigma\bar{\tau}_2) \otimes \Gamma(\sigma b),$$

as an algebra.

Proof. We want to show that we have no other differentials than the usual ones in this spectral sequence. All the algebra generators are in filtration 0, 1 and p^k for $k \geq 1$, so any first differential has to go from filtration p^k for some k. Furthermore, the E^2-term is flat over $H_*(j;\mathbb{Z}/p)$, so we have a coproduct, and this differential has to hit a coalgebra primitive.

The Γ-algebras (on $\sigma\bar{\tau}_k$ and σb) have no primitive elements in filtration > 1, and all the other generators are in filtration 0 and 1. Thus there can be no primitive elements in filtration > 1.

This means that there are no differentials before d^{p-1}. $|d^{p-1}(\gamma_p(\sigma b))| = 2p^2(p-1) - 1$, and it has to be A_*-comodule primitive. The A_* comodule primitives in filtration 1 are σb, $b\sigma b$, $\sigma\bar{\xi}_k$ and $b\sigma\bar{\xi}_k$, $k \geq 3$, and none of these have the right degree, so the differentials are generated by those we get from the map $THH(j) \longrightarrow THH(\ell)$, induced by $j \longrightarrow \ell$, i.e., all the d^{p-1}-differentials are generated by the d^{p-1}-differentials on $\gamma_q(\sigma\bar{\tau}_k)$ for $q \geq p$. This shows that the E^p term is

$$E^p_{**}(j) = H_*(j; \mathbb{Z}/p) \otimes E(\sigma\bar{\xi}_1^p, \sigma\bar{\xi}_2) \otimes P_p(\sigma\bar{\tau}_k | k \geq 2) \otimes \Gamma(\sigma b).$$

We have the same criteria as before for what the next differential can be, because the flatness hypothesis still holds. Because $\sigma\bar{\xi}_k$ has been killed for $k \geq 3$, the following is a complete list of A_*-comodule primitives not involving $\gamma_k(\sigma b)$ for $k \geq 2$.

$$\big(\mathbb{Z}/p\{1, \sigma b, \sigma b\sigma\bar{\xi}_1^p\} \oplus \mathbb{Z}/p\{\sigma b\sigma\bar{\xi}_1^p\sigma\bar{\xi}_2\} \otimes P_p(\sigma\bar{\tau}_2)\big) \otimes E(b) \otimes P_p(\sigma\bar{\tau}_k | k \geq 3).$$

The only generators in filtration $> p$ are $\gamma_{p^k}(\sigma b)$ for $k \geq 2$, so all we have to check is that $d^{p^k-1}(\gamma_{p^k}(\sigma b))$ is not in the same degree as σb, $b\sigma b$, $\sigma\bar{\tau}_m$ or $b\sigma\bar{\tau}_m$. This is easily verified, so the spectral sequence collapses, and $E^p = E^\infty$.

As usual, the truncated polynomial ring on $\sigma\bar{\tau}_k$ is a polynomial ring on $\sigma\bar{\tau}_2$ in homology, so to get the above result as an algebra, we only need to show that $\Gamma(\sigma b)$ is still a Γ-algebra in homology.

Write γ_n for $\gamma_n(\sigma b)$, and assume per induction that $\gamma_n^p = 0$ for $n < p^k$. Look at the commutative diagram

$$\begin{array}{ccc}
H_*(THH(j); \mathbb{Z}/p) & \longrightarrow & H_*(THH(\ell); \mathbb{Z}/p) \\
\downarrow \nu & & \downarrow \nu \\
A_* \otimes H_*(THH(j); \mathbb{Z}/p) & \longrightarrow & A_* \otimes H_*(THH(\ell); \mathbb{Z}/p)
\end{array}$$

$\nu(\gamma_{p^k})$ cannot survive to $A_* \otimes H_*(THH(\ell); \mathbb{Z}/p)$, and because only monomials with a b, $\sigma\bar{\xi}_1^p$ or γ_n-factor are killed, we can write

$$\nu(\gamma_{p^k}) = \sum x_i' \otimes bx_i'' + \sum y_i' \otimes \sigma\bar{\xi}_1^p y_i'' + \sum z_i' \otimes \gamma_{n_i} z_i'',$$

and we get $\nu(\gamma_{p^k}^p) = 1 \otimes \gamma_{p^k}^p$ by the induction hypothesis, so $\gamma_{p^k}^p$ is primitive.

Our modified list of A_*-comodule primitives not involving γ_n for $n \geq 2$ looks like this.

$$\big(\mathbb{Z}/p\{1, \sigma b, \sigma b\sigma\bar{\xi}_1^p\} \oplus \mathbb{Z}/p\{\sigma b\sigma\bar{\xi}_1^p\sigma\bar{\xi}_2\} \otimes P_p(\sigma\bar{\tau}_2)\big) \otimes E(b) \otimes P((\sigma\bar{\tau}_2)^p).$$

Now assume that $\gamma_{p^k}^p \neq 0$. $|\gamma_{p^k}^p| = p^{k+2}q$, and the only element in this list in that degree is $(\sigma\bar{\tau}_2)^{(p-1)p^k}$. If $\gamma_{p^k}^p = (\sigma\bar{\tau}_2)^{(p-1)p^k} + \ldots$, we can replace γ_{p^k} with $\gamma_{p^k} + (\sigma\bar{\tau}_2)^{(p-1)p^{k-1}}$, so we can assume that $\gamma_{p^k}^p = \sum x_i$, where $x_i = x_i' x_i''$ and $x_i' \in \Gamma(\sigma b)$.

Let $\psi(\gamma_{p^k}) = \sum_{i+j=p^k} \gamma_i \otimes \gamma_j + \sum y_i \otimes z_i$. Then we get $\psi(\gamma_{p^k}^p) = \gamma_{p^k}^p \otimes 1 + 1 \otimes \gamma_{p^k}^p + \sum y_i^p \otimes z_i^p$, but we also have $\psi(\gamma_{p^k}^p) = x_1' \otimes x_1'' + \ldots$, with $x_1 \in \Gamma(\sigma b)$, which is impossible. $\square$

The case j for $p = 2$ remains. Following the same strategy as before, we can try to calculate $H_*(j; \mathbb{Z}/2)$ by using the fiber sequence defining j,

$$j \longrightarrow ko \xrightarrow{\psi^3-1} bspin.$$

See [8] for a calculation of $H^*(j; \mathbb{Z}/2)$.

We have $H^*(ko; \mathbb{Z}/2) = A//A_1$ and $H^*(bspin; \mathbb{Z}/2) = \Sigma^4 A/A\{Sq^1, Sq^5\}$. In a similar way as in proposition 6.3 we find that $(\psi^3 - 1)^*$ induces right multiplication with Sq^4. To determine the kernel and cokernel of this map, we can write $H^*(ko; \mathbb{Z}/2)$ as an extension of 8 copies of $A//A_2$, and $H^*(bspin; \mathbb{Z}/2)$ as an extension of 24 copies of $A//A_2$. We find that $coker(\psi^3 - 1)^* = A//A_2$, while $ker(\psi^3 - 1)^*$ is 17-dimensional as an $A//A_2$-module. More precisely, we find that $ker(\psi^3 - 1) \cong \Sigma^8 A \otimes_{A_2} A_2/I$, where I is the ideal in A_2 generated by Sq^1, Sq^7 and $Sq^{10} + Sq^8 Sq^2$.

Thus we find that $H_*(j; \mathbb{Z}/2)$ sits in a short exact sequence

$$0 \longrightarrow \Sigma^7 K \longrightarrow H_*(j; \mathbb{Z}/2) \longrightarrow (A//A_2)_* \longrightarrow 0,$$

where K is an extension of 17 copies of $(A//A_2)_*$. This short exact sequence is not split, and even if we could write down $HH_*(H_*(j; \mathbb{Z}/2))$, it would be so complicated that we cannot hope to determine the differentials in the BSS converging to $H_*(THH(j); \mathbb{Z}/2)$.

7 Homotopy of $THH(ku)$

Much of the material in this section comes from McClure and Staffeldt's article [15], which treats the case $THH(\ell)$ for odd primes.

If we try to calculate $\pi_*(THH(ku))$ and $\pi_*(THH(\ell))$ using the ASS, there are too many possible differentials for us to handle, at least for now, but if we smash with the $\mathbb{Z}/p$ Moore spectrum, $S^0 \cup_p e^1 = S^0/p$, the homotopy groups are more manageable. We will write $\pi_*(X; \mathbb{Z}/p)$ for $\pi_*(X \wedge S^0/p)$.

What we will do is to show that there is only one infinite v_1-tower in $\pi_*(THH(ku); \mathbb{Z}/2)$, and then show that the differentials in the ASS converging to $\pi_*(THH(ku); \mathbb{Z}/2)$ are uniquely determined by this; there is only one way to kill all the v_1-towers except the first one.

Note that it is normal to talk about v_1^4 when using $\mathbb{Z}/2$-coefficients, because $v_1^4 \in \pi_8(S^0/2)$, but there is no v_1 in $\pi_2(S^0/2)$. However, in this situation we can get away with using the v_1 in $\pi_2(ku)$, because most of the spectra in this chapter will be ku-modules. This is unproblematic, because $v_1^4 \wedge 1$ and $1 \wedge v_1^4$ in $\pi_8(ku \wedge S^0/2)$ are homotopic, which follows because $\pi_*(ku \wedge S^0/2) \cong \mathbb{Z}/2[v_1]$.

Recall that

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots],$$

where $|v_k| = 2(p^k - 1)$. We also have

$$BP_*BP = BP_*[t_1, t_2, \ldots],$$

where $|t_k| = 2(p^k - 1)$, and we have left and right units $BP_* \longrightarrow BP_*BP$, which we will denote by η_L and η_R, respectively. $\eta_L(v_k) = v_k$, while $\eta_R(v_k)$ is more complicated, see [20, Appendix 2].

We need to use Morava K-theory, and the following theorem recapitulates what we need to know.

Theorem 7.1. *For each $n \geq 1$ there is a ring spectrum $K(n)$, the n'th Morava K-theory, with $K(n)_* = \mathbb{Z}/p[v_n, v_n^{-1}]$. The associated homology theory satisfies the strict Künneth formula*

$$K(n)_*(X \wedge Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y).$$

$K(n)$ is the direct limit of the connective ring spectrum $k(n)$ under multiplications with v_n, and $k(n)_* = \mathbb{Z}/p[v_n]$. There is a map $BP \longrightarrow k(n)$, which induces the quotient map $\mathbb{Z}_{(p)}[v_1, v_2, \ldots] \longrightarrow \mathbb{Z}/p[v_n]$. Also, $k(1) \cong ku \wedge S^0/2$ for $p = 2$ and $k(1) \cong \ell \wedge S^0/p$ for odd p.

We automatically have a ring structure on $THH(\ell) \wedge S^0/p$, because S^0/p is a ring spectrum. But $S^0/2$ is not a ring spectrum, so we need to show that we still have a multiplication on $THH(ku) \wedge S^0/2$ to get a multiplication on the ASS.

By a μ-spectrum we will mean a spectrum with a multiplication, not necessarily associative or unital. This is all we need to get a multiplication on the ASS later, as stated in theorem 2.2.

We begin with a proposition similar to [18, Theorem 1.5], though this is slightly easier.

Proposition 7.2. *Let R be a left and right unital μ-spectrum and let $S^k \xrightarrow{f} S^0 \longrightarrow M$ be a cofiber sequence such that $f : \Sigma^k R \wedge M \longrightarrow R \wedge M$ is nullhomotopic. Then there is a multiplication $\mu : (R \wedge M) \wedge (R \wedge M) \longrightarrow R \wedge M$ making $R \wedge M$ into a left and right unital μ-spectrum.*

Proof. Look at the cofiber sequence

$$\Sigma^k R \wedge M \xrightarrow{f} R \wedge M \xrightarrow{1 \wedge 1 \wedge i} R \wedge M \wedge M \xrightarrow{1 \wedge 1 \wedge \pi} \Sigma^{k+1} R \wedge M.$$

There is a splitting of $1 \wedge 1 \wedge i$, because $f : \Sigma^k R \wedge M \longrightarrow R \wedge M$ is nullhomotopic. Thus we get a map $m_1 : R \wedge M \wedge M \longrightarrow R \wedge M$ such that $m_1(1 \wedge 1 \wedge i) = id_{R \wedge M}$.

We need to make sure that also $m_1(1 \wedge i \wedge 1) = id_{R\wedge M}$. From the cofiber sequence $\Sigma^k R \xrightarrow{f} R \xrightarrow{1\wedge i} R \wedge M \longrightarrow \Sigma^{k+1} R$ we get the following long exact sequence:

$$\ldots \longrightarrow [\Sigma^{k+1}R, R \wedge M] \xrightarrow{\pi^*} [R \wedge M, R \wedge M] \xrightarrow{i^*} [R, R \wedge M] \longrightarrow \ldots$$

Let $i_1 = 1 \wedge 1 \wedge i$ and $i_2 = 1 \wedge i \wedge 1 : R \wedge M \longrightarrow R \wedge M \wedge M$. Consider the element $m_1(i_1 - i_2) \in [R \wedge M, R \wedge M]$. It maps to zero in $[R, R \wedge M]$, so there is an element $\phi \in [\Sigma^{k+1}R, R \wedge M]$ that maps to $m_1(i_1 - i_2)$.

The next observation is that under the conditions stated in the proposition, the map $(1 \wedge i)^* : [\Sigma^{k+1}R \wedge M, R \wedge M] \longrightarrow [\Sigma^{k+1}R, R \wedge M]$ is surjective. $(1 \wedge i)^*$ being surjective is equivalent to $f^* : [\Sigma^{k+1}R, R \wedge M] \longrightarrow [\Sigma^{2k+1}R, R \wedge M]$ being the zero map. Using that the Spanier-Whitehead dual of M is equivalent to $\Sigma^{-(k+1)}M$ this is the same as saying that $f^* : [R \wedge M, R] \longrightarrow [\Sigma^k R \wedge M, R]$ is the zero map, which is true because $f : \Sigma^k R \wedge M \longrightarrow R \wedge M$ is nullhomotopic.

Because $(1 \wedge i)^*$ is surjective, there is an element $\psi \in [\Sigma^{k+1}R \wedge M, R \wedge M]$ mapping to ϕ. Thus we have the following diagram:

$$\begin{tikzcd}
R \wedge M \arrow[r, "\pi"] \arrow[d, "i_1 - i_2"] & \Sigma^{k+1} R \arrow[r, "i"] & \Sigma^{k+1} R \wedge M \arrow[ddll, "\psi"] \\
R \wedge M \wedge M \arrow[urr, dashed, "1\wedge 1 \wedge \pi"] \arrow[d, "m_1"] & & \\
R \wedge M & &
\end{tikzcd}$$

Now we can adjust m_1 by setting $m = m_1 - \psi(1 \wedge 1 \wedge \pi)$. This map satisfies $mi_1 = id_{R\wedge M}$ because $(1 \wedge 1 \wedge \pi) i_1 = 0$ and $mi_2 = id_{R \wedge M}$ because $(1 \wedge 1 \wedge \pi) i_2 = i\pi$. Using this map we can make a left and right unital multiplication on $R \wedge M$,

$$(R \wedge M) \wedge (R \wedge M) \xrightarrow{1\wedge\tau\wedge 1} R \wedge R \wedge M \wedge M \xrightarrow{1 \wedge m} R \wedge R \wedge M \xrightarrow{\mu \wedge 1} R \wedge M.$$

$\square$

From [2, Theorem 1.1] we get

Lemma 7.3. $2 : S^0/2 \longrightarrow S^0/2$ *factors as*

$$S^0/2 \xrightarrow{\pi} S^1 \xrightarrow{\eta} S^0 \xrightarrow{i} S^0/2.$$

Proposition 7.4. *$THH(ku) \wedge S^0/2$ is a left and right unital μ-spectrum.*

Proof. Let $T = THH(ku)$. Because $\eta : \Sigma T \longrightarrow T$ factors as

$$\Sigma T \cong S^1 \wedge T \xrightarrow{\eta \wedge 1} ku \wedge T \longrightarrow T,$$

and $\pi_1(ku) = 0$, we can conclude that $2 : T \wedge S^0/2 \longrightarrow T \wedge S^0/2$ is nullhomotopic, and the result follows from proposition 7.2. $\square$

Recall that

$$H_*(THH(ku); \mathbb{Z}/2) = H_*(ku; \mathbb{Z}/2) \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3)$$

as an A_*-comodule algebra, where $H_*(ku; \mathbb{Z}/2) = P(\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_k | k \geq 3)$.

Proposition 7.5. *Let $\bar{A}_* = P(\bar{\xi}_1, \bar{\xi}_2^2, \bar{\xi}_k | k \geq 3)$. Then there is an A_*-comodule isomorphism*

$$H_*(THH(ku) \wedge S^0/2; \mathbb{Z}/2) \cong \bar{A}_* \otimes E(e_3, e_7) \otimes P(e_8),$$

where the e_i are A_*-comodule primitives. This isomorphism preserves the multiplicative structure.

Proof. We will construct this isomorphism in two steps. We know that $H_*(S^0/2; \mathbb{Z}/2) = \mathbb{Z}/2\{1, \bar{\xi}_1\}$, so we get

$$H_*(T \wedge S^0/2; \mathbb{Z}/2) = H_*(T; \mathbb{Z}/2) \otimes \mathbb{Z}/2\{1, \bar{\xi}_1\}.$$

There is an isomorphism $(A//E_1)_* \otimes \mathbb{Z}/2\{1, \bar{\xi}_1\} \cong \bar{A}_*$ of A_*-comodules, so we get an isomorphism

$$H_*(T \wedge S^0/2; \mathbb{Z}/2) \cong \bar{A}_* \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3)$$

of A_*-comodule, and to check that this isomorphism preserves the multiplicative structure, we only need to show that $\mu_*((x \otimes \bar{\xi}_1) \otimes (y \otimes \bar{\xi}_1)) = xy\bar{\xi}_1^2 \otimes 1$, the other cases follow by unitality. To see this, we use the A_*-comodule action. We have that

$$\nu((1 \otimes \bar{\xi}_1) \otimes (1 \otimes \bar{\xi}_1)) = \bar{\xi}_1^2 \otimes (1 \otimes 1) \otimes (1 \otimes 1) + \dots,$$

which gives

$$(1 \otimes \mu_*)(\nu((1 \otimes \bar{\xi}_1) \otimes (1 \otimes \bar{\xi}_1)) = \bar{\xi}_1^2 \otimes 1 \otimes 1 + \dots$$

This means that $\mu_*((1 \otimes \bar{\xi}_1) \otimes (1 \otimes \bar{\xi}_1)) \neq 0$, so $\mu_*((1 \otimes \bar{\xi}_1) \otimes (1 \otimes \bar{\xi}_1)) = \bar{\xi}_1^2 \otimes 1$ is the only possibility.

Next we use that the multiplication on $T \wedge S^0/2$ was made by using the multiplication on T, i.e., that μ is a T-map, so the diagram

$$\begin{tikzcd} T \wedge T \wedge (T \wedge S^0/2) \wedge (T \wedge S^0/2) \arrow[r] \arrow[d] & (T \wedge S^0/2) \wedge (T \wedge S^0/2) \arrow[d] \\ T \wedge (T \wedge S^0/2) \arrow[r] & T \wedge S^0/2 \end{tikzcd}$$

commutes. Seeing what happens in homology when we start with $x \otimes y \otimes (1 \otimes \bar{\xi}_1) \otimes (1 \otimes \bar{\xi}_1)$ in the upper left corner gives us that $\mu_*((x \otimes \bar{\xi}_1) \otimes (y \otimes \bar{\xi}_1)) = xy\bar{\xi}_1^2 \otimes 1$.

Next we have an isomorphism

$$\bar{A}_* \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3) \cong \bar{A}_* \otimes E(e_3, e_7) \otimes P(e_8),$$

which sends $\sigma\bar{\xi}_1^2$ to e_3, $\sigma\bar{\xi}_2^2$ to e_7 and $\sigma\bar{\xi}_3 + \bar{\xi}_1\sigma\bar{\xi}_2^2$ to e_8. Using the A_*-comodule action described in theorem 5.5 we can check that the e_i are primitive. For example, $\nu(e_8) \cong \nu(\sigma\bar{\xi}_3) + \nu(\bar{\xi}_1\sigma\bar{\xi}_2^2) = 1 \otimes \sigma\bar{\xi}_3 + \bar{\xi}_1 \otimes \sigma\bar{\xi}_2^2 + 1 \otimes \bar{\xi}_1\sigma\bar{\xi}_2^2 + \bar{\xi}_1 \otimes \sigma\bar{\xi}_2^2 = 1 \otimes (\sigma\bar{\xi}_3 + \bar{\xi}_1\sigma\bar{\xi}_2^2) \cong 1 \otimes e_8$. □

Note that $\bar{A}_*$ is dual to $A//E(Q_1)$, and according to lemma 2.3 we get

$$Ext_A(A//E(Q_1), \mathbb{Z}/2) = Ext_{E(Q_1)}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2[v_1].$$

Now we can calculate the E^2-term of the ASS converging to $\pi_*(THH(ku); \mathbb{Z}/2)$, and we get

Proposition 7.6. *The E^2-term of the ASS converging to $\pi_*(THH(ku); \mathbb{Z}/2)$ is given by*

$$\hat{E}^2_{**}(THH(ku) \wedge S^0/2) = E(\lambda_1, \lambda_2) \otimes P(\mu) \otimes \mathbb{Z}/2[v_1]$$

for generators λ_1, λ_2 and μ in degree 3, 7 and 8, respectively. The multiplication on this spectral sequence coming from the multiplication on $THH(ku) \wedge S^0/2$ is the one suggested by the notation.

Proof. We have already calculated that the A_*-comodule primitives in $H_*(T \wedge S^0/2; \mathbb{Z}/2)$ are $E(e_3, e_7) \otimes P(e_8)$, and the multiplicative structure follows from the multiplication in homology. □

For odd primes we have a similar result, see [15].

$$\hat{E}^2_{**}(THH(\ell) \wedge S^0/p) = E(\lambda_1, \lambda_2) \otimes P(\mu) \otimes \mathbb{Z}/p[v_1],$$

where λ_1, λ_2 and μ are in degree $2p-1$, $2p^2-1$ and $2p^2$.

To show that we only have one v_1-tower in $\pi_*(THH(ku) \wedge S^0/2)$, we use Morava K-theory.

Proposition 7.7.

i. $K(1)_*ku = \mathbb{Z}/2[v_1, v_1^{-1}][t_1, t_2, \ldots]/(t_k^2 = v_1^{2^k-1}t_k | k \geq 1)$.

ii. $K(1)_*\ell = \mathbb{Z}/p[v_1, v_1^{-1}][t_1, t_2, \ldots]/(t_k^p = v_1^{p^k-1}t_k | k \geq 1)$.

Proof. The odd-primary case is treated in [15]; the $K(1)$-homology of ku follows in the same way.

We know that $BP_*BP = BP_* \otimes_{BP_*} BP_*BP$ from Landweber's exact functor theorem. Using the map $BP \longrightarrow k(1)$ we then get $k(1)_*BP = k(1)_* \otimes_{BP_*} BP_*BP$, and taking the direct limit under multiplications with v_1 gives us

$$K(1)_*BP = K(1)_* \otimes_{BP_*} BP_*BP \cong \mathbb{Z}/p[v_1, v_1^{-1}][t_1, t_2, \ldots].$$

Using the map $BP \longrightarrow ku$, which kills all the v_i for $i \geq 2$ we get that $K(1)_*(ku) = K(1)_*(BP)/(\eta_R(v_2), \eta_R(v_3), \ldots)$. From [20, formula 6.1.13] with $p = 2$ we get $\eta_R(v_{k+1}) \equiv v_1 t_k^2 - v_1^{2^k} t_k \pmod{\eta_R(v_2), \ldots, \eta_R(v_k)}$, so the result follows. □

Now let us look at the BSS with $K(1)$-homology instead of $\mathbb{Z}/p$-homology. We will only treat the $p = 2$ case; the odd case is similar and can be found in [15].

Proposition 7.8. *The Hochschild homology of $K(1)_*(ku)$ over the ground ring $K(1)_* = \mathbb{Z}/2[v_1, v_1^{-1}]$ is just $K(1)_*(ku)$ in Hochschild degree 0 and zero elsewhere.*

Proof. First note that $K(1)_*(ku) \cong K(1)_* \otimes K(1)_0(ku)$, where $K(1)_0(ku) = \mathbb{Z}/2[u_k | k \geq 1]/(u_k^2 = u_k)$ and $u_k = v_1^{-(2^k-1)} t_k$.

Then we can calculate the Hochschild homology by changing rings,

$$HH_*^{K(1)_*}(K(1)_* \otimes K(1)_0(ku)) \cong K(1)_* \otimes HH_*^{\mathbb{Z}/2}(K(1)_0(ku)).$$

The result then follows by calculating $HH_*^{\mathbb{Z}/2}(K(1)_0(ku))$. Note that $K(1)_0(ku)$ is the direct limit of $\mathbb{Z}/2[u_1, u_2, \ldots, u_n]/(u_k^2 = u_k)$, and that $\mathbb{Z}/2[u_1, u_2, \ldots, u_n]/(u_k^2 = u_k) \cong \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \ldots \oplus \mathbb{F}_2$, with 2^n summands. Of course $HH_*(\mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \ldots \oplus \mathbb{F}_2) = \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \ldots \oplus \mathbb{F}_2$ concentrated in degree zero. Using that Hochschild homology commutes with direct limits, we get $HH_*^{\mathbb{Z}/2}(K(1)_0(ku)) = K(1)_0(ku)$ concentrated in degree 0, and thus $HH_*^{K(1)_*}(K(1)_*(ku)) = K(1)_*(ku)$ concentrated in Hochschild degree 0. □

This proposition gives us that the BSS with $E^2_{**}(ku) = HH_*^{K(1)_*}(K(1)_*(ku))$ converging to $K(1)_*(THH(ku))$ collapses, so we get

Corollary 7.9.
$$K(1)_*(ku) \longrightarrow K(1)_*(THH(ku))$$
is an isomorphism.

Look at the commutative diagram

$$\begin{tikzcd}
\pi_*(ku \wedge ku; \mathbb{Z}/2) \arrow[r] \arrow[d] & \pi_*(ku \wedge THH(ku); \mathbb{Z}/2) \arrow[d] \\
\pi_*(ku; \mathbb{Z}/2) \arrow[r] & \pi_*(THH(ku); \mathbb{Z}/2)
\end{tikzcd}$$

where the horizontal maps come from the inclusion $ku \longrightarrow THH(ku)$ and the vertical maps from $ku \wedge ku \longrightarrow ku$ and $ku \wedge THH(ku) \longrightarrow THH(ku)$. This is a diagram of ku-modules.

First note that the vertical maps are surjections, because the compositions $ku \xrightarrow{i \wedge 1} ku \wedge ku \longrightarrow ku$ and $THH(ku) \xrightarrow{i \wedge 1} ku \wedge THH(ku) \longrightarrow THH(ku)$ are the identity. Using that $ku \wedge S^0/2 \cong k(1)$ and taking direct limit with respect to multiplications with v_1 everywhere, we get

$$\begin{tikzcd}
K(1)_*ku \arrow[r] \arrow[d] & K(1)_*THH(ku) \arrow[d] \\
v_1^{-1}\pi_*(ku; \mathbb{Z}/2) \arrow[r] & v_1^{-1}\pi_*(THH(ku); \mathbb{Z}/2)
\end{tikzcd}$$

The upper horizontal map is an isomorphism and the vertical maps are surjections, so it follows that the lower horizontal map is surjective. It is also (split) injective, so we get

Theorem 7.10. *The map*
$$v_1^{-1}\pi_*(ku; \mathbb{Z}/2) \longrightarrow v_1^{-1}\pi_*(THH(ku); \mathbb{Z}/2)$$
is an isomorphism.

$\pi_*(ku; \mathbb{Z}/2) = \mathbb{Z}/2[v_1]$, so $v_1^{-1}\pi_*(ku; \mathbb{Z}/2) = \mathbb{Z}/2[v_1, v_1^{-1}]$, and we can conclude that

Corollary 7.11. *There is only one infinite v_1-tower in $\pi_*(THH(ku); \mathbb{Z}/2)$. This v_1-tower starts in degree 0.*

Let
$$r(n) = \begin{cases} 2^n + 2^{n-2} + \ldots + 2 \text{ for } n \geq 1 \text{ odd} \\ 2^n + 2^{n-2} + \ldots + 4 \text{ for } n \geq 2 \text{ even} \end{cases}$$
and
$$s(n) = \begin{cases} 2^n + 2^{n-2} + \ldots + 2^3 + 3 \text{ for } n \geq 1 \text{ odd} \\ 2^n + 2^{n-2} + \ldots + 2^4 + 7 \text{ for } n \geq 2 \text{ even.} \end{cases}$$

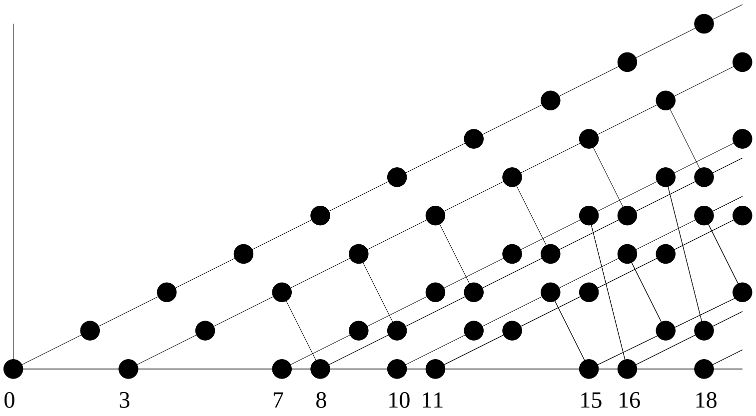

Figure 4: The ASS converging to $\pi_*(THH(ku);\mathbb{Z}/2)$.

Theorem 7.12. *As a $\mathbb{Z}/2[v_1]$-module, $\pi_*(THH(ku);\mathbb{Z}/2)$ is generated by elements 1, $x_{n,m}$ and $x'_{n,m}$, $n \geq 1$, $m \geq 0$. Here $|x_{n,m}| = s(n) + m2^{n+3}$ and $|x'_{n,m}| = s(n) + s(n+1) + m2^{n+3}$. The only relations are $v_1^{r(n)} x_{n,m} = 0$ and $v_1^{r(n)} x'_{n,m} = 0$*

Proof. We use the ASS with $\hat{E}^2_{**}(T \wedge S^0/2) = E(\lambda_1, \lambda_2) \otimes P(\mu) \otimes \mathbb{Z}/2[v_1]$. First note that the v_1-tower starting in μ^m cannot be hit by any differential. To see this, suppose that $d_r(a) = v_1^k \mu^m$ for some a. Because $v_1^k \mu^m$ is in even total degree, we have to have $a = \lambda_1 b$ or $a = \lambda_2 b$ for some b. But because of the multiplication on this spectral sequence we get

$$d_r(\lambda_i b) = d_r(\lambda_i) b + \lambda_i d_r(b) = \lambda_i d_r(b) \neq v_1^k \mu^m,$$

for $i = 1$, 2.

Thus the only way for the v_1-tower starting with μ to disappear is for it to support a differential, so $d_2(v_1^k \mu) = v_1^{k+2} \lambda_1$ for some k, and this means that $d_2(\mu) = v_1^2 \lambda$, again because of the multiplicative structure. We also get $d_2(\lambda_2 \mu) = v_1^2 \lambda_1 \lambda_2$, and $d_2(\mu^{2k+1}) = v_1^2 \lambda_1 \mu^{2k}$ for all k. Let $\lambda_i = \mu^{2^{i-3}} \lambda_{i-2}$ for $i \geq 3$. After some bookkeeping we get

$$\hat{E}^3_{**}(T \wedge S^0/2) = E(\lambda_2, \lambda_3) \otimes P(\mu^2) \otimes \mathbb{Z}/2[v_1] \oplus v_1\text{-torsion},$$

where the v_1-torsion is generated by the elements $x_{1,m}$ and $x'_{1,m}$ over $\mathbb{Z}/2[v_1]$.

Continuing by induction we can show that

$$\hat{E}^{r(n)+1} = \hat{E}^{r(n+1)} = E(\lambda_{n+1}, \lambda_{n+2}) \otimes P(\mu^{2^n}) \otimes \mathbb{Z}/2[v_1] \oplus v_1\text{-torsion},$$

and the result follows by computing the v_1-torsion. $\square$

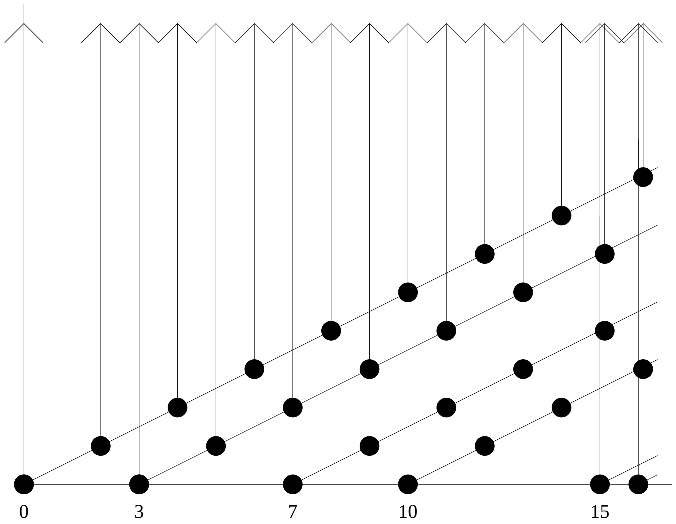

Figure 5: The ASS converging to $\pi_*(THH(ku))$.

In theory, we are only one Bockstein spectral sequence away from $\pi_*(THH(ku))$, but since the $\mathbb{Z}/2$-homotopy is so complicated, getting the "final" answer does not seem feasible. However, it would also be interesting to know the result in low dimensions, and we can actually determine $\pi_*(THH(ku))$ for $* \leq 14$, see figure 5. There is a $\mathbb{Z}$-summand in each degree except degree 1, and $\mathbb{Z}/2$-torsion in degree 7, 9, 10, 11, 12, 13 and 14.

The answer for ℓ at odd primes is very similar, see [15]. We state it here for completeness. If we redefine $r(n)$ and $s(n)$, setting

$$r(n) = \begin{cases} p^n + p^{n-2} + \ldots + p \text{ for } n \geq 1 \text{ odd} \\ p^n + p^{n-2} + \ldots + p^2 \text{ for } n \geq 2 \text{ even} \end{cases}$$

and

$$s(n) = \begin{cases} p^n + p^{n-2} + \ldots + p^3 + 2p - 1 \text{ for } n \geq 1 \text{ odd} \\ p^n + p^{n-2} + \ldots + p^4 + 2p^2 - 1 \text{ for } n \geq 2 \text{ even} \end{cases}$$

we get

Theorem 7.13. *As a $\mathbb{Z}/p$-module, $\pi_*(THH(\ell);\mathbb{Z}/p)$ is generated by elements 1, $x_{n.m}$ and $x'_{n,m}$, $n \geq 1$ and $m \geq 0$, $m \not\equiv p-1 \mod p$, where $|x_{n,m}| = s(n) + mp^{n+2}$ and $|x'_{n,m}| = s(n) + s(n+1) + mp^{n+2}$. The only relations are $v_1^{r(n)} x_{n,m} = 0$ and $v_1^{r(n)} x'_{n,m} = 0$.*

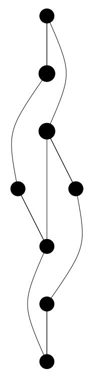

Figure 6: A_1. The lines represent multiplication with Sq^1 and Sq^2.

8 Homotopy of $THH(ko)$

Like $THH(ku)$, $THH(ko)$ is so complicated that we cannot determine the homotopy type completely with these techniques. So we will do the same thing as for $THH(ku)$, we will introduce coefficients from a finite CW-spectrum. A dualization of proposition 5.1 gives us that

$$H^*(ko;\mathbb{Z}/2) = A//A_1,$$

so if we can find a finite CW-spectrum X with $H^*(X;\mathbb{Z}/2) = A_1$, we get $H^*(ko\wedge X;\mathbb{Z}/2) = A//A_1 \otimes A_1 \cong A$, and thus $\pi_*(ko;A_1) = \pi_*(ko\wedge X) = \mathbb{Z}/2$ in degree 0 and zero elsewhere from the ASS.

From Davis and Mahowald's article [9, Proposition 2.1] we get the following.

Proposition 8.1. *There are precisely 4 homotopy classes of finite CW-spectra X_i, $i = 1, 2, 3, 4$, with $H^*(X_i;\mathbb{Z}/2) = A_1$. These are distinguished by the action of Sq^4 on the cells in dimension 0 and 2.*

Proposition 8.2. *We have*

$$H_*(THH(ko)\wedge X_i;\mathbb{Z}/2) \cong A_* \otimes E(e_5, e_7) \otimes P(e_8)$$

as an A_*-comodule for $i = 1, 2, 3$ and 4, where the e_i are A_*-comodule primitive. Thus the A_1-homotopy of $THH(ko)$ is well defined up to isomorphism, and

$$\pi_*(THH(ko);A_1) = E(e_5, e_7) \otimes P(e_8)$$

additively.

Proof. Note that we do not say anything about any multiplicative structure here, $E(e_5, e_7) \otimes P(e_8)$ is just notation. Let $T = THH(ko)$. Using the isomorphism $\bar{\bar{A}}_* \otimes (A_1)_* = (A//A_1)_* \otimes (A_1)_* \cong A_*$ we get an isomorphism

$$H_*(T \wedge X_i; \mathbb{Z}/2) \cong A_* \otimes E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3).$$

of A_*-comodules.

Next we define an isomorphism

$$A_* \otimes E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3) \cong A_* \otimes E(e_5, e_7) \otimes P(e_8)$$

by sending $\sigma\bar{\xi}_1^4$ to e_5, $\sigma\bar{\xi}_2^2 + \bar{\xi}_1^2\sigma\bar{\xi}_1^4$ to e_7 and $\sigma\bar{\xi}_3 + \bar{\xi}_2\sigma\bar{\xi}_1^4 + \bar{\xi}_1\sigma\bar{\xi}_2^2$ to e_8. Using the A_*-comodule structure on $H_*(T; \mathbb{Z}/2)$ described in theorem 5.5, we can check that the e_i are primitive. $\square$

We can also do some more. By using that $ko \wedge C_\eta \cong ku$ we get $ko \wedge Y \cong k(1)$, so it is tempting to look at the $K(1)_*$-homology of ko and $THH(ko)$. Let us first establish that $THH(ko) \wedge Y$ is a μ-spectrum.

Lemma 8.3. *The map* $\eta : \Sigma C_\eta \longrightarrow C_\eta$ *factors as*

$$\Sigma C_\eta \xrightarrow{\pi} S^3 \xrightarrow{\nu} S^0 \xrightarrow{i} C_\eta.$$

Proof. An ASS calculation shows that $\pi_i(C_\eta) = \mathbb{Z}$, 0, $\mathbb{Z}$ and $\mathbb{Z}/4$ for $i = 0$, 1, 2 and 3, and another ASS calculation shows that $[\Sigma^i C_\eta, C_\eta] = \mathbb{Z}$, 0, $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}/2$ for $i = -2, -1$, 0 and 1. We know that $\eta : \Sigma C_\eta \longrightarrow C_\eta$ is essential, because Sq^4 acts nontrivially on its mapping cone $C_\eta \wedge C_\eta$, so it represents the nontrivial element in $[\Sigma C_\eta, C_\eta]$. Look at the diagram

$$\begin{tikzcd}
S^0 \wedge S^1 \arrow[r] & C_\eta \wedge S^1 \arrow[r] \arrow[d, "\eta"] & S^2 \wedge S^1 \\
S^0 \wedge S^0 \arrow[r] & C_\eta \wedge S^0 \arrow[r] & S^2 \wedge S^0
\end{tikzcd}$$

where the horizontal maps are parts of the cofiber sequence defining C_η. We get the following long exact sequence.

$$\ldots \longleftarrow [S^1, C_\eta] \longleftarrow [\Sigma C_\eta, C_\eta] \longleftarrow [S^3, C_\eta] \longleftarrow \ldots$$

And this looks like

$$\ldots \longleftarrow 0 \longleftarrow \mathbb{Z}/2 \longleftarrow \mathbb{Z}/4 \longleftarrow \ldots,$$

so $\eta : \Sigma C_\eta \longrightarrow C_\eta$ factors through the generator in $\pi_3(C_\eta)$.

$$\begin{tikzcd} S^0 \wedge S^1 \arrow[r] & C_\eta \wedge S^1 \arrow[r] \arrow[d, "\eta"] & S^2 \wedge S^1 \arrow[dl] \\ S^0 \wedge S^0 \arrow[r] & C_\eta \wedge S^0 \arrow[r] & S^2 \wedge S^0 \end{tikzcd}$$

Next look at the long exact sequence

$$\ldots \longrightarrow [S^3, S^0] \longrightarrow [S^3, C_\eta] \longrightarrow [S^3, S^2] \longrightarrow [S^3, S^1] \longrightarrow \ldots$$

If we calculate these groups we get

$$\ldots \longrightarrow \mathbb{Z}/8 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow \ldots,$$

where the map $\mathbb{Z}/2 \longrightarrow \mathbb{Z}/2$ is an isomorphism, so $[S^3, S^0] \longrightarrow [S^3, C_\eta]$ is surjective, and so the following diagram commutes, where the map $S^2 \wedge S^1 \longrightarrow S^0 \wedge S^0$ is ν.

$$\begin{tikzcd} S^0 \wedge S^1 \arrow[r] & C_\eta \wedge S^1 \arrow[r] \arrow[d] & S^2 \wedge S^1 \arrow[dll] \\ S^0 \wedge S^0 \arrow[r] & C_\eta \wedge S^0 \arrow[r] & S^2 \wedge S^0 \end{tikzcd}$$

□

Proposition 8.4. *$THH(ko) \wedge Y$ is a left and right unital μ-spectrum.*

Proof. We first make a multiplication on $T \wedge C_\eta$ and then on $(T \wedge C_\eta) \wedge S^0/2 \cong T \wedge Y$ by using proposition 7.2 twice. Because $\nu : \Sigma^3 T \longrightarrow T$ factors as $\Sigma^3 T \cong S^3 \wedge T \xrightarrow{\nu \wedge 1} ko \wedge T \longrightarrow T$ and $\pi_3(ko) = 0$, we find that $\eta : \Sigma T \wedge C_\eta \longrightarrow T \wedge C_\eta$ is nullhomotopic, and $T \wedge C_\eta$ is a left and right unital μ-spectrum by 7.2.

Using that $\eta : \Sigma T \wedge C_\eta \longrightarrow T \wedge C_\eta$ is nullhomotopic once more, we get that $2 : (T \wedge C_\eta) \wedge S^0/2 \longrightarrow (T \wedge C_\eta) \wedge S^0/2$ is nullhomotopic by lemma 7.3, which gives us a left and right unital multiplication on $(T \wedge C_\eta) \wedge S^0/2 \cong T \wedge Y$. □

Proposition 8.5. *$H_*(THH(ko) \wedge Y; \mathbb{Z}/2) \cong \bar{A}_* \otimes E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3)$ as an A_*-comodule, with multiplication as suggested by the notation. The A_*-comodule action on $\sigma\bar{\xi}_1^4$, $\sigma\bar{\xi}_2^2$ and $\sigma\bar{\xi}_3$ is the same as stated in theorem 5.5.*

Figure 7: Part of $H^*(THH(ko);\mathbb{Z}/2)$. Each dot represents $A//A_1$.

Proof. We prove this in two steps. First look at $H_*(T\wedge C_\eta;\mathbb{Z}/2) = H_*(T;\mathbb{Z}/2) \otimes\mathbb{Z}/2\{1,\bar{\xi}_1^2\}$. Using the A_*-comodule isomorphism $\bar{\bar{A}}_*\otimes\mathbb{Z}/2\{1,\bar{\xi}_1^2\} \cong (A//E_1)_*$ we get an A_*-comodule isomorphism

$$H_*(T\wedge C_\eta;\mathbb{Z}/2) \cong (A//E_1)_* \otimes E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3),$$

and we only need to show that this isomorphism is multiplicative, i.e., we need to show that $\mu(1\otimes\bar{\xi}_1^2\otimes 1\otimes\bar{\xi}_1^2) = \bar{\xi}_1^4\otimes 1$. This follows by using the coproduct. In $H_*(T\wedge C_\eta\wedge T\wedge C_\eta;\mathbb{Z}/2)$ we get

$$\nu(1\otimes\bar{\xi}_1^2\otimes 1\otimes\bar{\xi}_1^2) = \bar{\xi}_1^4\otimes 1\otimes 1\otimes 1\otimes 1 + \ldots,$$

and by using $\nu\mu = (1\otimes\mu)\nu$ we get that $\mu(1\otimes\bar{\xi}_1^2\otimes 1\otimes\bar{\xi}_1^2) = \bar{\xi}_1^4\otimes 1$. The rest follows from unitality and that the multiplication on $T\wedge C_\eta$ is a T-map.

Next we look at $H_*(T\wedge Y;\mathbb{Z}/2) = H_*(T\wedge C_\eta;\mathbb{Z}/2)\otimes\mathbb{Z}/2\{1,\bar{\xi}_1\}$ Using that $(A//E_1)_*\otimes\mathbb{Z}/2\{1,\bar{\xi}_1\} \cong \bar{A}_*$ we only need to check that $\mu(1\otimes\bar{\xi}_1\otimes 1\otimes\bar{\xi}_1) = \bar{\xi}_1^2\otimes 1$, which follows in the same way using the coproduct. □

From [23] we get

Proposition 8.6. *Let bo be the* 0*-connected cover of ko, and let bspin be the* 3*-connected cover of ko. Then*

i. $H^*(bo;\mathbb{Z}/2) = \Sigma A//A_1\otimes\mathbb{Z}/2\{1, Sq^1, Sq^2Sq^1\}$, *or dually* $H_*(bo;\mathbb{Z}/2) = \Sigma\bar{\bar{A}}_*\otimes\mathbb{Z}/2\{1,\bar{\xi}_1,\bar{\xi}_2\}$.

ii. $H^*(bspin;\mathbb{Z}/2) = \Sigma^4 A//A_1\otimes\mathbb{Z}/2\{1, Sq^2, Sq^3\}$, *or dually* $H_*(bspin;\mathbb{Z}/2) = \Sigma^4\bar{\bar{A}}_*\otimes\mathbb{Z}/2\{1,\bar{\xi}_1^2,\bar{\xi}_2\}$.

Using the A_*-comodule structure in $H_*(THH(ko);\mathbb{Z}/2)$ found in theorem 5.5 we see that $H_*(THH(ko);\mathbb{Z}/2)$ has copies of $H_*(ko;\mathbb{Z}/2)$ in degree 0 and 20, a copy of $\Sigma^{-1}H_*(bo;\mathbb{Z}/2)$ in degree 12 and a copy of $\Sigma^{-4}H_*(bspin;\mathbb{Z}/2)$ in degree 5, see figure 7. Also, $(\sigma\bar{\xi}_3)^2$ is primitive, so the A_*-comodule structure is 16-periodic.

Proposition 8.7. *The* E^2*-term of the ASS converging to* $\pi_*(THH(ko);Y)$ *is given by*

$$\hat{E}^2_{**}(THH(ko)\wedge Y) = \big(E(\lambda_1,\lambda_2)\otimes\mathbb{Z}/2[v_1]\oplus\mathbb{Z}/2\{f_1,f_2\}\big)\otimes P(\mu)$$

where $|\lambda_1| = 7$, $|\lambda_2| = 13$, $|f_1| = 5$, $|f_2| = 12$ *and* $|\mu| = 16$. *The multiplication on this ASS, which comes from the multiplication on* $THH(ko)\wedge Y$, *is the one suggested by the notation. In addition,* $f_1^2 = f_2^2 = f_1 f_2 = 0$, *and* $\lambda_1 f_1 = f_2$.

Proof. The formula for the E^2-term follows directly from the A_*-comodule structure found in proposition 8.5. We can check that f_1 is represented by $\sigma\bar{\xi}_1^4$, λ_1 by $\bar{\xi}_1^2\sigma\bar{\xi}_1^4 + \sigma\bar{\xi}_2^2$, f_2 by $\sigma\bar{\xi}_1^4\sigma\bar{\xi}_2^2$ and λ_2 is represented by $\bar{\xi}_1\sigma\bar{\xi}_1^4\sigma\bar{\xi}_2^2 + \sigma\bar{\xi}_1^4\sigma\bar{\xi}_3$. Also, μ is represented by $(\sigma\bar{\xi}_3)^2$. Using the multiplication in homology found in proposition 8.5, we can check that the multiplication on the ASS is the right one. □

Next we want to show that we have only one infinite v_1-tower in $\pi_*(THH(ko);Y)$, and we show this in a similar way as for $THH(ku)\wedge S^0/2$ with Morava K-theory.

Proposition 8.8. *The* $K(1)_*$*-homology of* ko *is given by*

$$K(1)_*(ko) = \mathbb{Z}/2[v_1, v_1^{-1}][t_2, t_3, \ldots]/(t_k^2 = v_1^{2^k-1}t_k | k \geq 2).$$

Proof. Let us start by calculating $K(1)_*(ku)$ in a different way, using the ASS instead of BP_*BP. For any spectrum R we get

$$Ext_A^{**}(H^*(k(1)\wedge R;\mathbb{Z}/2),\mathbb{Z}/2) = Ext_{E(Q_1)}^{**}(H^*(R;\mathbb{Z}/2),\mathbb{Z}/2),$$

so it boils down to a question of how Q_1 acts on $H^*(R;\mathbb{Z}/2)$, or equivalently, when $\nu : H_*(R;\mathbb{Z}/2) \longrightarrow A_*\otimes H_*(R;\mathbb{Z}/2)$ involves $\bar{\xi}_2$. Whenever Q_1 acts trivially we get an infinite v_1-tower, if Q_1 acts nontrivially we get a $\mathbb{Z}/2$ in filtration zero.

By looking closely at the A_*-comodule structure on $H_*(ku;\mathbb{Z}/2)$ we get

$$\hat{E}^2_{**}(k(1)\wedge ku) = Ext_{E(Q_1)}^{**}(H^*(ku;\mathbb{Z}/2),\mathbb{Z}/2) = \mathbb{Z}/2[v_1]\otimes E(\bar{\xi}_k^2|k\geq 1)\oplus\text{torsion}.$$

Everything except the torsion in filtration 0 is in even total degree, so there can be no differentials. By taking direct limit with multiplications with v_1 we get the same result as in proposition 7.7, at least additively, and we see that $\bar{\xi}_k^2$ corresponds to t_k. We do not see that $t_k^2 \neq 0$ in the E^∞-term of this ASS, so we could not have used this method, at least not in its present form, to prove proposition 7.7.

Now we can use the same method for ko, we can calculate $Ext_{E(Q_1)}^{**}(H^*(ko;\mathbb{Z}/2),\mathbb{Z}/2)$ by looking at the A_*-comodule structure. We get almost the same answer as for ku, except that $\bar{\xi}_1^2 \notin H_*(ko;\mathbb{Z}/2)$, so we get

$$\hat{E}^2_{**}(k(1)\wedge ko) = \hat{E}^\infty_{**}(k(1)\wedge ko) = \mathbb{Z}/2[v_1]\otimes E(\bar{\xi}_k^2|k\geq 2)\oplus\text{torsion}.$$

This means that after inverting v_1, we get the same result as for ku, except that we do not have t_1.

□

Now it follows in the same way as for $THH(ku)$ that the BSS with $K(1)_*$-homology is trivial in positive degrees, so that

$$K(1)_*(ko) \longrightarrow K(1)_*(THH(ko))$$

is an isomorphism.

Look at the diagram

$$\begin{tikzcd}
\pi_*(ko \wedge ko; Y) \arrow[r] \arrow[d] & \pi_*(ko \wedge THH(ko); Y) \arrow[d] \\
\pi_*(ko; Y) \arrow[r] & \pi_*(THH(ko); Y)
\end{tikzcd}$$

The vertical maps are surjections, and we note that $\pi_*(ko \wedge R; Y) \cong k(1)_*(R)$ for $R = ko$ and $R = THH(ko)$. The next thing we want to do is to take the direct limit with multiplications with v_1 everywhere, but there is no obvious way to define multiplication with v_1 here, because $v_1 \notin \pi_2(ko)$. However, $v_1^4 \in \pi_8(ko)$, and the diagram is a diagram of ko-modules. Of course there is no difference between taking the direct limit with multiplications with v_1 or with v_1^4, so we get a new commutative diagram.

$$\begin{tikzcd}
K(1)_*(ko) \arrow[r] \arrow[d] & K(1)_*(THH(ko)) \arrow[d] \\
v_1^{-1}\pi_*(ko; Y) \arrow[r] & v_1^{-1}\pi_*(THH(ko); Y)
\end{tikzcd}$$

The upper horizontal map is an isomorphism and the vertical maps are surjections, so the lower horizontal map is surjective. It is also injective, because it is induced by $ko \longrightarrow THH(ko) \longrightarrow ko$, so we have proved the following theorem.

Theorem 8.9. *The map*

$$v_1^{-1}\pi_*(ko; Y) \longrightarrow v_1^{-1}\pi_*(THH(ko); Y)$$

is an isomorphism.

Let

$$r(n) = \begin{cases} 2^{n+1} + 2^{n-1} + \ldots + 2^4 + 4 \text{ for } n \geq 1 \text{ odd} \\ 2^{n+1} + 2^{n-1} + \ldots + 2^3 + 1 \text{ for } n \geq 0 \text{ even} \end{cases}$$

and

$$s(n) = \begin{cases} 2^{n+1} + 2^{n-1} + \ldots + 2^4 + 7 \text{ for } n \geq 1 \text{ odd} \\ 2^{n+1} + 2^{n-1} + \ldots + 2^3 + 5 \text{ for } n \geq 0 \text{ even.} \end{cases}$$

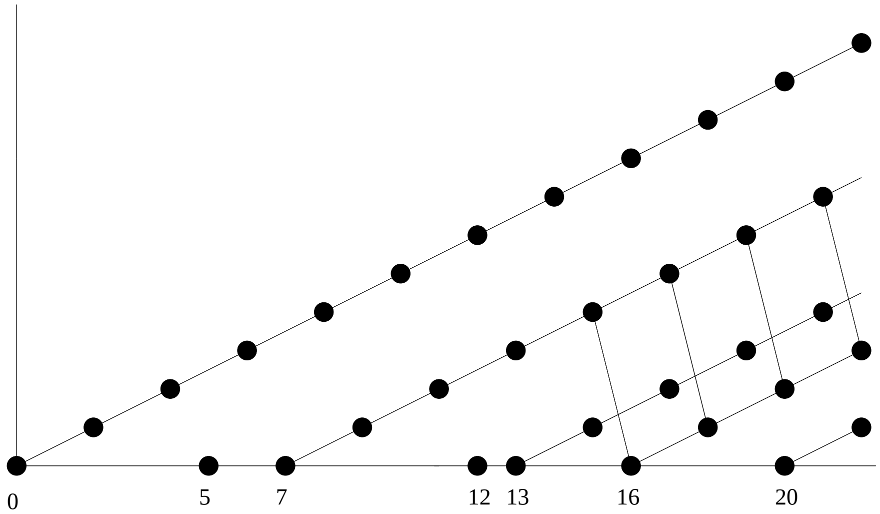

Figure 8: The ASS converging to $\pi_*(THH(ko);Y)$.

Theorem 8.10. *As a $\mathbb{Z}/2[v_1]$-module, $\pi_*(THH(ko);Y)$ is generated by elements 1, $x_{n,m}$ and $x'_{n,m}$ for n, $m \geq 0$, where $|x_{n,m}| = s(n) + m2^{n+4}$ and $|x'_{n,m}| = s(n) + s(n+1) + m2^{n+4}$. The only relations are $v_1^{r(n)} x_{n,m} = 0$ and $v_1^{r(n)} x'_{n,m} = 0$.*

Proof. The proof goes in exactly the same way as for theorem 7.12. The first differential has to be $d_4(\mu) = v_1^4 \lambda_1$, and we can prove by induction that

$$\hat{E}^{r(n)+1} = \hat{E}^{r(n+1)} = E(\lambda_{n+1}, \lambda_{n+2}) \otimes P(\mu^{2^n}) \otimes \mathbb{Z}/2[v_1] \oplus \text{torsion}$$

for $n \geq 1$, where $\lambda_i = \mu^{2^{i-3}} \lambda_{i-2}$ for $i \geq 3$. $\square$

As for $THH(ku)$, we can determine $\pi_*(THH(ko))$ for small $*$ by just looking at the ASS. For $* \leq 10$ we get a $\mathbb{Z}$ summand in degree 0, 4, 5, 8 and 9, a $\mathbb{Z}/2$ summand in degree 1, 2 and 9, and $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ in degree 10, see figure 9.

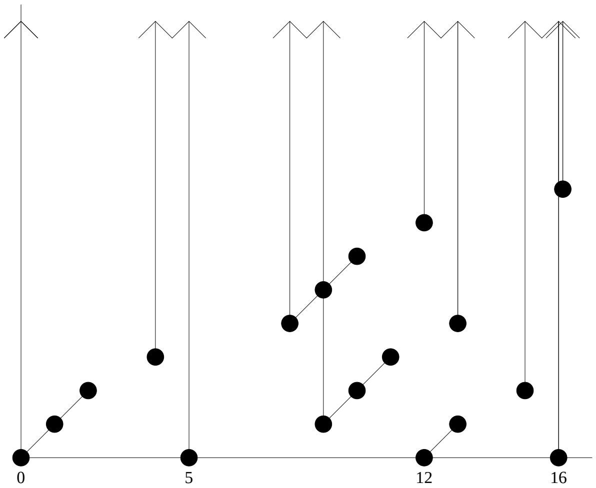

Figure 9: The ASS converging to $\pi_*(THH(ko))$.

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